Exercise 11.1

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23~47. Determine whether the sequence converges or diverges. If it converges, find the limit.

23. $a_n = 1 - (0.2)^n$

The sequence converges, and the limit is 1.

Proof. proof(1) :

$$a_{n+1} - a_n = 1 - (0.2)^{n-1} - 1 + (0.2)^n = (1 - 0.2) \times (0.2)^n > 0$$

so a_n is increasing.

$$a_n = 1 - (0.2)^n < 1$$

so a_n is bounded above.
By the Monotonic Sequence Theorem, the sequence is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 - \lim_{n\to\infty} (0.2)^n = 1 - 0 = 1$$

so the limit of the sequence is 1.

□

29. $a_n = \tan(\frac{2n\pi}{1+8n})$

The sequence converges, and the limit is also 1.

Proof. let $b_n = \frac{2n\pi}{1+8n}$, so

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2\pi}{8 + \frac{1}{n}} = \frac{\pi}{4}$$

Let $L = \frac{\pi}{4}$, and by the Henie Theorem, we have

$$\lim_{n\to\infty} a_n = \lim_{x\to L} \tan x = 1$$

□

30. $a_n = \sqrt{\frac{n+1}{9n+1}}$

The sequence converges, and its limit is 1.

Proof.

$$
\begin{aligned}
\lim_{n\to\infty} a_n &= \sqrt{\lim_{n\to\infty} \frac{n + \frac{1}{9} + \frac{8}{9}}{9n+1}} \\
&= \sqrt{\lim_{n\to\infty} (\frac{1}{9} + \frac{8}{9(9n+1)})} \\
&= \sqrt{(\lim_{n\to\infty} \frac{1}{9} + \lim_{n\to\infty} \frac{8}{9(9n+1)})} \\
&= \sqrt{\frac{1}{9}} \\
&= \frac{1}{3}
\end{aligned}
$$

□

32. $a_n = e^{2n/(n+2)}$

The sequence converges, and the limit is e^2.

Proof. Let $b_n = \frac{2n}{n+2}$, and we have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2}{1+\frac{2}{n}} = \frac{2}{1+0} = 2$$

Let $L = 2$, and by Henie Theorem,

$$\lim_{n\to\infty} a_n = \lim_{x\to L} e^x = e^2$$

□

33. $a_n = \frac{(-1)^n}{2\sqrt{n}}$

Proof. The sequence converges, and the limit is 0.

When n is odd,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-1}{2\sqrt{n}} = 0$$

And when n is even,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$$

So for all n, the limit of the sequence is 0. □

38. $\{\frac{\ln n}{\ln 2n}\}$

Proof. The sequence converges, and the limit is 1.

$$\frac{\ln n}{\ln 2n} = \frac{\ln 2n - \ln 2}{\ln 2n} = 1 - \frac{\ln 2}{\ln 2n}$$

So

$$\lim_{n\to\infty} \frac{\ln n}{\ln 2n} = \lim_{n\to\infty} (1 - \frac{\ln 2}{\ln 2n}) = 1 - 0 = 1$$

□

40. $a_n = \frac{\tan^{-1} n}{n}$

The sequence diverges.

Proof. Let $f(x) = \frac{\tan^{-1} x}{x}$, and let $\tan y = x (x \neq 0)$, so

$$\frac{\tan^{-1} x}{x} = \frac{y}{\tan y} (y \neq 0)$$

Let $g(y) = \frac{y}{\tan y}$, and

$$\lim_{y\to\infty} g(y) = \frac{\lim_{y\to\infty} y}{\frac{\pi}{2}} = \infty$$

Because of the relationship of $f(x)$ and $g(y)$, we can get

$$\lim_{x\to\infty} f(x) = \infty$$

So obviously,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \infty$$

□

45. $a_n = n \sin(1/n)$

Proof. We first prove $\lim_{x\to 0} \frac{\sin x}{x} = 1$

When $n \to \infty$, $\frac{1}{n} \to 0$

Let $x = \frac{1}{n}$, so

$$\lim_{n\to\infty} n \sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

□

47. $a_n = (1 + \frac{2}{n})^n$

The sequence converges, and its limit is e^2.

Proof. For the definition of the natural constant e, we know

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

Let $x = \frac{n}{2}$, and we can get

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}(1+\frac{1}{\frac{n}{2}})^{\frac{n}{2}+\frac{n}{2}} \\ &= \lim_{n\to\infty}((1+x)^x)^2 = \lim_{n\to\infty}^{2}(1+x)^x \\ &= e^2\end{aligned}$$

□

81. Show that the sequence defined by

$$a_1 = 1 \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

Proof.

