Exercise 11.2

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March 4, 2021

35. $\sum_{n=1}^{\infty} \ln(\frac{n^2+1}{2n^2+1})$

$$\because \lim_{n\to\infty} \ln(\frac{n^2+1}{2n^2+1}) = \lim_{n\to\infty} \ln(1-\frac{1}{2+\frac{1}{n^2}}) = \ln\frac{1}{2} \neq 0$$

$\therefore$ This series is divergent

39. $\sum_{n=1}^{\infty} \arctan n$

$$\because \lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$$

$\therefore$ This series is divergent

40. $\sum_{n=1}^{\infty} (\frac{3}{5^n} + \frac{2}{n})$

$$\because \sum_{n=1}^{\infty} \frac{3}{5^n} = \sum_{n=1}^{\infty} \frac{3}{5}(\frac{1}{5})^{n-1}, |\frac{1}{5}| < 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{3}{5^n} \text{ is convergent and its sum is } \frac{\frac{3}{5}}{1-\frac{1}{5}} = \frac{3}{4}$$

$$\because \sum_{n=1}^{\infty} \frac{2}{n} \text{ is divergent}$$

$$\therefore \sum_{n=1}^{\infty} (\frac{3}{5^n} + \frac{2}{n}) \text{ is divergent}$$

41. $\sum_{n=1}^{\infty}(\frac{1}{e^n}+\frac{1}{n(n+1)})$

$$\because \sum_{n=1}^{\infty}\frac{1}{e^n}=\sum_{n=1}^{\infty}\frac{1}{e}(\frac{1}{e})^{n-1}, |\frac{1}{e}|<1$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{e^n}=\frac{\frac{1}{e}}{1-\frac{1}{e}}=\frac{1}{e-1}$$

$$\because \sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+1})=\lim_{n\to\infty}(\frac{1}{1}-\frac{1}{n+1})=1$$

$$\therefore \sum_{n=1}^{\infty}(\frac{1}{e^n}+\frac{1}{n(n+1)}) \text{ is convergent and its sum is } \frac{e}{e-1}.$$

42. $\sum_{n=1}^{\infty}\frac{e^n}{n^2}$

$$\because \lim_{n\to\infty}\frac{e^n}{n^2}=\lim_{n\to\infty}\frac{e^n}{2n}=\lim_{n\to\infty}\frac{e^n}{2}\neq 0$$

$$\therefore \sum_{n=1}^{\infty}\frac{e^n}{n^2} \text{ is divergent}$$

48. $\sum_{n=2}^{\infty}\frac{1}{n^3-n}$

$$\begin{aligned}\because \sum_{n=2}^{\infty}\frac{1}{n^3-n} &= \sum_{n=2}^{\infty}\frac{1}{2}(\frac{1}{n(n-1)}-\frac{1}{n(n+1)})\\ &= \frac{1}{2}(\frac{1}{2}-\frac{1}{6}+\frac{1}{6}-\frac{1}{12}+\cdots+\frac{1}{n(n-1)}-\frac{1}{n(n+1)})\\ &= \lim_{n\to\infty}\frac{1}{2}(\frac{1}{2}-\frac{1}{n(n+1)})\\ &= \frac{1}{4}\end{aligned}$$

$\therefore \sum_{n=2}^{\infty}\frac{1}{n^3-n}$ is convergent and its sum is $\frac{1}{4}$.

49. Let $x=0.99999\cdots$

(a) I think $x=1$.

(b) Solution:

$$
\begin{aligned}
x &= 0.99999\cdots \\
&= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{9}{10}(\frac{1}{10})^{n-1} \\
&= \frac{\frac{9}{10}}{1 - \frac{1}{10}} \\
&= 1
\end{aligned}
$$

(c) Two decimal representations, 1 and $0.\dot{9}$, respectively.

(d) Finite repeating decimals.

50. A sequence of terms is defined by

$$a_1 = 1 \qquad a_n = (5 - n)a_{n-1}$$

Calculate $\sum_{n=1}^{\infty} a_n$.

