

# Unit 7: Multiple linear regression

## 1. Introduction to multiple linear regression

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GOVT 3990 - Spring 2017

Cornell University

## 1. Housekeeping

## 2. Main ideas

1. In MLR everything is conditional on all other variables in the model
2. Categorical predictors and slopes for (almost) each level
3. Inference for MLR: model as a whole + individual slopes
4. Adjusted  $R^2$  applies a penalty for additional variables
5. Avoid collinearity in MLR
6. Model selection criterion depends on goal: significance vs. prediction
7. Conditions for MLR are (almost) the same as conditions for SLR

## 3. Summary

- ▶ Project questions?

# Outline

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## 3. Summary

## (1) In MLR everything is conditional on all other variables in the model

- ▶ All estimates in a MLR for a given variable are conditional on all other variables being in the model.
- ▶ **Slope:**
  - Numerical  $x$ : *All else held constant*, for one unit increase in  $x_i$ ,  $y$  is expected to be higher / lower on average by  $b_i$  units.
  - Categorical  $x$ : *All else held constant*, the predicted difference in  $y$  for the baseline and given levels of  $x_i$  is  $b_i$ .

A random sample of 783 observations from the 2012 ACS.

1. **income**: Yearly income (wages and salaries)
2. **employment**: Employment status, not in labor force, unemployed, or employed
3. **hrs\_work**: Weekly hours worked
4. **race**: Race, White, Black, Asian, or other
5. **age**: Age
6. **gender**: gender, male or female
7. **citizens**: Whether respondent is a US citizen or not
8. **time\_to\_work**: Travel time to work
9. **lang**: Language spoken at home, English or other
10. **married**: Whether respondent is married or not
11. **edu**: Education level, hs or lower, college, or grad
12. **disability**: Whether respondent is disabled or not
13. **birth\_qrtr**: Quarter in which respondent is born, jan thru mar, apr thru jun, jul thru sep, or oct thru dec

## Activity: MLR interpretations

1. Interpret the intercept.
2. Interpret the slope for hrs\_work.
3. Interpret the slope for gender.

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	-15342.76	11716.57	-1.31	0.19
hrs_work	1048.96	149.25	7.03	0.00
raceblack	-7998.99	6191.83	-1.29	0.20
raceasian	29909.80	9154.92	3.27	0.00
raceother	-6756.32	7240.08	-0.93	0.35
age	565.07	133.77	4.22	0.00
genderfemale	-17135.05	3705.35	-4.62	0.00
citizenyes	-12907.34	8231.66	-1.57	0.12
time_to_work	90.04	79.83	1.13	0.26
langother	-10510.44	5447.45	-1.93	0.05
marriedyes	5409.24	3900.76	1.39	0.17
educollege	15993.85	4098.99	3.90	0.00
edugrad	59658.52	5660.26	10.54	0.00
disabilityyes	-14142.79	6639.40	-2.13	0.03
birth_qrtrapr thru jun	-2043.42	4978.12	-0.41	0.68
birth_qrtrjul thru sep	3036.02	4853.19	0.63	0.53
birth_qrthroct thru dec	2674.11	5038.45	0.53	0.60



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## 3. Summary

## (2) Categorical predictors and slopes for (almost) each level

- ▶ Each categorical variable, with  $k$  levels, added to the model results in  $k - 1$  parameters being estimated.
- ▶ It only takes  $k - 1$  columns to code a categorical variable with  $k$  levels as 0/1s.

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Citizen: yes / no ( $k = 2$ )

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Citizen: yes / no ( $k = 2$ )

Baseline: no

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Citizen: yes / no ( $k = 2$ )

Baseline: no

Respondent		citizen:yes
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Baseline: no

Respondent	citizen:yes
1, Citizen	1

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Citizen: yes / no ( $k = 2$ )

Baseline: no

Respondent	citizen:yes
1, Citizen	1
2, Not-citizen	0

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Citizen: yes / no ( $k = 2$ )

Baseline: no

Race: ( $k = 4$ )

Respondent	citizen:yes
1, Citizen	1
2, Not-citizen	0



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Citizen: yes / no ( $k = 2$ )  
Baseline: no

Race: ( $k = 4$ )  
Baseline: White

Respondent	citizen:yes
1, Citizen	1
2, Not-citizen	0

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Citizen: yes / no ( $k = 2$ )  
Baseline: no