$$a_2 = 3 - \frac{1}{a_1} = 3 - 1 = 2$$

So $a_1 < a_2$
Suppose that $a_n < a_{n+1}$, then

$$a_{n+2} = 3 - \frac{1}{a_{n+1}} > 3 - \frac{1}{a_n} = a_{n+1}$$

By the mathematical induction, a_n is increasing.
Because $a_1 = 1 < 3$ and a_n is increasing, $a_n >= a_1 = 1 > 0$
Also because

$$a_{n+1} = 3 - \frac{1}{a_n} < 3$$

So every term of a_n is less than 3, which also means a_n is bounded above.
By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent.
Denote the limit of $\{a_n\}$ as L.
Since $a_n \to L$, $a_{n+1} \to L$, by the recurrence formula, we have

$$L = 3 - \frac{1}{L}$$

We can get

$$L = \frac{3+\sqrt{5}}{2} \quad or \quad \frac{3-\sqrt{5}}{2}$$

But because $a_n \geq a_1 = 1$, $\frac{3-\sqrt{5}}{2} < 1$, this solution should be abandoned.
So

$$L = \frac{3+\sqrt{5}}{2}$$

□

82. Show that the sequence defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3-a_n}$$

satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

Proof.

$$a_2 = \frac{1}{3-a_1} = \frac{1}{3-2} = 1 < a_1$$

so we can get $a_1 < a_2$
Suppose that $a_n < a_{n+1}$, then

$$a_{n+2} = \frac{1}{3-a_{n+1}} < \frac{1}{3-a_n} = a_{n+1}$$

By the mathematical induction, $\{a_n\}$ is decreasing.
Because $\{a_n\}$ is decreasing, $a_n \leq a_1 = 2$, which means

$$3 - a_n > 1 > 0$$

So

$$a_{n+1} = \frac{1}{3-a_n} > 0$$

So every term of $\{a_n\}$ satisfies $0 < a_n \leq 2$.
By the Monotonic Sequence Theorem, the sequence is convergent.
Let the limit of a_n be L.
Since $a_n \to L$, $a_{n+1} \to L$, and by the recurrence formula, we have

$$L = \frac{1}{3-L}$$

Solving this equation, we know

$$L = \frac{3+\sqrt{5}}{2} \quad or \quad \frac{3-\sqrt{5}}{2}$$

But because $\frac{3+\sqrt{5}}{2} > \frac{4}{2} = 2$, $a_n \leq 2$, this solution should be abandoned.
So

$$L = \frac{3-\sqrt{5}}{2}$$

□

89. Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty}(a_n b_n) = 0$

Proof. $\because$ $\{b_n\}$ is bounded, $\therefore$ $\{b_n\}$ has both least upper bound A and greatest lower bound B, such that

$$A \le b_n \le B$$

So the limit of b_n can only be either A or B.
but no matter what value b_n is,

$$\begin{aligned}\lim_{n\to\infty}(a_n b_n) &= \lim_{n\to\infty} a_n \times \lim_{n\to\infty} b_n \\ &= 0 \times a \quad or \quad 0 \times b\end{aligned}$$

No matter what the bound of b_n is,

$$\lim_{n\to\infty}(a_n b_n) = 0$$

□

91. Let a and b be postive numbers with $a > b$. Let a_1 be their arithmetic mean and b_1 their geometric mean:

$$a_1 = \frac{a+b}{2} \qquad b_1 = \sqrt{ab}$$

Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n}$$

(a) Use mathematical induction to show that

$$a_n > a_{n+1} > b_{n+1} > b_n$$

Proof. **Method 1: mathematical induction**

When $n = 1$,
by the mean inequality, we know

$$\frac{a+b}{2} \ge \sqrt{ab}$$

only when $a = b$ the euqality holds.
It also means

$$a_1 > b_1, a_2 > b_2$$

$\because a_2 = \frac{a_1+b_1}{2} < \frac{a_1+a_1}{2}, b_2 = \sqrt{a_1 b_1} > \sqrt{b_1 b_1}$
$\therefore$

$$a_1 > a_2 > b_2 > b_1$$

When $n \geq 2$, suppose when $n = k(k \in N_+)$, we have

$$a_k > a_{k+1} > b_{k+1} > b_k$$

Obviously,

$$a_{k+1} > b_{k+1}, a_{k+2} > b_{k+2}$$

$\because a_{k+2} = \frac{a_{k+1}+b_{k+1}}{2} < \frac{a_{k+1}+a_{k+1}}{2}, b_{k+2} = \sqrt{a_{k+1}b_{k+1}} > \sqrt{b_{k+1}b_{k+1}} = b_{k+1}$
$\therefore$

$$a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1}$$

So by the mathematical induction,

$$a_n > a_{n+1} > b_{n+1} > b_n$$

Method 2: properties of inequality

By the mean inequality, we know

$$\frac{a+b}{2} \geq \sqrt{ab}$$

only when $a = b$ the euqality holds.
$\because a \neq b \quad \therefore a_1 > b_1$
Let $a = a_n, b = b_n$, and we can get

$$a_n > b_n, a_{n+1} > b_{n+1}$$

So

$$a_n = \frac{a_n + a_n}{2} > \frac{a_n + b_n}{2} = a_{n+1}$$

$$b_{n+1} = \sqrt{a_n b_n} > \sqrt{b_n b_n} = b_n$$

So we can get

$$a_n > a_{n+1} > b_{n+1} > b_n$$

for any $n \in N_+$, the inequality holds. □

(b) Deduce that both $\{a_n\}$ and $\{b_n\}$ are convergent.

