Tipsy Cop and Tipsy Robber on Vertex and Edge Symmetric Graphs

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Presented by Giancarlo Arcese

Research Description, Objective

- This research studies a variation of the well-known "cops and robbers" game in graph theory.
- The main difference lies in that the cop and robber occasionally move at random (hence the "tipsy"). This random movement is governed by the parameters p and q.
- The goal is to calculate the **expected number of movements** until the game ends as a function of p and q.
- Today I will present our solutions for three graphs/families of graphs: the Petersen graph, cycle graphs, and hypercube graphs.

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4. **Zombie Move Rule:** With probability q, the zombie moves to an adjacent vertex that minimizes its shortest-path distance from the survivor. If multiple vertices minimize the distance, one is chosen uniformly at random. Otherwise, (with probability $1 - q$), the zombie moves to an adjacent vertex uniformly at random.

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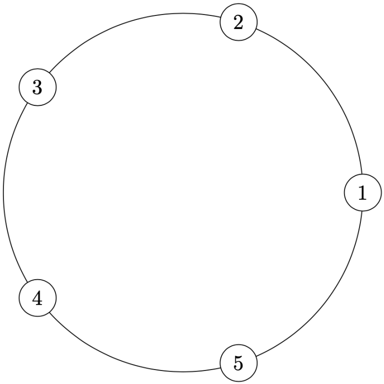
5. The game ends when one agent moves onto the same vertex as the other agent. If that occurs on the n^{th} move, the game is said to have lasted n moves.

Game Example

Let's see an example of the game being played on the 5-cycle graph:



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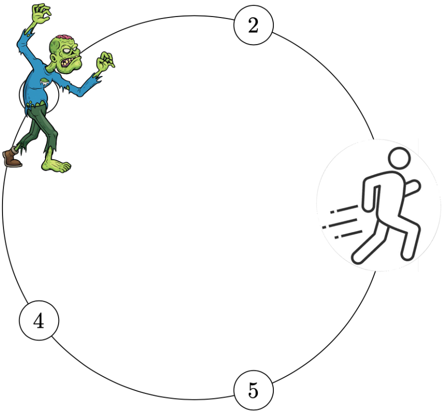
Let $p = q = 0.5$

Move	Sroll	Zroll	Spos	Zpos
0	-	-	1	3



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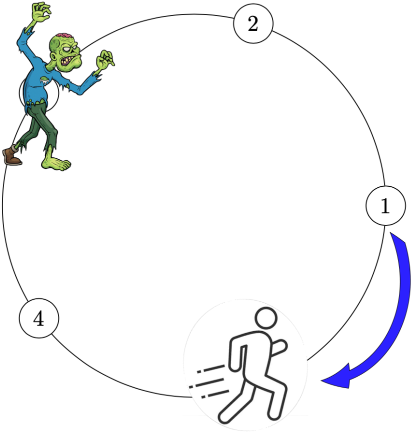
Move	Sroll	Zroll	Spos	Zpos
0	-	-	1	3
1	0.92	-	5	3



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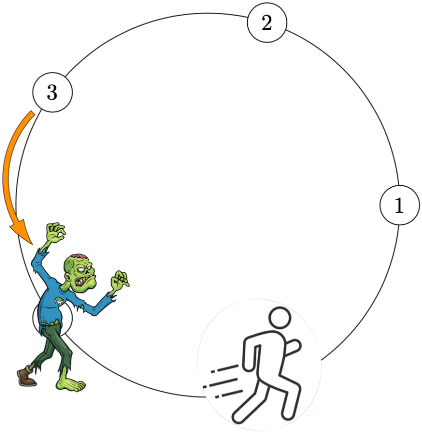
Let $p = q = 0.5$

