Chapter 3 Third Exercise

Wang Yue from CS Elite Class

November 3, 2020

Exercise 3.5

27. $x^2 + xy + y^2 = 3, (1, 1)$ (ellipse)

Take derivative on both sides, and we get

$$2x + y + xy' + 2yy' = 0$$

$$y' = -\frac{2x + y}{x + 2y}$$

When $x = 1, y = 1$, $y' = -\frac{3}{3} = -1$
So the slope of the tangent line is -1.

28. $x^2 + 2xy - y^2 + x = 2, (1, 2)$ (hyperbola)

Take derivative on both sides, and we get

$$2x + 2y + 2xy' - 2yy' + 1 = 0$$

$$y' = \frac{-1 - 2x - 2y}{2x - 2y}$$

When $x = 1, y = 2$, $y' = \frac{-1-2-4}{-2} = \frac{7}{2}$
So the slope of the tangent line is $\frac{7}{2}$.

29. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2, (0, \frac{1}{2})$, (cardioid)

Take derivative on both sides, and we get

$$2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$$

$$y' = -\frac{8x^3 - 6x^2 + 8xy^2 - 2y^2}{8x^2y + 8y^3 - 4xy - y}$$

When $x = 0, y = \frac{1}{2}, y' = -\frac{-2y^2}{8y^3 - y} = \frac{2y}{8y^2 - 1} = \frac{1}{2-1} = 1$

30. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4, (-3\sqrt{3}, 1)$ (astroid)

Taking derivative on both sides, and we get

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0$$

$$y' = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\sqrt[3]{\frac{x}{y}}$$

When $x = -3\sqrt{3}, y = 1, y' = -\sqrt[3]{-3\sqrt{3}} = -(-\sqrt{3}) = \sqrt{3}$

31. $2(x^2 + y^2)^2 = 25(x^2 - y^2), (3, 1)$ (lemniscate)

Taking derivative on both sides, and we get

$$4(x^2 + y^2)(2x + 2yy') = 25(2x - 2yy')$$

$$y' = \frac{x[25 - (4x^2 + 4y^2)]}{y[25 + (4x^2 + 4y^2)]}$$

When $x = 3, y = 1, y' = 3\frac{25-40}{25+40} = -\frac{3}{13}$

32. $y^2(y^2 - 4) = x^2(x^2 - 5), (0, -2)$ (devil's curve)

Taking derivative on both sides, and we get

$$4y^3y' - 8yy' = 4x^3 - 10x$$

$$y' = \frac{2x^3 - 5x}{2y^3 - 4y}$$

When $x = 0, y = -2,\ y' = \frac{0}{-16+8} = 0$

36. $\sqrt{x} + \sqrt{y} = 1$

Taking derivative on both sides, and we get

$$\frac{1}{2}(x^{-\frac{1}{2}} + y^{-\frac{1}{2}}y') = 0$$

$$y' = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}$$

$$y'' = -\frac{-\frac{1}{2}x^{-\frac{3}{2}}y^{-\frac{1}{2}} + x^{-\frac{1}{2}}\frac{1}{2}y^{-\frac{3}{2}}y'}{(-\frac{1}{2}y^{-\frac{3}{2}})^2}$$

$$y'' = 2(x^{-\frac{3}{2}}y^{\frac{5}{2}} + x^{-1}y^2)$$

51. $y = \sin^{-1}(2x+1)$

$\because \frac{d \arcsin x}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\therefore$

$$y' = 2\frac{1}{\sqrt{-4x^2-4x}} = \frac{1}{\sqrt{-x^2-x}}$$

57. $y = x\sin^{-1}x + \sqrt{1-x^2}$

$$y' = x\frac{1}{\sqrt{1-x^2}} + \arcsin x + \frac{1}{2}\frac{1}{\sqrt{1-x^2}}(-2) = \arcsin x - \frac{\sqrt{1-x^2}}{x+1}$$

