Exercise 7_4

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11. $\int_0^1 \frac{2}{2x^2+3x+1}dx$

$$\begin{aligned}\int_0^1 \frac{2}{2x^2+3x+1}dx &= \int_0^1 \frac{2}{(2x+1)(x+1)}dx \\ &= \int_0^1 (\frac{A}{2x+1} + \frac{B}{x+1})dx\end{aligned}$$

$$A(x+1) + B(2x+1) = (A+2B)x + (A+B) = 2$$

Solving $\begin{cases} A+2B=0 \\ A+B=2 \end{cases}$, we get $\begin{cases} A=4 \\ B=-2 \end{cases}$

$$\begin{aligned}\therefore \int_0^1 \frac{2}{2x^2+3x+1}dx &= \int_0^1 (\frac{4}{2x+1} - \frac{2}{x+1})dx \\ &= (2\ln(2x+1) - 2\ln(x+1))\Big|_0^1 \\ &= 2(\ln 3 - \ln 2 - \ln 1 + \ln 1) \\ &= 2\ln\frac{3}{2}\end{aligned}$$

19. $\int \frac{x^2+1}{(x-3)(x-2)^2}dx$

$$\int \frac{x^2+1}{(x-3)(x-2)^2}dx = \int (\frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2})dx$$

$$A(x-2)^2 + B(x-3)(x-2) + C(x-3) = x^2+1$$

Let $x=2$, then $-C=5 \iff C=-5$
Let $x=3$, then $A=10$
Let $x=4$, then $40+2B-5=17 \iff B=-9$

$$\therefore \int \frac{x^2+1}{(x-3)(x-2)^2}dx = \int (\frac{10}{x-3} - \frac{9}{x-2} - \frac{5}{(x-2)^2})dx$$
$$= 10\ln|x-3| - 9\ln|x-2| - \int \frac{5}{(x-2)^2}d(x-2)$$
$$= 10\ln|x-3| - 9\ln|x-2| + \frac{5}{x-2} + K$$

24. $\int \frac{x^2-x+6}{x^3+3x}dx$

$$\int \frac{x^2-x+6}{x^3+3x}dx = \int (\frac{A}{x} + \frac{Bx+C}{x^2+3})dx$$

$$A(x^2+3) + (Bx+C)x = (A+B)x^2 + Cx + 3A = x^2 - x + 6$$

$$\therefore \begin{cases} A = 2 \\ B = -1 \\ C = -1 \end{cases}$$

$$\therefore \int \frac{x^2-x+6}{x^2+3x}dx = \int (\frac{2}{x} + \frac{-x-1}{x^2+3})dx$$
$$= 2\ln|x| - \int \frac{x}{x^2+3}dx - \int \frac{1}{x^2+3}dx$$
$$= 2\ln|x| - \frac{1}{2}\int \frac{dx^2}{x^2+3} - \int \frac{1}{x^2+3}dx$$
$$= 2\ln|x| - \frac{1}{2}\ln|x^2+3| - \int \frac{1}{x^2+3}dx$$

Let $x = \sqrt{3}u, x^2 = 3u^2$, then

$$\int \frac{1}{x^2+3}dx = \frac{\sqrt{3}}{3}\int \frac{1}{u^2+1}du$$
$$= \frac{1}{\sqrt{3}}\arctan u + K$$
$$= \frac{1}{\sqrt{3}}\arctan \frac{x}{\sqrt{3}} + K$$

$$\therefore \int \frac{x^2-x+6}{x^2+3x}dx = 2\ln|x| - \frac{\ln|x^2+3|}{2} - \frac{1}{\sqrt{3}}\arctan \frac{x}{\sqrt{3}} + K$$

26. $\int \frac{x^2+x+1}{(x^2+1)^2}dx$

$$\begin{aligned}\int \frac{x^2+x+1}{(x^2+1)^2}dx &= \int (\frac{1}{x^2+1} + \frac{1}{(x^2+1)^2})dx \\ &= \int \frac{1}{x^2+1}dx + \int \frac{1}{(x^2+1)^2}dx \\ &= \arctan x + \frac{1}{2}\int \frac{d(x^2+1)}{(x^2+1)^2} \\ &= \arctan x - \frac{1}{2(x^2+1)} + C\end{aligned}$$

28. $\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx$

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx = \int (\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1})dx$$

$$A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2 = x^2-2x-1$$

$$(A+C)x^3+(-A+B-2C+D)x^2+(A+B+C-2D)x+(-A+B+D) = x^2-2x-1$$

Solving $\begin{cases} A+C=0 \\ -A+B-2C+D=1 \\ A+B+C-2D=-2 \\ -A+B+D=-1 \end{cases}$, we have $\begin{cases} A=1 \\ B=-\frac{2}{3} \\ C=-1 \\ D=\frac{2}{3} \end{cases}$

$$\begin{aligned}\therefore \int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx &= \int (\frac{1}{x-1} - \frac{\frac{2}{3}}{(x-1)^2} + \frac{-x+\frac{2}{3}}{x^2+1})dx \\ &= \int \frac{1}{x-1}dx - \int \frac{\frac{2}{3}}{(x-1)^2}dx + \int \frac{-x+\frac{2}{3}}{x^2+1}dx \\ &= \ln|x-1| - \frac{2}{3}\int \frac{d(x-1)}{(x-1)^2} - \int \frac{x}{x^2+1}dx + \frac{2}{3}\int \frac{1}{x^2+1}dx \\ &= \ln|x-1| + \frac{2}{3(x-1)} - \frac{1}{2}\int \frac{dx^2}{x^2+1} + \frac{2}{3}\arctan x \\ &= \ln|x-1| + \frac{2}{3(x-1)} - \frac{\ln|x^2+1|}{2} + \frac{2}{3}\arctan x + K\end{aligned}$$

