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1 Introduction

The Black-Scholes equations are PDEs from mathematical finance describing the price of European type options. The equation is given as

$$u_t - \frac{1}{2}\sigma^2 x^2 u_{xx} - rxu_x + cu = 0, x \in \mathbb{R}^+, t \in (0,T) \tag{1}$$

$$u_t - \frac{1}{2}x^2\varphi(u_{xx})u_{xx} = 0, x \in \mathbb{R}^+, t \in (0,T) \tag{2}$$

in linear and nonlinear 1-D. Where $\sigma, r, c \gneq 0$, are volatility, interest rate, dividends and correlation, and

$$\varphi(x) = \sigma_1^2 + \frac{\sigma_2^2 - \sigma_1^2}{2}\left(1 + \frac{2}{\pi}\text{arctan}x\right).$$

In this report, we will solve both types of the Black-Scholes equation using the Finite Difference method. We will use central differences in space, and Forward Euler, Backward Euler and Crank-Nicolson in space. Different choices of initial conditions for this PDE is European put, butterfly spread and Binary call, which are respectively

$$\begin{aligned} u(x,0) &= (K-x)^+ \\ u(x,0) &= (x-K)^+ - 2(x-(K+H))^+ + (x-(K+2H))^+ \\ u(x,0) &= \text{sgn}^+(x-K), \end{aligned}$$

where K and H are strike prices. For the boundary conditions, we will in $x=0$, use the solution of the ODE that the PDE turns into. This yields $u(0,t) = Ce^{-ct}$, where C is the initial condition $u(0,0)$. For the right boundary we will use artificial boundary conditions. We will look at $u(R,t) = 0$, $u_x(R,t) = 0$ and $u_x(R,t) = u_x(R,0)$, where $R > 0$ is the right boundary.

2 Finite difference schemes

2.1 Forward Euler

Now we derive the finite diffence schemes to solve 1. We start by writing up the scheme for (FE) in time,

$$\frac{1}{k}\Delta_t u_m^n = \frac{1}{2}\sigma^2 x^2 \frac{1}{h^2}\delta_x^2 u_m^n + rx\frac{1}{2h}\delta_x u_m^n - cu_m^n \tag{3}$$

$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{1}{2}\sigma^2 x^2 \frac{1}{h^2}\left(U_{m-1}^n - 2U_m^n + U_{m+1}^n\right) + rx\frac{1}{2h}(U_{m+1}^n - U_{m-1}^n) - cU_m^n \tag{4}$$

carrying out the algebra in this equation yields

$$U_m^{n+1} = \alpha_{FE}U_{m-1}^n + \beta_{FE}U_m^n + \gamma_{FE}U_{m+1}^n, \tag{5}$$

$$\alpha_{FE} = \frac{k}{h^2}\frac{1}{2}\sigma^2 x^2 - \frac{k}{2h}rx,\ \beta_{FE} = 1 - \frac{k}{h^2}\sigma^2 x^2 - kc,\ \gamma_{FE} = \frac{k}{h^2}\frac{1}{2}\sigma^2 x^2 + \frac{k}{2h}rx. \tag{6}$$

This can also be written in matrix form as

$$U_m^{n+1} = CU_m^n + q^n, \tag{7}$$

where C is a tridiagonal matrix, $C = \text{tridiag}\{\alpha_{FE}, \beta_{FE}, \gamma_{FE}\}$, and q^n is a vector with boundary conditions.

2.2 Backward Euler

The scheme for (BE) in time is

$$\frac{1}{k}\nabla_t u_m^n = \frac{1}{2}\sigma^2 x^2 \frac{1}{h^2}\delta_x^2 u_m^n + rx\frac{1}{2h}\delta_x u_m^n - cu_m^n \tag{8}$$

$$\frac{U_m^{n+1}-U_m^n}{k} = \frac{1}{2}\sigma^2 x^2 \frac{1}{h^2}(U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}) + rx\frac{1}{2h}(U_{m+1}^{n+1} - U_{m-1}^{n+1}) - cU_m^{n+1} \tag{9}$$

carrying out the algebra in this case yields

$$U_m^n = -\alpha_{BE}U_{m-1}^{n+1} + \beta_{BE}U_m^{n+1} - \gamma_{BE}U_{m+1}^{n+1}, \tag{10}$$

$$\alpha_{BE} = \frac{1}{2}\frac{k\sigma^2 x^2}{h^2} - \frac{rkx}{2h},\ \beta_{BE} = 1 + \frac{k\sigma^2 x^2}{h^2} + kc,\ \gamma_{BE} = \frac{1}{2}\frac{k\sigma^2 x^2}{h^2} + \frac{rkx}{2h}. \tag{11}$$

