On the Linear Aspects of Horse Racing from the Bookmaker's Perspective
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1. Introduction and condensed summary of main idea:

Suppose you are a bookmaker laying bets on the winner of an n horse race. Define the following n-dimensional vectors over the real numbers as follows:

$h = (h_1, h_2, ..., h_n)^T$ where h_i is the probability that the i^{th} horse will win the race

(note if we ignore the possibility of a dead heat, we can assume that $h_1 + ... + h_n = 1$)

$w = (w_1, w_2, ..., w_n)^T$ where w_i is the total amount wagered on the i^{th} horse in dollars

$p = (p_1, p_2, ..., p_n)^T$ where p_i is the net profit earned by the bookmaker in case the i^{th} horse wins

$a = (a_1, a_2, ..., a_n)^T$ where a_i is the decimal odds offered by the bookmaker on the i^{th} horse

By "decimal odds" we mean that if the i^{th} horse wins the race, the bookmaker will pay a total of a_i dollars for every \$1 wagered on the i^{th} horse (a_i includes both the original \$1 stake plus winnings so $a_i > 1$ whenever $1 \le i \le n$)

We call h the **<u>probability</u>** vector, w the **<u>wager</u>** vector, p the **<u>profit</u>** vector and a the **<u>odds</u>** vector

If there is a fixed "vig" i.e., commission percentage v_0, let $v = 1 + v_0$ then h and a should ideally have the following relation: for each i, $a_i = 1/(h_i v)$

That is given v_0, we have $a = (a_1, a_2, ..., a_n)^T = (1/(h_1 v), 1/(h_2 v), ..., 1/(h_n v))^T$

Let M_0, M_1, M and N be n x n matrices such that

(i) For all $i, j \in \{1, 2, ..., n\}$ $M_0(i, j) = 1$ (we call M_0 the *inflow* matrix and M_1 the *outflow* matrix)

(ii) For all $i, j \in \{1, 2, ..., n\}$ $M_1(i, i) = a_i$ and if $i \neq j$ then $M_1(i, j) = 0$

(iii) $M = M_0 - M_1$ (we call M the **<u>odds</u>** matrix)

(iv) $N = M^{-1}$

Let EV denote the *expected value* of the race from the point of view of the bookmaker and let ● indicate the dot product operation on n-dimensional vectors. Upon these definitions we have the equations

(I) $Mw = p$ (profit vector = odds matrix times wager vector, inverse exists)

(II) $w = Np$ (wager vector = inverse of odds matrix times profit vector)

(III) $EV = p \bullet h = (Mw) \bullet h$ (expected value = dot product of the profit vector and probability vector)

2. Expand and construct from first principles:

(i) $h = (h_1, h_2, ..., h_n)^T$ where h_i is the probability that the i^{th} horse wins the race

(ii) $v = 1 + v_0$ where $0 < v_0 < 1$ and v_0 is the bookmaker's commission (hence $v > 1$)

(iii) $w = (w_1, w_2, ..., w_n)^T$ where w_i is the total amount wagered on the i^{th} horse in dollars

define $w_\Sigma = w_1 + w_2 + ... + w_n$

Then w_Σ is the bookmaker's gross revenue on the bets laid. That is, w_Σ is the total amount of **in-flowing** money to the bookmaker (regardless of what horse wins the race). Now, whatever the odds may be, suppose (iv) $a = (a_1, a_2, ..., a_n)^T$ where a_i is the decimal odds for the i^{th} horse

This is to say that the total **out-flowing** money to the bookmaker is $a_i w_i$ just in case the i^{th} horse wins the race.

So we defined two n x n matrices earlier, M_0 and M_1 whenever $1 \le i, j \le n$

(v) $M_0(i, j) = 1$, (vi) $M_1(i, i) = a_i$ and if $i \neq j$ then $M_1(i, j) = 0$, and (vii) $M = M_0 - M_1$

Notice that every entry of the vector $M_0 w$ is w_Σ

$M_0 w = (w_\Sigma, w_\Sigma, ..., w_\Sigma)^T$ is the vector whose i^{th} entry is the total **in-flowing** money if the i^{th} horse wins

$M_1 w = (a_1 w_1, a_2 w_2, ..., a_n w_n)^T$ is the vector whose i^{th} entry is the total **out-flowing** money if the i^{th} horse wins

Letting $M = M_0 – M_1$ the vector whose i^{th} entry is the net profit to the bookmaker if the i^{th} horse wins is

$$p = M_0 w - M_1 w = (M_0 – M_1)w = Mw$$

We define two predicates with regard to wager vectors given fixed vector h and scalar v_0 on the assumptions that (i) $v = 1 + v_0$, (ii) for each i, the decimal odds $a_i = 1/(vh_i)$ and (iii) $\bullet$ denotes the dot product operator

We say that the wager vector w is **<u>Dutch</u>** when $p_i \ge 0$ for every i and $Mw = p = (p_1, p_2, ..., p_n)$

We say that the wager vector w is **<u>sound</u>** when $(Mw) \bullet h \ge 0$

That is, a wager vector is Dutch when the profit is non-negative regardless of which horse wins the race and the vector is sound when the expected value of the profit for the race is non-negative.

If X is a set of wager vectors, we say that X is Dutch just in case every member of X is Dutch, and we say that X is sound just in case every member of X is sound

Clearly every Dutch wager is sound and every Dutch set is sound (but not vice versa).

