AALBORG UNIVERSITY

Lecture 2
The Wigner-Ville Distribution

Time-frequency analysis, adaptive filtering and source separation

José Biurrun Manresa

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Time-Frequency representations

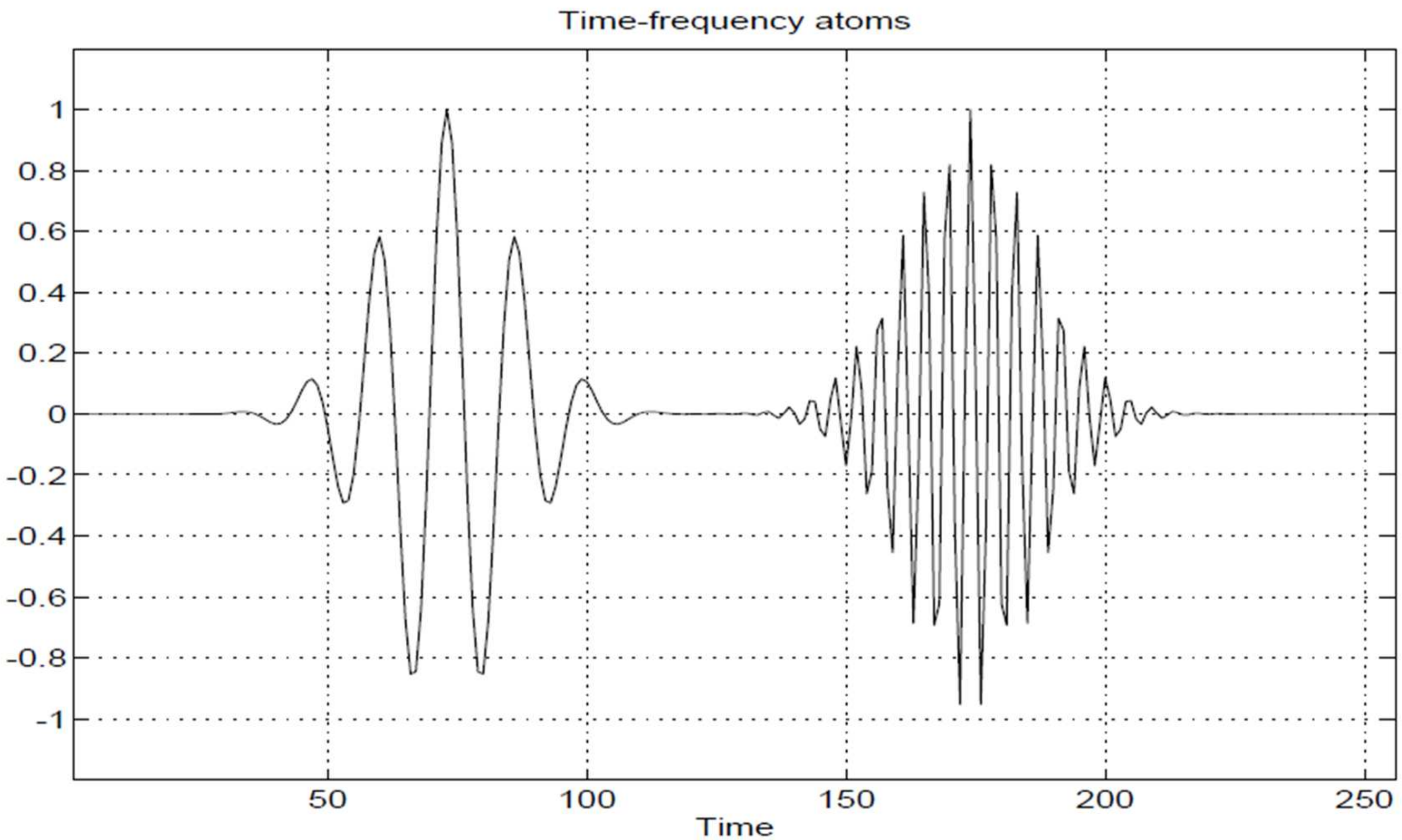
- The Short-Time Fourier Transform (STFT) takes a **linear** approach for a time-frequency representation

- It decomposes the signal on elementary components, called **atoms**

$$h_{t,\omega}(\tau) = h(\tau - t)e^{-j\omega\tau}$$

- Each atom is obtained from the window $h(t)$ by a translation in time and a translation in frequency (modulation)

Time-Frequency representations



Time-Frequency representations

- Alternatively, in the frequency domain

$$H_{t,\omega}(\theta) = H(\omega - \theta)e^{-j\theta t}$$

- The STFT can be considered as the result of passing the signal through a bank of band-pass filters whose frequency response is $H(\omega - \theta)$ and is deduced from a mother filter $H(\omega)$ by a translation of θ

- Each filter in the bank has a constant bandwidth

Time-Frequency representations

- If we consider the square modulus of the STFT, we get the **spectrogram**, which is th spectral energy density of the locally windowed signal $s_t(\tau) = s(\tau)h(\tau - t)$

