Chapter 3 Second Exercise

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Exercise 3.3

9. $y = \frac{x}{2-\tan x}$

$$y' = \frac{(2-\tan x) - x(-\sec^2 x)}{(2-\tan x)^2}$$

17. Prove that $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

Proof.

$$(\csc x)' = (\frac{1}{\sin x})' = \frac{-\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x}\frac{1}{\sin x} = -\csc x \cot x$$

□

18. Prove that $\frac{d}{dx}(\sec x) = \sec x \tan x$.

Proof.

$$(\sec x)' = (\frac{1}{\cos x})' = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

□

42. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta}$

Solution:

By the equivalent infinitesimal,

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\sin\theta} = \lim_{x\to 0}\frac{-\frac{1}{2}x^2}{x} = 0$$

44. $\lim_{x\to 0} \frac{\sin 3x \sin 5x}{x^2}$

Solution:

By the equivalent infinitesimal,

$$\lim_{x\to 0}\frac{\sin 3x \sin 5x}{x^2} = \lim_{x\to 0}\frac{3x \times 5x}{x^2} = 15$$

45. $\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}$

Solution:

By the equivalent infinitesimal,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{x\to 0}xx+x=\frac{1}{2}$$

47. $\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{\sin x-\cos x}$

Solution:

$$\begin{aligned}lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{\sin x-\cos x}&=lim_{x\to\frac{\pi}{4}}1-\frac{\sin x}{\cos x}\sin x-\cos x\\&=\lim_{x\to\frac{\pi}{4}}\cos x-\sin x(\sin x-\cos x)\cos x\\&=\lim_{x\to\frac{\pi}{4}}-1\cos x\\&=-\sqrt{2}\end{aligned}$$

54.

$$|PQ|=2\times 10\sin\frac{\theta}{2}=20\sin\frac{\theta}{2}(\mathrm{cm})$$

$$A(\theta)=\frac{1}{2}\pi(\frac{|PQ|}{2})^2=50\pi\sin^2\frac{\theta}{2}(\mathrm{cm}^2)$$

$$B(\theta)=\frac{1}{2}\times 10\times 10\times\sin\theta=50\sin\theta(\mathrm{cm}^2)$$

$$\begin{aligned}\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)}&=\lim_{\theta\to 0^+}\frac{\pi\sin^2\frac{\theta}{2}}{\sin\theta}\\&=\lim_{\theta\to 0^+}\frac{(1-\cos\theta)\pi}{2\sin\theta}\\&=\lim_{\theta\to 0^+}\frac{\frac{1}{2}\theta^2}{2\theta}\\&=\lim_{\theta\to 0^+}\frac{\theta}{4}\\&=0\end{aligned}$$

Exercise 3.4

28. $y=\frac{e^u-e^{-u}}{e^u+e^{-u}}$

Solution:

$$
\begin{aligned}
y' &= \frac{(e^u + e^{-u})^2 - (e^u - e^{-u})^2}{(e^u + e^{-u})^2} \\
&= \frac{(2e^u)(2e^{-u})}{(e^u + e^{-u})^2} \\
&= \frac{4}{(e^u + e^{-u})^2}
\end{aligned}
$$

0.1 29. $F(t) = e^{t \sin 2t}$

Solution:

$$
\begin{aligned}
F'(t) &= e^{t \sin 2t}(\sin 2t + t \times 2 \cos 2t) \\
&= e^{t \sin 2t}(\sin 2t + 2t \cos 2t)
\end{aligned}
$$

42. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

Solution:

$$
\begin{aligned}
y' &= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}(x + \sqrt{x + \sqrt{x}})' \\
&= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}(1 + \frac{1}{2\sqrt{x + \sqrt{x}}}(1 + \frac{1}{2\sqrt{x}}))
\end{aligned}
$$

46. $y = [x + (x + \sin^2 x)^3]^4$

Solution:

$$
\begin{aligned}
y' &= 4[x + (x + \sin^2 x)^3]^3[x + (x + \sin^2 x)^3]' \\
&= 4[x + (x + \sin^2 x)^3]^3[1 + 3(x + \sin^2 x)^2(1 + 2 \sin x \cos x)]
\end{aligned}
$$

49. $y = e^{ax} \sin \beta x$

Solution:

$$
\begin{aligned}
y' &= (ae^{ax}) \sin \beta x + e^{ax} \beta \cos \beta x \\
&= e^{ax}(a \sin \beta x + \beta \cos \beta x)
\end{aligned}
$$

91. Use the Chain Rule to prove the following.

(a) The derivative of an even function is an odd funciton.

Solution:

Let $f(x)$ is an even function, which means $f(-x) = f(x)$

$$-f'(-x) = f'(x) \iff f'(-x) = -f'(x)$$

So $f'(x)$ is an odd function.

