Unit 2: Probability and distributions

1. Probability and conditional probability

GOVT 3990 - Spring 2018

Cornell University

Outline

1. Main ideas
   1. Disjoint and independent do not mean the same thing
   2. Application of the addition rule depends on disjointness of events
   3. Bayes' theorem works for all types of events
   4. Probability trees are useful for conditional probability calculations
   5. Bayesian inference: start with a prior, collect data, calculate posterior, make a decision or iterate
   6. Posterior probability and p-value do not mean the same thing
2. Summary

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1. Disjoint and independent do not mean the same thing

- *Disjoint (mutually exclusive) events* cannot happen at the same time
  - A voter cannot register as a Democrat and a Republican at the same time
  - But they might be a Republican and a Moderate at the same time – *non-disjoint events*
  - For disjoint A and B: $P(A\ and\ B) = 0$

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  - But they might be a Republican and a Moderate at the same time – *non-disjoint events*
  - For disjoint A and B: $P(A \ and \ B) = 0$
- If A and B are *independent events*, having information on A does not tell us anything about B (and vice versa)
  - If A and B are independent:
    - $P(A \mid B) = P(A)$
    - $P(A \ and \ B) = P(A) \times P(B)$

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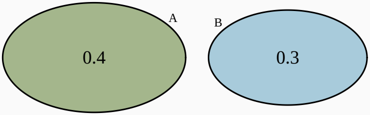

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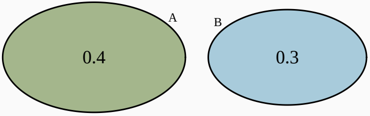
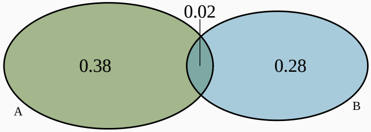


non-disjoint events:

P(A or B)

= P(A) + P(B) - P(A and B)

= 0.4 + 0.3 - 0.02 = 0.68



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Application exercise: 2.1 Probability and conditional probability

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1. Probability trees are useful for conditional probability calculations

- Probability trees are useful for organizing information in conditional probability calculations
- They're especially useful in cases where you know P(A | B), along with some other information, and you're asked for P(B | A)

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- You can iterate this process.

Dice game

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- Ultimate goal: come to a class consensus about whether the die on the left or the die on the right is the "good die"
- We will start with priors, collect data, and calculate posteriors, and make a decision or iterate until we're ready to make a decision

Prior probabilities

- At each roll I tell you whether you won or not (win = ≥ 4)
  - P(win | 6-sided die) = 0.5 $\rightarrow$ bad die
  - P(win | 12-sided die) = 0.75 $\rightarrow$ good die

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- Since initially you have no idea which is true, you can assign equal *prior probabilities* to the hypotheses

  P(H_1 is true) = 0.5

  P(H_2 is true) = 0.5

Rules of the game

- You won't know which die I'm holding in which hand, left (L) or right (R). left = YOUR left
- You pick die (L or R), I roll it, and I tell you if you win or not, where winning is getting a number ≥ 4. If you win, you get a piece of candy. If you lose, I get to keep the candy.
- We'll play this multiple times with different contestants.
- I will not swap the sides the dice are on at any point.
- You get to pick how long you want play, but there are costs associated with playing longer.

	Truth	
Decision	L good, R bad	L bad, R good
Pick L	*You get candy!*	*You lose all the candy :(*
Pick R	*You lose all the candy :(*	*You get candy!*

Sampling isn't free!

At each trial you risk losing pieces of candy if you lose (the die comes up < 4). Too many trials means you won't have much candy left. And if we spend too much class time and we may not get through all the material.

	Choice (L or R)	Result (win or loss)
Roll 1		
Roll 2		
Roll 3		
Roll 4		
Roll 5		
Roll 6		
Roll 7		
...		

What is your decision? How did you make this decision?

