MA4_2 Exercise

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Exercise 4.4

20. $\lim_{x\to\infty} \frac{\ln\ln x}{x}$

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln\ln x}{x} &= \lim_{x\to\infty} \frac{\frac{1}{x\ln x}}{1} \\ &= \frac{1}{x\ln x} \\ &= 0\end{aligned}$$

23. $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x} &= \lim_{x\to 0} \frac{1+2x-(1-4x)}{x(\sqrt{1+2x}+\sqrt{1+4x})} \\ &= \lim_{x\to 0} \frac{6}{\sqrt{1+2x}+\sqrt{1-4x}} \\ &= \frac{6}{1+1} \\ &= 3\end{aligned}$$

25. $\lim_{x\to 0} \frac{e^x-1-x}{x^2}$

$$\begin{aligned}\lim_{x\to 0} \frac{e^x-1-x}{x^2} &= \lim_{x\to 0} \frac{e^x-1}{2x} \\ &= \lim_{x\to 0} \frac{e^x}{2} \\ &= \frac{1}{2}\end{aligned}$$

35. $\lim_{x\to 1} \frac{1-x+\ln x}{1+\cos \pi x}$

$$\begin{aligned}\lim_{x\to 1} \frac{1-x+\ln x}{1+\cos \pi x} &= \lim_{x\to 1} \frac{-1+\frac{1}{x}}{-\pi \sin \pi x}\\ &= \lim_{x\to 1} \frac{x-1}{\pi x \sin \pi x}\\ &= \lim_{x\to 1} \frac{1}{\pi(\sin \pi x + \pi x \cos \pi x)}\\ &= \frac{1}{-\pi^2}\end{aligned}$$

44. $\lim_{x\to 0^+}(\sin x \ln x)$

$$\begin{aligned}\lim_{x\to 0^+} \sin \ln x &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}}\\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}}\\ &= \lim_{x\to 0^+} \frac{-\tan x}{x \cos x}\\ &= \lim_{x\to 0^+} \frac{-x}{x \cos x}\\ &= \lim_{x\to 0^+} \frac{-1}{\cos x} = -1\end{aligned}$$

49. $\lim_{x\to 1}(\frac{x}{x-1} - \frac{1}{\ln x})$

$$\begin{aligned}\lim_{x\to 1}(\frac{x}{x-1} - \frac{1}{\ln x}) &= \lim_{x\to 1} \frac{x\ln x - x + 1}{(x-1)\ln x}\\ &= \lim_{x\to 1} \frac{\ln x + 1 - 1}{\ln x + \frac{x-1}{x}}\\ &= \lim_{x\to 1} \frac{\ln x}{\ln x + 1 - \frac{1}{x}}\\ &= \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}}\\ &= \lim_{x\to 1} \frac{x}{x+1}\\ &= \frac{1}{2}\end{aligned}$$

52. $\lim_{x\to 0^+}(\cot x - \frac{1}{x})$

$$\begin{aligned}
\lim_{x\to 0^+}(\cot x - \frac{1}{x}) &= \lim_{x\to 0^+}(\frac{\cos x}{\sin x} - \frac{1}{x}) \\
&= \lim_{x\to 0^+}\frac{x\cos x - \sin x}{x\sin x} \\
&= \lim_{x\ to 0^+}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x} \\
&= \lim_{x\to 0^+}\frac{-\sin x}{\frac{\sin x}{x} + \cos x} \\
&= \lim_{x\to 0^+}\frac{-\sin x}{1+\cos x} \\
&= 0
\end{aligned}$$

