MA4_3 Exercise

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Exercise 4.3

50. $f(x) = x - \frac{1}{6}x^2 - \frac{2}{3}\ln x$

(a) Find the vertical and horizontal asymptotes.

$\because$

$$\lim_{x\to\infty}(x - \frac{1}{6}x^2 - \frac{2}{3}\ln x) = -\infty$$

$$\lim_{x\to 0^+}(x - \frac{1}{6}x^2 - \frac{2}{3}\ln x) = \infty$$

$\therefore$ there is a vertical asymptote $x = 0$, but no horizontal asymptotes of f.

(b) Find the intervals of increase or decrease.

$\because f'(x) = 1 - \frac{1}{3}x - \frac{2}{3x} = \frac{-x^2+3x-2}{3x} = -\frac{(x-1)(x-2)}{3x}$

$\therefore$ we have the table below:

intervals	$(0,1)$	1	$(1,2)$	2	$(2,\infty)$
f'	-	0	+	0	-
f	decreasing	local minimum	increasing	local maximum	decreasing

$\therefore$ increasing interval is $(1,2)$ and decreasing intervals are $(0,1),(2,\infty)$.

(c) Find the local maximum and minimum values.

$\because$ by the conclusion of problem (b), f attains its local minimum at $x = 1$ and local maximum at $x = 2$

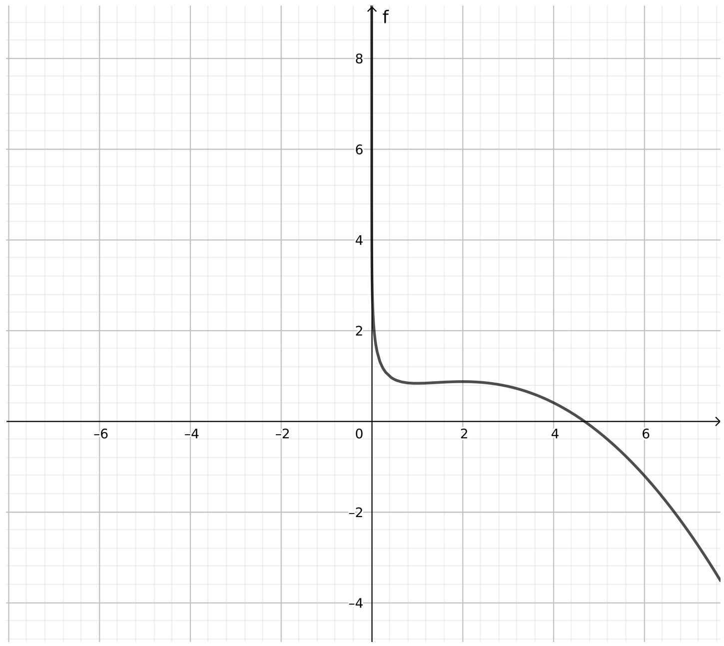
$\therefore$ the local minimum value is $f(1) = 1 - \frac{1}{6} - 0 = \frac{5}{6}$, and the local maximum value is $f(2) = 2 - \frac{2}{3} - \frac{2}{3}\ln 2 = \frac{2}{3}(2 - \ln 2)$

(d) Find the intervals of concavity and the inflection points.

$\because f''(x) = \frac{\mathrm{d}f'(x)}{\mathrm{d}x} = \frac{(-2x+3)(3x)-3(-x^2+3x-2)}{9x^2} = \frac{2-x^2}{3x^2}$

$\therefore$ we still have the table below:

(e) Use the information from parts (a)-(d) to sketch the graph of f.



81. Prove that if $(c, f(c))$ is a point of inflection of the graph of f and f'' exists in an open interval that contains c, then $f''(c) = 0$.

Proof. Without loss of generality, let the open interval be (a, b) with c inside.

Introduce a new function

$$F(x) = \begin{cases} \lim_{x\to a^+} f(x) & \text{if } x = a \\ f(x) & \text{if } a < x < b \\ \lim_{x\to b^-} f(x) & \text{if } x = b \end{cases}$$

$\because f''$ exists in (a, b)
$\therefore f'$ is differentiable in (a, b)
$\therefore F'$ is differentiable in (a, b) and continuous on $[a, b]$
$\because (c, f(c))$ is an inflection point in (a, b)
$\therefore f'(x)$ attains local maximum or minimum at $x = c$ in (a, b)
$\therefore$ by the Rolle Theorem, $f''(c) = 0$ $\square$

intervals	$(0, \sqrt{2})$	$\sqrt{2}$	$(\sqrt{2}, \infty)$
f”	+	0	-
concavity	concave upward	inflection point	concave downward

82. Show that if $f(x) = x^4$, then $f''(0) = 0$, but $(0, 0)$ is not an inflection point of the graph of f.

Proof. $\because f(0) = 0, f'(x) = 4x^3, f''(x) = 12x^2$
$\therefore f''(0) = 12 \times 0 = 0$
$\because$ when $x < 0$, $f''(x) = 12x^2 > 0$; when $x > 0$, $f''(x) = 12x^2 > 0$
$\therefore f(x)$ is both concave upward on $(-\infty, 0)$ and $(0, \infty)$
$\therefore (0, 0)$ is not an inflection point though $f''(0) = 0$ □

83. Show that the function $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.

Proof. $\because g(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

$\therefore g'(x) = \begin{cases} 2x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -2x & \text{if } x < 0 \end{cases}$

$\therefore g''(x) = \begin{cases} 2 & \text{if } x > 0 \\ -2 & \text{if } x < 0 \end{cases}$

$\because$ when $x \in (-\infty, 0), g''(x) < 0; x \in (0, \infty), f''(x) > 0$
$\therefore g(x)$ is concave downward in $(-\infty, 0)$ and concave upward $(0, \infty)$
$\therefore (0, 0)$ is an inflection point.
However, $\because \lim_{x \to 0^-} \frac{g'(x) - g'(0)}{x - 0} = -2 \neq \lim_{x \to 0^+} \frac{g'(x) - g'(0)}{x - 0} = 2$
$\therefore g''(0)$ does not exist.

□