This can also be written in matrix form as

$$AU_m^{n+1} = U_m^n + q^{n+1} \tag{12}$$

where $A = \text{tridiag}\{-\alpha_{BE}, \beta_{BE}, -\gamma_{BE}\}$, and the vector q as before.

2.3 Crank-Nicolson

The scheme for (CN) in time is

$$\begin{aligned}\frac{U_m^{n+1}-U_m^n}{k} = \frac{1}{2}\Big[&\frac{1}{2}\sigma^2 x^2 \frac{1}{h^2}\left(U_{m-1}^n - 2U_m^n + U_{m+1}^n\right) + rx\frac{1}{2h}(U_{m+1}^n - U_{m-1}^n) - cU_m^n + \\ &\frac{1}{2}\sigma^2 x^2 \frac{1}{h^2}\left(U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}\right) + rx\frac{1}{2h}(U_{m+1}^{n+1} - U_{m-1}^{n+1}) - cU_m^{n+1}\Big].\end{aligned}$$

Rearranging this yields

$$\begin{aligned}&-\frac{1}{2}\left(\frac{1}{2}\frac{\sigma^2 x^2 k}{h^2} - \frac{rxk}{2h}\right)U_{m-1}^{n+1} + (1 + \frac{\sigma^2 x^2 k}{2h^2} + \frac{ck}{2})U_m^{n+1} - \frac{1}{2}\left(\frac{1}{2}\frac{\sigma^2 x^2 k}{h^2} + \frac{rxk}{2h}\right)U_{m+1}^{n+1} \\ &= \frac{1}{2}\left(\frac{1}{2}\frac{\sigma^2 x^2 k}{h^2} - \frac{rxk}{2h}\right)U_{m-1}^n + (1 - \frac{\sigma^2 x^2 k}{2h^2} - \frac{ck}{2})U_m^n + \frac{1}{2}\left(\frac{1}{2}\frac{\sigma^2 x^2 k}{h^2} + \frac{rxk}{2h}\right)U_{m+1}^n.\end{aligned}$$

We can simplify this to

$$-\alpha_{CN}U_{m-1}^{n+1} + (1+\beta_{CN})U_m^{n+1} - \gamma_{CN}U_{m+1}^{n+1} = \alpha_{CN}U_{m-1}^n + (1-\beta_{CN})U_m^n + \gamma_{CN}U_{m+1}^n \tag{13}$$

$$\alpha_{CN} = \frac{1}{2}\left(\frac{1}{2}\frac{\sigma^2 x^2 k}{h^2} - \frac{rxk}{2h}\right),\ \beta_{CN} = \frac{\sigma^2 x^2 k}{h^2} + \frac{ck}{2},\ \gamma_{CN} = \frac{1}{2}\left(\frac{1}{2}\frac{\sigma^2 x^2 k}{h^2} + \frac{rxk}{2h}\right) \tag{14}$$

This can be written in matrix form as

$$AU_m^{n+1} = BU_m^n + q, \tag{15}$$

where $A = \text{tridiag}\{-\alpha_{CN}, 1+\beta_{CN}, -\gamma_{CN}\}$ and $B = \text{tridiag}\{\alpha_{CN}, 1-\beta_{CN}, \gamma_{CN}\}$.

