Libraria Calculosis

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Contents

Chapter 1

Approximating $\sqrt{95}$ Using Newton's Method

Let $f(x) = x^2 - 95$, then $f'(x) = 2x$. Newton's Update Function then produces

$$x - \frac{f(x)}{f'(x)}$$

$$= x\frac{x^2 - 95}{2x}$$

$$= \frac{2x^2}{2x} - \frac{x^2 - 95}{2x}$$

$$= \frac{x^2 + 95}{2x}$$

Thus, we can define a recursive sequence $x_{n+1} = \frac{x_n^2+95}{2x_n}$. Using a starting point of $x_0 = 10$, since $\sqrt{95}$ is close to $\sqrt{100} = 10$. we get the following

x_0	10
x_1	9.75
x_2	9.764764

Notice that even for x_2, we get an approximation correct to 6 digits.

1.0.1 If Newton's method converges, does it always converge to a root?

Another way of phrasing this is if $\{x_n\} \to L$, is $f(L) = 0$?.

Since the sequence is convergent, we can replace x_{n+1} and x_n with L, and solve $L = L - \frac{f(L)}{f'(L)}$ which implies that $f(L) = 0$. Therefore, if Newton's Method converges, we're guaranteed that it converges to a root.

Chapter 2

Derivatives of Inverse Functions

We still haven't seen $\frac{\mathrm{d}}{\mathrm{dx}}[\ln(x)]$. Our goal is to relate the derivative of a function with the derivative of its inverse. Given a function $f(x)$, we can define the linearization at a point a as $\mathcal{L}_a^f$. Then, if we invert the linearization function, we get $\left(\mathcal{L}_a^f\right)^{-1} = a + \frac{1}{f'(a)}(x - f(a))$. If we take a point $f(a) = b$, then we get that $a = f^{-1}(b)$. Thus $\left(\mathcal{L}_a^f\right)^{-1} = f^{-1}(b) + \frac{x-b}{f'(f^{-1}(b))}$. Therefore, we know that the inverse linearization of $f(x)$ at a point $x = a$, is the same as $\mathcal{L}_b^{f^{-1}}$. That is to say, the inverse linearization of a function at a point a is the same as the linearization of the inverse function at a point $f(a)$. Thus, $\mathcal{L}_b^{f^{-1}} = f^{-1}(b) + \frac{\mathrm{d}}{\mathrm{dx}}\left[f^{-1}\right]|_{x=b} \cdot (x - b)$. Thus, by equality we know that $\frac{\mathrm{d}}{\mathrm{dx}}\left[f^{-1}\right]|_{x=b} = \frac{1}{f'(f^{-1}(b))}$

Theorem: Inverse Function Theorem

If f is an invertible function on $[c, d]$, differentiable on (c, d), and that $f'(a) \neq 0$, then f^{-1} is differentiable at $x = f(a)$ with the derivative $\frac{\mathrm{d}}{\mathrm{dx}}\left[f^{-1}\right]|_{x=b} = \frac{1}{f'(f^{-1}(b))}$. Along with that $\left(\mathcal{L}_a^f\right)^{-1} = \mathcal{L}_{f(a)}^{f^{-1}}$

2.0.1 Proof sketch

If f is an invertible function, then $f(f^{-1}(x)) = x$. Thus, since they're equal $\frac{\mathrm{d}}{\mathrm{dx}}\left[f(f^{-1}(x))\right] = \frac{\mathrm{d}}{\mathrm{dx}}[x]$

$$\frac{\mathrm{d}}{\mathrm{dx}}\left[f(f^{-1}(x))\right] = f'(f^{-1}(x)) \cdot \frac{\mathrm{d}}{\mathrm{dx}}\left[f^{-1}(x)\right] = 1$$

$$\frac{\mathrm{d}}{\mathrm{dx}}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

$\mathcal{QED}$

2.0.2 Testing Inverse Function Theorem

Let $f(x) = x^5$, then $f^{-1}(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$. Thus, we know that $\frac{\mathrm{d}}{\mathrm{dx}}\left[f^{-1}(x)\right] = \frac{1}{5}x^{-4}5 = \frac{1}{5 \cdot \sqrt[5]{x^4}}$.
Using IFT, we get that the derivative is $\frac{1}{f'(f^{-1}(x))} = \frac{1}{5(f^{-1}(x))^4} = \frac{1}{5(x)^{\frac{4}{5}}}$

