MA5_2 Exercise

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Exercise 5.3

13. $h(x) = \int_1^{e^x} \ln t dt$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^{e^x} \ln t dt = e^x \ln e^x = xe^x$$

18. $y = \int_{\sin x}^1 \sqrt{1+t^2} dt$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(-\int_1^{\sin x} \sqrt{1+t^2} dt) \\ &= -\cos x\sqrt{1+\sin^2 x}\end{aligned}$$

72.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{g(x)}^{h(x)} f(t)dt &= \frac{\mathrm{d}}{\mathrm{d}x}(\int_a^{h(x)} f(t)dt - \int_a^{g(x)} f(t)dt) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\int_a^{h(x)} f(t)dt - \frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)dt \\ &= f(h(x))h'(x) - f(g(x))g'(x)\end{aligned}$$

57. $F(x) = \int_x^{x^2} e^{t^2} dt$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}F(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(\int_a^{x^2} e^{t^2} dt - \int_a^{x} e^{t^2} dt) \\ &= 2xe^{x^4} - e^{x^2}\end{aligned}$$

58. $F(x) = \int_{\sqrt{x}}^{2x} \arctan t dt$

$$\begin{aligned}\frac{\mathrm{d}F(x)}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(\int_a^{2x} \arctan t dt - \int_a^{\sqrt{x}} \arctan t dt) \\ &= 2\arctan 2x - \frac{\arctan\sqrt{x}}{2\sqrt{x}}\end{aligned}$$

59. $y = \int_{\cos x}^{\sin x} \ln(1+2v) dv$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{\cos x}^{\sin x} \ln(1+2v)dv &= \frac{\mathrm{d}}{\mathrm{d}x}(\int_a^{\sin x} \ln(1+2v)dv - \int_a^{\cos x} \ln(1+2v)dv) \\ &= \cos x \ln(1+2\sin x) + \sin x \ln(1+2\cos x)\end{aligned}$$

62. If $f(x) = \int_0^x (1-t^2)e^{t^2} dt$, on what interval is f increasing?

$\because f'(x) = (1-x^2)e^{x^2}$

$\therefore$ when $1 - x^2 \geq 0 \iff x \in [-1, 1]$, $f'(x) \geq 0$, and f is increasing

29. $\int_1^9 \frac{x-1}{\sqrt{x}} dx$

$$\because \frac{x-1}{\sqrt{x}} = \sqrt{x} - \frac{1}{\sqrt{x}}$$

$$\therefore \int_1^9 \frac{x-1}{\sqrt{x}} dx = (\frac{2}{3}x^{\frac{3}{2}} - 2\sqrt{x})\Big|_1^9 = (18-6) - (\frac{2}{3} - 2) = 12 + \frac{4}{3} = \frac{40}{3}$$

32. $\int_0^{\pi/4} \sec\theta \tan\theta d\theta$

$$\because (\sec\theta)' = \sec\theta\tan\theta$$

$$\therefore \int_0^{\frac{\pi}{4}} \sec\theta\tan\theta d\theta = \sec\theta\Big|_0^{\frac{\pi}{4}} = \sqrt{2} - 1$$

41. $\int_{-1}^1 e^{u+1} du$

$$\int_{-1}^1 e^{u+1} du = e^{u+1}\Big|_{-1}^1 = e^2 - e^0 = e^2 - 1$$

42. $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx$

$$\because \frac{\mathrm{d} \arcsin x}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{4}{\sqrt{1-x^2}} dx = 4 \arcsin x \Big|_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} = 4(\frac{\pi}{4} - \frac{\pi}{6}) = \frac{\pi}{3}$$

44. $\int_{-2}^{2} f(x)dx$ **where** $f(x) = \begin{cases} 2 & \textbf{if } -2 \le x \le 0 \\ 4 - x^2 & \textbf{if } 0 < x \le 2 \end{cases}$

$$\begin{aligned} \int_{-2}^{2} f(x)dx &= \int_{-2}^{0} f(x)dx + \int_{0}^{2} f(x)dx \\ &= 2 \times (0+2) + (4x - \frac{1}{3}x^3)\Big|_0^2 \\ &= 4 + (8 - \frac{8}{3}) - (0) \\ &= \frac{28}{3} \end{aligned}$$

46.

The equation is wrong because x cannot be 0, thus $y = x^{-4}$ is not continuous on $[-1, 2]$.

In order to calculate the definite integral, we need to divide the intergral interval into $[-1, 0)$ and $(0, 2]$.

47.

The equation is also wrong because $y = \sec\theta \tan\theta$ is not defined at $\theta = \frac{\pi}{2}$, thus the function is not continuous on $[\frac{\pi}{3}, \pi]$.

In order to correct the equation, we need to split this interval into $[\frac{\pi}{3}, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi]$.

70. $\lim_{n\to\infty} \frac{1}{n}(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}})$

$$\because \lim_{n\to\infty} \frac{1}{n}(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i}{n}}$$

which is the same as the Riemman sum of $f(x) = \sqrt{x}$ from 0 to 1, and $\Delta x = \frac{1}{n}$

$$\therefore \lim_{n\to\infty} \frac{1}{n}(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}) = \int_0^1 \sqrt{x} dx = (\frac{2}{3}x^{\frac{3}{2}})|_0^1 = \frac{2}{3}$$