3 Monotonicity

A numerical scheme is monotone if it can be written as

$$\alpha_m^{n+1}U_m^{n+1} - \sum_l \sum_{\substack{k\leq n+1\\(k,l)\neq(n+1,0)}} \beta_{m,l}^k U_{m+l}^k + c_m^l = 0, \begin{cases} n = 0,\dots,N-1 \\ m = 1,\dots,M-1 \\ c_m \in \mathbb{R}\end{cases} \tag{16}$$

where

$$\alpha_m^n > 0,\ \beta_m^n \geq 0,\ \alpha_m^n \geq \sum \sum_{\substack{l,k\leq n\\(k,l)\neq(n,0)}} \beta_m^k.$$

If we rearrange the scheme FE scheme 5, we get

$$U_m^{n+1} - \left(\alpha_{FE} U_{m-1}^n + \beta_{FE} U_m^n + \gamma_{FE}\right) = 0$$

and this is on the same form as 16. We then check if the conditions for monotonicity holds for the (FE) scheme. First, $\alpha_m^{n+1} = 1 > 0$, so the first condition is satisfied. Then, we need to check for $\alpha_{FE}, \beta_{FE}, \gamma_{FE} \geq 0$. We have, using $x = mh$,

$$\begin{aligned}
\alpha_{FE} &= \frac{1}{2} k \left(\sigma^2 m^2 - rm\right) \implies \alpha_{FE} \geq 0 \iff \sigma^2 > r \\
\beta_{FE} &= 1 - k\left(\sigma^2 m^2 + c\right) \geq 0 \iff k\left(\sigma^2 m^2 + c\right) \leq 1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \gamma_{FE} \geq 0.
\end{aligned}$$

as γ_{FE} is the sum of positive elements. Lastly, we need to check the last condition:

$$\begin{aligned}
\alpha_{FE} + \beta_{FE} + \gamma_{FE} &= \frac{1}{2}\frac{k\sigma^2 x^2}{h^2} - \frac{rkx}{2h} + 1 - \frac{k\sigma^2 x^2}{h^2} - kc + \frac{1}{2}\frac{k\sigma^2 x^2}{h^2} + \frac{rkx}{2h} \\
&= 1 - kc < 1 = \alpha_m^{n+1}.
\end{aligned}$$

Thus, the Forward Euler scheme for the Black-Scholes equation is monotone if $\sigma^2 > r$ and under the CFL-condition

$$k\left(\sigma^2 m^2 + c\right) \leq 1.$$

Rearranging the Backward Euler scheme 10, to be on the same form as the monotonicity equation 16, gives us the equation

$$\beta_{BE} U_m^{n+1} - (\alpha_{BE} U_{m-1}^{n+1} + \gamma_{BE} U_{m+1}^{n+1}) - U_m^n = 0,$$

We know from 11, that when $\sigma^2 \geq r$

$$\beta_{BE} \geq 0, \ \alpha_{BE} \geq 0, \ \gamma_{BE} \geq 0,$$

and by filling in the values we can check the last condition for a monotone scheme,

$$\begin{aligned}
\beta_{BE} &\geq \alpha_{BE} + \gamma_{BE} \\
1 + \frac{k\sigma^2 x^2}{h^2} + kc &\geq \frac{1}{2}\frac{k\sigma^2 x^2}{h^2} - \frac{rkx}{2h} + \frac{1}{2}\frac{k\sigma^2 x^2}{h^2} + \frac{rkx}{2h} \\
1 + kc &\geq 0 \\
&\rightarrow \beta_{BE} \geq 0.
\end{aligned}$$

Thus, the Backward Euler scheme is monotone.

For the Crank-Nicolson scheme, we write equation 13 as 16 and get the equation

$$(1+\beta_{CN})U_m^{n+1} - (\alpha_{CN} U_{m-1}^{n+1} + \gamma_{CN} U_{m+1}^{n+1}) - \alpha_{CN} U_{m-1}^n - (1-\beta_{CN}) U_m^n - \gamma_{CN} U_{m+1}^n = 0$$