Move	Sroll	Zroll	Spos	Zpos
0	-	-	1	3
1	0.92	-	5	3
2	-	0.32	5	4



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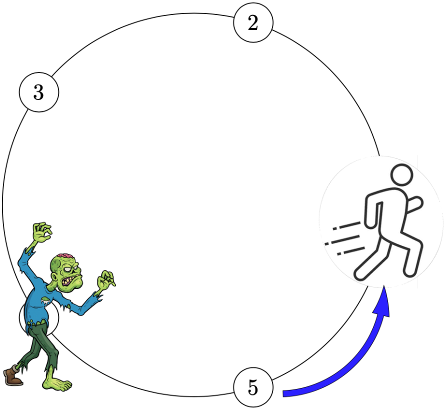
Move	Sroll	Zroll	Spos	Zpos
0	-	-	1	3
1	0.92	-	5	3
2	-	0.32	5	4
3	0.57	-	1	4



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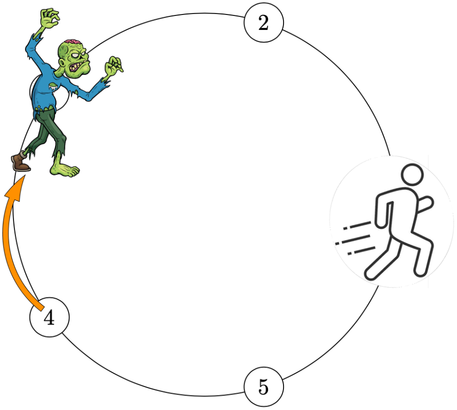

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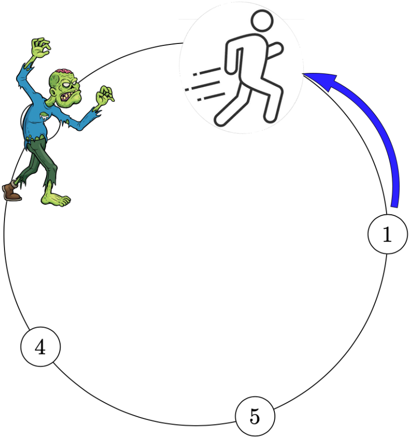
3	0.57	-	1	4
4	-	0.37	1	3
5	0.11	-	2	3



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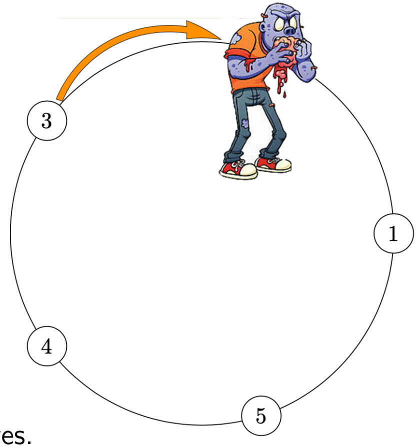
1	0.92	-	5	3
2	-	0.32	5	4
3	0.57	-	1	4
4	-	0.37	1	3
5	0.11	-	2	3
6	-	0.79	2	2



In this example, the game lasted 6 moves.

Solving Strategy

- The general strategy used to solve for expected length involves encoding the game as a **Markov chain** where each state of the transition matrix is a possible distance between the agents.
- The state denoting a distance of 0 corresponds to the game ending. This is the chain's sole **absorbing state**, and is denoted "End."
- Expected length is calculated by way of the row sums of the associated **fundamental matrix**.
- We therefore have the relatively straightforward plan of:

Construct transition matrix $\rightarrow$ **Calculate fundamental matrix**

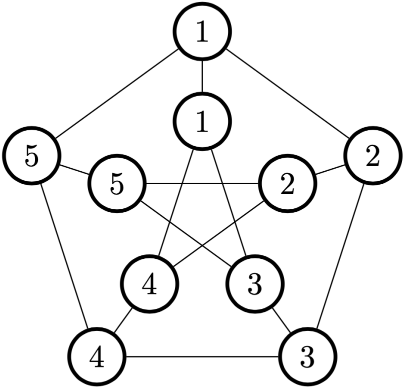
for the three graphs/families of graphs of interest.

Graph Choice Rationale

- Not all graphs can be modeled with a distance-stated Markov chain; only those that are **vertex and edge symmetric** are able to.
- Other graphs can still be modeled as a Markov chain by using the **positioning of the agents**, but this creates large, irregularly-patterned transition matrices.
- This makes solving for the fundamental matrix difficult, since that requires inverting a large, irregularly-patterned matrix.
- For that reason, this research chooses to focus on graphs like the Petersen graph, cycle-graphs, and the hypercube graphs, since they are all symmetric.