$\because$ Obviously, we can get

$$a_2 = 3 \times 1 = 3, a_3 = 2 \times 3 = 6, a_4 = 1 \times 6 = 6, a_5 = 0 \times 6 = 0, a_6 = 0, \cdots$$

$$\therefore \sum_{n=1}^{\infty} a_n = 1 + 3 + 6 + 6 = 16$$

64. We have seen that the harmonic series is a divergent series whose terms approach 0. Show that

$$\sum_{n=1}^{\infty} \ln(1 + \frac{1}{n})$$

is another series with this property.

$$
\begin{aligned}
&\because \lim_{n\to\infty} \ln(1 + \frac{1}{n}) = \ln 1 = 0 \\
&\because \sum_{n=1}^{\infty} \ln(1 + \frac{1}{n}) = \ln(\frac{2}{1} \times \frac{3}{2} \times \cdots \times \frac{n+1}{n}) = \ln(n+1) = \infty \\
&\therefore \sum_{n=1}^{\infty} \ln(1 + \frac{1}{n}) \text{ is another series with this property.}
\end{aligned}
$$

67. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+1}$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

$$\text{If } n \geq 2, a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{n-2}{n} = \frac{2}{n(n+1)}$$

$$\text{If } n = 1, a_1 = s_1 = 0$$

$$\therefore a_n = \begin{cases} 0 & \text{if } n = 1 \\ \frac{2}{n(n+1)} & \text{if } n \geq 2 \end{cases}$$

$$\because \lim_{n\to\infty} s_n = \lim_{n\to\infty} (1 - \frac{2}{n+1}) = 1$$

$$\therefore \sum_{n=1}^{\infty} a_n = 1$$

68. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = 3 - n2^{-n}$, find a_n and $\sum_{n=1}^{\infty} a_n$.

$$\text{If } n \geq 2, a_n = s_n - s_{n-1} = \frac{n-1}{2^{n-1}} - \frac{n}{2^n} = \frac{n-2}{2^n}$$

$$\text{If } n = 1, a_1 = s_1 = 3 - \frac{1}{2} = \frac{5}{2}$$

$$\therefore a_n = \begin{cases} \frac{5}{2} & \text{if } n = 1 \\ \frac{n-2}{2^n} & \text{if } n \geq 2 \end{cases}$$

$$\because \lim_{n\to\infty} s_n = \lim_{n\to\infty} (3 - \frac{n}{2^n}) = 3$$

$$\therefore \sum_{n=1}^{\infty} a_n = 3$$

79. What is wrong with the following calculation?

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= 1 + (-1+1) + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + 0 + \cdots = 1 \end{aligned}$$

Let $a_n = 1, b_n = -1$, and

$$\sum_{n=1}^{\infty} a_n = 1 + 1 + \cdots = \sum_{n=1}^{n} 1, \quad \sum_{n=1}^{\infty} b_n = -1 - 1 - \cdots = \sum_{n=1}^{n} (-1)$$

Obviously, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent.

$$\because 0 = 1 - 1 = a_n + b_n$$
$$\therefore \sum_{n=1}^{\infty} 0 = \sum_{n=1}^{\infty}(a_n + b_n) = 0$$
$$\because \text{Though } \sum_{n=1}^{\infty}(a_n + b_n) \text{ is convergent, } \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n \text{ are divergent}$$
$$\therefore \sum_{n=1}^{\infty}(a_n + b_n) \neq \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

In other words,

$$\begin{aligned} 0 &= (1-1) + (1-1) + (1-1) + \cdots \\ &\neq 1 + 1 + 1 + \cdots - 1 - 1 - 1 - \cdots = 1 - 1 + 1 - 1 + 1 - 1 + \cdots \end{aligned}$$

89. Consider the series $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

(a) Solution:

$$s_1 = \frac{1}{2}, s_2 = s_1 + \frac{2}{6} = \frac{5}{6}, s_3 = s_2 + \frac{3}{24} = \frac{23}{24}, s_4 = s_3 + \frac{4}{120} = \frac{119}{120}$$

The denominators are factorial forms.

And we can guess

$$s_n = 1 - \frac{1}{(n+1)!}$$

(b) Solution:

If $n = 1$, $s_1 = \frac{1}{2} = 1 - \frac{1}{(1+1)!}$, which satisfies the hypothesis.

If $n \geq 2$, suppose the hypothesis holds when $n = k (k \in N_+)$. That is to say

$$s_k = 1 - \frac{1}{(k+1)!}$$

$$s_{k+1} = s_k + \frac{k+1}{(k+2)!} = \frac{(k+2)! - (k+2) + (k+1)}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$

$\therefore$ the hypothesis also holds when $n = k + 1$

$\therefore$ we can conclude that

$$s_n = 1 - \frac{1}{(n+1)!}$$

(c) *Proof.*

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}(1 - \frac{1}{(n+1)!}) = 1$$

$\therefore$ the given infinite series is convergent, and its sum is 1. $\square$