We see from 14 that when $\sigma^2 \geq r$,

$$\alpha_{CN} \geq 0, \ \beta_{CN} \geq 0, \ \gamma_{CN} \geq 0,$$

and we check the last monotonicity condition,

$$\begin{aligned}
1 + \beta_{CN} &\geq \alpha_{CN} U_{m-1}^{n+1} + \gamma_{CN} U_{m+1}^{n+1} \\
1 + \frac{\sigma^2 x^2 k}{h^2} + \frac{ck}{2} &\geq \frac{1}{2}\left(\frac{1}{2}\frac{\sigma^2 x^2 k}{h^2} - \frac{rxk}{2h}\right) + \frac{1}{2}\left(\frac{1}{2}\frac{\sigma^2 x^2 k}{h^2} + \frac{rxk}{2h}\right) \\
1 + \frac{1}{2}\frac{\sigma^2 x^2 k}{h^2} + \frac{ck}{2} &\geq 0
\end{aligned}$$

Thus, the Crank-Nicolson scheme is also monotone.

4 L^∞-stability

We now want to show L^∞-stability for our difference methods. Using the matrix form of the Forward Euler scheme 7, we have L^∞-stability if there exists a μ independent of h and k such that

$$\|C\|_\infty \leq 1 + \mu k.$$

So to show L stability for Forward Euler, we only have to bound C by its norm,

$$\|C\|_\infty = |\alpha_{FE}| + |\beta_{FE}| + |\gamma_{FE}| = 1 - kc < 1 + kc$$

where we assume that the conditions for a monotone scheme holds. So we have L^∞-stability for the Forward Euler method with respect to a arbitrary right hand side. To show L^∞-stability for Backward Euler, we rewrite the scheme 10 to

$$-\mathcal{L}_h U_m^n = -\left(\frac{1}{2}\frac{k\sigma^2 x^2}{h^2} - \frac{rkx}{2h}\right) U_{m-1}^{n+1} + \left(1 + \frac{k\sigma^2 x^2}{h^2} + kc\right) U_m^{n+1} - \left(\frac{1}{2}\frac{k\sigma^2 x^2}{h^2} + \frac{rkx}{2h}\right) U_{m+1}^{n+1} = 0$$
$$x \in \mathbb{R}^+, t \in (0,T)$$

We need to find stability with respect to a right hand side, so we introduce a solution V_m^n, such that $V(0,t) = V(R,t) = 0$. Then

$$-\mathcal{L}_h V_m^n = f.$$

In order to use the Discrete maximum principle (DMP) for this equation, we need to remove f. We introduce the special function $\psi = \frac{1}{c}$. This function has the property that

$$-\mathcal{L}\psi = 1.$$

Then, $W_m^n = V_m^n - \|\vec{f}\|_{L^\infty}\psi$ satisfies

$$-\mathcal{L}_h W_m^n = -\mathcal{L}_h V_m^n - \|\vec{f}\|_{L^\infty}(-\mathcal{L}_h\psi) = f - \|\vec{f}\|_{L^\infty} \leq 0.$$

As W_m^n is monotone, it satisfies DMP, and this yields

$$W_m^n \leq \max_{m=0,M} W_m^n \leq 0.$$

From this we easily see that

$$V_m^n \leq \psi\|\vec{f}\|_{L^\infty}.$$

We can then do the exact same analysis but let $(V,f) \to (-V,-f)$, then

$$-V_m^n \leq \psi\|\vec{f}\|_{L^\infty}.$$

In conclusion, we have

$$\max|V_m^n| \leq \|\vec{f}\|_{L^\infty} \max\psi,$$

and we have proven that the Backward Euler scheme is L^∞-stable. We can use a very similar proof to show that the Crank-Nicolson scheme is L^∞-stable, but this is left as an exercise to the reader.

5 Consistency

A scheme is consistent if the truncation error $\tau \to 0$ as $k,h \to 0$. The truncation error is given as

$$\tau = U - u, \tag{17}$$

with u as equation 1 and rewriting equation 4 to U,

$$U = \frac{U_m^{n+1} - U_m^n}{k} - \frac{1}{2}\sigma^2 x^2 \frac{1}{h^2}(U_{m-1}^n - 2U_m^n + U_{m+1}^n) - \frac{1}{2} r x \frac{1}{h}(U_{m+1}^n - U_{m-1}^n) + cU_m^n.$$