Section Preface

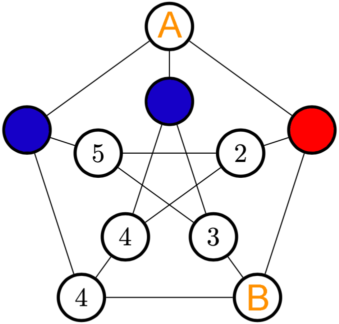
The Petersen Graph



Deriving Transition Matrix

Two useful observations about the Petersen:

1. The max distance between vertices is 2.
2. Relative to a vertex B, a vertex A borders **2** vertices that are the max distance from B, and **1** that is closer to B.



From this we formulate $P_s(C)$, $P_z(C)$, the probability the agents have moved closer after the survivor or zombie's move, respectively:

$$P_s(C) = \frac{1-p}{3} \qquad P_z(C) = \frac{2q+1}{3}$$

Their converses will be denoted as $P_s(F)$ and $P_z(F)$.

Transition Matrix

Using the last slide's results, we obtain the following transition matrix T:

$$T = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccc} \text{End} & S_1 & Z_1 & S_2 & Z_2 & \\ \end{array}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ P_s(C) & 0 & 0 & 0 & P_s(F) \\ P_z(C) & 0 & 0 & P_z(F) & 0 \\ 0 & 0 & P_s(C) & 0 & P_s(F) \\ 0 & P_z(C) & 0 & P_z(F) & 0 \end{pmatrix} \begin{array}{l} \text{End} \\ S_1 \\ Z_1 \\ S_2 \\ Z_2 \end{array}$$

Where the entry (i, j) entry denotes the probability of transitioning to state j having started in state i.

From here, the fundamental matrix N is easily calculated as $(I - Q)^{-1}$, where Q is the 4×4 submatrix formed by the transient states of T.

Inverting (I-Q)

Let $a = P_z(C)$, $b = P_s(C)$, $c = P_z(F)$, $d = P_s(F)$. Then $(I - Q)$ and its determinant D are the following:

$$(I-Q) = \begin{array}{c c} \begin{array}{cccc} S_1 & Z_1 & S_2 & Z_2 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & -d \\ 0 & 1 & -c & 0 \\ 0 & -b & 1 & -d \\ -a & 0 & -c & 1 \end{pmatrix} & \begin{array}{c} S_1 \\ Z_1 \\ S_2 \\ Z_2 \end{array} \end{array} \qquad D = (1-ad)(1-bc) - ab$$

Multiplying $\frac{1}{D}$ by the adjugate matrix of $(I - Q)$ yields the fundamental matrix N:

$$N = \frac{1}{D} \times \begin{array}{c c} \begin{array}{cccc} S_1 & Z_1 & S_2 & Z_2 \end{array} & \\ \begin{pmatrix} 1-ab-cb & abd & ad & d(1-cb) \\ abc & 1-ab-ad & c(1-ad) & bc \\ ab & b(1-ad) & 1-ad & b \\ a(1-bc) & ab & a & 1-bc \end{pmatrix} & \begin{array}{c} S_1 \\ Z_1 \\ S_2 \\ Z_2 \end{array} \end{array}$$

Solution

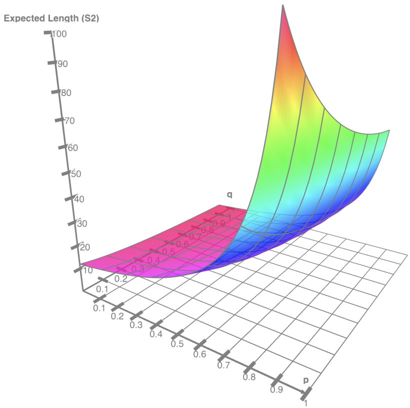
The row sums of N give the expected length of the game for the different possible starting conditions. For example, the row sum of the S_2 row yields the expected length when the agents start at a distance of two and the survivor moves first. In terms of p and q, we get:

$$\mathbf{E}(S_2) = \frac{3(2p^2q + p^2 + 2pq + p - 4q + 52)}{(p-1)(2q+1)(2pq - 2p + 4q - 13)}$$