Race: ( $k = 4$ )  
Baseline: White

Respondent	citizen:yes
1, Citizen	1
2, Not-citizen	0

Respondent	race:black	race:asian	race:other
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Citizen: yes / no ( $k = 2$ )

Baseline: no

Respondent	citizen:yes
1, Citizen	1
2, Not-citizen	0

Race: ( $k = 4$ )

Baseline: White

Respondent	race:black	race:asian	race:other
1, White	0	0	0

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Citizen: yes / no ( $k = 2$ )

Baseline: no

Respondent	citizen:yes
1, Citizen	1
2, Not-citizen	0

Race: ( $k = 4$ )

Baseline: White

Respondent	race:black	race:asian	race:other
1, White	0	0	0
2, Black	1	0	0

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Respondent	citizen:yes
1, Citizen	1
2, Not-citizen	0

Race: ( $k = 4$ )

Baseline: White

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1, White	0	0	0
2, Black	1	0	0
3, Asian	0	1	0

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Citizen: yes / no ( $k = 2$ )

Baseline: no

Respondent	citizen:yes
1, Citizen	1
2, Not-citizen	0

Race: ( $k = 4$ )

Baseline: White

Respondent	race:black	race:asian	race:other
1, White	0	0	0
2, Black	1	0	0
3, Asian	0	1	0
4, Other	0	0	1

## Your turn

All else held constant, how do incomes of those born January thru March compare to those born April thru June?

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	-15342.76	11716.57	-1.31	0.19
hrs_work	1048.96	149.25	7.03	0.00
raceblack	-7998.99	6191.83	-1.29	0.20
raceasian	29909.80	9154.92	3.27	0.00
raceother	-6756.32	7240.08	-0.93	0.35
age	565.07	133.77	4.22	0.00
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time_to_work	90.04	79.83	1.13	0.26
langother	-10510.44	5447.45	-1.93	0.05
marriedyes	5409.24	3900.76	1.39	0.17
educollege	15993.85	4098.99	3.90	0.00
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disabilityyes	-14142.79	6639.40	-2.13	0.03
birth_qrtrapr thru jun	-2043.42	4978.12	-0.41	0.68
birth_qrtrjul thru sep	3036.02	4853.19	0.63	0.53
birth_qrthroct thru dec	2674.11	5038.45	0.53	0.60

All else held constant, those born Jan thru Mar make, on average,

- (a) \$2,043.42 less      (b) \$2,043.42 more      (c) \$4978.12 less more      (d) \$4978.12

## Your turn

All else held constant, how do incomes of those born January thru March compare to those born April thru June?

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- (a) \$2,043.42 less      (b) **\$2,043.42 more**      (c) \$4978.12 less      more  
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### (3) Inference for MLR: model as a whole + individual slopes

- Inference for the model as a whole: F-test,  $df_1 = p$ ,  
 $df_2 = n - k - 1$

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_k = 0$$

$H_A$  : At least one of the  $\beta_i \neq 0$

### (3) Inference for MLR: model as a whole + individual slopes

- Inference for the model as a whole: F-test,  $df_1 = p$ ,  
 $df_2 = n - k - 1$

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_k = 0$$

$H_A$  : At least one of the  $\beta_i \neq 0$

- Inference for each slope: T-test,  $df = n - k - 1$

- HT:

- $H_0 : \beta_1 = 0$ , when all other variables are included in the model

- $H_A : \beta_1 \neq 0$ , when all other variables are included in the model

- CI:  $b_1 \pm T_{df}^* SE_{b_1}$

## Model output

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )	
(Intercept)	-15342.76	11716.57	-1.309	0.190760	
hrs_work	1048.96	149.25	7.028	4.63e-12	***
raceblack	-7998.99	6191.83	-1.292	0.196795	
raceasian	29909.80	9154.92	3.267	0.001135	**
raceother	-6756.32	7240.08	-0.933	0.351019	
age	565.07	133.77	4.224	2.69e-05	***
genderfemale	-17135.05	3705.35	-4.624	4.41e-06	***
citizenyes	-12907.34	8231.66	-1.568	0.117291	
time_to_work	90.04	79.83	1.128	0.259716	
langother	-10510.44	5447.45	-1.929	0.054047	.
marriedyes	5409.24	3900.76	1.387	0.165932	
educcollege	15993.85	4098.99	3.902	0.000104	***
edugrad	59658.52	5660.26	10.540	< 2e-16	***
disabilityyes	-14142.79	6639.40	-2.130	0.033479	*
birth_qrtrapr thru jun	-2043.42	4978.12	-0.410	0.681569	
birth_qrtrjul thru sep	3036.02	4853.19	0.626	0.531782	
birth_qrthroct thru dec	2674.11	5038.45	0.531	0.595752	

Residual standard error: 48670 on 766 degrees of freedom

(60 observations deleted due to missingness)

Multiple R-squared: 0.3126, Adjusted R-squared: 0.2982

F-statistic: 21.77 on 16 and 766 DF, p-value: < 2.2e-16

### Your turn

True / False: The F test yielding a significant result means the model fits the data well.