After Taylor expanding U, and subtracting u, we get the truncation error:

$$\tau = \frac{k}{2}u^n_{tt,m} - \frac{1}{24}\sigma^2 x^2 h^2 u^n_{xxxx,m} - \frac{1}{6}rxh^2 u^2_{xxx,m} + O(k^2 + h^4)$$

We know from Lax Equivalence Theorem that if a scheme is stable and consistent, then it also is convergent, hence the Forward Euler scheme is L^∞ convergent. The error bound is given as

$$\tau \leq \frac{k}{2}\max_{t\in(0,T)}|u_{tt}| + \frac{1}{24}\sigma^2 R^2 h^2 \max_{x\in\mathbb{R}^+}|u^n_{xxxx,m}| + \frac{1}{6}rRh^2 \max_{x\in\mathbb{R}^+}|u^n_{xxx,m}|,$$

where R is the largest x-value.

6 Numerical testing

We check the CFL-condition for the Forward Euler scheme numerically, by running the code twice. First with parameters satisfying the CFL-condition, and then with parameters not satisfying the condition.

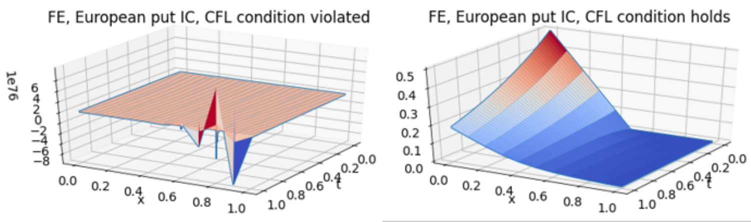

Figure 1: Violated and not violated CFL-condtion in Forward Euler scheme. Used European put as initial condition, and $U(R,t) = 0$ as boundary condition.

As we can see from figure 1, when the CFL-condition is violated, we do not get a valid solution of equation 1 using the Forward Euler scheme.
To check the convergence order of our schemes, we introduce a right hand side to equation 1, and compares it to an exact solution. We choose

$$f(x,t) = e^{-t}\sin(\pi x) \tag{18}$$

as our exact solution. To find the right hand side of our scheme, we put equation 18 in equation 1. We also modify the initial conditions and boundary conditions to fit those of equation 18.

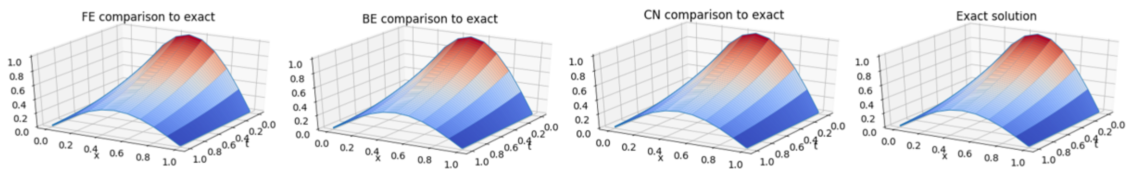

Figure 2: Comparison of the numerical solution with FE, BE and CN finite difference schemes and the exact solution 18.

From figure 2, we can see that the numerical solution seems to replicate the exact solution well for all schemes. If we run the schemes multiple times, and varying the grid in x and t, we can compute the error of the numerical schemes. We plot the error as loglog-plots, and these can be seen in figure 3.

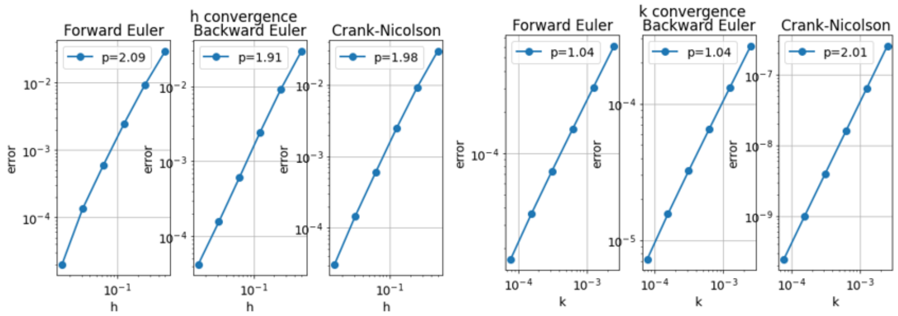

Figure 3: Loglog plots of the error to check convergence rates in k and h for all schemes.