Evaluated for $p = q = 0.5$ yields:

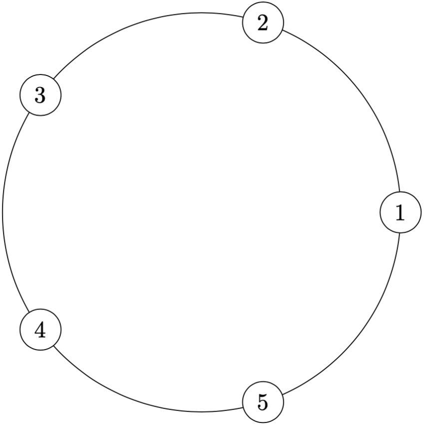
$$\mathbf{E}(S_2) = 13.43478$$

Graph


Expected Length (S2)
100
90
80
70
60
50
40
30
20
10
q
p
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

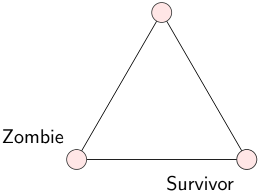
Section Preface

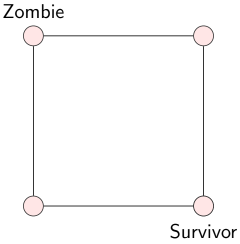
Cycle Graphs



Deriving Transition Matrix

- The max distance m is $\frac{n}{2}$ for n even and $\frac{n-1}{2}$ for n odd.
- Given a pair of vertices A and B at a distance d:
    - **If $\mathbf{d \neq m}$:** A borders **1** vertex that is closer to B and **1** that is further
    - **If $\mathbf{d = m}$ and n is even:** A borders **2** vertices that are closer to B.
    - **If $\mathbf{d = m}$ and n is odd:** A borders **1** vertex closer to B and one at the same distance d.





- This means our transition probabilities $(P_s(C), P_z(C), P_s(F), P_z(F))$ are equivalent across all distances except the max distance m.

Transition Matrix Precursor

- Unlike the Petersen graph, the transition matrices for n-cycles will be on a two-move scale instead of a one-move scale. This helps reduce the number of states and simplify the matrix structure.
- Furthermore, the structure of the transition matrices depends on n. There are three different cases:
  1. $n = 4k,\ k \in \mathbb{N}_1$
  2. $n = 4k + 2,\ k \in \mathbb{N}_1$
  3. n odd
- Thus, we will need to investigate all three cases in order to solve for all n-cycles.
- Fortunately, all cases can be solved in an analogous fashion. This presentation will only cover case 1.

Case 1: $n = 4k$

Transition matrix and $(I - Q)$ for the 12-cycle:

$$T = \begin{array}{cc} \begin{array}{cccc} \text{End} & S_2 & S_4 & S_6 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -B & 1+A+B & -A & 0 \\ 0 & -B & 1+A+B & -A \\ 0 & 0 & -C & 1+C \end{pmatrix} & \begin{array}{c} \text{End} \\ S_2 \\ S_4 \\ S_6 \end{array} \end{array}$$

$$(I - Q) = \begin{array}{cc} \begin{array}{ccc} S_2 & S_4 & S_6 \end{array} & \\ \begin{pmatrix} -(A+B) & A & 0 \\ B & -(A+B) & A \\ 0 & C & -C \end{pmatrix} & \begin{array}{c} S_2 \\ S_4 \\ S_6 \end{array} \end{array}$$

Where:

$$A = -P_s(F)P_z(F) \qquad B = -P_s(C)P_z(C) \qquad C = -P_z(C)$$

Computing the Inverse - Determinant Recursion (1)

Define $\mathcal{M}_k$ the family of $(I - Q)$ matrices when $n = 4k$, $k \in \mathbb{N}_1$.

