Exercise 2.7

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28. $f(t) = 2t^3 + t$**, find** $f'(a)$**.**

$$\begin{aligned}
f'(a) &= \lim_{t\to a} \frac{f(t) - f(a)}{t - a} \\
&= \lim_{t\to a} (\frac{2(t^3 - a^3)}{t - a} + 1) \\
&= \lim_{t\to a} [2(t^2 + at + a^2) + 1] \\
&= 6a^2 + 1
\end{aligned}$$

30. $f(x) = x^{-2}$**, find** $f'(a)$**.**

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{(\frac{1}{x})^2 - (\frac{1}{a})^2}{x - a} \\
&= \lim_{x\to a} \frac{(\frac{1}{x} - \frac{1}{a})(\frac{1}{x} + \frac{1}{a})}{x - a} \\
&= \lim_{x\to a} \frac{(\frac{a-x}{ax}\frac{a+x}{ax})}{x - a} \\
&= \lim_{x\to a} \frac{a + x}{(ax)^2} \\
&= \frac{2a}{a^4} \\
&= 2a^{-3}
\end{aligned}$$

31. $f(x)=\sqrt{1-2x}$**, find** $f'(a)$**.**

$$\begin{aligned} f'(a) &= \lim_{x\to a}\frac{\sqrt{1-2x}-\sqrt{1-2a}}{x-a} \\ &= \lim_{x\to a}\frac{(1-2x)-(1-2a)}{(x-a)(\sqrt{1-2x}+\sqrt{1-2a})} \\ &= \lim_{x\to a}\frac{-2}{\sqrt{1-2x}+\sqrt{1-2a}} \\ &= \frac{-2}{2\sqrt{1-2a}} \\ &= \frac{-1}{\sqrt{1-2a}} \end{aligned}$$

34. $\lim_{h\to 0}\frac{\sqrt[4]{16+h}-2}{h}$

Compared to $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$, we can know

$$f(x_0)=\sqrt[4]{16}=2, \quad f(x_0+h)=\sqrt[4]{16+h}$$

So $f=\sqrt[4]{x}, a=2$

35. $\lim_{x\to 5}\frac{2^x-32}{x-5}$

Compared to $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$, we can know

$$f(5)=2^5=32, \quad f(x)=2^x$$

So $f=2^x, a=5$

37. $\lim_{h\to 0}\frac{\cos(\pi+h)+1}{h}$

Compared to $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$, we can know

$$f(\pi)=\cos\pi=-1, \quad f(\pi+h)=\cos(\pi+h)$$

So $f=\cos x, a=\pi$

53 54. Determine whether $f'(0)$ exists.

53.

$$f(x)=\begin{cases} x\sin\frac{1}{x} & \textbf{if } x\neq 0 \\ 0 & \textbf{if } x=0 \end{cases}$$

$$
\begin{aligned}
f'(0) &= lim_{x\to 0}\frac{f(x)-f(0)}{x} \\
&= \lim_{x\to 0}\sin\frac{1}{x} \\
&= \sin\lim_{x\to 0}\frac{1}{x}
\end{aligned}
$$

Since $\lim_{x\to 0}\frac{1}{x}=\infty$, $f'(0)=\sin\infty$ does not exists.

54.

$$
f(x)=\begin{cases} x^2\sin\frac{1}{x} & \textbf{if } x\neq 0 \\ 0 & \textbf{if } x=0 \end{cases}
$$

$$
\begin{aligned}
f'(0) &= \lim_{x\to 0}(x^2\sin\frac{1}{x}) \\
&= \lim_{x\to 0}x^2\times\lim_{x\to 0}\sin\frac{1}{x}
\end{aligned}
$$

$\because x^2$ is infinitesimal as $x\to 0$, and $\sin\frac{1}{x}$ is bounded on $[-1,1]$ as $x\to 0$,
$\therefore \lim_{x\to 0}f(x)=\lim_{x\to 0}x^2\sin\frac{1}{x}=0=f(0)$
$\therefore f(x)$ is continuous everywhere.