In compliance with the truncation error of the Forward Euler scheme, we can see from figure 3 that we have quadratic convergence in h and linear convergence in k. We have the convergence orders for Backward Euler. This is in compliance with known theory. For the Crank-Nicolson scheme, we get quadratic convergence in both time and space, this is also in compliance with known theory.

To check computational time for our code, we ran all of the difference schemes 100 times, and timed the code using the time library in python. From this we find that the FE is fastest, and CN is a bit slower than BE. This is as we would expect, as both implicit methods (BE and CN) solves a linear system in each iteration step while the explicit scheme (FE) does not.

For the Forward Euler scheme and Butterfly spread initial condition, we tried with different boundary conditions. For the left boundary, we always used the condition specified in the introduction, but we varied the right boundary (at $x = R$). We used $u(R,t) = 0$, $u(R,t) = u(R,0)$ and $u_x(R,t) = 0$. All these plots can be seen in figure 4.

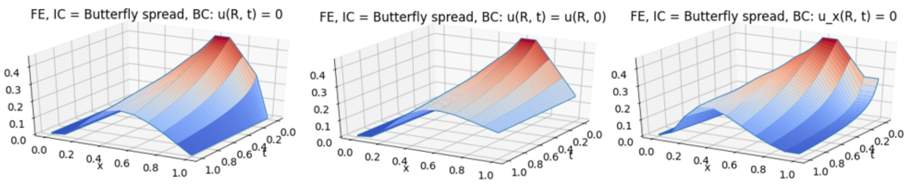

Figure 4: Forward Euler with Butterfly spread initial condition and different boundary conditions.

For the case where we have $u_x(R,t) = 0$, we implement this by using backward differences, $\frac{U_M^0 - U_{M-1}^0}{h} = 0 \implies U_M^0 = U_{M-1}^0$. As expected, the solution is equal for all schemes except at the boundaries. For the last scheme, we can see that the derivative at the boundary is zero. In addition, we wrote code that checked how the error scaled with increasing R. This showed that increasing R gives increasing error. This makes sense, as we do not change number of gridpoints, so both h and k increases, and the less refined grid yields larger error.

7 1-D Nonlinear Black-Scholes

To solve the the nonlinear Black-Scholes equation 2, we introduce two finite difference schemes. These are

$$\frac{1}{k}\nabla_t U_m^{n+1} = \frac{1}{2}x_m^2 \varphi\left(\frac{1}{h^2}\delta_x^2 U_m^n\right)\frac{1}{h^2}\delta_x^2 U_m^{n+1} \tag{19}$$

$$\frac{1}{k}\nabla_t U_m^{n+1} = \frac{1}{2}x_m^2 \varphi\left(\frac{1}{h^2}\delta_x^2 U_m^{n+1}\right)\frac{1}{h^2}\delta_x^2 U_m^{n+1}. \tag{20}$$

and are called IMEX and Backward Euler respectively.
Explicitly writing out the IMEX equation 19 yields

$$\left(\frac{1}{k} + \frac{x_m^2 \varphi_{IMEX}}{h^2}\right)U_m^{n+1} = \frac{x_m^2 \varphi_{IMEX}}{2h^2}U_{m-1}^{n+1} + \frac{x_m^2 \varphi_{IMEX}}{2h^2}U_{m+1}^{n+1} + \frac{1}{k}U_m^n. \tag{21}$$

If we want to solve equation 2 with an exact solution we have to implement a right hand side (RHS) f. We find this by using 2 on our exact solution. In this example we use $u(x,t) = sin(t) + x \Rightarrow f(x,t) = cos(t)$, where $u(x,t)$ is a non-convex exact solution. Inserting a RHS in the IMEX scheme 21 yields

$$-\frac{x_m^2 \varphi_{IMEX}}{2h^2} U_{m-1}^{n+1} + \left(1 + \frac{x_m^2 \varphi_{IMEX}}{h^2}\right) U_m^{n+1} - \frac{x_m^2 \varphi_{IMEX}}{2h^2} U_{m+1}^{n+1} = U_m^n + k f_m^n. \tag{22}$$