$$\mathcal{M}_k = \begin{pmatrix} -(A+B) & A & 0 & \dots & 0 & 0 & 0 \\ B & -(A+B) & A & \dots & 0 & 0 & 0 \\ 0 & B & -(A+B) & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -(A+B) & A & 0 \\ 0 & 0 & 0 & \dots & B & -(A+B) & A \\ 0 & 0 & 0 & \dots & 0 & C & -C \end{pmatrix}$$

The determinant of $\mathcal{M}_k$ is given recursively as:

$$\det(\mathcal{M}_k) = -(A+B)\det(\mathcal{M}_{k-1}) - AB\det(\mathcal{M}_{k-2}) \quad k \geq 2$$

We define $\det(\mathcal{M}_0) = \dfrac{C}{B}$ so that $\det(\mathcal{M}_2) = BC$ agrees with the formula.

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The determinant of $\mathcal{M}_k$ is given recursively as:

$$\det(\mathcal{M}_k) = \underbrace{-(A+B)\det(\mathcal{M}_{k-1})}_{\text{Pivotting on (1,1)}} - (AB)\det(\mathcal{M}_{k-2}) \quad k \geq 2$$

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Computing the Inverse - Determinant Recursion (2)

We obtain the following characteristic equation from this recursion:

$$X^2 + (A+B)X + AB = 0$$
$$\rightarrow (X+A)(X+B) = 0$$

Therefore, using Binet's formula, we have that:

$$\det(\mathcal{M}_k) = \alpha(-A)^k + \beta(-B)^k$$

Plugging in $k = 0{,}1$ to solve for α, β:

$$k=0: \ \frac{C}{B} = \alpha + \beta \qquad\qquad k=1: \ -C = -\alpha A - \beta B$$

$$\rightarrow \beta = \frac{C}{B}, \ \alpha = 0$$

Plugging in α, β gives a closed form expression for $\det(\mathcal{M}_k)$

$$\det(\mathcal{M}_k) = \frac{C}{B}(-B)^k$$

Computing the Inverse - Matrix Minors

Next we need to compute the minors of $\mathcal{M}_k$. We'll investigate the minors of $\mathcal{M}_7$ and use its example to help generalize to $\mathcal{M}_k$.

$$\mathcal{M}_7 = \begin{pmatrix} -(A+B) & A & 0 & 0 & 0 & 0 & 0 \\ B & -(A+B) & A & 0 & 0 & 0 & 0 \\ 0 & B & -(A+B) & A & 0 & 0 & 0 \\ 0 & 0 & B & -(A+B) & A & 0 & 0 \\ 0 & 0 & 0 & B & -(A+B) & A & 0 \\ 0 & 0 & 0 & 0 & B & -(A+B) & A \\ 0 & 0 & 0 & 0 & 0 & C & -C \end{pmatrix}$$

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The minor of a diagonal entry yields a block diagonal matrix. The bottom right block is simply a matrix of $\mathcal{M}$. The top left block is a unaberrated version of $\mathcal{M}$. We will define this unaberrated family of matrices as $\mathcal{U}$.

Like $\mathcal{M}_k$, $\mathcal{U}_k$, the determinant of $\mathcal{U}_k$ is easily derived by way of recursion.

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The minor of a non-diagonal entry yields a block triangular matrix with three blocks: one of $\mathcal{M}$, $\mathcal{U}$, and a middle block which is itself triangular.

Whether the 'middle block' contains B or A along the diagonal depends on whether the minor being computed is above or below the main diagonal.