49. $\int \frac{\sec^2 t}{\tan^2 t+3\tan t+2}dt$

$$\begin{aligned}\int \frac{\sec^2 t}{\tan^2 t+3\tan t+2}dt &= \int \frac{d\tan t}{(\tan t+1)(\tan t+2)}\\ &= \int (\frac{1}{\tan t+1}-\frac{1}{\tan t+2})d\tan t\\ &= \ln|\tan t+1|-\ln|\tan t+2|+C\end{aligned}$$

50. $\int \frac{e^x}{(e^x-2)(e^{2x}+1)}dx$

Let $u=e^x, x=\ln u$, then

$$\begin{aligned}\int \frac{e^x}{(e^x-2)(e^{2x}+1)}dx &= \int \frac{1}{(u-2)(u^2+1)}du\\ &= \int (\frac{A}{u-2}+\frac{Bu+C}{u^2+1})du\end{aligned}$$

$$A(u^2+1)+(Bu+C)(u-2)=(A+B)u^2+(C-2B)u+A-2C=1$$

Solving $\begin{cases} A+B=0\\ C-2B=0\\ A-2C=1\end{cases}$, we can get $\begin{cases} A=\frac{1}{5}\\ B=-\frac{1}{5}\\ C=-\frac{2}{5}\end{cases}$

$$\begin{aligned}\therefore \int \frac{e^x}{(e^x-2)(e^{2x}+1)}dx &= \frac{1}{5}\int (\frac{1}{u-2}-\frac{u+2}{u^2+1})du\\ &= \frac{1}{5}(\ln|u-2|-\int \frac{u}{u^2+1}du-\int \frac{2}{u^2+1}du)\\ &= \frac{1}{5}(\ln|u-2|-\frac{1}{2}\int \frac{du^2}{u^2+1}-2\arctan u)\\ &= \frac{1}{5}\ln|u-2|-\frac{1}{10}\ln|u^2+1|-\frac{2}{5}\arctan u+K\\ &= \frac{1}{5}\ln|e^x-2|-\frac{1}{10}\ln|e^{2x}+1|-\frac{2}{5}\arctan e^x+K\end{aligned}$$

61. $\int \frac{1}{3\sin x-4\cos x}dx$

Let $t=\tan\frac{x}{2}, x=2\arctan t$, then $\sin x=\frac{2t}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2}$, then

$$\begin{aligned}
\int \frac{1}{3\sin x - 4\cos x}dx &= \int \frac{1}{\frac{6t-4(1-t^2)}{1+t^2}}\frac{2}{1+t^2}dt \\
&= \int \frac{1}{2t^2+3t-2}dt \\
&= \int \frac{1}{(2t-1)(t+2)}dt \\
&= \int (\frac{A}{2t-1}+\frac{B}{t+2})dt
\end{aligned}$$

Solving $At+2A+2Bt-B=(A+2B)t+2A-B=1$, we get

$$A=\frac{2}{5}, B=-\frac{1}{5}$$

$$\begin{aligned}
\therefore \int \frac{1}{3\sin x - 4\cos x}dx &= \frac{1}{5}\int (\frac{2}{2t-1}-\frac{1}{t+2})dt \\
&= \frac{1}{5}(\ln|2t-1| - \ln|t+2|) + C \\
&= \frac{1}{5}\ln|\frac{2t-1}{t+2}| + C
\end{aligned}$$

62. $\int_{\pi/3}^{\pi/2} \frac{1}{1+\sin x - \cos x}dx$

Let $t=\tan\frac{x}{2}, x=2\arctan t$, then $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$, then

$$\begin{aligned}
\int_{\pi/3}^{\pi/2} \frac{1}{1+\sin x - \cos x}dx &= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{1+\frac{2t-1+t^2}{1+t^2}}\frac{2}{1+t^2}dt \\
&= \int_{\frac{\sqrt{3}}{3}}^{1} \frac{2}{2t^2+2t}dt \\
&= \int_{\frac{\sqrt{3}}{3}}^{1} \frac{1}{t} - \frac{1}{t+1}dt \\
&= (\ln|\frac{t}{t+1}|)\Big|_{\frac{\sqrt{3}}{3}}^{1} \\
&= \ln\frac{1}{2} - \ln\frac{1}{1+\sqrt{3}} \\
&= \ln\frac{1+\sqrt{3}}{2}
\end{aligned}$$

72. If f is a quadratic function such that $f(0)=1$ and

$$\int \frac{f(x)}{x^2(x+1)^3}dx$$

is a rational function, find the value of $f'(0)$.

Let $f(x) = ax^2 + bx + 1$, then $f'(x) = 2ax + b, f'(0) = b$, so we only need to evaluate the value of b.

$$\int \frac{ax^2 + bx + 1}{x^2(x+1)^3} dx = \int (\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}) dx$$

To make $\int \frac{ax^2+bx+1}{x^2(x+1)^3} dx$ is a rational function, $\frac{A}{x}$ and $\frac{C}{x+1}$ cannot appear in the integrand, which means

$$A = C = 0$$

$$\therefore \int \frac{ax^2 + bx + 1}{x^2(x+1)^3} dx = \int (\frac{B}{x^2} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}) dx$$

$$\begin{aligned} B(x+1)^3 + D(x+1)x^2 + Ex^2 &= (B+D)x^3 + (3B+D+E)x^2 + (3B)x + B \\ &= ax^2 + bx + 1 \end{aligned}$$

Solving $\begin{cases} B + D = 0 \\ 3B + D + E = a \\ 3B = b \\ B = 1 \end{cases}$, we can know $b = 3$.

$\therefore f'(0) = b = 3$