Note that on the assumption that $v > 1$, every wager vector is sound.
Consider the vector Mw

$Mw = ((1\text{-}1/(vh_1))w_1 + \ldots + w_n),\ w_1 + (1\text{-}1/(vh_2))w_2 + \ldots + w_n,\ \ldots,\ w_1 + \ldots + (1\text{-}1/(vh_n))w_n)^T$ and

So

$(Mw) \bullet h = h_1((1\text{-}1/(vh_1))w_1 + w_2 + \ldots + w_n) + \ldots + h_n(w_1 + w_2 + \ldots + (1\text{-}1/(vh_n))w_n)$

$(Mw) \bullet h = (h_1 + \ldots + h_n)w_1 + \ldots + (h_1 + \ldots + h_n)w_n - (w_1/v + \ldots + w_n/v)$

$(Mw) \bullet h = w_1 + \ldots + w_n - (1/v)(w_1 + \ldots + w_n)$

So as long as $v > 1$, the dot product $(Mw) \bullet h > 0$ QED.

3. Ideal wager distributions

If it is the case that for all i , $w_i = w_\Sigma h_i$ and $a_i = 1/(vh_i)$ then we say that w is an **ideal** wager vector with respect to h

That is, w is ideal when for each i the portion of the sum of all wagers wagered on the i^{th} horse is equal to the probability that the i^{th} horse will win. Observe that in this case profit is constant no matter which horse wins:

Let $w_i = w_\Sigma h_i$ for all i. Then $Mw = M(w_\Sigma h) = w_\Sigma Mh$

The i^{th} entry of the vector Mh is

$$h_1(1) + \ldots + h_i(1 - 1/(vh_i)) + \ldots + h_n(1) = h_1 + \ldots + h_i - 1/v + \ldots + h_n = h_1 + \ldots + h_n - 1/v = 1 - 1/v$$

so is the same for all i. Hence

$Mw = w_\Sigma Mh = (w_\Sigma(1 - 1/v), w_\Sigma(1 - 1/v), \ldots, w_\Sigma(1 - 1/v))^T$

Practically speaking: a book of wagers is ideal when the profit is both positive and constant for all possible winners of the race. Notice that an ideal wager is Dutch.

Notice that if $w_\Sigma > 0$ (which is assumed) and $v = 1 + v_0$ then

(i) $w_\Sigma(1 - 1/v) > 0$ if and only if $v_0 > 0$

(ii) $w_\Sigma(1 - 1/v) = 0$ if and only if $v_0 = 0$

That is, given an ideal wager vector the bookmaker ensures a positive and a highly stable/predictable net profit regardless of which horse wins the race. This constant profit is zero precisely when the commission rate is zero, and the constant net profit is positive when the commission rate is positive. In practice no bookmaker could operate with a non-positive commission rate. Since it is always uncertain which horse will win in practice, an ideal wager is desirable from the bookmaker's point of view with regards to risk avoidance and/or management.

So if we assume a bookmaker's commission $v_0 > 0$ then every ideal wager is Dutch. Both ideal and Dutch wager vectors correspond to the impossibility of a negative net profit, however this is generally not the case for wager vectors.

In practice, a precisely ideal book will be very rare however we can simulate a plausible expectation by considering a distribution of wagers which deviates from an ideal distribution by way of some random noise. The greater this noise, the less likely the wager vector will be Dutch.

4. From odds first instead of probabilities

Suppose that we are not given the vector *h* assigning a probability of victory to each of the *n* horses directly but instead we are given the odds vector $a = (a_1, a_2, ..., a_n)^T$ Based on some reasonable assumptions such as the rationality of the bookmaker, we can assume that

$$a_i = 1/(vh_i)$$

$$1/a_i = vh_i$$

$$1/(va_i) = h_i$$

$$(1/v)(1/a_i) = h_i$$

$$(1/v)(1/a_1 + ... + 1/a_n) = h_1 + ... + h_n$$

$$(1/v)(1/a_1 + ... + 1/a_n) = 1$$

$$1/a_1 + ... + 1/a_n = v$$

From this calculation of v we can use the odds to determine the implied probability of victory for the i^{th} horse namely $h_i = 1/(va_i)$

In this respect we can speak of a wager being ideal or Dutch with respect to the odds *a* as opposed to the probabilities *h*

5. On the inverse of the odds Matrix

To prove that M is invertible, we suppose M^{-1} does not exist and derive a contradiction. If this is the case then there exists a vector

$x = (x_1, x_2, \dots , x_n)^T$ which is non-zero such that $Mx = 0$.

Consider the vector Mx

$Mx = (x_1 - x_1/(vh_1) + x_2 + \dots + x_n,\ x_1 + x_2 - x_2/(vh_2) + \dots + x_n, \dots, x_1 + x_2 + \dots + x_n - x_n/(vh_n))^T$

So if $Mx = 0$ then

For all *i*	$x_1 + x_2 + \dots + x_n - x_i/(vh_i) = 0$
So for all *i*	$x_1 + x_2 + \dots + x_n = x_i/(vh_i)$
So for all *i*	$vh_i(x_1 + x_2 + \dots + x_n) = x_i$
Therefore	$v(h_1 + \dots + h_n)(x_1 + \dots + x_n) = x_1 + \dots + x_n$
Therefore on the assumption that $x \neq 0$	$v(h_1 + \dots + h_n) = 1$
Therefore	$v(1) = 1$
Therefore	$v = 1$

Which contradicts the assumption of a positive commission of the bookmaker, namely $v > 1$. QED.

Notice that $M^T = M$ and therefore $N^T = N$ where $N = M^{-1}$