2.0.3 Derivative of $\ln(x)$

Using IFT, we can now calculate the derivative of $\ln(x)$.

$$\frac{\mathrm{d}}{\mathrm{dx}}[\ln(x)] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

Furthermore, we can generalize this to know that $\frac{\mathrm{d}}{\mathrm{dx}}[\log_a(x)] = \frac{1}{x\ln(a)}$

2.0.4 Derivative of Inverse Trig Functions

We know that $\frac{\mathrm{d}}{\mathrm{dx}}[\sin(x)] = \cos(x)$.

$$\frac{\mathrm{d}}{\mathrm{dx}}[\arcsin x] = \frac{1}{\cos(\arcsin(x))}$$

Let $\theta = \arcsin(x)$, then $\sin(\theta) = x$. Using a triangle with a hypotenuse of 1 and an opposite side to θ of x. Then by the pythagorean theorem, we know the final side a can be found as $a = \sqrt{1-x^2}$. Thus, $\cos(\arcsin(x)) = \cos(\theta) = \sqrt{1-x^2}$

Therefore, the derivative of the $\arcsin(x)$ function is $\frac{1}{\sqrt{1-x^2}}$. Similarly, $\frac{\mathrm{d}}{\mathrm{dx}}[\arccos(x)] = \frac{-1}{\sqrt{1-x^2}}$ and $\frac{\mathrm{d}}{\mathrm{dx}}[\arctan(x)] = \frac{1}{x^2+1}$.

Chapter 3

Implicit Differentiation

We understand how to take the derivative of explicitly defined functions, as in $y = f(x)$. However, there are ways to differentiate implicitly defined functions, such as $x^2 + y^2 = 1$. That expression implicitly defines two functions. An implicit function is an equation where y is a function of x, thus we can (poorly) rewrite the equation for the unit circle as $x^2 + f(x)^2 = 1$.

To take the derivative of an implicit function, we take the derivative of both sides

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dx}}\left[x^2 + y^2 = 1\right] \\ \implies &\frac{\mathrm{d}}{\mathrm{dx}}\left[x^2 + y^2\right] = \frac{\mathrm{d}}{\mathrm{dx}}[1] \\ \implies &\frac{\mathrm{d}}{\mathrm{dx}}\left[x^2\right] + \frac{\mathrm{d}}{\mathrm{dx}}\left[y^2\right] = 0 \\ \implies &2x + 2y\frac{\mathrm{dy}}{\mathrm{dx}} = 0 \\ \implies &\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{-x}{y} \end{aligned}$$

Often with implicitly defined functions, the result of differentiation will also be implicit.

If we wanted to know the slope(s) of the unit circle when $x = \frac{1}{2}$, we can that the two points we get are

$$\begin{aligned} &\left(\frac{1}{2}\right)^2 + y^2 = 1 \\ \implies &y^2 = \frac{3}{4} \\ \implies &y = \sqrt{\frac{3}{4}} = \frac{\pm\sqrt{3}}{2} \end{aligned}$$

From this, we can use the implicit derivative on both points to get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\frac{1}{2}}{\frac{-\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

Thus, we see that the two slopes of the unit circle at $x = \frac{1}{2}$ are $\frac{\pm 1}{\sqrt{3}}$.

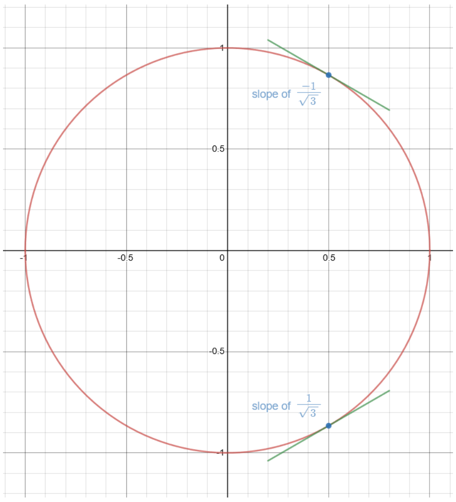

Figure 3.1: Slopes of the unit circle at $x = \frac{1}{2}$

Trying this again using $x^2 + y^2 = -1$, we see that the implicit derivative is the same as $x^2 + y^2 = 1$, however, the derivative doesn't actually make sense, as the sum of two squares is always nonnegative. Thus, $x^2 + y^2 = -1$ does not define a function in $\mathbb{R}^2$, as there are no pairs of real numbers (x, y) that satisfy $x^2 + y^2 = -1$.