- The spectrogram is a **quadratic** or **bilinear** representation

- If the energy of the windows is selected to be one, the energy of the spectrogram is equal to the energy of the signal

- Thus, it can be interpreted as a measure of the energy of the signal contained in the time-frequency domain centered on the point (t, ω)

Time-Frequency representations

- **Linear**
  - **STFT**
  - Wavelet
- **Bilinear or Quadratic**
  - **Cohen's class**
    - **Spectrogram**
    - **Wigner-Ville**
    - **Choi-Williams**
    - **...**
  - Affine distributions

The energy distributions

- The purpose of the energy distributions is to distribute the energy of the signal over time and frequency

- The energy of a signal $s(t)$ can be deduced from the squared modulus of either the signal or its Fourier transform

$$E_s = \int |s(t)|^2 \, dt = \int |S(\omega)|^2 \, d\omega$$

- $|s(t)|^2$ and $|S(\omega)|^2$ can be interpreted as energy densities in time and frequency, respectively

The energy distributions

- It is natural to look for a **joint** time and frequency energy density $\rho_s(t,\omega)$ such that

$$E_s = \iint \rho_s(t,\omega)dtd\omega$$

- As the energy is a quadratic function of the signal, the time-frequency energy distributions will be in general quadratic representations

The energy distributions

- Two properties that an energy density should satisfy are the time and frequency **marginal** conditions

$$\int \rho_s(t,\omega)dt = |S(\omega)|^2$$

$$\int \rho_s(t,\omega)d\omega = |s(t)|^2$$

- If the time-frequency energy density is integrated along one variable, the result is the energy density corresponding to the other variable

Cohen's class

- There are many distributions that statisfy the properties mentioned before
- Therefore, it is possible to impose additional constraints to $\rho_s(t, \omega)$ that would result in desirable properties
- Among these properties, the **covariance** principles are of fundamental importance
- The Cohen's class is the family of time-frequency energy distributions **covariant by translations in time and frequency**

Cohen's class

- The spectrogram is an element of the Cohen's class, since it is a quadratic, time- and frequency-covariant, and preserves energy

$$x(t) = s(t - t_0) \;\Rightarrow\; P_x(t,\omega) = P_s(t-t_0,\omega)$$

$$x(t) = s(t)e^{-j\omega_0 t} \;\Rightarrow\; P_x(t,\omega) = P_s(t,\omega - \omega_0)$$

$$P(t,\omega) = |S(\omega)|^2 = |s(t)|^2$$

- Taking the square modulus of an atomic decomposition is only a restrictive possibility to define a quadratic representation

The Wigner-Ville distribution

- The approach is based on the use of the autocorrelation function for calculating the power spectrum

- To construct the autocorrelation function, the signal is compared to itself for all possible relative shifts, or lags

$$r_{ss}(\tau) = \int s(t)s(t+\tau)dt$$

where τ is the shift of the signal with respect to itself

The Wigner-Ville Distribution

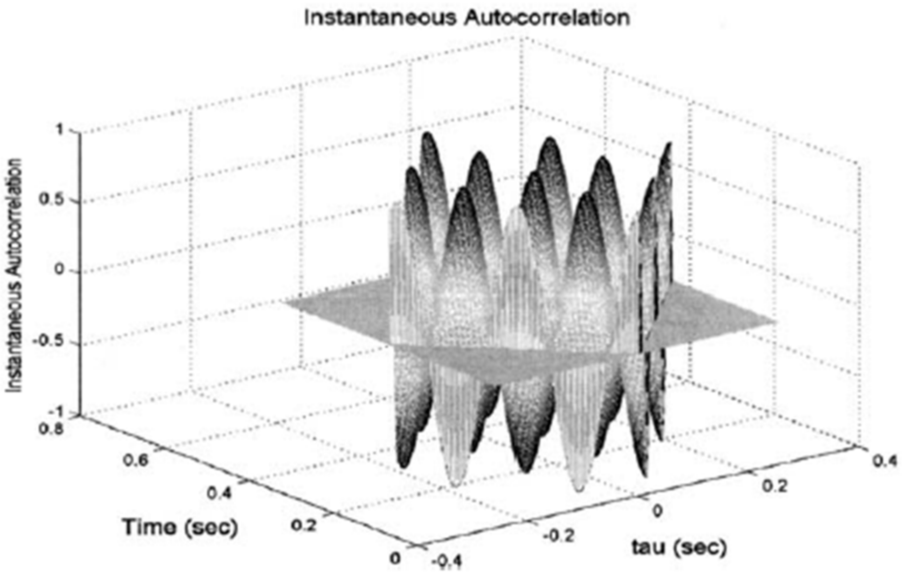
- In the standard autocorrelation function, time is integrated out of the result, and r_{ss} is only a function of the time lag τ
- The Wigner-Ville (and all of Cohen's class of distribution) uses a variation of the autocorrelation function wher time remains in the result, called **instantaneous autocorrelation function**