55. $\lim_{x\to 0^+} x^{\sqrt{x}}$

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\sqrt{x}\ln x}$$

$$\begin{aligned}
\because \lim_{x\to 0^+}\sqrt{x}\ln x &= \lim_{x\to 0^+}\frac{\ln x}{x^{-\frac{1}{2}}} \\
&= \lim_{x\to 0^+} -2\sqrt{x} \\
&= 0
\end{aligned}$$

$$\therefore \lim_{x\to 0^+}\sqrt{x}\ln x = e^0 = 1$$

58. $\lim_{x\to\infty}(1+\frac{a}{x})^{bx}$

$$\lim_{x\to\infty}(1+\frac{a}{x})^{bx} = \lim_{x\to\infty} e^{bx\ln(1+\frac{a}{x})}$$

$$\begin{aligned}
\because \lim_{x\to\infty} bx\ln(1+\frac{a}{x}) &= \lim_{x\to\infty}\frac{\ln(1+\frac{a}{x})}{\frac{1}{bx}} \\
&= \lim_{x\to\infty}\frac{\frac{-\frac{a}{x^2}}{1+\frac{a}{x}}}{-\frac{1}{bx^2}} \\
&= \lim_{x\to\infty}\frac{ab}{1+\frac{a}{x}} \\
&= \frac{ab}{1+0} = ab
\end{aligned}$$

$$\therefore \lim_{x\to\infty}(1+\frac{a}{x})^{bx} = e^{ab}$$

62. $\lim_{x\to\infty}(e^x+x)^{\frac{1}{x}}$

$$\lim_{x\to\infty}(e^x+x)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(e^x+x)}$$

$$\begin{aligned}\because \lim_{x\to\infty}\frac{\ln(e^x+x)}{x} &= \lim_{x\to\infty}\frac{\frac{e^x+1}{e^x+x}}{1}\\ &= \lim_{x\to\infty}\frac{e^x}{e^x+1}\\ &= \lim_{x\to\infty}(1-\frac{1}{e^x+1})\\ &= 1\end{aligned}$$

$$\therefore \lim_{x\to\infty}(e^x+x)^{\frac{1}{x}} = e^1 = e$$

66. $\lim_{x\to\infty}(\frac{2x-3}{2x+5})^{2x+1}$

$$\lim_{x\to\infty}(\frac{2x-3}{2x+5})^{2x+1} = \lim_{x\to\infty} e^{(2x+1)\ln\frac{2x-3}{2x+5}}$$

$$\begin{aligned}\lim_{x\to\infty}(2x+1)\ln\frac{2x-3}{2x+5} &= \lim_{x\to\infty}\frac{\ln\frac{2x-3}{2x+5}}{\frac{1}{2x+1}}\\ &= \lim_{x\to\infty}\frac{\frac{2x+5}{2x-3}\frac{2(2x+5)-2(2x+3)}{(2x+5)^2}}{\frac{-2}{(2x+1)^2}}\\ &= \lim_{x\to\infty}\frac{-2(2x+1)^2}{(2x-3)(2x+5)}\\ &= \lim_{x\to\infty}\frac{-2\times 2(2x+1)\times 2}{2(2x+5)+2(2x-3)}\\ &= \lim_{x\to\infty}\frac{-8(2x+1)}{2(2x+1)}\\ &= -4\end{aligned}$$

