Find all values of r such that $8 \cdot 10^r + 1$ is a perfect square

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November 15, 2019

Assuming $8 \cdot 10^r + 1$ is a perfect square, let $n^2 = 8 \cdot 10^r + 1$. We can rearrange to get $8 \cdot 10^r = (n+1)(n-1)$. If we divide by 8, we find that $10^r = \frac{1}{8}(n-1)(n+1)$. From there, we can use base 10 logarithms to isolate for r. This can be written as a function of n.

$$f(n) = \log\left(\frac{1}{8}(n-1)(n+1)\right) \tag{1}$$

The question then becomes, what values of n will $f(n)$ return a positive integer. Either $n-1$ or $n+1$ must divide into 8. The remainder of that division multiplied by the compliment must be a power of 10. If $8|n-1$ then $\frac{(n-1)}{8}(n+1)$ must be representable as 10^k for $\{k \in \mathbb{Z}^+\}$.
How can integers n and q in the form $q(n+1) = 10^k$ for $\{k \in \mathbb{Z}^+\}$?
Let $q = \frac{n-1}{8}$.
Both $n+1$ and q must be in the form 10^k. If $n+1 = 10^a$ and $q = 10^b$ then $q(n+1)$ will be in the form 10^{a+b} which is in the form 10^k. This would mean, that all digits composing n must be 9. If all digits of n must be 9. Then $n-1$ will only be divisible by 8 if $n = 9$. For all integer values of $\alpha > 2$, $(10^\alpha - 2 \bmod 8)$ are equal to 6 and $(10^2 - 2 \bmod 8) = 2$. We know that the only existing value for n is 9. $f(9) = 1$.
The second case is if $(n+1)$ divides by 8. If $(n+1)$ divides into 8, then $(n-1)$ must be in the form of 10^k which means that n is in the form of $10^k + 1$ and thus $n+1$ is in the form of $10^k + 2$. $(10^0 + 2 \bmod 8) = 3$, $(10^1 + 2 \bmod 8) = 4$, $(10^2 + 2 \bmod 8) = 6$ and finally, $(10^k + 2 \bmod 8) = 2$ for $\{k \in \mathbb{Z}^+ | k > 2\}$. Since there would be no value for n in which $n+1$ could divide into 8, we know that the second case provides no solutions. Thus, the only values available for r come from the first case, and are $r = 0$ and $r = 1$.

$$\sqrt{8 \cdot 10^0 + 1} = \sqrt{9} = 3$$
$$\sqrt{8 \cdot 10^1 + 1} = \sqrt{81} = 9$$