

Exercise 11.4

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20. $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$

Let $a_n = \frac{n+4^n}{n+6^n}$, $b_n = \frac{4^n}{6^n}$, and we can get

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n+4^n}{n+6^n}}{\frac{4^n}{6^n}} = \lim_{n \rightarrow \infty} \frac{\frac{n}{4^n} + 1}{\frac{n}{6^n} + 1} = 1$$

$\therefore \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (\frac{2}{3})^n$, which is convergent

$\therefore \sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$ is also convergent

21. $\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$

Let $a_n = \frac{\sqrt{n+2}}{2n^2+n+1}$, $b_n = \frac{\sqrt{n}}{n^2}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n+2}}{\sqrt{n}} \frac{2n^2+n+1}{n^2} = \lim_{n \rightarrow \infty} \sqrt{1+\frac{2}{n}} (2+\frac{1}{n}+\frac{1}{n^2}) = 2 \in (0, \infty)$$

$$\therefore b_n = \frac{1}{n^{\frac{3}{2}}}, \frac{3}{2} > 1$$

$\therefore \sum_{n=1}^{\infty} b_n$ is convergent

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$ is also convergent

27. $\sum_{n=1}^{\infty} (1+\frac{1}{n})^2 e^{-n}$

If we restrict $n \geq 2$, then

$$\sum_{n=2}^{\infty} (1+\frac{1}{n})^2 e^{-n} \leq \sum_{n=2}^{\infty} \frac{(1+\frac{1}{n})^n}{e^n}$$

Let $a_n = \frac{(1+\frac{1}{n})^n}{e^n}$, then

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$$

$\therefore \sum_{n=2}^{\infty} \frac{(1+\frac{1}{n})^n}{e^n}$ is convergent

$\therefore \sum_{n=2}^{\infty} (1+\frac{1}{n})^2 e^{-n}$ is convergent

$\therefore \sum_{n=1}^{\infty} (1+\frac{1}{n})^2 e^{-n}$ is convergent

30. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Let $a_n = \frac{n!}{n^n} > 0$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges

31. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Let $a_n = \sin\left(\frac{1}{n}\right)$, $b_n = \frac{1}{n}$, so

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 \in (0, \infty)$$

$\therefore \sum_{n=1}^{\infty} b_n$ is divergent

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ is also divergent

32. $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

Let $a_n = \frac{1}{n^{1+\frac{1}{n}}}$, $b_n = \frac{1}{n}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{n^{\frac{1}{n}}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}}$$

Let $t = \sqrt[n]{n} - 1$, then $(t+1)^n = n$, which is equivalent to

$$1 + nt + \frac{n(n-1)}{2}t^2 + \dots = n$$

$$\begin{aligned} \therefore \frac{n(n-1)}{2}t^2 &< n \\ \therefore 0 &\leq t^2 < \frac{2}{n-1} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{2}{n-1} = \lim_{n \rightarrow \infty} 0 = 0$$

\therefore by the squeeze theorem, $\lim_{n \rightarrow \infty} t^2 = 0$, which means

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1 \neq 0$$

$$\therefore \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1 \in (0, \infty)$$

$$\therefore \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \text{ also diverges}$$

37.

Proof. Let $a_n = \frac{d_n}{10^n}$, where d_n is one of the numbers $0, 1, \dots, 9$

And let $b_n = \frac{1}{10^n}$, obviously we know $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{10^n}$ converges.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} d_n \in \{0, 1, 2, \dots, 9\}$$

If $\lim_{n \rightarrow \infty} d_n = 0$, $\because \sum_{n=1}^{\infty} b_n$ converges, $\therefore \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ converges.

If $\lim_{n \rightarrow \infty} d_n \neq 0$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ converges as $\sum_{n=1}^{\infty} \frac{1}{10^n}$ converges. \square

39. Prove that if $a_n \geq 0$ and $\sum a_n$ converges, then $\sum a_n^2$ also converges.

Proof.

$$\lim_{n \rightarrow \infty} \frac{a_n^2}{a_n} = \lim_{n \rightarrow \infty} a_n$$

If $\lim_{n \rightarrow \infty} a_n > 0$, either both $\sum a_n^2$ and $\sum a_n$ are convergent or both divergent. Therefore, $\sum a_n^2$ is convergent.

In conclusion, $\sum a_n^2$ is convergent. \square

40.TODO

(a) *Proof.* TODO \square

(b) (i) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Proof. Let $a_n = \frac{\ln n}{n^3}$, $b_n = \frac{1}{n^2}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

$\because \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p -series

\therefore by the conclusion of (a), $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ is also convergent \square

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}e^n}$

Proof. Let $a_n = \frac{\ln n}{\sqrt{n}e^n}$, $b_n = \frac{1}{e^n}$

$\because \frac{1}{e} < 1 \therefore \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{e^n}$ is convergent

$$\therefore \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{n}} = 0$$

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}e^n}$ is convergent \square

41.

(a) *Proof.* □

(b) (i) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$
Let $a_n = \frac{1}{\ln n}$, $b_n = \frac{1}{n}$, and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{\ln n} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n}} = \infty$$

$\therefore \sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n}$ is divergent
 $\therefore \sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ is also divergent

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$
Let $a_n = \frac{\ln n}{n}$, $b_n = \frac{1}{n}$, and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \ln n = \infty$$

$\therefore \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent
 $\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n}$ is also divergent

43. Show that if $a_n > 0$ and $\lim_{n \rightarrow \infty} na_n \neq 0$, then $\sum a_n$ is divergent.

Proof.

$$\lim_{n \rightarrow \infty} na_n = \lim_{n \rightarrow \infty} \frac{a_n}{\frac{1}{n}} \neq 0$$

If $\lim_{n \rightarrow \infty} \frac{a_n}{\frac{1}{n}} = \infty$
 $\therefore \sum \frac{1}{n}$ is divergent
 \therefore by the conclusion of 41(a), $\sum a_n$ is also divergent
If $\lim_{n \rightarrow \infty} \frac{a_n}{\frac{1}{n}} \in (0, \infty)$, they are either both convergent or both divergent
 $\therefore \sum \frac{1}{n}$ is divergent
 $\therefore \sum a_n$ is also divergent □

44. Show that if $a_n > 0$ and $\sum a_n$ is convergent, then $\sum \ln(1 + a_n)$ is convergent.

Proof. $\therefore x > \ln(x + 1)$ when $x > 0 \quad \therefore a_n \geq \ln(1 + a_n)$

$$\therefore \sum a_n \geq \sum \ln(1 + a_n)$$

$\therefore \sum a_n$ is convergent
 $\therefore \sum \ln(1 + a_n)$ is also convergent □

45. If $\sum a_n$ is a convergent series with positive terms, is it true that $\sum \sin(a_n)$ is also convergent?

It is true. And the reason is as follows.

$$\because x > \sin x \text{ when } x > 0 \quad \therefore a_n > \sin(a_n)$$

$$\therefore \sum a_n > \sum \sin(a_n)$$

$$\because \sum a_n \text{ is convergent}$$

$$\therefore \sum \sin(a_n) \text{ is also convergent}$$