This can be written as a matrix equation with a tridiagonal matrix $A = \text{tridiag}\{-\frac{x_m^2 \varphi_{IMEX}}{2h^2}, (1 + \frac{x_m^2 \varphi_{IMEX}}{h^2}), -\frac{x_m^2 \varphi_{IMEX}}{2h^2}\}$ which is strictly diagonal dominant and is therefore invertible and the scheme can be written as

$$U^{n+1} = A^{-1}(U^n + kf^n).$$

Using this method with an exact solution yields the plots in figure 5,

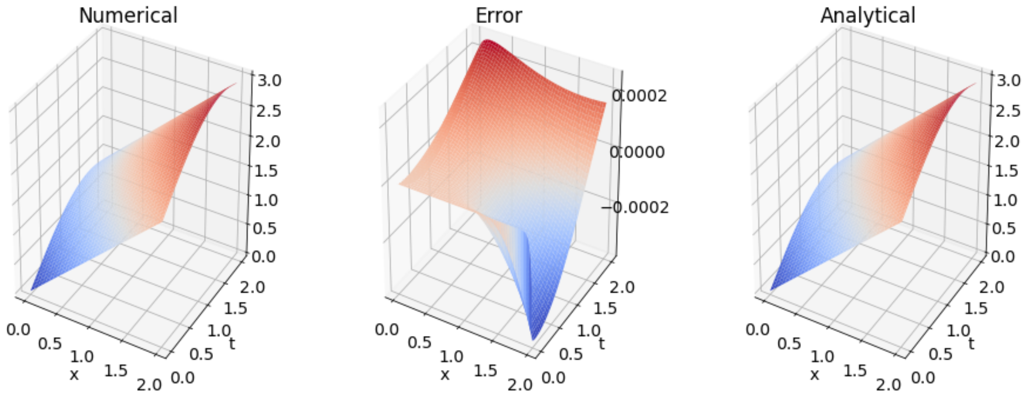

Figure 5: IMEX scheme on equation 2 with an exact solution. From the left the plots are the numerical solution, the error between the exact and numerical solution and the exact solution

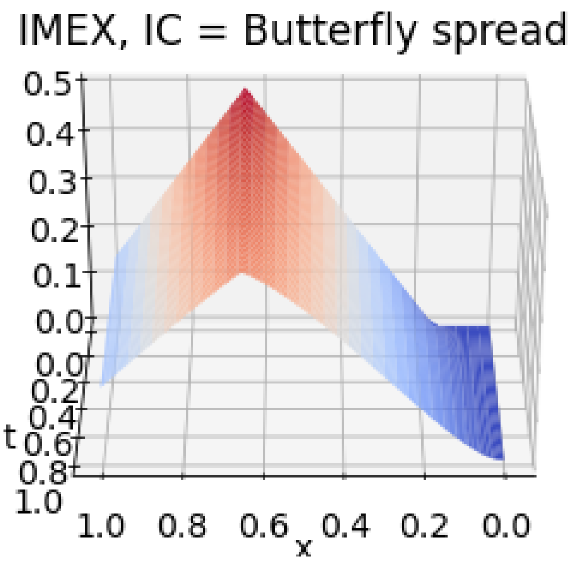

Figure 6: IMEX scheme on equation 2 with Butterfly spread initial conditions

Computing the truncation error of the schemes, using equation 17, with U for IMEX as a rewritten equation of 19 and BE as the same rewriting of equation 20. Then we use forward difference in time, central difference in space, and taylor expansion of U, and end up with the truncation errors,

$$\tau_{IMEX} \leq \frac{k}{2}(u^n_{tt,m} - x^2\varphi(u^n_{xx,m})u^n_{xxt,m}) - \frac{1}{24}R^2h^2(u^{n+1}_{xxxx,m}\varphi(u^n_{xx,m}) + u^n_{xxxx,m}u^{n+1}_{xx,m}(\frac{\sigma_2^2-\sigma_1^2}{\pi})) + O(h^3+k^2) \tag{23}$$