- (a) True
- (b) False

## Your turn

True / False: The F test yielding a significant result means the model fits the data well.

(a) True

(b) *False*

*The F test yielding a significant result doesn't mean the model fits the data well, it just means at least one of the  $\beta$ s is non-zero. Whether or not the model fit the data well is evaluated based on model diagnostics.*

### Your turn

True / False: The F test not yielding a significant result means individual variables included in the model are not good predictors of  $y$ .

- (a) True
- (b) False

### Your turn

True / False: The F test not yielding a significant result means individual variables included in the model are not good predictors of  $y$ .

(a) True

(b) **False**

*The F test not yielding a significant result doesn't mean individual variables included in the model are not good predictors of  $y$ , it just means that the combination of these variables doesn't yield a good model.*



Significance also depends on what else is in the model

Model 1:	Estimate	Std. Error	t value	Pr(> t )
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Model 2:	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	-22498.2	8216.2	-2.738	0.00631
hrs_work	1149.7	145.2	7.919	7.60e-15
raceblack	-7677.5	6350.8	-1.209	0.22704
raceasian	38600.2	8566.4	4.506	7.55e-06
raceother	-7907.1	7116.2	-1.111	0.26683
age	533.1	131.2	4.064	5.27e-05
genderfemale	-15178.9	3767.4	-4.029	6.11e-05
marriedyes	8731.0	3956.8	2.207	0.02762 <----

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#### (4) Adjusted $R^2$ applies a penalty for additional variables

- ▶ When any variable is added to the model  $R^2$  increases.

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- ▶ But if the added variable doesn't really provide any new information, or is completely unrelated, adjusted  $R^2$  does not increase.

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- ▶ But if the added variable doesn't really provide any new information, or is completely unrelated, adjusted  $R^2$  does not increase.

#### Adjusted $R^2$

$$R_{adj}^2 = 1 - \left( \frac{SS_{Error}}{SS_{Total}} \times \frac{n - 1}{n - k - 1} \right)$$

where  $n$  is the number of cases and  $k$  is the number of sloped estimated in the model.

# Analysis of Variance Table

Response: income

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
hrs_work	1	3.0633e+11	3.0633e+11	129.3025	< 2.2e-16 ***
race	3	7.1656e+10	2.3885e+10	10.0821	1.608e-06 ***
age	1	7.6008e+10	7.6008e+10	32.0836	2.090e-08 ***
gender	1	4.8665e+10	4.8665e+10	20.5418	6.767e-06 ***
citizen	1	1.1135e+09	1.1135e+09	0.4700	0.49319
time_to_work	1	3.5371e+09	3.5371e+09	1.4930	0.22213
lang	1	1.2815e+10	1.2815e+10	5.4094	0.02029 *
married	1	1.2190e+10	1.2190e+10	5.1453	0.02359 *
edu	2	2.7867e+11	1.3933e+11	58.8131	< 2.2e-16 ***
disability	1	1.0852e+10	1.0852e+10	4.5808	0.03265 *
birth_qrtr	3	3.3060e+09	1.1020e+09	0.4652	0.70667
Residuals	766	1.8147e+12	2.3691e+09		
Total	782	2.6399e+12			

$$R^2_{adj} = 1 - \left( \frac{1.8147e + 12}{2.6399e + 12} \times \frac{783 - 1}{783 - 16 - 1} \right) \approx 1 - 0.7018 = 0.2982$$



### Your turn

True / False: For a model with at least one predictor,  $R_{adj}^2$  will always be smaller than  $R^2$ .

- (a) True
- (b) False

## Your turn

True / False: For a model with at least one predictor,  $R_{adj}^2$  will always be smaller than  $R^2$ .