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$\det(\mathcal{U})$ Recursion, Adjugate Formulas

Then $\det(\mathcal{U}_k)$ is given recursively as:

$$\det(\mathcal{U}_k) = -(A+B)\det(\mathcal{M}_{k-1}) - AB\det(\mathcal{U}_{k-2}) \quad k \geq 2$$

Solving in an analogous fashion to $\det(\mathcal{M}_k)$, we get:

$$\det(\mathcal{U}_k) = (-1)^k \frac{A^{k+1} - B^{k+1}}{A - B}.$$

With $\det(\mathcal{U}_k)$ established, we can formulate a closed form expression for adjugate$(M_k)_{ij}$. For the sake of notational succinctness, the boldface $\boldsymbol{\mathcal{M}}$, $\boldsymbol{\mathcal{U}}$ will denote the determinants of $\mathcal{M}$, $\mathcal{U}$.

$$\text{adjugate}(\mathcal{M}_k)_{ij} = \begin{cases} \boldsymbol{\mathcal{U}}_{i-1}\boldsymbol{\mathcal{M}}_{k-j} & \text{for } i = j \\ (-1)^{i+j}A^{j-i}\boldsymbol{\mathcal{U}}_{i-1}\boldsymbol{\mathcal{M}}_{k-j} & \text{for } i < j \\ (-1)^{i+j}B^{i-j}\boldsymbol{\mathcal{U}}_{j-1}\boldsymbol{\mathcal{M}}_{k-i} & \text{for } i > j \end{cases}$$

Adjugate Matrix

The fundamental matrix for the 20-cycle, N_{20}, is given as the following:

$$N_{20} = \frac{1}{\mathcal{M}_5} \times \begin{pmatrix} \mathcal{M}_4 & -A\mathcal{M}_3 & A^2\mathcal{M}_2 & -A^3\mathcal{M}_1 & A^4 \\ -B\mathcal{M}_3 & \mathcal{U}_1\mathcal{M}_3 & -A\mathcal{U}_1\mathcal{M}_2 & A^2\mathcal{U}_1\mathcal{M}_1 & -A^3\mathcal{U}_1 \\ B^2\mathcal{M}_2 & -B\mathcal{U}_1\mathcal{M}_2 & \mathcal{U}_2\mathcal{M}_2 & -A\mathcal{U}_2\mathcal{M}_1 & A^2\mathcal{U}_2 \\ -B^3\mathcal{M}_1 & B^2\mathcal{U}_1\mathcal{M}_1 & -B\mathcal{U}_2\mathcal{M}_1 & \mathcal{U}_3\mathcal{M}_1 & -A\mathcal{U}_3 \\ B^4 & -B^3\mathcal{U}_1 & B^2\mathcal{U}_2 & -B\mathcal{U}_3 & \mathcal{U}_4 \end{pmatrix}$$

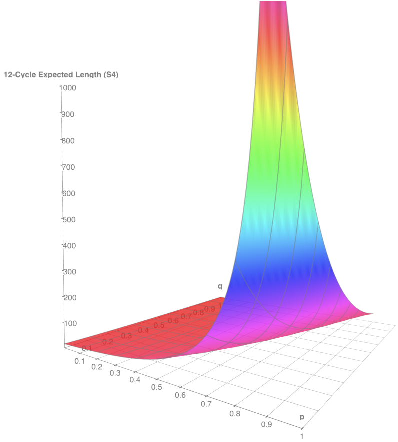
The third row sum of N_{20} is given by:

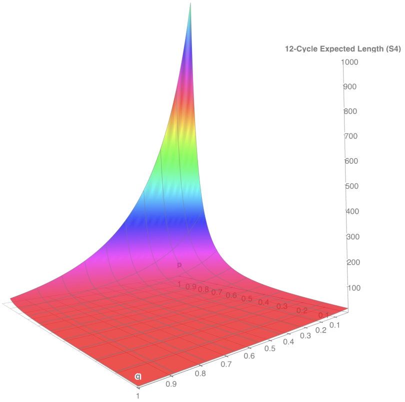
$$(B^2\mathcal{M}_2 - B\mathcal{U}_1\mathcal{M}_2 + \mathcal{U}_2\mathcal{M}_2 - A\mathcal{U}_2\mathcal{M}_1 + A^2\mathcal{U}_2)$$

Generalizing to the i^{th} row sum of N_{4k}:

$$\left[\left(\sum_{x=1}^{i}(-1)^{i+x}B^{i-x}\mathcal{U}_{x-1}\mathcal{M}_{k-i} + \sum_{y=0}^{k-i}A^y\mathcal{U}_{i-1}\mathcal{M}_{k-i-y}\right) - \mathcal{U}_{i-1}\mathcal{M}_{k-i}\right]$$