$$\therefore \lim_{x\to\infty}(\frac{2x-3}{2x+5})^{2x+1} = e^{-4}$$

87. If f' is continuous, use l'Hospital's Rule to show that

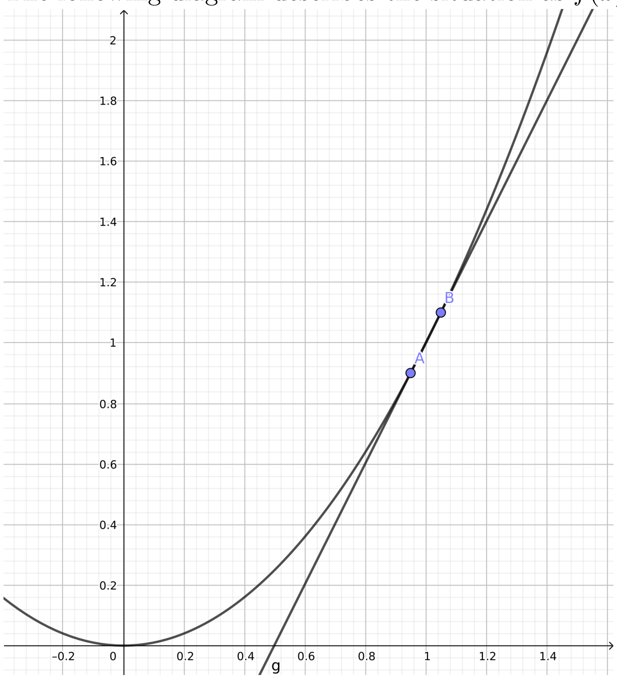
$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} = f'(x)$$

Explain the meaning of this equation with the aid of a diagram.

Use l'Hospital Rule with respect to h, and we get

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} = \lim_{h\to 0}\frac{f'(x+h)+f'(x-h)}{2}$$
$$= f'(x)$$

The meaning of this equation is the slope of the tangent line to f.
The following diagram describes the situation as $f(x)=x^2, x=1, h=0.05$.



88. If f'' is continuous, show that

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} = f''(x)$$

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} &= \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} \\
&= \lim_{h\to 0}\frac{f'(x+h)-f'(x)+f'(x)-f'(x-h)}{2h} \\
&= \lim_{h\to 0}\frac{1}{2}\left(\frac{f'(x+h)-f'(x)}{h}+\frac{f'(x)-f'(x-h)}{h}\right) \\
&= \frac{2f''(x)}{2} = f''(x)
\end{aligned}$$

89. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \textbf{if } x \neq 0 \\ 0 & \textbf{if } x = 0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$.

$$\begin{aligned}
f'(0) &= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \\
&= \lim_{x\to 0} \frac{e^{-x^{-2}}}{x} \\
&= \lim_{x\to 0} \frac{\frac{1}{x}}{e^{x^{-2}}} \\
&= \lim_{x\to 0} \frac{-\frac{1}{x^2}}{-2e^{x^{-2}}x^{-3}} \\
&= \lim_{x\to 0} \frac{x}{2e^{x^{-2}}} \\
&= 0
\end{aligned}$$

(b) Show that f has derivatives of all orders that are defined on R.

Proof. Suppose

$$f^{(n)}(x) = \begin{cases} 0 & \text{if } x = 0 \\ y(n)e^{-x^{-2}} & \text{if } x \neq 0 \end{cases}$$

in which $y(n)$ is a polymonial with respect to x.

We will prove it by mathematical induction.

$4e^{-x^{-2}}x^{-3} - 6e^{-x^{-2}}x^{-4} = e^{-x^{-2}}(4x^{-3} - 6x^{-4})$

When $n = 1$,

$$f^{(1)}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2e^{-x^{-2}}x^{-3} & \text{if } x \neq 0 \end{cases}$$

which means that when $n = 1$, $y(1) = x^{-3}$, $f^{(n)}(x)$ exists for all $x \in R$.

When $n \geq 2$, suppose that $f^{(n)}(x)$ exists when $n = k, k \in N_+$.

If $x = 0$,

$$f^{(k+1)}(x) = (f^{(k)}(x))' = (0)' = 0$$

If $x \neq 0$,

$$\begin{aligned}
f^{(k+1)}(x) &= (f^{(k)}(x))' \\
&= (y(k)e^{-x^{-2}})' \\
&= (y'(k) + 2x^{-3})e^{-x^{-2}} \\
&= y(k+1)e^{-x^{-2}}
\end{aligned}$$

Since $y(k+1)$ is a polymonial, $y(k+1)e^{-x^{-2}}$ can be defined when $x \neq 0$.

Therefore $f^{(k+1)}(x)$ can be defined on R.

In all, $f^{(n)}(x)$ can be defined on R.

$\square$