MA3_2 Exercise

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Exercise 10.2

11. $x = t^2 + 1, y = t^2 + t$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{2t+1}{2t} = 1 + \frac{1}{2t}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}y}{\mathrm{d}x})}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{-\frac{1}{2t^2}}{2t} = -\frac{1}{4t^3}\end{aligned}$$

13. $x = e^t, y = te^{-t}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{e^{-t} - te^{-t}}{e^t} = (1-t)e^{-2t}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}y}{\mathrm{d}x})}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{-e^{-2t} - 2(1-t)e^{-2t}}{e^t} = (2t-3)e^{-3t}\end{aligned}$$

15. $x = 2\sin t, y = 3\cos t, 0 < t < 2\pi$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{-3\sin t}{2\cos t} = -\frac{3}{2}\tan t\end{aligned}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}y}{\mathrm{d}x})}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$
$$= \frac{-\frac{3}{2}\sec^2 t}{2\cos t} = \frac{-3}{4\cos^3 t}$$

Exercise 10.3

55. $r = 2\sin\theta, \theta = \frac{\pi}{6}$

$\because \begin{cases} x = r(\theta)\cos\theta \\ y = r(\theta)\sin\theta \end{cases}$

$\therefore$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$
$$= \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$
$$= \frac{2\cos\theta\sin\theta + 2\sin\theta\cos\theta}{2\cos^2\theta - 2\sin^2\theta}$$
$$= \frac{2\sin 2\theta}{2cos2\theta}$$
$$= \tan 2\theta$$

When $\theta = \frac{\pi}{6}$, the slope of tangent line is $\tan\frac{\pi}{3} = \sqrt{3}$

59. $r = \cos 2\theta, \theta = \frac{\pi}{4}$

By the conclusion above,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\sin 2\theta\sin\theta + \cos 2\theta\cos\theta}{-2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}$$

When $\theta = \frac{\pi}{4}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\times\frac{1}{\sqrt{2}} + 0}{-2\times\frac{1}{\sqrt{2}} - 0} = 1$$

60. $r = 1 + 2\cos\theta, \theta = \frac{\pi}{3}$

By the conclusion above,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\sin\theta\sin\theta + (1+2\cos\theta)\cos\theta}{-2\sin\theta\cos\theta - (1+2\cos\theta)\sin\theta}$$
$$= \frac{2\cos 2\theta + \cos\theta}{-2\sin 2\theta - \sin\theta}$$

When $\theta = \frac{\pi}{3}$, the slope of tangent line is

$$\frac{2 \times (-\frac{1}{2}) + \frac{1}{2}}{-2 \times \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{9}$$