(b) The derivative of an odd function is an even function.

Solution:

Let $f(x)$ is an odd function, which means $f(-x) = -f(x)$

$$-f'(-x) = -f'(x) \iff f'(-x) = f'(x)$$

So $f'(x)$ is an even function.

92. Use the Chain Rule and the Product Rule to give an alternative proof of the Quotient Rule.

Proof.

$$\begin{aligned}(\frac{f(x)}{g(x)})' &= (f(x)g(x)^{-1})' \\ &= f'(x)g(x)^{-1} + f(x)(-g(x)^{-2}g'(x)) \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\end{aligned}$$

□

95. Use the Chain Rule to show that if θ is measured in degrees, then

$$\frac{d}{d\theta}(\sin\theta) = \frac{\pi}{180}\cos\theta$$

Solution:

Let $x = \frac{\pi}{180}\theta$, and x represents the same angle as θ in radians.
$\because \sin\theta = \sin x, \cos x = \cos\theta$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\frac{\pi}{180}\theta) = \frac{\pi}{180}\cos x = \frac{\pi}{180}\cos\theta$$

96. (a)

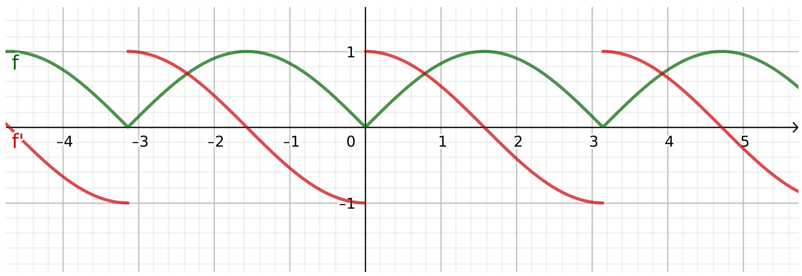
Solution:

$$\begin{aligned}\frac{d}{dx}|x| &= (\sqrt{x^2})' \\ &= \frac{1}{2\sqrt{x^2}}2x \\ &= \frac{x}{|x|}\end{aligned}$$

96. (b)

By the conclusion of problem (a), we can get

$$f'(x) = \frac{\sin x}{|\sin x|}\cos x$$

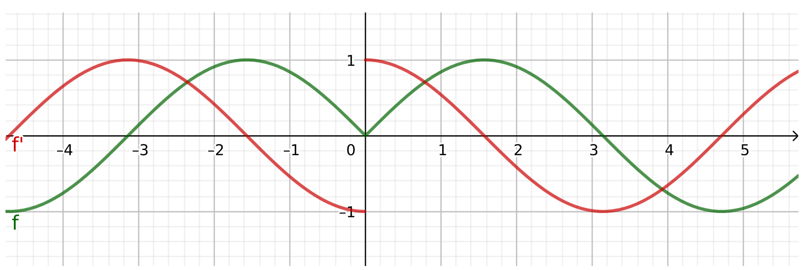


The green line represents $f(x)$, and the red line represents $f'(x)$.
f is not differentable at $x = k\pi$, where $k \in Z$.

96. (c)

By the conclusion of problem (a), we can get

$$g'(x) = \cos|x|(|x|)' = \cos|x|\frac{x}{|x|}$$



The green line represents $g(x)$ and the red line represents $g'(x)$.
g is not differentable at $x = 0$.

97. If $y = f(u)$ and $u = g(x)$, where f and g are twice differentable functions, show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}u^2}(\frac{\mathrm{d}u}{\mathrm{d}x})^2 + \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}$$

$\because u = g(x), f'(u) = f'(u)g'(x)$

$$\therefore f''(x) = [f'(u)]' = f''(u)g'(x) + f'(u)g''(x) = f''(u)u' + f'(u)u''$$

$\therefore$ rewrite the formula in Leibniz's form, and we get

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}u^2}(\frac{\mathrm{d}u}{\mathrm{d}x})^2 + \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}$$

98. If $y = f(u)$ and $u = g(x)$, where f and g possess third derivatives, find a formula for $\mathrm{d}^3 y/\mathrm{d}x^3$ similar to the one given in Exercise 97.

By the conclusion of Exercise 97, we know

$$f''(x) = [f'(u)]' = f''(u)g'(x) + f'(u)g''(x)$$

$$\therefore f'''(x) = f'''(u)g'(x) + 2f''(u)g''(x) + f'(u)g'''(x)$$

$\therefore$ rewrite the formula in Leibniz's form, and we get

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = \frac{\mathrm{d}^3 y}{\mathrm{d}u^3}\frac{\mathrm{d}u}{\mathrm{d}x} + 2\frac{\mathrm{d}^2 y}{\mathrm{d}u^2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}^3 u}{\mathrm{d}x^3}$$