Exercise 2.3

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Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

9. $\lim_{x\to 2} \frac{2x^2+1}{3x-2}$

$$\begin{aligned}
\lim_{x\to 2} \frac{2x^2+1}{3x-2} &= \sqrt{\lim_{x\to 2} \frac{2x^2+1}{3x-2}} \\
&= \sqrt{\frac{\lim_{x\to 2} 2x^2+1}{\lim_{x\to 2} 3x-2}} \\
&= \sqrt{\frac{9}{4}} \\
&= \frac{3}{2}
\end{aligned}$$

Evaluate the limit, if it exists.

14. $\lim_{x\to -1} \frac{x^2-4x}{x^2-3x-4}$

$$\begin{aligned}
\lim_{x\to -1} \frac{x^2-4x}{x^2-3x-4} &= \lim_{x\to -1} x(x-4)(x-4)(x+1) \\
&= \lim_{x\to -1} \frac{x}{x+1} \\
&= \lim_{x\to -1} (1 - \frac{1}{x+1})
\end{aligned}$$

$\because$ when $x \to -1^-$, $\frac{1}{x+1} \to -\infty$, when $x \to -1^+$, $\frac{1}{x+1} \to \infty$

$\therefore \lim_{x\to -1} \frac{1}{x+1}$ does not exist.

$\therefore \lim_{x\to -1} \frac{x^2-4x}{x^2-3x-4}$ does not exist.

18. $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$

$$\begin{aligned}
\lim_{h\to 0} \frac{(2+h)^3-8}{h} &= \lim_{h\to 0} \frac{(h^2+4h+4)(h+2)}{h} \\
&= \lim_{h\to 0} \frac{h^3+6h^2+12h+8-8}{h} \\
&= \lim_{h\to 0} (h^2+6h+12) \\
&= \lim_{h\to 0} h^2 + \lim_{h\to 0} 6h + \lim_{h\to 0} 12 \\
&= 12
\end{aligned}$$

22. $\lim_{u\to 2} \frac{\sqrt{4u+1}-3}{u-2}$

$$\begin{aligned}
\lim_{u\to 2} \frac{\sqrt{4u+1}-3}{u-2} &= \lim_{u\to 2} \frac{4u+1-3^2}{(u-2)(\sqrt{4u+1}+3)} \\
&= \lim_{u\to 2} \frac{4(u-2)}{(u-2)(\sqrt{4u+1}+3)} \\
&= \lim_{u\to 2} \frac{4}{\sqrt{4u+1}+3} \\
&= \frac{4}{3+3} = \frac{2}{3}
\end{aligned}$$

25. $\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$

$$\begin{aligned}
\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t} &= \lim_{t\to 0} \frac{(\sqrt{1+t}-\sqrt{1-t})(\sqrt{1+t}+\sqrt{1-t})}{t(\sqrt{1+t}+\sqrt{1-t})} \\
&= \lim_{t\to 0} \frac{1+t-1+t}{t(\sqrt{1+t}+\sqrt{1-t})} \\
&= \lim_{t\to 0} \frac{2}{\sqrt{1+t}+\sqrt{1-t}} \\
&= \frac{2}{\lim_{t\to 0} \sqrt{1+t}+\sqrt{1-t}} \\
&= \frac{2}{1+1} \\
&= 1
\end{aligned}$$

29. $\lim_{t\to 0}(\frac{1}{t\sqrt{1+t}} - \frac{1}{t})$

$$\begin{aligned}\lim_{t\to 0}(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}) &= \lim_{t\to 0}\frac{1-\sqrt{1+x}}{x\sqrt{1+x}}\\ &= \lim_{t\to 0}\frac{1-(1+x)}{x\sqrt{1+x}(1+\sqrt{1+x})}\\ &= \lim_{t\to 0}\frac{-1}{\sqrt{1+x}(1+\sqrt{1+x})}\\ &= \lim_{t\to 0}\frac{-1}{1+x+\sqrt{1+x}}\\ &= \frac{-1}{1+0+1}\\ &= -\frac{1}{2}\end{aligned}$$

39. Prove that $\lim_{x\to 0} x^4 \cos\frac{2}{x} = 0$

Proof. $\because -1 < \cos\frac{2}{x} < 1$
$\therefore -x^4 < x^4\cos\frac{2}{x} < x^4$
$\because \lim_{x\to 0} -x^4 = \lim_{x\to 0} x^4 = 0$
$\therefore$ by the squeeze theorem,

$$\lim_{x\to 0} x^4\cos\frac{2}{x} = 0$$

□

40. Prove that $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$

Proof. $\because -1 < \sin(\frac{\pi}{x}) < 1$
$\therefore \frac{\sqrt{x}}{e} < \sqrt{x}e^{\sin(\frac{\pi}{x})} < e\sqrt{x}$
$\because \lim_{x\to 0}\frac{\sqrt{x}}{e} = \lim_{x\to 0} e\sqrt{x} = 0$
$\therefore$ by the squeeze theorem,