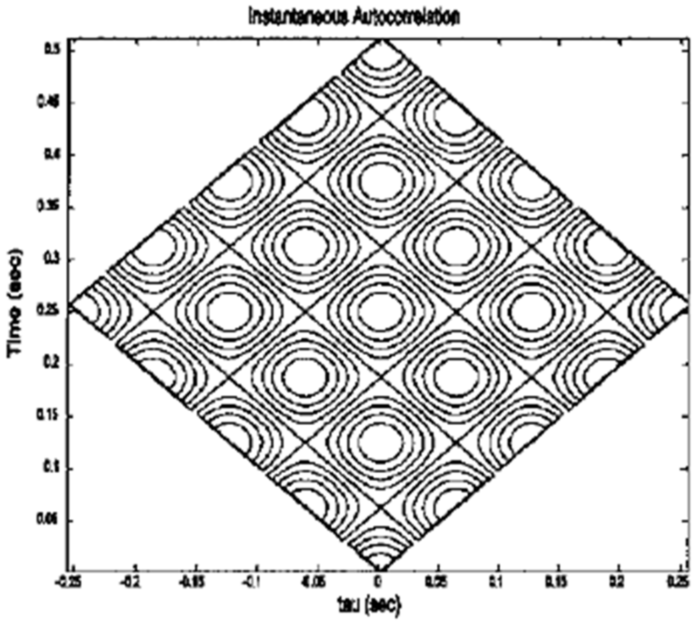
$$R_{ss}(t,\tau) = s\left(t + {}^{\tau}/_{2}\right)s^{*}\left(t - {}^{\tau}/_{2}\right)$$

Where τ is the time lag and * represents the complex conjugate of the signal s.

The Wigner-Ville Distribution

- Instantaneous autocorrelation of four cycle sine plots





The Wigner-Ville Distribution

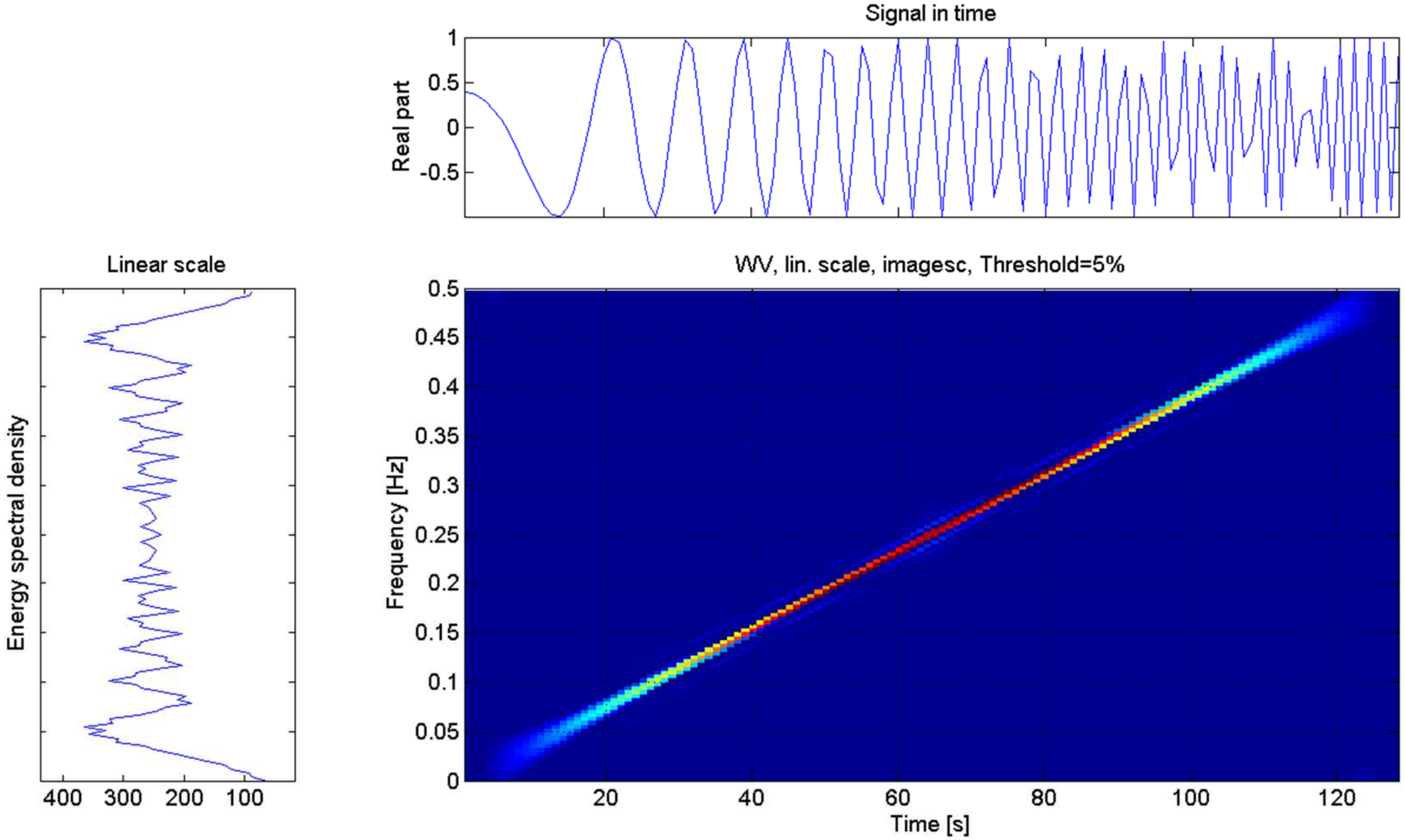
- The Wigner-Ville Distribution (WVD) is defined as

