Exercise 11.3

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16. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

Let $f(x) = \frac{x^2}{x^3+1}$, so

$$f'(x) = \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

$\therefore$ when $x \geq 2, f'(x) < 0, f(x)$ is decreasing.

$$\begin{aligned}
\int_2^{\infty} \frac{x^2}{x^3+1} \mathrm{d}x &= \int_2^{\infty} \frac{1}{3(x^3+1)} \mathrm{d}x^3 \\
&= \lim_{t\to\infty} \frac{\ln|x^3+1|}{3}\bigg|_2^t \\
&= \lim_{t\to\infty} \frac{1}{3} \ln|\frac{t^3+1}{9}| \\
&= \infty
\end{aligned}$$

$\because \int_2^{\infty} \frac{x^2}{x^3+1} \mathrm{d}x$ is divergent
$\therefore$by integral test, $\sum_{n=2}^{\infty} \frac{n^2}{n^3+1}$ is also divergent.
$\therefore \sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ is also divergent.

22. $\lim_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Let $f(x) = \frac{1}{x(\ln x)^2}, x \geq 2$, so

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln x)^2} \mathrm{d}x &= \int_2^{\infty} \frac{1}{(\ln x)^2} \mathrm{d}\ln x \\
&= \lim_{t\to\infty} (-\frac{1}{\ln x})\bigg|_2^t \\
&= \lim_{t\to\infty} (\frac{1}{\ln 2} - \frac{1}{\ln t}) \\
&= \frac{1}{\ln 2}
\end{aligned}$$

$\because f(x)$ is obviously decreasing
$\therefore \int_2^\infty \frac{1}{x(\ln x)^2}\mathrm{d}x$ is convergent
$\therefore \sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ is also convergent.

23. $\sum_{n=1}^\infty \frac{e^{\frac{1}{n}}}{n^2}$

Let $f(x) = \frac{e^{\frac{1}{x}}}{x^2}, x > 0$, and let $t = \frac{1}{x} > 0$, then $f(x)$ can be rewritten as

$$f(t) = t^2 e^t$$

$\because f'(t) = (t^2 + 2t)e^t = ((t+1)^2 - 1)e^t > 0$
$\therefore \forall t \in (0, \infty), f(t)$ is increasing
$\therefore \forall x \in (0, \infty), f(x)$ is decreasing

$$\begin{aligned}\because \int_1^\infty \frac{e^{\frac{1}{x}}}{x^2}\mathrm{d}x &= \int_1^\infty -e^{\frac{1}{x}}\mathrm{d}\frac{1}{x} \\ &= \lim_{t\to\infty}(-e^{\frac{1}{x}})\Big|_1^t \\ &= lim_{t\to\infty}(e^1 - e^{\frac{1}{t}}) \\ &= e - 1\end{aligned}$$

which means $\int_1^\infty \frac{e^{\frac{1}{x}}}{x^2}$ is convergent
$\therefore \sum_{n=1}^\infty \frac{e^{\frac{1}{n}}}{n^2}$ is also convergent

24. $\sum_{n=3}^\infty \frac{n^2}{e^n}$

Let $f(x) = \frac{x^2}{e^x}, x \geq 3$, then

$$f'(x) = \frac{2x - x^2}{e^x} = \frac{x(2-x)}{e^x} < 0$$

$\therefore f(x)$ is decreasing
$\because f(x) > 0$
$\therefore$ by the monotonic sequence theorem, $f(x)$ is convergent
$\sum_{n=3}^\infty \frac{n^2}{e^n}$ is also convergent

26. $\sum_{n=1}^\infty \frac{n}{n^4+1}$

Let $f(x) = \frac{x}{x^4+1}, x \geq 1$, so

$$f'(x) = \frac{x^4 + 1 - 4x^4}{(x^4+1)^2} = \frac{1 - 3x^4}{(x^4+1)^2} < 0$$

$\therefore f(x)$ is decreasing

$$
\begin{aligned}
\int_1^\infty \frac{x}{x^4+1}\mathrm{d}x &= \int_1^\infty \frac{1}{2(x^4+1)}\mathrm{d}x^2 \\
&= \lim_{t\to\infty} \left.\frac{\arctan x^2}{2}\right|_1^t \\
&= \lim_{t\to\infty} \frac{\arctan t^2 - \frac{\pi}{4}}{2} \\
&= \frac{\pi}{4}
\end{aligned}
$$

$\because \int_1^\infty \frac{x}{x^4+1}\mathrm{d}x$ is convergent
$\therefore \sum_{n=1}^\infty \frac{n}{n^4+1}$ is also convergent

33. The Riemann zeta-function ζ is defined by

$$\zeta(x) = \sum_{n=1}^\infty \frac{1}{n^x}$$

and is used in number theory to study the distribution of prime numbers. What is the domain of ζ ?

$\because \zeta(x)$ is used to study the distribution of prime numbers
$\therefore \zeta(x)$ must be integrable
$\therefore \zeta(x)$ must be convergent
$\therefore$ by the p-series, $x > 1$
$\therefore$ the domain of ζ is $(1, \infty)$

Leonhard Euler was able to calculate the exact sum of the p-series with $p = 2$:

$$\zeta(2) = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$$

Use this fact to find the sum of each series.

(a) $\sum_{n=2}^\infty \frac{1}{n^2}$

$$\sum_{n=2}^\infty \frac{1}{n^2} = \sum_{n=1}^\infty \frac{1}{n^2} - \frac{1}{1^2} = \frac{\pi^2}{6} - 1$$

(b) $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{1}{(n+1)^2} &= \sum_{n=4}^{\infty} \frac{1}{n^2} \\
&= (\sum_{n=1}^{\infty} \frac{1}{n^2}) - \frac{1}{1} - \frac{1}{4} - \frac{1}{9} \\
&= \frac{\pi^2}{6} - \frac{49}{36}
\end{aligned}$$

(c) $\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

$$\because \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent}$$

$$\begin{aligned}
\therefore \sum_{n=1}^{\infty} \frac{1}{(2n)^2} &= \sum_{n=1}^{\infty} \frac{1}{4n^2} \\
&= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{\pi^2}{24}
\end{aligned}$$