Similarly, the relation $2x = x$ defines a point rather than a function, and thus differentiating doesn't yield anything reasonable, as the result is $2 = 1$.

3.1 Example

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ if $x^3y^5 + 3x = 8y^3 + 1$

$$\frac{\mathrm{d}}{\mathrm{dx}}\left[x^3y^5 + 3x = 8y^3 + 1\right]$$

$$\implies \frac{\mathrm{d}}{\mathrm{dx}}\left[x^3y^5 + 3x\right] = 24y^2\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\implies \frac{\mathrm{d}}{\mathrm{dx}}\left[x^3y^5\right] + 3 = 24y^2\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\implies 3x^2y^5 + 5y^4x^3\frac{\mathrm{dy}}{\mathrm{dx}} + 3 = 24y^2\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\implies \frac{\mathrm{dy}}{\mathrm{dx}}\left(5x^3y^4 - 24y^2\right) = -3x^2y^5 - 3$$

$$\implies \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{-3x^2y^5 - 3}{5x^3y^4 - 24y^2}$$

Chapter 4

Logarithmic Differentiation

Logarithmic Differentiation is a trick in which you take the natural logarithm of both sides before implicit differentiation.

Differentiation with a logarithm gives means of dealing with things like $y = f(x)^{g(x)}$. Along with that, since $\ln(ab) = \ln(a) + \ln(b)$, using logarithmic differentiation allows us to skip using the product rule.

4.0.1 Example 1

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ for $y = x^x$

$$\begin{aligned}
y &= x^x \\
\implies \ln(y) &= \ln(x^x) \\
\implies \ln(y) &= x\ln(x) \\
\implies \frac{\mathrm{d}}{\mathrm{dx}}&\left[\ln(y) = x\ln(x)\right] \\
\implies \frac{\mathrm{d}}{\mathrm{dx}}\left[\ln(y)\right] &= \frac{\mathrm{d}}{\mathrm{dx}}\left[x\ln(x)\right] \\
\implies \frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}} &= \ln(x) + 1 \\
\implies \frac{\mathrm{dy}}{\mathrm{dx}} &= y\left(\ln(x) + 1\right)
\end{aligned}$$

Since the original function was defined explicitly, the derivative should be explicitly defined as well, thus we substitute $y = x^x$ to get

$$\frac{\mathrm{dy}}{\mathrm{dx}} = x^x\left(\ln(x) + 1\right)$$

4.0.2 Example 2

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ for $y = \frac{\sin(x)e^x x^3}{\ln(x)}$

$$\begin{aligned}
y &= \frac{sin(x)e^x x^3}{\ln(x)} \\
\ln(y) &= \ln\left(\frac{\sin(x)e^x x^3}{\ln(x)}\right) \\
\implies \ln(y) &= \ln(\sin(x)) + \ln(e^x) + \ln(x^3) - \ln(\ln(x)) \\
\implies \ln(y) &= \ln(\sin(x)) + 3\ln(x) - \ln(\ln(x)) + x \\
\implies \frac{\mathrm{d}}{\mathrm{dx}}&\left[\ln(y) = \ln(\sin(x)) + 3\ln(x) - \ln(\ln(x)) + x\right] \\
\implies \frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}} &= \frac{\cos(x)}{\sin(x)} + \frac{3}{x} + 1 - \frac{1}{x\ln(x)} \\
\implies \frac{\mathrm{dy}}{\mathrm{dx}} &= y\left(\frac{\cos(x)}{\sin(x)} + \frac{3}{x} + 1 - \frac{1}{x\ln(x)}\right)
\end{aligned}$$

