MA 5_4 Exercise

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Exercise 5.5

30. $\int \frac{\tan^{-1} x}{1+x^2} dx$

$$\begin{aligned}\int \frac{\arctan x}{1+x^2} dx &= \int \arctan x d \arctan x \\ &= \frac{\arctan^2 x}{2} + C\end{aligned}$$

31. $\int e^{\tan x} \sec^2 x dx$

$$\begin{aligned}\int e^{\tan x} \sec^2 x dx &= \int e^{\tan x} d \tan x \\ &= e^{\tan x} + C\end{aligned}$$

34. $\int \frac{\cos(\frac{\pi}{x})}{x^2} dx$

$$\begin{aligned}\int \frac{\cos(\frac{\pi}{x})}{x^2} dx &= -\frac{1}{\pi} \int \cos(\frac{\pi}{x}) d\frac{\pi}{x} \\ &= -\frac{1}{\pi}(-\sin\frac{\pi}{x}) + C \\ &= \frac{1}{\pi} \sin\frac{\pi}{x} + C\end{aligned}$$

46. $\int x^2\sqrt{2+x} dx$

Let $t = x + 2$, then

$$\begin{aligned}\int x^2\sqrt{2+x} dx &= \int (t-2)^2\sqrt{t} dt \\ &= \int (t^{\frac{5}{2}} - 4t^{\frac{3}{2}} + 4t^{\frac{1}{2}}) dt \\ &= \frac{2}{7}x^{\frac{7}{2}} - \frac{8}{5}x^{\frac{5}{2}} + \frac{8}{3}x^{\frac{3}{2}} + C\end{aligned}$$

66. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} x^4 \sin x dx$

$\because x \in [-\frac{\pi}{3}, \frac{\pi}{3}], y = x^4 \sin x$ is an odd function

$\therefore \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} x^4 \sin x dx = 0$

69. $\int_{e}^{e^4} \frac{dx}{x\sqrt{\ln x}}$

$$\begin{aligned}\int_{e}^{e^4} \frac{dx}{x \ln \sqrt{x}} &= \int_{e}^{e^4} \frac{2d \ln \sqrt{x}}{\ln \sqrt{x}} \\ &= (2 \ln \ln \sqrt{x})\Big|_{e}^{e^4} \\ &= 2(\ln 4 - \ln 1) \\ &= 4 \ln 2\end{aligned}$$

85. If f is continuous and $\int_0^4 f(x)dx = 10$, find $\int_0^2 f(2x)dx$

Let $u = 2x$, then

$$\int_0^2 f(2x)dx = \frac{1}{2}\int_0^4 f(u)du = \frac{10}{2} = 5$$

86. If f is continuous and $\int_0^9 f(x)dx = 4$, find $\int_0^3 xf(x^2)dx$

Let $u = x^2$, then

$$\int_0^3 xf(x^2)dx = 2\int_0^9 f(u)du = 2 \times 4 = 8$$

Exercise 7.2

1. $\int \sin^3 x \cos^2 x dx$

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin^3 x(1 - \sin^2 x)dx \\ &= \int \sin^3 x - \sin^5 x dx \\ &= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C\end{aligned}$$

6. $\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}} dx$

$$\begin{aligned}\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}} dx &= \int 2\sin^3(\sqrt{x})\frac{1}{2\sqrt{x}} dx \\ &= \int 2\sin^3(\sqrt{x}) d\sqrt{x} \\ &= -\frac{2}{3}\cos^3(\sqrt{x}) + C\end{aligned}$$

16. $\int x\sin^3 x dx$

$$\begin{aligned}\int x\sin^3 x dx &= \int x\sin x(1-\cos^2 x)dx \\ &= \int (x\sin x - x\cos^2 x\sin x)dx \\ &= \int x\sin x dx - \int x\cos^2 x\sin x dx \\ &= -\int x d\cos x + \frac{1}{3}\int x d\cos^3 x \\ &= -x\cos x + \int \cos x dx + \frac{1}{3}(x\cos^3 x - \int cos^3 x dx) \\ &= -x\cos x + \sin x + \frac{1}{3}x\cos^3 x - \frac{1}{3}\int (\cos x - \cos x\sin^2 x)dx \\ &= -x\cos x + \sin x + \frac{1}{3}x\cos^3 x - \frac{1}{3}(\sin x - \frac{1}{3}\sin^3 x) + C \\ &= \frac{1}{3}x\cos^3 x + \frac{2}{3}\sin x - x\cos x + \frac{1}{9}\sin^3 x + C\end{aligned}$$

21. $\int \tan x\sec^3 x dx$

$$\begin{aligned}\int \tan x\sec^3 x dx &= \int \sec^2 x(\tan x\sec x dx) \\ &= \int \sec^2 x d\sec x \\ &= 2\sec^2 x + C\end{aligned}$$

23. $\int \tan^2 x dx$

$$\begin{aligned}\int \tan^2 x dx &= \int (\sec^2 x - 1)dx \\ &= \tan x - x + C\end{aligned}$$

32. $\int \tan^2 x \sec x dx$

We first solve $\int \sec^3 x dx$:

$$\begin{aligned}
\int \sec^3 x dx &= \int \sec x (\sec^2 x dx) \\
&= \int \sec x d \tan x \\
&= \sec x \tan x - \int \tan x d \sec x \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \ln|\sec x + \tan x|
\end{aligned}$$

$$\therefore \int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$$

$$\begin{aligned}
\int \tan^2 x \sec x dx &= \int \sec^3 x - \sec x dx \\
&= \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C
\end{aligned}$$