(a) *True*

(b) False

*Because  $k$  is never negative,  $R_{adj}^2$  will always be smaller than  $R^2$ .*

$$R_{adj}^2 = 1 - \left( \frac{SS_{Error}}{SS_{Total}} \times \frac{n-1}{n-k-1} \right)$$

### Your turn

True / False: Adjusted  $R^2$  tells us the percentage of variability in the response variable explained by the model.

- (a) True
- (b) False

## Your turn

True / False: Adjusted  $R^2$  tells us the percentage of variability in the response variable explained by the model.

(a) True

(b) *False*

*$R^2$  tells us the percentage of variability in the response variable explained by the model, adjusted  $R^2$  is only useful for model selection.*

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## 2. Main ideas

1. In MLR everything is conditional on all other variables in the model
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## 3. Summary

## (5) Avoid collinearity in MLR

- ▶ Two predictor variables are said to be collinear when they are correlated, and this *collinearity* (also called *multicollinearity*) complicates model estimation.

*Remember:* Predictors are also called explanatory or independent variables, so they should be independent of each other.

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- ▶ In addition, addition of collinear variables can result in unreliable estimates of the slope parameters.
- ▶ While it's impossible to avoid collinearity from arising in observational data, experiments are usually designed to control for correlated predictors.

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- ▶ Either way, can use *backward elimination* or *forward selection*.
- ▶ Expert opinion and focus of research might also demand that a particular variable be included in the model.

## Your turn

Using the p-value approach, which variable would you remove from the model first?

	Estimate	Std. Error	t value	Pr(>  t )
(Intercept)	-15342.76	11716.57	-1.31	0.19
hrs_work	1048.96	149.25	7.03	0.00
raceblack	-7998.99	6191.83	-1.29	0.20
raceasian	29909.80	9154.92	3.27	0.00
raceother	-6756.32	7240.08	-0.93	0.35
age	565.07	133.77	4.22	0.00
genderfemale	-17135.05	3705.35	-4.62	0.00
citizenyes	-12907.34	8231.66	-1.57	0.12
time_to_work	90.04	79.83	1.13	0.26
langother	-10510.44	5447.45	-1.93	0.05
marriedyes	5409.24	3900.76	1.39	0.17
educollege	15993.85	4098.99	3.90	0.00
edugrad	59658.52	5660.26	10.54	0.00
disabilityyes	-14142.79	6639.40	-2.13	0.03
birth_qrtrapr thru jun	-2043.42	4978.12	-0.41	0.68
birth_qrtrjul thru sep	3036.02	4853.19	0.63	0.53
birth_qrthroct thru dec	2674.11	5038.45	0.53	0.60

(a) race:other

(b) race

(c) time\_to\_work

(d) birth\_qrtr:apr thru jun

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Using the p-value approach, which variable would you remove from the model next?

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	-14022.48	11137.08	-1.26	0.21
hrs_work	1045.85	149.05	7.02	0.00
raceblack	-7636.32	6177.50	-1.24	0.22
raceasian	29944.35	9137.13	3.28	0.00
raceother	-7212.57	7212.25	-1.00	0.32
age	559.51	133.27	4.20	0.00
genderfemale	-17010.85	3699.19	-4.60	0.00
citizenyes	-13059.46	8219.99	-1.59	0.11
time_to_work	88.77	79.73	1.11	0.27
langother	-10150.41	5431.15	-1.87	0.06
marriedyes	5400.41	3896.12	1.39	0.17
educollege	16214.46	4089.17	3.97	0.00
edugrad	59572.20	5631.33	10.58	0.00
disabilityyes	-14201.11	6628.26	-2.14	0.03

(a) married

(b) race

(c) race:other

(d) race:black

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- ▶ Constant variability of residuals (*homoscedasticity*) → no fan shape in the residuals plot
- ▶ Independence of residuals (and hence observations) → depends on data collection method, often violated for time-series data
- ▶ Also important to make sure that your explanatory variables are not *collinear*

### Your turn

Which of the following is the appropriate plot for checking the homoscedasticity condition in MLR?

- (a) scatterplot of residuals vs.  $\hat{y}$
- (b) scatterplot of residuals vs.  $x$
- (c) histogram of residuals
- (d) normal probability plot of residuals
- (e) scatterplot of residuals vs. order of data collection

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- (e) scatterplot of residuals vs. order of data collection

*Plotting residuals against  $\hat{y}$  (predicted, or fitted, values of  $y$ ) allows us to evaluate the whole model as a whole as opposed to homoscedasticity with regards to just one of the explanatory variables in the model.*



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