Chapter 5

Extrema

def^{n}: A point c is a local maximum of a function f if there exists an open interval $\mathcal{I}$ containing the point c such that $f(x) \leq f(c) \forall x \in \mathcal{I}$. Similarly, c is a local minimum if $f(x) \geq f(c) \forall x \in \mathcal{I}$. If $f : \mathbb{R} \to \mathcal{I}$ then c is a global extrema. Thus, all global extrema are also local extrema. It is also true that many local extrema can exist

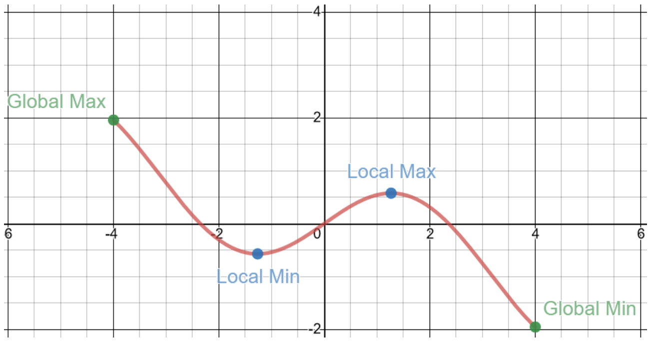

Figure 5.1: Examples of local and global extrema

Theorem: Local Extrema Theorem

If c is a local extrema and $f'(c)$ exists, then we know $f'(c) = 0$.

5.0.1 proof

Notice that if c is a local min of f, we can define a new function $g(x) = -f(x)$ and thus c is a local max of g. Now, suppose WLOG suppose c is a local max of f. Therefore, there exists an open interval $\mathcal{I} = (a, b)$ containing c, such that $f(x) \leq f(c) \forall x \in \mathcal{I}$. Furthermore, suppose that $f'(c)$ exists. Therefore, the following is true:

$$f(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

If $h > 0$ but is small enough that $a < c + h < b$ then we know that $f(c+h) \leq f(c)$ because c is a max of f in the interval $\mathcal{I}$. Therefore, we get that $\frac{f(c+h)-f(x)}{h} \leq 0$ and thus $f'(c) \leq 0$

If $h < 0$ but small ehough that $a < c + h < b$ then we know that $\frac{f(c+h)-f(c)}{h} \geq 0$ since division by a negative flips the inequality sign. Therefore, we get that $f'(c) \geq 0$

Therefore, since $f'(c) \leq 0$ and $f'(c) \geq 0$ we get that $f'(c) = 0$

$\mathcal{QED}$

5.0.2 Converse

Consider $f(x) = x^3$. Notice now that $f'(x) = 3x^2$ and thus $f'(0) = 0$, and $f(0)$ isn't a local extrema. Therefore, the converse isn't true. If we also look at $f(x) = |x|$, we know that $f(0)$ is a global min, however the derivative at $x = 0$ doesn't exist, and thus the theorem doesn't apply.

5.1 Finding Global Extrema

def$^{\text{n}}$: A point c is a critical point of $f'(c) = 0$ or $f'(c)$ doesn't exist.

Let's say that all max/mins occur at critical points. Combining this with the extreme value theorem gives us an algorithm for finding extrema.

Consider a differentiable function f on the closed interval $[a, b]$. To find all global extrema, we do the following:

1. find all critical points of f on $[a, b]$, let $\mathcal{C}$ be the set of all x-values of critical points.
2. Evaluate $f(a)$, $f(b)$ and $f(c) \forall c \in \mathcal{C}$.
3. Choose the biggest and smallest values of $f(a)$, $f(b)$ and $f(c) \forall c \in \mathcal{C}$
   The biggest is the global max, and the smallest is the global min.

5.1.1 Example

Find the max/min of $f(x) = e^{x^3 - 2x^2 - 7x}$ on $[0, 4]$

Finding critical points: We know that $\frac{\text{df}}{\text{dx}}$ is defined everywhere, and thus there will be no places on the interval where the derivative is undefined.