53. $\int \sin 3x \sin 6x dx$

$$\begin{aligned}
\int \sin 3x \sin 6x dx &= \int \sin 3x (2 \sin 3x \cos 3x) dx \\
&= \int 2 \sin^2 3x \cos 3x dx \\
&= \int 2(\cos 3x - \cos^3 3x) dx \\
&= \frac{2}{3} \sin 3x - \int 2 \cos^3 3x \\
&= \frac{2}{3} \sin 3x - 2 \int (\cos 3x - \sin^2 3x \cos 3x) dx \\
&= \frac{2}{3} \sin 3x - \frac{2}{3} \sin 3x + \frac{2}{9} \sin^3 3x + C \\
&= \frac{2}{9} \sin^3 3x + C
\end{aligned}$$

Exercise 7.3

6. $\int_0^3 \frac{x}{\sqrt{36-x^2}}dx$

$$\int_0^3 \frac{x}{\sqrt{36-x^2}}dx = \frac{1}{2}\int_0^3 \frac{1}{\sqrt{36-x^2}}dx^2$$

let $x = 6u$, then

$$\begin{aligned}
\frac{1}{2}\int_0^3 \frac{1}{\sqrt{36-x^2}}dx^2 &= \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{36-36u^2}}d36u^2 \\
&= \int_0^{\frac{1}{2}} \frac{72}{2\times 6}\frac{1}{\sqrt{1-u^2}}du \\
&= 6\arcsin u\Big|_0^{\frac{1}{2}} \\
&= 6(\frac{\pi}{6}-0) \\
&= \pi
\end{aligned}$$

7. $\int_0^a \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}, \quad a > 0$

Let $x = a\tan t$, and $t = \arctan\frac{x}{a}$, then

$$\begin{aligned}
\int_0^a \frac{1}{(a^2+x^2)^{\frac{3}{2}}}dx &= \int_0^{\frac{\pi}{4}} \frac{1}{(a^2\sec^2 t)^{\frac{3}{2}}}a\sec^2 t dt \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{a^2\sec t}dt \\
&= \int_0^{\frac{\pi}{4}} \frac{\cos t}{a^2}dt \\
&= \frac{\sin t}{a^2}\Big|_0^{\frac{\pi}{4}} \\
&= \frac{\sqrt{2}}{2a^2}
\end{aligned}$$

8. $\int \frac{dt}{t^2\sqrt{t^2-16}}$

Let $t = 4\sec\theta$, then

$$\begin{aligned}
\int \frac{dt}{t^2\sqrt{t^2-16}} &= \int \frac{1}{16\sec^2\theta 4\tan\theta} 4\sec\theta\tan\theta d\theta \\
&= \int \frac{1}{16\sec\theta} d\theta \\
&= \int \frac{\cos\theta}{16} d\theta \\
&= \frac{\sin\theta}{16} + C \\
&= \frac{1}{16}\sqrt{t^2-16t^2} + C
\end{aligned}$$

13. $\int \frac{\sqrt{x^2-9}}{x^3} dx$

Let $x = 3\sec t$, then

$$\begin{aligned}
\int \frac{\sqrt{x^2-9}}{x^3} dx &= \int \frac{3\tan t}{27\sec^3 t} 3\sec t\tan t dt \\
&= \int \frac{\tan^2 t}{3\sec^2 t} dt \\
&= \int \frac{\sin^2 t}{3} dt \\
&= \int \frac{1-\cos 2t}{6} dt \\
&= \frac{1}{6}t - \frac{\sin 2t}{3} + C \\
&= \frac{1}{6}\arccos\frac{3}{x} - \frac{2}{x}\sqrt{1-\frac{9}{x^2}} + C
\end{aligned}$$

23. $\int \sqrt{5+4x-x^2} dx$

$$\int \sqrt{5+4x-x^2} dx = \int \sqrt{9-(x-2)^2} d(x-2)$$

Let $y = x-2$, $y = 3\sin t$, $t \in (0, \pi)$, then

$$
\begin{aligned}
\int \sqrt{5+4x-x^2}dx &= \int \sqrt{9-y^2}dy \\
&= 3\int \cos t 3\cos t dt \\
&= \int 9\frac{1+\cos 2t}{2}dt \\
&= \frac{9}{2}t + \frac{9}{4}\sin 2t + C \\
&= \frac{9}{2}\arcsin\frac{x-2}{3} + \frac{3}{2}(x-2)\sqrt{1-\frac{(x-2)^2}{9}} + C
\end{aligned}
$$

25. $\int \frac{x}{\sqrt{x^2+x+1}}dx$

$$
\int \frac{x}{\sqrt{x^2+x+1}}dx = \int \frac{x}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}dx
$$

Let $u = x + \frac{1}{2}$, then

$$
\int \frac{x}{\sqrt{x^2+x+1}}dx = \int \frac{u-\frac{1}{2}}{\sqrt{u^2+\frac{3}{4}}}du
$$

Now substitue $u = \frac{\sqrt{3}}{2}\tan\theta$, so

$$
\begin{aligned}
\int \frac{x}{\sqrt{x^2+x+1}}dx &= \int \frac{\frac{\sqrt{3}}{2}\tan\theta - \frac{1}{2}}{\frac{\sqrt{3}}{2}\sec\theta}\frac{\sqrt{3}}{2}\sec^2\theta d\theta \\
&= \int (\frac{\sqrt{3}}{2}\tan\theta\sec\theta - \frac{1}{2}\sec\theta)d\theta \\
&= \frac{\sqrt{3}}{2}\sec\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C
\end{aligned}
$$