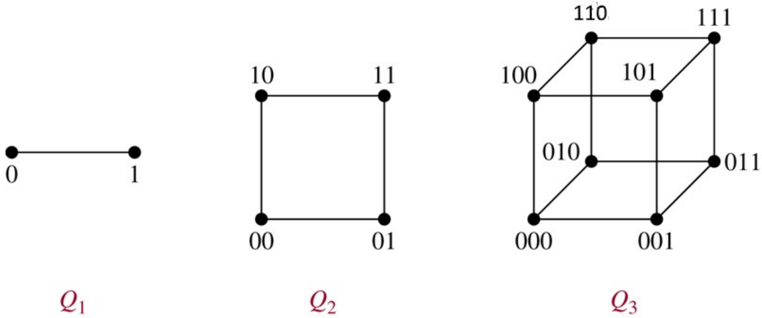
Graph - 12-cycle





Section Preface

Hypercube Graphs



Deriving Transition Matrix

As we saw on the section preface, we can represent the vertices of an n-dimension hypercube as length n binary words, allowing for initial 0's.

With this, the following properties of an n-dimension hypercube follow:

- Each vertex is adjacent to n vertices.
- The distance between two vertices equals the number of distinct positions in their corresponding binary words.
- The maximum distance between two vertices is n.

$$\underbrace{\{1111\} \qquad \{0000\}}_{\text{Max distance}} \qquad\qquad \underbrace{\{1010\} \qquad \{0011\}}_{\text{Distance} = 2}$$

Therefore we see that our transition probabilities will always depend on the current distance between the agents. A superscript will denote the current distance (e.g. $P_s^3(C)$).

(I-Q)

The transition matrices for the hypercube graphs depend on two cases:

1. n even (analogous to n-cycle case 1)
2. n odd (analogous to n-cycle case 2)

We will focus on the first of these cases. Define $\mathcal{Q}_k$ the family of $(I - Q)$ matrices when $n = 2k, k \in \mathbb{N}_1$. The general $\mathcal{Q}_k$ is given below:

$$\mathcal{Q}_k = \begin{array}{c c} \begin{array}{ccccc} S_2 & S_4 & \dots & S_{2k-2} & S_{2k} \end{array} & \\ \begin{pmatrix} -(B_2+A_2) & A_2 & \dots & 0 & 0 \\ B_4 & -(B_4+A_4) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -(B_{2k-2}+A_{2k-2}) & A_{2k-2} \\ 0 & 0 & \dots & C_{2k} & -C_{2k} \end{pmatrix} & \begin{array}{c} S_2 \\ S_4 \\ \vdots \\ S_{2k-2} \\ S_{2k} \end{array} \end{array}$$

Where: $A_i = -P_s^i(F)P_z^{i+1}(F)$ $B_i = -P_s^i(C)P_z^{i-1}(C)$ $C_{2k} = -P_z^{2k-1}(C)$

Determinant Recursion

Below is $\mathcal{Q}_k$. While is is possible to construct submatrices in the same fashion as the *n*-cycle, the submatrices do not yield previous iterations of $\mathcal{Q}_k$. **The red submatrix is not $\mathcal{Q}_{k-1}$, since the entries depend on k.**

$$
\begin{pmatrix}
-(A_2+B_2) & A_2 & 0 & \dots & 0 & 0 & 0 \\
B_4 & -(A_4+B_4) & A_4 & \dots & 0 & 0 & 0 \\
0 & B_6 & -(A_6+B_6) & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & -(A_{2k-4}+B_{2k-4}) & A_{2k-4} & 0 \\
0 & 0 & 0 & \dots & B_{2k-2} & -(A_{2k-2}+B_{2k-2}) & A_{2k-2} \\
0 & 0 & 0 & \dots & 0 & C_{2k} & -C_{2k}
\end{pmatrix}
$$

This inhibits us from using Binet's formula to calculate $\det(\mathcal{Q}_k)$. Fortunately, we are able to find a closed form solution using a different technique.