By the chain rule we know $\frac{\text{df}}{\text{dx}} = e^{x^3 - 2x^2 - 7x}(3x^2 - 4x - 7)$. Since $e^{p(x)}$ will never be zero, we know that $\frac{\text{df}}{\text{dx}}$ will be zero only when the quadratic term is 0. Since the quadratic factors to $(x + 1)(3x - 7)$, and since we know that $-1 \notin [0, 4]$, we know that the only critical point will be $\frac{7}{3}$.

$\left\{f(0), f(4), f\left(\frac{7}{3}\right)\right\} = \left\{1, e^4, e^{\frac{-392}{27}}\right\}$

From here, we can see that the global max and min are given by the points $(4, e^4)$, $\left(\frac{7}{3}, e^{\frac{-392}{27}}\right)$. Therefore, the global max is 4 and the global min is $\frac{7}{3}$.

This topic will be revisited upon the genesis of the curve sketching unit.

Chapter 6

The Mean Value Theorem

Theorem: Rolle's Theorem

If f is a continuous function on $[a, b]$ and differentiable on (a, b), and if $0 = f(a) = f(b)$, then $\exists c \in (a, b)$ such that $f'(c) = 0$.

6.0.1 Proof

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) with $f(a) = f(b) = 0$.

Case 1

$\exists x_0 \in (a_b)$ with $f(x_0) > 0$. The Extreme Value Theorem, shows that $f(x)$ achieves a max value on $[a, b]$. We know that the maximum point $c \in (a, b)$ is thus not at the endpoints. Hence, since $f'(c)$ exists, by the Local Extrema Theorem we know that $f'(c) = 0$.

Case 2

$\exists x_0 \in (a, b)$ with $f(x_0) < 0$. The Extreme Value Theorem shows that $f(x)$ achieves a min value on $[a, b]$. We know that the minimum point $c \in (a, b)$ is thus not at the endpoints. Hence, since $f'(c)$ exists, by the Local Extrema Theorem we know that $f'(c) = 0$.

Case 3

$f : [a, b] \to 0$, hence $f' : (a, b) \to 0$ and thus there are lots of choices for c.

Theorem: The Mean Value Theorem

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then, $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Rolle's Theorem is a special case of Mean Value Theorem.

6.0.2 Proof

Let $h(x) = f(x) - f(a) - \left(\frac{f(b)-f(a)}{b-a}\right)(x - a)$. Notice that $h(a) = h(b) = 0$. Since h is continuous on $[a, b]$ and differentiable on (a, b). Thus, since the conditions of Rolle's Theorem are met, we know that $\exists c \in (a, b)$ such that $h'(c) = 0$.

$$\frac{\mathrm{d}}{\mathrm{dx}}\left[h'(x)\right]|_{x=c} = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

Therefore, we get that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

6.0.3 Examples

Find all points c that satisfy the mean value theorem for $f(x) = x^3 + 2x^2 - x$ on $[-1, 2]$

By MVT we know that $f'(c) = \frac{f(2)-f(-1)}{2-(-1)} = \frac{14-2}{3} = 4$. Therefore, there is a point c where the derivative is equal to 4. $f'(x) = 3x^2 + 4x - 1$

$$f'(x) = 3x^2 + 4x - 1 = c$$

$$x = \frac{-4 \pm \sqrt{76}}{6}$$

Since $\frac{-4-\sqrt{76}}{6} \notin [-1, 2]$, we know that it is not a proper value of c. Therefore, the only value of c for this interval is $c = \frac{\sqrt{76}-4}{6}$

Suppose that a function f is continuous and differentiable everywhere. Furthermore, suppose f has two roots. Then show f′ has at least one root

Let a, b be distinct roots of f, then $f(a) = f(b) = 0$. Thus, by Rolle's Theorem, we know that $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$ and thus there is at least one root of f'.

6.1 Applications of MVT

Def^{n}: A function F is called an antiderivative of f is $F'(x) = f(x) \forall x \in \mathbb{R}$

$F(x) = \frac{x^2}{2}$ is an antiderivative of $f(x)$, since $F'(x) = \frac{2x}{2} = x = f(x)$

Theorem: The Constant Function Theorem

If $f'(x) = 0 \ \forall x \in \mathcal{I}$ then $f(x) = c \ \forall x \in \mathcal{I}$.