Proof. $\because a_n > a_{n+1} > b_{n+1} > b_n > b_1$
$\therefore a_n > b_1$, which means $\{a_n\}$ is bounded below.
$\because \{a_n\}$ is decreasing, $\therefore \{a_n\}$ is convergent.
$\because a_1 > a_n > a_{n+1} > b_{n+1} > b_n$
$\therefore b_n < a_1$, which means $\{b_n\}$ is bounded above.
$\because \{b_n\}$ is increasing, $\therefore \{b_n\}$ is convergent.

□

(c) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, Gauss called the common value of these limits the **arithmetic-geometric mean** of the numbers a and b.
First let us prove $a_{n+1} - b_{n+1} < \frac{1}{2}(a_n - b_n)$

Proof.

$$\begin{aligned} a_{n+1} - b_{n+1} - \frac{1}{2}a_n + \frac{1}{2}b_n &= \frac{a_n + b_n}{2} - \frac{a_n}{2} + \frac{b_n}{2} - b_{n+1} \\ &= b_n - b_{n+1} < 0 \end{aligned}$$

$\therefore a_{n+1} - b_{n+1} < \frac{1}{2}(a_n - b_n)$ □

With the first conclusion proved above, let us prove the problem (c).

Proof. let $c_n = a_n - b_n$, so we have

$$c_{n+1} < \frac{1}{2}c_n$$

And then

$$c_n < \frac{1}{2}c_{n-1} < \cdots < \frac{1}{2^{n-1}}c_1$$

$\because c_n = a_n - b_n \geq 0$
$\because c_1 = \frac{a+b}{2} - \sqrt{ab} \geq 0$ because when $a > b$ the equality never holds.
Obviously, $\lim_{n\to\infty} \frac{1}{2^{n-1}}c_1 = 0$
So

$$\lim_{n\to\infty} 0 < \lim_{n\to\infty} c_n < \lim_{n\to\infty} \frac{1}{2^{n-1}}c_1$$

By the squeeze theorem, we can get

$$\lim_{n\to\infty} c_n = 0$$

, which is equivlant to

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

□

92.

(a) Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.

Proof. (1) When n is odd, let $n = 2k + 1$, so

$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{2k+1} = L$$

(2) When n is evem let $n = 2k$, so

$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} a_{2k} = L$$

Above all,

$$\lim_{n\to\infty} a_n = L$$

□

(b) If $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1+a_n}$$

find the first eight terms of the sequence $\{a_n\}$. Then use part(a) to show that $\lim_{n\to\infty} a_n = \sqrt{2}$. This gives the **continued fraction expansion**

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+...}}$$

$$a_2 = 1 + \frac{1}{1+a_1} = \frac{3}{2}$$

$$a_3 = 1 + \frac{1}{1+a_2} = \frac{7}{5}$$

$$a_4 = 1 + \frac{1}{1+a_3} = \frac{17}{12}$$

$$a_5 = 1 + \frac{1}{1+a_4} = \frac{41}{29}$$

$$a_6 = 1 + \frac{1}{1+a_5} = \frac{99}{70}$$

$$a_7 = 1 + \frac{1}{1+a_6} = \frac{239}{169}$$

$$a_8 = 1 + \frac{1}{1+a_7} = \frac{577}{408}$$

Proof.

$$a_{n+2} = \frac{1}{1+a_{n+1}} = \frac{1}{1+1+\frac{1}{1+a_n}} = \frac{a_n+1}{2a_n+3} + 1$$

(1) When n is odd, let $n = 2k+1$.
Let $\lim_{k\to\infty} a_{2k+1} = L$, by $\lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} a_{2k+3} = L$, so

$$L = \frac{1}{1+L}$$

Solving the equation, we can get $L = \sqrt{2}$.
We can know a_n is increasing. and $a_1 = 1 < \sqrt{2}$
So

$$\lim_{k\to\infty} a_{2k+1} = \sqrt{2}$$

(2) When n is even, let $n = 2k$.
Let $\lim_{k\to\infty} a_{2k} = L$, by $\lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} a_{2k+2} = L$, so

$$L = \frac{1}{1+L}$$

Solving the equation, we can get $L = \sqrt{2}$.
We can know a_n is decreasing. and $a_2 = \frac{3}{2} > \sqrt{2}$
So

$$\lim_{k\to\infty} a_{2k} = \sqrt{2}$$

Above all, $\lim_{n\to\infty} a_n = \sqrt{2}$

$\square$