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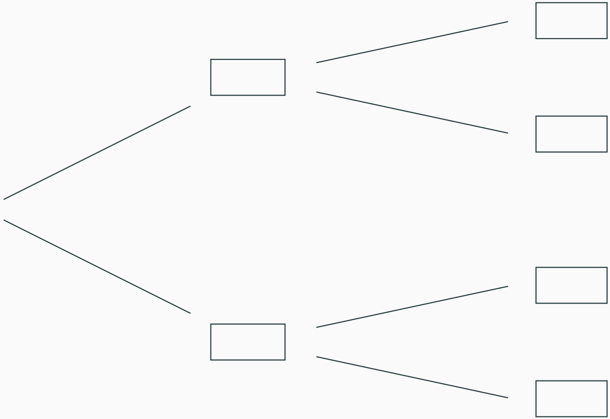
\begin{aligned}
P(hypothesis \mid data) &= \frac{P(hypothesis\ and\ data)}{P(data)} \\
&= \frac{P(data \mid hypothesis) \times P(hypothesis)}{P(data)}
\end{aligned}
$$

Calculate the posterior probability for the hypothesis chosen in the first roll, and discuss how this might influence your decision for the next roll.



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- You are the hiring manager at a company that drug tests their employees. You have recently decided to hire a new employee. What is the prior probability that this employee is a user of this drug? (You may assume that this prospective employee is a randomly drawn person from the population.)

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  - *P(drug user) = 0.05*

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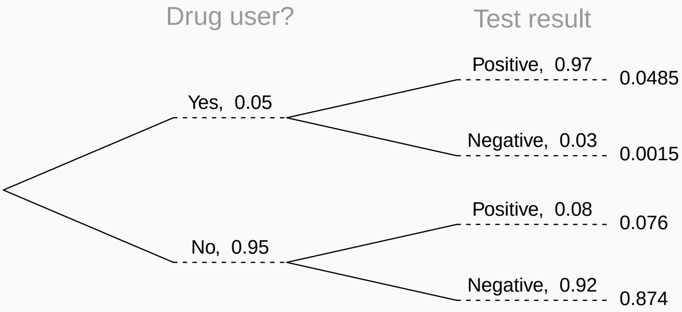

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- The employee tests positive again in the second test. Should the new probability of them actually being a user of this drug be higher or lower than what you calculated before, or the same? Answer this question before you actually complete the calculations.
  - *Higher.*

- Finally, calculate the new updated probability that this employee is a user of this drug. When answering this question sketch a probability tree, take a picture, and upload to Sakai as an attachment.
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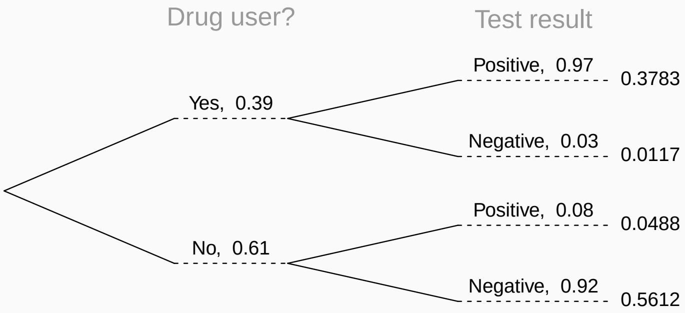

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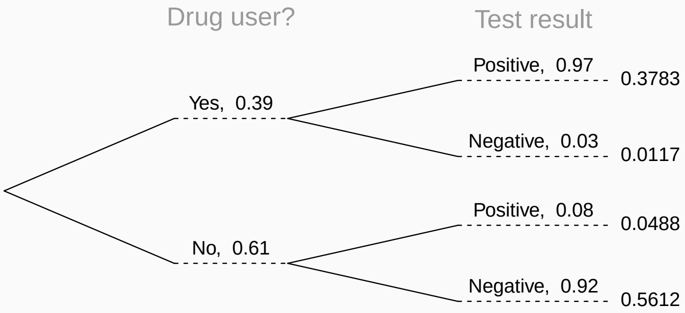

- Based on these results, would you hire this employee? Why or why not?
  - *No, quite likely that the employee is a drug user.*

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3. Posterior probability and p-value do not mean the same thing

- *p-value* : P(observed or more extreme outcome | null hypothesis is true)
  - This is more like P(data | hyp) than P(hyp | data).
- *posterior* : P(hypothesis | data)
- Bayesian approach avoids the counter-intuitive Frequentist p-value for decision making, and more advanced Bayesian techniques offer flexibility not present in Frequentist models
- *Watch out!*
  - *Bayes*: A good prior helps, a bad prior hurts, but the prior matters less the more data you have.
  - *p-value*: It is really easy to mess up p-values: Goodman, 2008

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Summary of main ideas

1. Probability trees are useful for conditional probability calculations
2. Bayesian inference: start with a prior, collect data, calculate posterior, make a decision or iterate
3. Posterior probability and p-value do not mean the same thing