$$W_s(t,\omega) = \frac{1}{2\pi}\int s\left(t + {}^{\tau}\!/_{2}\right) s^*\left(t - {}^{\tau}\!/_{2}\right) e^{-j\omega\tau} d\tau$$

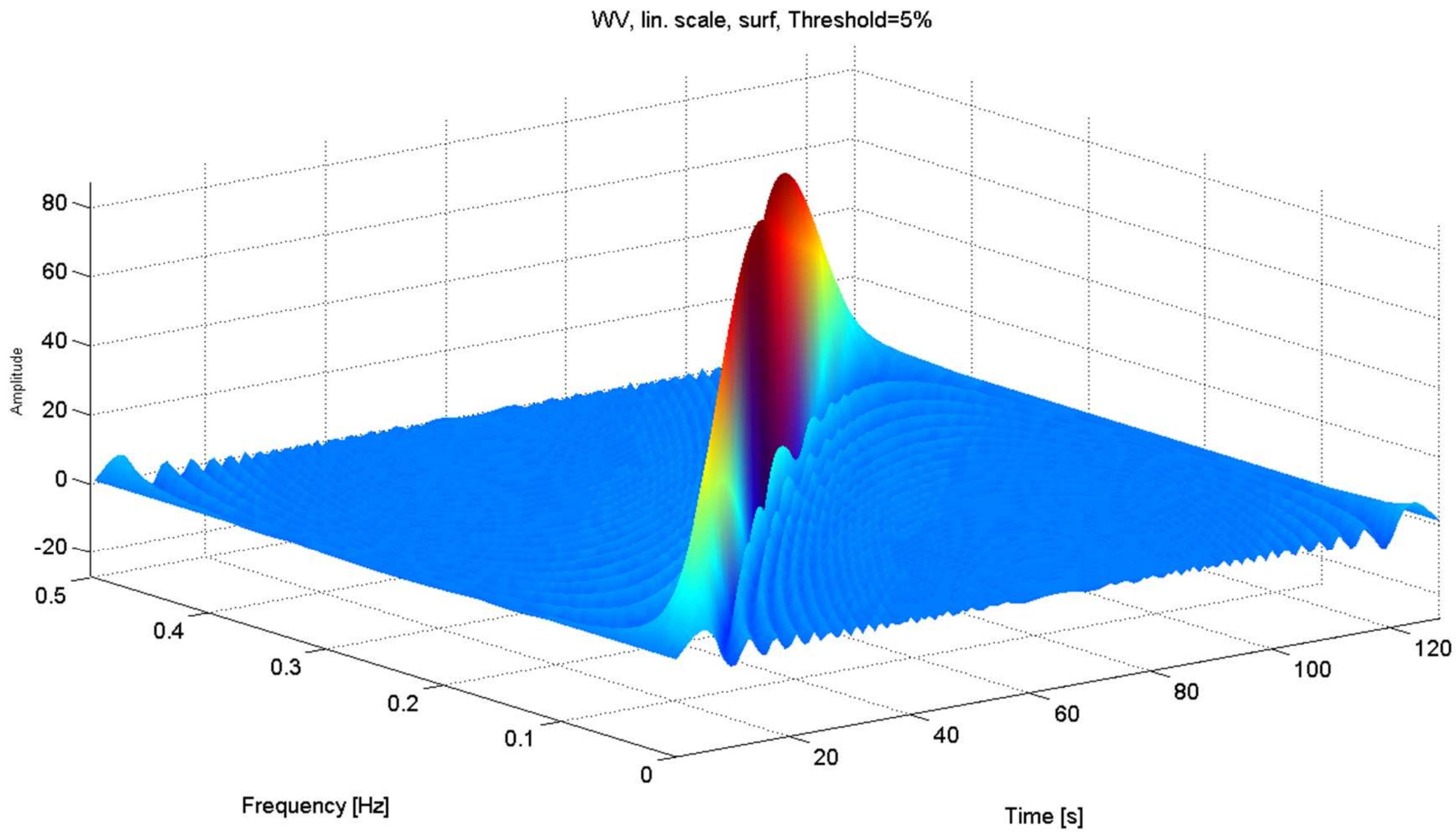
or equivalently

$$W_s(t,\omega) = \frac{1}{2\pi}\int S\left(\omega + {}^{\theta}\!/_{2}\right) S^*\left(t - {}^{\theta}\!/_{2}\right) e^{-j\theta t} d\theta$$

