MA4_1 Exercise

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Exercise 4.2

15. Let$f(x) = (x-3)^{-2}$. Show that there is no value of c in $(1,4)$ such that $f(4) - f(1) = f'(c)(4-1)$. Why does this not contradict the Mean Value Theorem?

$\because f'(x) = \frac{-2}{(x-3)^3}, x \in (-\infty, 3) \cup (3, \infty)$

$\therefore$

when $1 < x < 3$, $f'(x) > \frac{1}{4}$

when $3 < x < 4$, $f'(x) < -2$

$\because \frac{f(4)-f(1)}{4-1} = \frac{1-\frac{1}{4}}{3} = \frac{1}{4}$

$\therefore$ when $1 < x < 4$, there is no value of c in $(1,4)$ such that $f(4) - f(1) = f'(c)(4-1)$.

This does not contradict the Mean Value Theorem because $f(x)$ is not continuous at $[1,4]$.

18. $x^3 + e^x = 0$

Let $f(x) = x^3 + e^x$, and obviously

$$f'(x) = 3x^2 + e^x$$

$\because f(-1) = -1 + \frac{1}{e} < 0, f(1) = 1 + e > 0, f(-1)f(1) < 0$

$\therefore$ there exists at least one real root of $f(x) = 0$ as $x \in (-1,1)$

Suppose there exists two real roots x_1, x_2 of $f(x) = 0$.

$\because f(x)$ is continuous and differentiable on (x_1, x_2), $f(x_1) = f(x_2)$

$\therefore \exists \xi \in (x_1, x_2)$, such that

$$f'(\xi) = e^{\xi} + 3\xi^2 = 0$$

$\because e^x > 0, x^2 \geq 0$

$\therefore f'(\xi) > 0$, which contradicts $\exists \xi \in (x_1, x_2), f'(\xi) = 0$.

$\therefore$ there is exactly one real root of the equation $x^3 + e^x = 0$

27. Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ if $x > 0$.

Proof. Let $f(x) = \sqrt{1+x}$, and obviously

$$f'(x) = \frac{1}{2\sqrt{1+x}} < \frac{1}{2}$$

By the Lagrange Mean Value Theorem on $(0,1)$, $\exists \xi \in (0,x)$ such that

$$\sqrt{1+x} - \sqrt{1+0} = f'(\xi)(x-0)$$

$\therefore \exists \xi \in (0,x)$, such that

$$\sqrt{1+x} - 1 = f'(\xi)x < \frac{1}{2}x$$

$\therefore \sqrt{1+x} < 1 + \frac{1}{2}x$ if $x > 0$ $\square$

30. If $f'(x) = c$ for all x, use Corollary 7 to show that $f(x) = cx + d$ for some constant d.

Proof. Let $g(x) = cx$, so obviously

$$g'(x) = c = f'(x)$$

So by the Corollary 7, $\forall x \in R$, $f(x) = g(x) + d$ where d is a constant.
It is equivalent to $f(x) = cx + d$ for some constant d. $\square$

36. A number a is called a fixed point of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

Proof. We will prove it by contradiction.
Suppose that $\exists a, b \in R$ such that

$$f(a) = a, f(b) = b$$

By the Lagrange Mean Value Theorem, $\exists \xi \in (a,b)$ such that

$$f(b) - f(a) = b - a = f'(\xi)(b-a)$$

$\therefore \exists \xi \in (a,b)$ such that $f(\xi) = 1$, which contradict the hypothesis that $\forall x, f'(x) \neq 1$

$\therefore f$ has at most one fixed point. $\square$