Exercise 2.6 Homework

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Evaluate the limit and justify each step by indicating the appropriate properties of limits.

14. $\lim_{x\to\infty}\sqrt{\frac{12x^3-5x+2}{1+4x^2+3x^3}}$

$$\begin{aligned}\lim_{x\to\infty}\sqrt{\frac{12x^3-5x+2}{1+4x^2+3x^3}} &= \lim_{x\to\infty}\sqrt{\frac{12-\frac{5}{x^2}+\frac{2}{x^3}}{3+\frac{4}{x^2}+\frac{1}{x^3}}}\\ &= \frac{12-0+0}{3+0+0}\\ &= 4\end{aligned}$$

Find the limit or show that it does not exist.

26. $\lim_{x\to-\infty}(x+\sqrt{x^2+2x})$

$$\begin{aligned}\lim_{x\to-\infty}(x+\sqrt{x^2+2x}) &= \lim_{x\to-\infty}\frac{x^2-x^2-2x}{x-\sqrt{x^2+2x}}\\ &= \lim_{x\to-\infty}\frac{2x}{\sqrt{x^2+2x}-x}\\ &= \lim_{x\to-\infty}\frac{-2}{\sqrt{1-\frac{2}{x}}+1}\\ &= \frac{-2}{1+1} = -1\end{aligned}$$

27. $\lim_{x\to\infty}(\sqrt{x^2+ax}-\sqrt{x^2+bx})$

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{x^2+ax}-\sqrt{x^2+bx}) &= \lim_{x\to\infty}\frac{x^2+ax-x^2-bx}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}\\ &= \lim_{x\to\infty}\frac{(a-b)x}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}\\ &= \lim_{x\to\infty}\frac{a-b}{\sqrt{1+\frac{1}{a}}+\sqrt{1+\frac{1}{b}}}\\ &= \frac{a-b}{1+1}\\ &= \frac{a-b}{2}\end{aligned}$$

28. $\lim_{x\to\infty}\sqrt{x^2+1}$

The limit does not exist. Here is the reason:

$\forall\epsilon>0, \exists N=\lceil\sqrt{\epsilon^2-1}\rceil$

if $x>N$, then

$$\sqrt{x^2+1}>\sqrt{N^2+1}>\epsilon$$

So

$$\lim_{x\to\infty}\sqrt{x^2+1}=\infty$$

which means that the limit does not exist.

30. $\lim_{x\to\infty}(e^{-x}+2\cos 3x)$

The limit does not exist, and here is the reason:

Obviously, $\lim_{x\to\infty}e^{-x}=0$.

Let $a_n=\frac{2n\pi}{3}, b_n=(2n+3)\pi 3$,

and we know $a_n\to\infty, b_n\to\infty$ as $n\to\infty$

By the Henie Theorem,

$$\lim_{x\to\infty}2\cos 3x=\lim_{x\to\infty}2\cos 2n\pi=0$$

$$lim_{x\to\infty}2\cos 3x=\lim_{x\to\infty}(2n\pi+\pi)=-1$$

For these two limits are not equal, $\lim_{x\to\infty}2\cos 3x$ does not exist.

So $\lim_{x\to\infty}e^{-x}+2\cos 3x$ does not exist.

33. $\lim_{x\to-\infty}\arctan(e^x)$

Let $u=e^x$, $u_0=\lim_{x\to-\infty}e^x=0$. So

$$\lim_{x\to-\infty}\arctan(e^x)=\lim_{u\to u_0}\arctan(u)=0$$

36. $\lim_{x\to\infty} \frac{\sin^2 x}{x^2+1}$

Obviously,

$$\lim_{x\to\infty} \frac{1}{x^2+1} = 0$$

which means $\lim_{x\to\infty} \frac{1}{x^2+1}$ is infinitesimal.

And $\because$

$$\lim_{x\to\infty} \sin^2 x \in [0, 1]$$

which means $\lim_{x\to\infty} \sin^2 x$ is bounded.

By the properties of infinitesimal,

$$\lim_{x\to\infty} \frac{\sin^2 x}{x^2+1} = 0$$

37. $\lim_{x\to\infty}(e^{-2x} \cos x)$

Obviously,

$$\lim_{x\to\infty} e^{-2x} = 0$$

which means this value is infinitesimal.

And $\because$

$$\lim_{x\to\infty} \cos x \in [-1, 1]$$

which means this value is bounded.

By the properties of infinitesimal,

$$\lim_{x\to\infty} e^{-2x} \cos x = 0$$

51. A function f is a ratio of quadratic functions and has a vertical asymptote $x = 4$ and just one x-intercept, $x = 1$. It is known that f has a removable discountinuity at $x = -1$ and $\lim_{x\to-1} f(x) = 2$. Evaluate

(a) $f(0)$

Let $f(x) = \frac{g(x)}{h(x)}$.

$\because f(x)$ has a vertical asymptote $x = 4$

$\therefore h(x) = 0$ as $x = 4$, which means

$$h(x) = (x-4)^2$$

And we can establish a system of equations:

$$\begin{cases} f(1) = \frac{g(1)}{h(1)} = 0 \\ f(-1) = \frac{g(-1)}{h(-1)} = 2 \end{cases}$$

Solving this system, we can get $g(x) = x(x-1)$

Therefore, $f(0) = 0$

(b) $\lim_{x\to\infty} f(x)$

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x(x-1)}{(x-4)^2}\\ &= \lim_{x\to\infty} \frac{1-\frac{1}{x}}{(1-\frac{4}{x})^2}\\ &= \frac{1-0}{(1-0)^2}\\ &= 1\end{aligned}$$

71. Use Definition 8 to prove that $\lim_{x\to-\infty} \frac{1}{x} = 0$.

Proof. $\forall \epsilon > 0, \exists N = -\frac{1}{\epsilon}$

If $x < N$, then

$$|f(x) - 0| = -\frac{1}{x} < -\frac{1}{N} = \epsilon$$

□

75. Prove that

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f(1/t)$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{t\to 0^-} f(1/t)$$

if these limits exist.

Proof. $\because \lim_{x\to\infty} x = \infty = \lim_{t\to 0^+} \frac{1}{t} = \infty$

$\therefore \lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f(1/t)$

$\because \lim_{x\to-\infty} x = -\infty = \lim_{x\to 0^-} \frac{1}{t} = -\infty$

$\therefore \lim_{x\to-\infty} f(x) = \lim_{t\to 0^-} f(1/t)$

□