The Wigner-Ville Distribution



The Wigner-Ville Distribution



The Wigner-Ville Distribution

- In an analogy to the STFT, the window is basically a shifted version of the same signal
- It is obtained by comparing the information of the signal with its own information at other times and frequencies
- It possesses several interesting properties, described as follows

Properties of the WVD

- **Energy conservation**: by integrating the WVD of s all over the time-frequency plane, the energy of s is obtained

$$E_s = \iint W_s(t,\omega)dtd\omega$$

- **Real-valued**: the WVD is real-valued across time and frequency

$$W_s(t,\omega) \in \mathbb{R}, \qquad \forall\, t,\omega$$

Properties of the WVD

- **Marginal properties**: the energy spectral density and the instantaneous power can be obtained as marginal distributions of W_s

$$\int W_s(t,\omega)dt = |S(\omega)|^2$$

$$\int W_s(t,\omega)d\omega = |s(t)|^2$$

Properties of the WVD

- **Translation covariance**: the WVD is time- and frequency-covariant

$$x(t) = s(t - t_0) \;\Rightarrow\; W_x(t, \omega) = W_s(t - t_0, \omega)$$

$$x(t) = s(t)e^{-j\omega_0 t} \;\Rightarrow\; W_x(t, \omega) = W_s(t, \omega - \omega_0)$$

- **Dilation covariance**: the WVD also preserves dilation

$$x(t) = \sqrt{k}s(kt) \;;\; k > 0 \;\Rightarrow\; W_x(t, \omega) = W_s\left(kt, {}^{\omega}/_{k}\right)$$

Properties of the WVD

- **Compatibility with filterings**: it expresses the fact that if a signal x is the convolution of s and h, the WVD of x is the time-convolution between the WVD of h and the WVD of s

$$x(t) = \int s(\tau)\, h(t-\tau) d\tau \;\Rightarrow$$

$$W_x(t,\omega) = \int W_s(\tau,\omega) W_h(t-\tau,\omega) d\tau$$

Properties of the WVD

- **Compatibility with modulations :** this is the dual property of the previous one : if x is the modulation of s by a function m, the WVD of x is the frequency-convolution between the WVD of s and the WVD of m

$$x(t) = s(t)m(t) \Rightarrow$$
$$W_x(t,\omega) = \int W_s(\tau,\theta)W_m(t,\omega-\theta)d\theta$$

Properties of the WVD

- **Wide-sense support conservation :** if a signal has a compact support in time (respectively in frequency), then its WVD also has the same compact support in time (respectively in frequency). This is also called **weak finite support**

$$s(t) = 0, |t| > T \;\Rightarrow\; W_s(t,\omega) = 0, |t| > T$$

$$S(\omega) = 0, |\omega| > B \;\Rightarrow\; W_s(t,\omega) = 0, |\omega| > B$$

- However, the WVD does not have **strong finite support**

Properties of the WVD

- **Unitarity :** the unitarity property expresses the conservation of the scalar product from the time-domain to the time-frequency domain (apart from the squared)

$$\left| \int s(t)x^*(t)dt \right|^2 = \iint W_s(t,\omega){W_x}^*(t,\omega)dtd\omega$$

Properties of the WVD

- **Instantaneous frequency and group delay :** The instantaneous frequency characterizes a local frequency behaviour as a function of time. In a dual way, the local time behaviour as a function of frequency is described by the group delay

- In order to introduce these terms, the concept of **analytic signal** $s_a(t)$ must be defined first

Properties of the WVD

- For any real valued signal s(t), we associate a complex valued signal $s_a(t)$ defined as

$$s_a(t) = s(t) + jHT(s(t))$$

where $HT(s(t))$ is the **Hilbert transform** of $s(t)$

- $s_a(t)$ is called the analytic signal associated to $s(t)$

Properties of the WVD

- This definition has a simple interpretation in the frequency domain since S_a is a single-sided Fourier transform where the negative frequency values have been removed, the strictly positive ones have been doubled, and the DC component is kept unchanged

$$S_a(\omega) = 0 \quad \text{if} \quad \omega < 0$$

$$S_a(\omega) = S(0) \quad \text{if} \quad \omega = 0$$

$$S_a(\omega) = 2S(\omega) \quad \text{if} \quad \omega > 0$$

Properties of the WVD

- **Instantaneous frequency:** the instantaneous frequency of a signal s can be recovered from the WVD as its first order moment (or center of gravity) in frequency

$$f_s(t) = \frac{\int \omega W_{s_a}(t,\omega)d\omega}{\int W_{s_a}(t,\omega)d\omega}$$

Properties of the WVD

- **Group delay:** the group delay of a signal s can be recovered from the WVD as its first order moment (or center of gravity) in time

$$t_s(\omega) = \frac{\int t W_{s_a}(t,\omega)dt}{\int W_{s_a}(t,\omega)dt}$$

Interference in the WVD

- As the WVD is a bilinear function of the signal s, the quadratic superposition principle applies