New Determinant Formula (1)

Let's calculate the first few determinants of $\mathcal{Q}_k$:

$$\begin{aligned}
\det(\mathcal{Q}_1) &= -C_2 \\
\det(\mathcal{Q}_2) &= (C_4)(A_2 + B_2) - (A_2)(C_4) = C_4 B_2 \\
\det(\mathcal{Q}_3) &= -(A_6 + B_6)(B_4 C_6) - A_6(B_4 C_6) = -C_6 B_2 B_4
\end{aligned}$$

One can prove this cancellation between the (1,1) and (1,2) pivot always occurs, yielding the following formula for $\det(\mathcal{Q}_k)$:

$$\det(\mathcal{Q}_k) = (-1)^k C_{2k} B_2 B_4 \dots B_{2k-2}$$

Or:

$$\det(\mathcal{Q}_k) = (-1)^k C_{2k} \prod_{x=1}^{k-1} B_{2x}$$

New Determinant Formula (2)

Furthermore, one can show that:

$$B_i = \frac{[(k-1)q + i]\,(1-p)(i-1)}{k^2}$$
$$C_{2k} = \frac{-(1-p)(2k-1)}{2k}$$

Putting this together, we get that:

$$\det(\mathcal{Q}_k) = \frac{(-1)^{k+1}(2k-1)(1-p)^k \prod_{x=1}^{k-1}[(k-2x)q + 2x]\,(2x-1)}{2k^{2k-1}}$$

Matrix of Minors

With a formula for $\det(\mathcal{Q}_k)$, let's now try to calculate our matrix minors.

$$
\begin{pmatrix}
-(A_2+B_2) & A_2 & 0 & 0 & 0 & 0 & 0 \\
B_4 & -(A_4+B_4) & A_4 & 0 & 0 & 0 & 0 \\
0 & B_6 & -(A_6+B_6) & A_6 & 0 & 0 & 0 \\
0 & 0 & B_8 & -(A_8+B_8) & A_8 & 0 & 0 \\
0 & 0 & 0 & B_{10} & -(A_{10}+B_{10}) & A_{10} & 0 \\
0 & 0 & 0 & 0 & B_{12} & -(A_{12}+B_{12}) & A_{12} \\
0 & 0 & 0 & 0 & 0 & C_{14} & -C_{14}
\end{pmatrix}
$$

The 3-block pattern we saw for the n-cycle is present here in the hypercube as well. The middle block is still triangular. At this time, a formula for the upper, unaberrated block $\mathcal{U}$ is still unknown.

Adjugate Formulas

We can now construct our formula for the adjugate when $i > j$. $i < j$ follows analogously. Let's first remind ourselves of our $\det(\mathcal{Q}_k)$ formula:

$$\det(\mathcal{Q}_k) = \frac{(-1)^{k+1}(2k-1)(1-p)^k \prod_{x=1}^{k-1}[(k-2x)q + 2x]\,(2x-1)}{2k^{2k-1}}$$

The n-cycle formula may be helpful as well:

$$\text{adjugate}(M_k)_{ij} = (-1)^{i+j} B^{i-j} \boldsymbol{\mathcal{U}}_{j-1} \boldsymbol{\mathcal{M}}_{k-i} \qquad \text{for } i > j$$

Let $\text{U} = \det(\mathcal{U}_{j-1})$. We get that $\text{adjugate}(Q_k)_{ij}$ is equivalent to the following expression:

$$\frac{1}{U} \cdot \frac{(-1)^{k-i+1}(2k-1)(1-p)^{k-i} \prod_{x=1}^{k-i-1}[(k-2i-2x)q + 2i + 2x]\,(2i+2x-1)}{2k^{2k-2i-1}}$$

Final Thoughts

- We provided solutions for the expected length on the Petersen Graph, n-cycles, and hypercube graphs.
- We are still working on making our hypercube solution completely closed-form.
- We are still working on interpreting the graphs we produce.
- Note that not all symmetric graphs can be solved in such a way. A Toroidal graph, for example, is symmetric, but produces a much different transition matrix.

Thank you for watching! Questions?