$$\lim_{x\to 0}\sqrt{x}e^{\sin(\frac{\pi}{x})} = 0$$

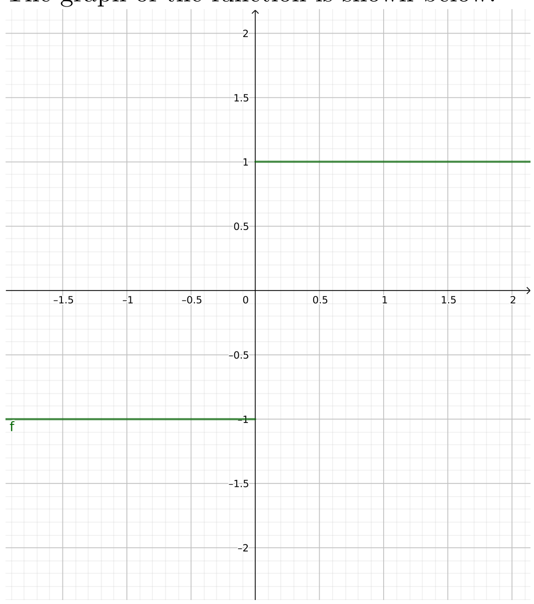
□

47. The $sigsum$**(or sign)**$function$**, denoted by sgn, is defined by**

$$sgnx = \begin{cases} -1 & \textbf{if x < 0} \\ 0 & \textbf{if x = 0} \\ 1 & \textbf{if x > 0} \end{cases}$$

(a) Sketch the graph of this function.

The graph of the function is shown below.



(b) Find each of the following limits or explain why it does not exist.

1. $\lim_{x\to0^+}$ sgn x The limit is 1.
2. $\lim_{x\to0^-}$ sgn x The limit is -1.
3. $\lim_{x\to0}$ sgn x By the answer of two problems above, we know that

$$\lim_{x\to0^+} sgnx \neq \lim_{x\to0^-} sgnx$$

So $\lim_{x\to0} sgnx$ does not exist.

4. $\lim_{x\to0}$ |sgn x| By the answer of two problems above, we know that

$$|\lim_{x\to0^+} sgnx| = |\lim_{x\to0^-} sgnx| = 1$$

So $\lim_{x\to0} |sgnx| = 1$

57. If $\lim_{x\to1} \frac{f(x)-8}{x-1} = 10$, find $\lim_{x\to1} f(x)$

$$\begin{aligned}\lim_{x\to1} f(x) &= 10(\lim_{x\to1}(x-1)) + 8\\ &= 8\end{aligned}$$

58. If $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$, find the following limits.

a. $\lim_{x\to 0} f(x)$

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{f(x)}{x^2} * \lim_{x\to 0} x^2 \\ &= 5 \times \lim_{x\to 0} x^2 \\ &= 0\end{aligned}$$

b. $\lim_{x\to 0} \frac{f(x)}{x}$

$$\begin{aligned}\lim_{x\to 0} \frac{f(x)}{x} &= \lim_{x\to 0} \frac{f(x)}{x^2} * \lim_{x\to 0} x \\ &= 5 \times 0 \\ &= 0\end{aligned}$$

59. If

$$f(x) = \begin{cases} x^2 & \textbf{if x is rational} \\ 0 & \textbf{if x is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x) = 0$

Proof. $\forall \epsilon > 0, \exists \delta = \sqrt{\epsilon}$

if $0 < |x - 0| < \delta$, then we can start our discussion.

1. If $\delta \in Q$, then

$$|f(x) - 0| < |f(\delta) - 0| = |\delta^2| = \epsilon$$

2. If $\delta \notin Q$, so $\exists \delta_0 \in Q$ and $\delta_0 < \delta$, so

$$|f(x) - 0| < |f(\delta_0)| = \delta_0^2 < \delta^2 = \epsilon$$

So no matter whether $\delta \in Q$, $\lim_{x\to 0} f(x) = 0$

$\square$

60. Show by means of an example that $\lim_{x\to a}[f(x) + g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

Proof. Let $f(x) = \frac{1}{x-a}$, $g(x) = -\frac{1}{x-a}$

Obviously, both the limit of $f(x)$ and $g(x)$ do not exist when x approach a. But

$$f(x) + g(x) = \frac{1}{x-a} - \frac{1}{x-a} = 0$$

So $\lim_{x\to a}[f(x) + g(x)]$ exists, and the value is 0. $\square$

61. Show by means of an example that $\lim_{x\to a}[f(x)g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

Proof. Let $f(x) = e^{\frac{1}{x-a}}$, $g(x) = e^{-\frac{1}{x-a}}$

Obviously both the limit of $f(x)$ and $g(x)$ do not exist when x approach a. But

$$f(x)g(x) = e^{\frac{1}{x-a}-\frac{1}{x-a}} = e^0 = 1$$

So $\lim_{x\to a}[f(x)g(x)]$ exists, and the value is 1. □

62. Evaluate $\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$

$$\begin{aligned}
\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \lim_{x\to 2}\frac{6-x-4}{(\sqrt{3-x}-1)(\sqrt{6-x}+2)} \\
&= \lim_{x\to 2}\frac{(2-x)(\sqrt{3-x}+1)}{(\sqrt{6-x}+2)(3-x-1)} \\
&= \lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} \\
&= \frac{2}{4} = \frac{1}{2}
\end{aligned}$$