60. $y = \arctan\sqrt{\frac{1-x}{1+x}}$

$$y' = \frac{1}{1+(\sqrt{\frac{1-x}{1+x}})^2}\frac{-1-x-1+x}{(1+x)^2} = \frac{1+x}{2}\frac{-2}{(1+x)^2} = \frac{-1}{1+x}$$

Exercise 3.6

16. $y = \ln|1+t-t^3|$

$$y' = \frac{1-3t^2}{1+t-t^3}$$

18. $y = \ln|\cos(\ln x)|$

$$y' = \frac{-\sin\ln x}{x\cos\ln x} = -\frac{\tan\ln x}{x}$$

25. $y = ln(x+\sqrt{1+x^2})$

$$y' = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}$$

$$\begin{aligned} y'' &= \frac{\frac{1}{\sqrt{1+x^2}}(x+\sqrt{1+x^2}) - (1+\frac{x}{\sqrt{1+x^2}})(1+\frac{x}{\sqrt{1+x^2}})}{(x+\sqrt{1+x^2})^2} \\ &= \frac{\frac{3x}{\sqrt{1+x^2}} + x^2 - \frac{x^2}{1+x^2}}{(x+\sqrt{1+x^2})^2} \\ &= \frac{x(3\sqrt{1+x^2}+1)}{(1+x^2)(x+\sqrt{1+x^2})^2} \end{aligned}$$

41. $y = \sqrt{\frac{x-1}{x^4+1}}$

$$\ln y = \frac{1}{2}\ln\frac{x-1}{x^4+1}$$

$$\frac{y'}{y} = \frac{1}{2}\frac{x^4+1}{x-1} \times \frac{4x^3+1}{(x^4+1)^2}$$

$$y' = \frac{4x^3+1}{2(x-1)(x^4+1)}\sqrt{\frac{x-1}{x^4+1}} = \frac{4x^3+1}{2}\sqrt{\frac{x^4+1}{x-1}}$$

45. $y = x^{\sin x}$

$$\ln y = \ln x^{\sin x} = \sin x \ln x$$

$$\frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x}$$

$$y' = x^{\sin x}(\cos x \ln x + \frac{\sin x}{x})$$

48. $y = (\sin x)^{\ln x}$

$$\ln y = \ln(\sin x)^{\ln x} = \ln x \ln \sin x$$

$$\frac{y'}{y} = \frac{\ln \sin x}{x} + \frac{\cos x \ln x}{\sin x}$$

$$y' = (\sin x)^{\ln x}(\frac{\ln \sin x}{x} + \frac{\cos x \ln x}{\sin x})$$

52. Find y' if $x^y = y^x$.

Take logarithm on both sides, and we get

$$\ln x^y = \ln y^x \iff y \ln x = x \ln y$$

Taking derivative on both sides, we have

$$y' \ln x + \frac{y}{x} = \ln y + x\frac{y'}{y}$$

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$$

53. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x-1)$

$\because f'(x) = \frac{1}{x-1}, f''(x) = \frac{-1}{(x-1)^2}, f'''(x) = \frac{2}{(x-1)^3}$

$\therefore$ we guess that

$$f^{(n)}(x) = \frac{(-1)^{n-1}((n-1)!}{(x-1)^n}$$

We will prove it by mathematical induction.

When $n = 1$, $f^{(1)}(x) = \frac{1}{x-1}$, which satisfies the assumption.

When $n \geq 2$, suppose the assumption holds when $n = k (k \in N_+)$, which means that

$$f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(x-1)^k}$$

$$\begin{aligned} f^{(k+1)}(x) &= \frac{-(-1)^{k-1}(k-1)! \times k(x+1)^{k-1}}{(x+1)^{2k}} \\ &= \frac{(-1)^k k!}{(x+1)^{k+1}} \end{aligned}$$

By the proof above, we can conclude that $f^{(n)}x = \frac{(-1)^{n-1}((n-1)!}{(x-1)^n}$