Exercise 11.4

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20. $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$

Let $a_n = \frac{n+4^n}{n+6^n}, b_n = \frac{4^n}{6^n}$, and we can get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+4^n}{n+6^n}}{\frac{4^n}{6^n}} = \lim_{n\to\infty} \frac{\frac{n}{4^n}+1}{\frac{n}{6^n}+1} = 1$$

$\because \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (\frac{2}{3})^n$, which is convergent
$\therefore \sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$ is also convergent

21. $\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$

Let $a_n = \frac{\sqrt{n+2}}{2n^2+n+1}, b_n = \frac{\sqrt{n}}{n^2}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n+2}}{\sqrt{n}} \frac{2n^2+n+1}{n^2} = \lim_{n\to\infty} \sqrt{1+\frac{2}{n}}(2+\frac{1}{n}+\frac{1}{n^2}) = 2 \in (0,\infty)$$

$\because b_n = \frac{1}{n^{\frac{3}{2}}}, \frac{3}{2} > 1$
$\therefore \sum_{n=1}^{\infty} b_n$ is convergent
$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$ is also convergent

27. $\sum_{n=1}^{\infty} (1+\frac{1}{n})^2 e^{-n}$

If we restrict $n \geq 2$, then

$$\sum_{n=2}^{\infty} (1+\frac{1}{n})^2 e^{-n} \leq \sum_{n=2}^{\infty} \frac{(1+\frac{1}{n})^n}{e^n}$$

Let $a_n = \frac{(1+\frac{1}{n})^n}{e^n}$, then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$$

$\therefore \sum_{n=2}^{\infty} \frac{(1+\frac{1}{n})^n}{e^n}$ is convergent
$\therefore \sum_{n=2}^{\infty} (1+\frac{1}{n})^2 e^{-n}$ is convergent
$\therefore \sum_{n=1}^{\infty} (1+\frac{1}{n})^2 e^{-n}$ is convergent

30. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Let $a_n = \frac{n!}{n^n} > 0$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} (\frac{n}{n+1})^n = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges

31. $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$

Let $a_n = \sin(\frac{1}{n}), b_n = \frac{1}{n}$, so

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1 \in (0, \infty)$$

$\because \sum_{n=1}^{\infty} b_n$ is divergent
$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \sin(\frac{1}{n})$ is also divergent

32. $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

Let $a_n = \frac{1}{n^{1+\frac{1}{n}}}, b_n = \frac{1}{n}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}}$$

Let $t = \sqrt[n]{n} - 1$, then $(t+1)^n = n$, which is equivalent to

$$1 + nt + \frac{n(n-1)}{2}t^2 + \cdots = n$$

$\therefore \frac{n(n-1)}{2}t^2 < n$
$\therefore 0 \leq t^2 < \frac{2}{n-1}$

$$\because \lim_{n\to\infty} \frac{2}{n-1} = \lim_{n\to\infty} 0 = 0$$

$\therefore$ by the squeeze theorem, $\lim_{n\to\infty} t^2 = 0$, which means

$$\lim_{n\to\infty} \sqrt[n]{n} = 1 \neq 0$$

$\therefore \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \lim_{n\to\infty} \frac{a_n}{b_n} = 1 \in (0, \infty)$
$\because \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges
$\therefore \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ also diverges

37.

Proof. Let $a_n = \frac{d_n}{10^n}$, where d_n is one of the numbers $0, 1, \cdots, 9$

And let $b_n = \frac{1}{10^n}$, obviously we know $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{10^n}$ converges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} d_n \in \{0, 1, 2, \cdots, 9\}$$

If $\lim_{n\to\infty} d_n = 0, \because \sum_{n=1}^{\infty} b_n$ converges, $\therefore \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ converges.