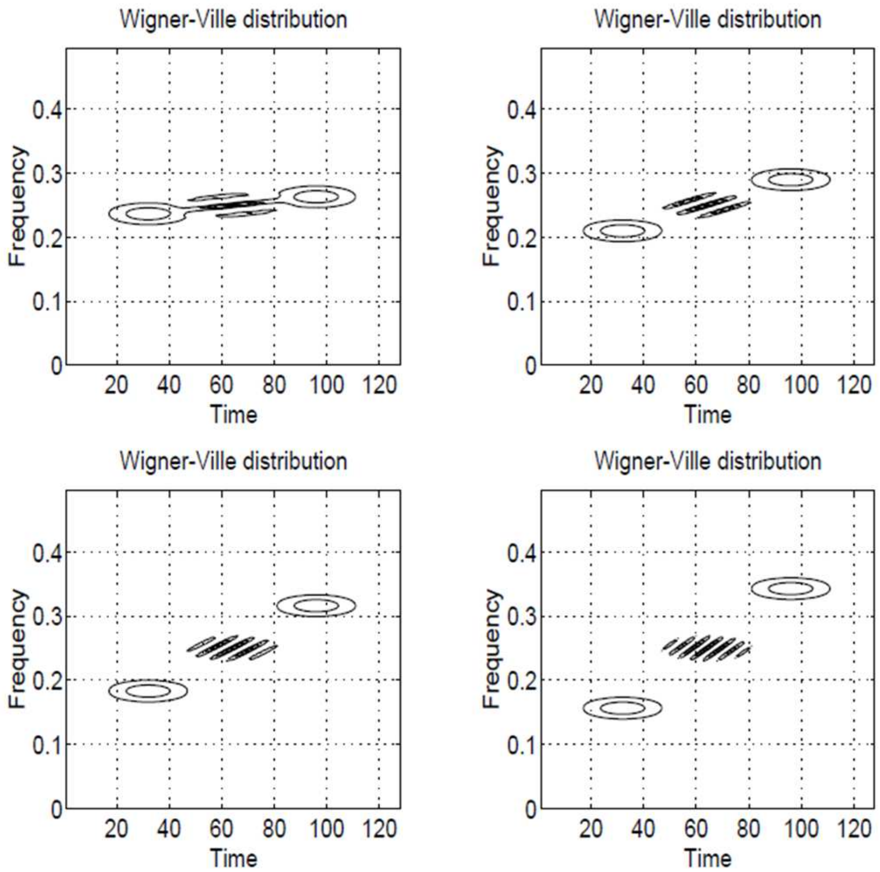
$$W_{s+x}(t,\omega) = W_s(t,\omega) + W_x(t,\omega) + 2\Re\{W_{s,x}(t,\omega)\}$$

where

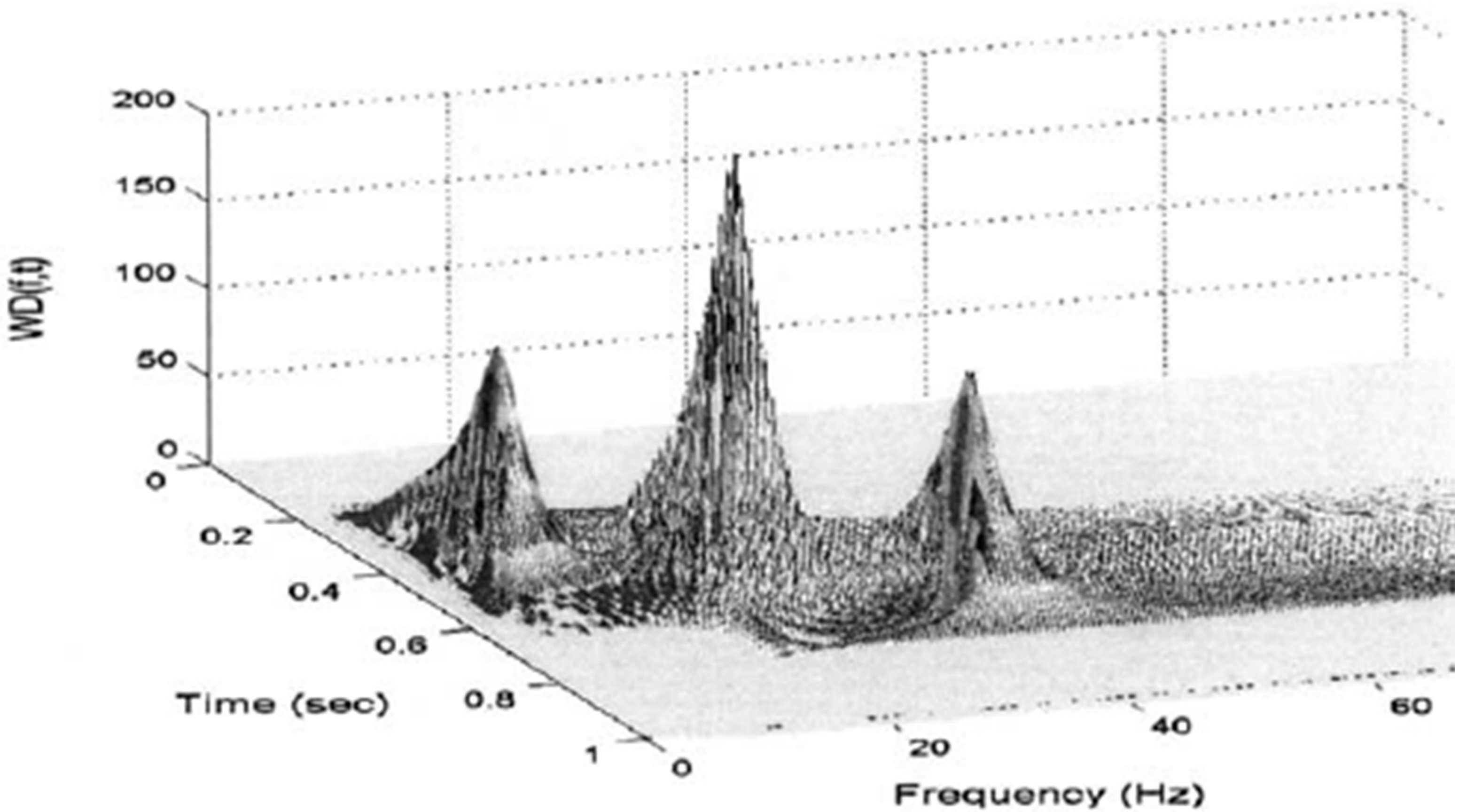
$$W_{s,x}(t,\omega) = \frac{1}{2\pi}\int s(t + {}^{\tau}/_{2})x^*(t - {}^{\tau}/_{2})e^{-j\omega\tau}d\tau$$