$$\tau_{BE} \leq \frac{k}{2}(u^n_{tt,m} - x^2_m\eta_t) - \frac{1}{2}R^2\frac{h^2}{12}u^{n+1}_{xxxx,m}(\varphi(u^{n+1}_{xx,m} + (\frac{\sigma_2^2-\sigma_1^2}{\pi})u^{n+1}_{xx,m})) + O(k^2+h^3) \tag{24}$$

$$\eta^n_m = u^n_{xx,m}\varphi(u^n_{xx,m}) \tag{25}$$

To show monotonicity we have to show that the methods can be written on the form

$$\alpha^{n+1}_m U^{n+1}_m - \sum_{i\neq 0}\beta^{n+1}_{m,i}U^{n+1}_{m+i} - \sum_i \beta^n_{m,i}U^n_{m+i} = 0, \alpha, \beta \geq 0, \alpha \geq \sum_{n,i}\beta > 0.$$

For IMEX we can se from 21 that we have

$$\alpha^{n+1}_m = \frac{1}{k} + \frac{x^2_m\varphi_{IMEX}}{h^2}, \beta^{n+1}_{m,-1} = \frac{x^2_m\varphi_{IMEX}}{2h^2}, \beta^{n+1}_{m,1} = \frac{x^2_m\varphi_{IMEX}}{2h^2}, \beta^n_{m,0} = \frac{1}{k}.$$

Further we see that $\alpha > 0$ since $k > 0$, $h > 0$, $x_m \in \mathbb{R}$, and $\varphi_{IMEX} > 0$. This also holds for all β by the same arguments. We can see that

$$\alpha^{n+1}_m = \beta^n_{m,0} + \beta^{n+1}_{m,-1} + \beta^{n+1}_{m,1} = \frac{1}{k} + \frac{x^2_m\varphi_{IMEX}}{h^2},$$

thus the IMEX is monotone.
The exact same arguments hold for BE. The only difference is that φ_{IMEX} is replaced with φ_{BE}, but this does not change any of the arguments since φ_{BE} is also defined to be strictly non negative.

To show L^∞-stability of IMEX we apply almost the same strategy as in section 4. First we rewrite the scheme to

$$-\mathcal{L}_h U^n_m = (\frac{1}{k} + \frac{x^2_m\varphi_{IMEX}}{h^2})U^{n+1}_m - (\frac{x^2_m\varphi_{IMEX}}{2h^2}U^{n+1}_{m-1} + \frac{x^2_m\varphi_{IMEX}}{2h^2}U^{n+1}_{m+1}) - \frac{1}{k}U^n_m = 0, x \in \mathbb{R}^+, t \in (0,T).$$

We then need to find stability w.r.t a right hand side, so we again introduce a solution V^n_m, s.t. V(0, t) = V(R, t) = 0. Then

$$-\mathcal{L}_h V^n_m = f.$$

In order to use DMP for this equation, we need to remove f. Therefore we introduce the special function $\psi = t$. Which satisfies

$$-\mathcal{L}\psi = 1.$$

Then, $W^n_m = V^n_m - ||f||_{L^\infty}\psi$ satisfies

$$-\mathcal{L}_h W^n_m = -\mathcal{L}_h V^n_m - ||f||_{L^\infty}(-\mathcal{L}\psi) = f - ||f||_{L^\infty} \leq 0.$$

Since W^n_m is monotone, it satisfies DMP, and gives

$$W^n_m \leq \max_{m=0,M} W^n_m \leq 0 \Rightarrow V^n_m \leq \psi||f||_{L^\infty}.$$

Repeating the analysis with $(V, f) \to (-V, -f) \Rightarrow -V^n_m \leq \psi||f||_{L^\infty}$. This gives that $\max|V^n_m| \leq ||f||_{L^\infty}\max\psi$, and thus (IMEX) is L^∞-stable.

8 Conclusion

In this report we have solved both the linear and nonlinear Black-Scholes partial differential eqaution with different finite difference schemes. We have looked at consistency, stability and convergence for most of these methods.