6.1.1 Proof

Let $\mathcal{I}$ be some interval, then suppose $f'(x) = 0 \ \forall x \in \mathcal{I}$. Note that f is continuous on $\mathcal{I}$. Let $x_1, x_2 \in \mathcal{I}$ where $x_1 \neq x_2$. Then, let $f(x_1) = k$. By the Mean Value Theorem, $\exists c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Since $(x_1, x_2) \subseteq \mathcal{I}$, we know that $f'(c) = 0$ and thus $\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0 \implies f(x_2) - f(x_1) = 0 \implies f(x_2) = f(x_1) = k$.

Since both x_1 and x_2 were arbitrary, we conclude that $f(x) = k \ \forall x \in \mathcal{I}$

Theorem: Uniqueness of Antiderivatives

Antiderivatives are not unique, since if we take $F(x) = h(x) + c$ and $f(x) = h'(x)$ then $F'(x) = h'(x) = f(x) \forall c \in \mathbb{R}$. However, outside of adding constants, antiderivatives are unique (although it is not straightforward). This can be shown using the Mean Value Theorem.

If $f'(x) = g'(x) \ \forall x \in \mathcal{I}$, and thus $f(x)$ and $g(x)$ are antiderivatives of the same function, then $\exists k \in \mathbb{R}$ such that $f(x) = g(x) + k \ \forall x \in \mathcal{I}$.

6.1.2 Proof

Let $h(x) = f(x) - g(x)$ and thus $h'(x) = f'(x) - g'(x) = 0 \ \forall x \in \mathcal{I}$. Hence, by The Constant Function Theorem, $\exists k \in \mathbb{R}$ such that $h(x) = k \ \forall x \in \mathcal{I}$. Therefore, $h(x) = f(x) - g(x) = k \ \forall k \in \mathcal{I} \implies f(x) = g(x) + k \ \forall x \in \mathcal{I}$.

6.1.3 Notation

The family of antiderivatives for a function $f(x)$ is denoted as follows, where $f(x)$ is referred to as the "integrand" and the dx is referred to as the "variable of integration".

$$F(x) = \int f(x) \ \mathrm{dx}$$

As an example, $\int x^2 \ \mathrm{dx} = \frac{x^2}{2}$

Theorem: Increasing/Decreasing Function Theorem

Let $\mathcal{I}$ be some interval with $x_1, x_2 \in \mathcal{I}$ where $x_1 < x_2$.

1. If $f'(x) > 0\,\forall x \in \mathcal{I}$ then, $f(x_2) > f(x_1)$ and we say f is *increasing* on $\mathcal{I}$
2. If $f'(x) \geq 0\,\forall x \in \mathcal{I}$ then, $f(x_2) \geq f(x_1)$ and we say f is *non-decreasing* on $\mathcal{I}$
3. If $f'(x) < 0\,\forall x \in \mathcal{I}$ then, $f(x_2) < f(x_1)$ then we say f is *decreasing* on $\mathcal{I}$
4. If $f'(x) \leq 0\,\forall x \in \mathcal{I}$ then, $f(x_2) \leq f(x_1)$ and we say f is *non-increasing* on $\mathcal{I}$

6.1.4 Proof of (1)

Let $\mathcal{I}$ be an interval such that $x_1 < x_2 \in \mathcal{I}$. Suppose that $f'(x) > 0\,\forall x \in \mathcal{I}$. Since f' exists, we know that f is continuous $\forall x \in \mathcal{I}$. We can thus apply The Mean Value Theorem to $[x_1, x_2]$.

$\exists c \in (x_1, x_2)$ such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$. Since we know $f'(c) > 0$ and $x_2 - x_1 > 0$, we know that $f(x_2) - f(x_1) > 0$. Thus, f is increasing over $\mathcal{I}$.

6.1.5 Converses

The converse of (2) and (4) are true (if $f(x)$ is non-increasing/non-decreasing on $\mathcal{I}$ then $f'(x) > 0\,\forall x \in \mathcal{I}$).

