Exercise 2.8

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Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

26. $f(x) = x + \sqrt{x}$

Solution:

The domain of $f(x)$ is $[0, \infty)$.

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{x - x_0 + \sqrt{x} - \sqrt{x_0}}{x - x_0} \\ &= 1 + \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} \\ &= 1 + \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} \\ &= \frac{1}{2\sqrt{x_0}} + 1 \end{aligned}$$

$\therefore f'(x) = \frac{1}{2\sqrt{x}} + 1$, and the domain of $f'(x)$ is $(0, \infty)$.

27. $g(x) = \sqrt{9 - x}$

Solution:

The domain of $g(x)$ is the solution set of $9 - x \geq 0$, which is $(-\infty, 9]$.

$$
\begin{aligned}
g'(x_0) &= \lim_{x\to x_0} \frac{\sqrt{9-x}-\sqrt{9-x_0}}{x-x_0} \\
&= \lim_{x\to x_0} \frac{9-x-(9-x_0)}{(x-x_0)(\sqrt{9-x}+\sqrt{9-x_0})} \\
&= \lim_{x\to x_0} \frac{-1}{\sqrt{9-x}+\sqrt{9-x_0}} \\
&= \frac{-1}{2\sqrt{9-x_0}}
\end{aligned}
$$

$\therefore g'(x) = \frac{-1}{2\sqrt{9-x}}$, and the domain of $g'(x)$ is $(-\infty, 9)$.

29. $G(t) = \frac{1-2t}{3+t}$

Solution:

The domain of $G(t)$ is $\{t \in R | t \neq -3\}$.
$G(t) = \frac{-6-2t+7}{3+t} = \frac{7}{t+3} - 2$

$$
\begin{aligned}
G'(t_0) &= \lim_{t\to t_0} \frac{\frac{7}{t+3}-\frac{7}{t_0+3}}{t-t_0} \\
&= 7\lim_{t\to t_0} \frac{\frac{t_0-t}{(t+3)(t_0+3)}}{t-t_0} \\
&= -7\lim_{t\to t_0} \frac{1}{(t+3)(t_0+3)} \\
&= \frac{-7}{(t_0+3)^2}
\end{aligned}
$$

$\therefore G'(t) = \frac{-7}{(t+3)^2}$, the domain of which is also $(-\infty,-3)\cup(-3,\infty)$

51. Let $f(x) = \sqrt[3]{x}$**.**

(a) If $a \neq 0$, use Equation 2.7.5 to find $f'(a)$.

Solution:

$$
\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\
&= \lim_{x\to a}\frac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a}\\
&= \lim_{x\to a}\frac{(x^{\frac{1}{3}}-a^{\frac{1}{3}})(x^{\frac{2}{3}}+(ax)^{\frac{1}{3}}+a^{\frac{2}{3}})}{(x-a))(x^{\frac{2}{3}}+(ax)^{\frac{1}{3}}+a^{\frac{2}{3}})}\\
&= \lim_{x\to a}\frac{1}{x^{\frac{2}{3}}+(ax)^{\frac{1}{3}}+a^{\frac{2}{3}}}\\
&= \frac{1}{3(a)^{\frac{2}{3}}}\\
&= \frac{1}{3}a^{-\frac{2}{3}}
\end{aligned}
$$

(b) Show that $f'(0)$ does not exist.

Solution:

$\because$ by problem (a), we can know the domain of $f'(a)$ is $(-\infty,0)\cup(0,\infty)$

$\therefore$ $f'(0)$ has no definition, which means $f'(0)$ does not exist.

(c) Show that $y=\sqrt[3]{x}$ has a vertical tangent line at $(0,0)$.

Solution:

$\because \lim_{x\to 0} f'(x) = \lim_{x\to 0}\frac{1}{3x^{\frac{2}{3}}}=\infty$

$\therefore$ the tangent line of $f(x)$ as $x\to 0$ is vertical

$\because \sqrt[3]{0}=0$

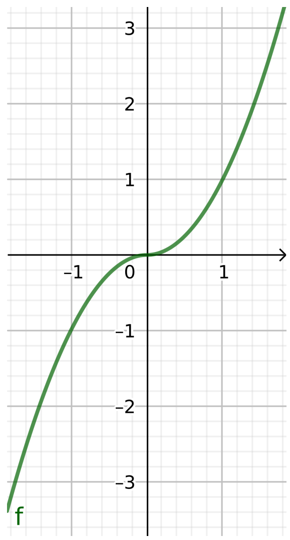
$\therefore y=\sqrt[3]{x}$ has a vertical tangent line at $(0,0)$

55.

(a) Sketch the graph of the function $f(x)=x|x|$.

Solution:

The graph of $f(x)=x|x|$ is below.



(b) For what values of x is f differentiable?

$\because f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

$\therefore f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$

So obviously, for any value of x, $f(x)$ is differentiable.

(c) Find a formula for f'.

By the problem (b), a formula for f' can be

$$f'(x) = 2|x|$$

57. Prove each of the following.

(a) The derivative of an even function is an odd function.

Proof. $\because f(x) = f(-x)$

$\therefore$ taking derivative on the both side, we can get

$$f'(x) = -f'(-x)$$

which means $f'(x)$ is an odd function. □

(b) The derivative of an odd function is an even function.

Proof. $\because f(x) = -f(-x)$
$\therefore$ taking derivative on the both side, we can get

$$f'(x) = f'(-x)$$

which means $f'(x)$ is an even function. $\square$