If $\lim_{n\to\infty} d_n \neq 0, \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ converges as $\sum_{n=1}^{\infty} \frac{1}{10^n}$ converges. □

39. Prove that if $a_n \geq 0$ and $\sum a_n$ converges, then $\sum a_n^2$ also converges.

Proof.

$$\lim_{n\to\infty} \frac{a_n^2}{a_n} = \lim_{n\to\infty} a_n$$

If $\lim_{n\to\infty} a_n > 0$, either both $\sum a_n^2$ and $\sum a_n$ are convergent or both divergent. Therefore, $\sum a_n^2$ is convergent.

In conclusion, $\sum a_n^2$ is convergent. □

40.

(a) *Proof.* $\because \lim_{n\to\infty} \frac{a_n}{b_n} = 0$

$\therefore \forall \epsilon > 0, \exists N > 0$, if $n > N$, $\frac{a_n}{b_n} < \epsilon$, which is equivalent to

$$a_n < \epsilon b_n$$

$\because \sum b_n$ is convergent when $n > N$

$\therefore \sum \epsilon b_n$ is convergent when $n > N$

$\therefore \sum a_n$ is also convergent when $n > N$

$\therefore \sum a_n$ is also convergent □

(b) (i) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Proof. Let $a_n = \frac{\ln n}{n^3}, b_n = \frac{1}{n^2}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$\because \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series

$\therefore$ by the conclusion of (a), $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ is also convergent □

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n} e^n}$

Proof. Let $a_n = \frac{\ln n}{\sqrt{n}e^n}, b_n = \frac{1}{e^n}$
$\because \frac{1}{e} < 1 \therefore \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{e^n}$ is convergent

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$$

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}e^n}$ is convergent

□

41.

(a) *Proof.* $\because \lim_{n\to\infty} \frac{a_n}{b_n} = 0$
$\therefore \forall M > 0, \exists N > 0$, if $n > N$, $\frac{a_n}{b_n} > M$, which is equivalent to

$$a_n > Mb_n$$

$\because \sum b_n$ is divergent when $n > N$
$\therefore \sum Mb_n$ is divergent when $n > N$
$\therefore \sum a_n$ is divergent when $n > N$
$\therefore \sum a_n$ is divergent □

(b) (i) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$
Let $a_n = \frac{1}{\ln n}, b_n = \frac{1}{n}$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \infty$$

$\because \sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n}$ is divergent
$\therefore \sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ is also divergent

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$
Let $a_n = \frac{\ln n}{n}, b_n = \frac{1}{n}$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \ln n = \infty$$

$\because \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent
$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n}$ is also divergent

43. Show that if $a_n > 0$ and $\lim_{n\to\infty} na_n \neq 0$, then $\sum a_n$ is divergent.

Proof.

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} \neq 0$$

If $\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = \infty$
$\because \sum \frac{1}{n}$ is divergent
$\therefore$ by the conclusion of 41(a), $\sum a_n$ is also divergent
If $\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} \in (0, \infty)$, they are either both convergent or both divergent
$\because \sum \frac{1}{n}$ is divergent
$\therefore \sum a_n$ is also divergent

□

44.Show that if $a_n > 0$ and $\sum a_n$ is convergent, then $\sum \ln(1 + a_n)$ is convergent.

Proof. $\because x > \ln(x+1)$ when $x > 0$ $\quad \therefore a_n \geq \ln(1 + a_n)$

$$\therefore \sum a_n \geq \sum \ln(1 + a_n)$$

$\because \sum a_n$ is convergent
$\therefore \sum \ln(1 + a_n)$ is also convergent □

45. If $\sum a_n$ is a convergent series with positive terms, is it true that $\sum \sin(a_n)$ is also convergent?

It is true. And the reason is as follows.
$\because x > \sin x$ when $x > 0$ $\quad \therefore a_n > \sin(a_n)$

$$\therefore \sum a_n > \sum \sin(a_n)$$

$\because \sum a_n$ is convergent
$\therefore \sum \sin(a_n)$ is also convergent