The converse of (1) and (3) are false. Consider $f(x) = x^3$. Where $f(x_2) > f(x_1) \forall x_2 > x_1$ However, $f'(0) = 0$.

6.2 Functions with Bounded Derivatives

What information can we derive from a function f if we know the bounds of $\frac{\mathrm{df}}{\mathrm{dx}}$?

Suppose $m \leq \frac{\mathrm{df}}{\mathrm{dx}} \leq M\ \forall x \in (a, b)$. If f is continuous on $[a, b]$ then we can use Mean Value Theorem.

Let $x \in (a, b)$. Applying MVT we get $\exists c \in (a, x)$ with $f'(c) = \frac{f(x)-f(a)}{x-a}$.

$$m \leq f'(c) = \frac{f(x) - f(a)}{x - a} \leq M$$

$$m(x - a) \leq f(x) - f(a) \leq M(x - a)$$

$$f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$$

This implies that $f(x)$ is bounded between two lines.

Theorem: Bounded Derivative Theorem

If a function f is continuous on ${}^a_b\mathcal{I}_o$ and differentiable on ${}^a_b\mathcal{I}_c$ with $m \leq f'(x) \leq M\ \forall x \in {}^a_b\mathcal{I}_o$ then $f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)\ \forall x \in {}^a_b\mathcal{I}_c$.

6.2.1 Example

Show that $\ln(3) \in \left[\frac{7-e}{4}, \frac{3}{e}\right]$

Let $f(x) = \ln(x)$. Hence, $f'(x) = \frac{1}{x}$. We need to bound $f'(x)$ to use the theorem. Thus, we need some interval ${}^a_b\mathcal{I}_c$ with $a < 3 < b$. We also need $f(a)$ to be easily computable. If we take $a = e$ then $\ln(a) = 1$. and $1 < 3$. Since we don't need $f(b)$ to be easily computable, we can take $b = 4$.

Note that $\frac{\mathrm{d}}{\mathrm{dx}}\left[\frac{1}{x}\right] = \frac{-1}{x^2}$. Thus, from the Increasing Function Theorem we know that $f'(x)$ is decreasing. Thus, $f'(4) \leq f'(x) \leq f'(e)$. Since the derivative can be bound between $\frac{1}{4}$ and $\frac{1}{e}$, the natural choice for a range would be $f'(x) \in \left[\frac{1}{4}, \frac{1}{e}\right]$.

$$f(a) + m(x - a) \leq f(x) \leq f(a) + M(x - a)$$

$$1 + \frac{1}{4}(x - e) \leq f(x) \leq f(a) + \frac{1}{e}(x - a)$$

$$1 + \frac{1}{4}(3 - e) \leq f(x) \leq 1 + \frac{1}{e}(x - e)$$

$$\frac{7 - e}{4} \leq f(x) \leq \frac{3}{e}$$

$\mathcal{QED}$

Check out the example in the textbook that uses Bounded Derivative Theorem to show the following;

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

6.3 Comparing Functions Using Derivatives

Suppose f and g are continuous at $x = a$ and $f(a) = g(a)$

1. If $f'(x) < g'(x)\ \forall x > a$ then $f(x) < g(x)\ \forall x > a$
2. If $f'(x) > g'(x)\ \forall x > a$ then $f(x) > g(x)\ \forall x > a$
3. If $f'(x) \leq g'(x)\ \forall x > a$ then $f(x) \leq g(x)\ \forall x > a$
4. If $f'(x) \geq g'(x)\ \forall x > a$ then $f(x) \geq g(x)\ \forall x > a$

6.3.1 Proof of (3)

Suppose f and g are continuous at $x = a$ with $f(a) = g(a)$ and $f'(x) \leq g'(x)\ \forall x > a$.

Let $h(x) = f(x) - g(x)$. Then $h(a) = f(a) - g(a) = 0$. Let $x > a$, $h'(x) = f'(x) - g'(x)$. By the hypothesis, we know that when $x > a$ that $f'(x) \leq g'(x)$, thus, by the Decreasing Function Theorem. Hence, $h(x) \leq 0 \implies f(x) - g(x) \leq 0 \implies f(x) \leq g(x)\ \forall x > a$.