is the cross-WVD of s and x

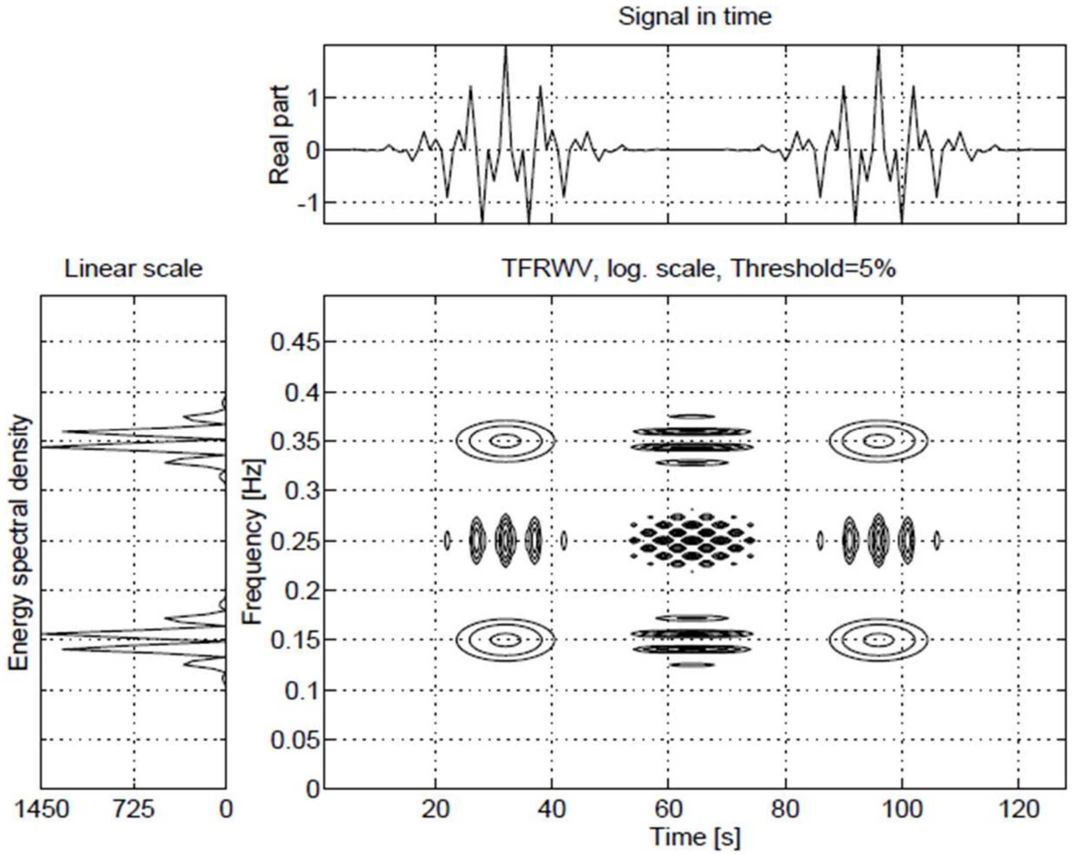
Interference in the WVD



Interference in the WVD



Interference in the WVD



Interference in the WVD

- These interference terms are troublesome since they may overlap with auto-terms (signal terms) and thus make it difficult to visually interpret the WVD image.
- It appears that these terms must be present or the good properties of the WVD (marginal properties, instantaneous frequency and group delay, localization, unitarity . . .) cannot be satisfied
- There is a trade-off between the quantity of interferences and the number of good properties

Pseudo-WVD

- The definition of the WVD requires the knowledge of

$$q_s(t,\omega) = s\left(t + {}^{\tau}/_{2}\right)s^*\left(t - {}^{\tau}/_{2}\right)$$

from $t = -\infty$ to $t = +\infty$, which can be a problem in practice

- Often a windowed version of $q_s(t,\omega)$ is used, leading to the Pseudo-WVD (PWVD)

$$PW_s(t,\omega) = \frac{1}{2\pi}\int s\left(t + {}^{\tau}/_{2}\right)s^*\left(t - {}^{\tau}/_{2}\right)h(t)e^{-j\omega\tau}d\tau$$

Pseudo-WVD

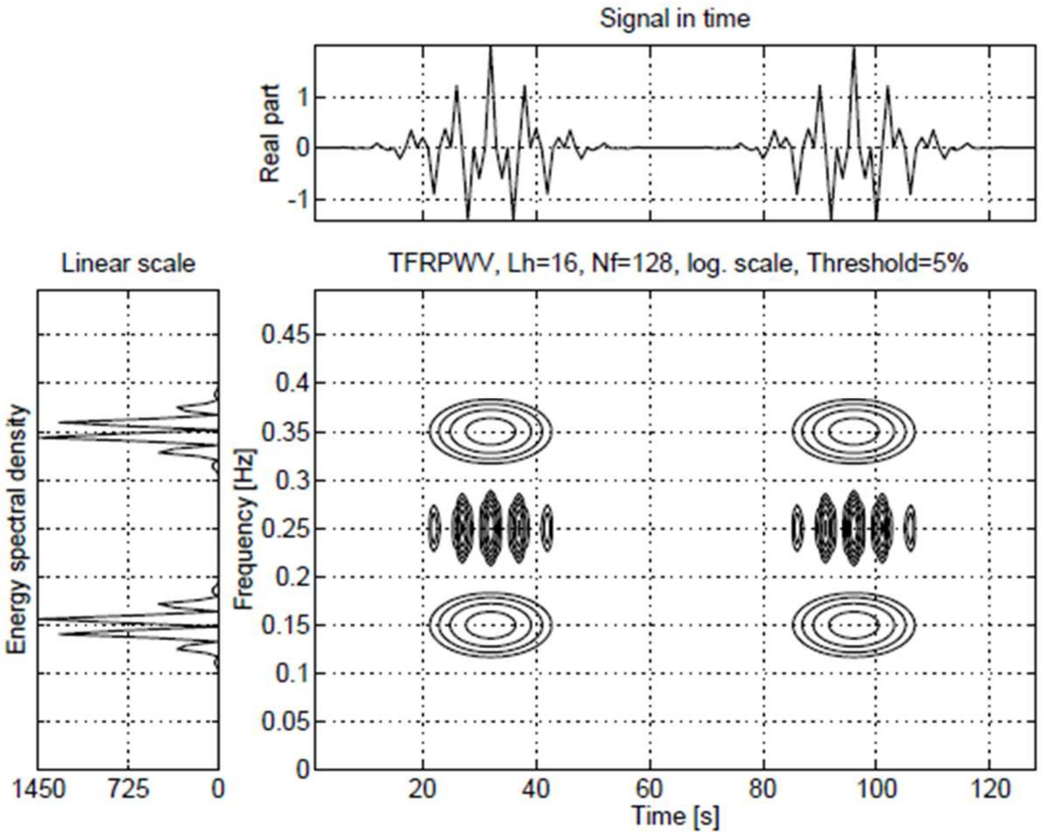
- This is equivalent to a frequency smoothing of the WVD, since

$$PW_s(t,\omega) = \int H(\omega-\theta)W_s(t,\theta)\,d\theta$$

where $H(\omega)$ is the Fourier transform of $h(t)$

- Because of their oscillating nature, the interferences will be attenuated in the pseudo-WVD compared to the WVD

Pseudo-WVD


Signal in time
Real part
1
0
-1
Linear scale
TFRPWV, Lh=16, Nf=128, log. scale, Threshold=5%
Energy spectral density
1450
725
0
Frequency [Hz]
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
20
40
60
80
100
120
Time [s]

Pseudo-WVD

- However, the consequence of this improved readability is that many properties of the WVD are lost:
  - The marginal properties
  - The unitarity
  - The frequency-support conservation
- The frequency-widths of the auto-terms are increased by this operation

References and further reading

- Time Frequency Analysis: Theory and Applications by Leon Cohen. Prentice Hall; 1994. Chapter 8
- Biosignal and Medical Image Processing, Second Edition by John L. Semmlow. CRC press; 2009. chapter 6 pp. 147-151
- The Time Frequency Toolbox tutorial (http://tftb.nongnu.org/tutorial.pdf)
- Material from *Signals and System* course, FI-UNER