Exercise 2.5

Wang Yue from CS Elite Class

October 22, 2020

37. $\lim_{x\to 1} e^{x^2-x}$

Let $f(x) = e^x, g(x) = x^2 - x$, and $f(x)$ and $g(x)$ are continuous.
By the property of continuous composite function,

$$\begin{aligned}\lim_{x\to 1} e^{x^2-x} &= \lim_{x\to 1} f(g(x)) \\ &= f(\lim_{x\to 1} g(x)) \\ &= f(g(1)) = f(0) = 1\end{aligned}$$

38. $\lim_{x\to 2} \arctan(\frac{x^2-4}{3x^2-6x})$

Obviously, $y = \arctan x$ and $y = \frac{x^2-4}{3x^2-6x}$ are continuous.
So by the property of continuous composite function,

$$\begin{aligned}\lim_{x\to 2} \arctan(\frac{x^2-4}{3x^2-6x}) &= \arctan(\lim_{x\to 2} \frac{(x+2)(x-2)}{3x(x-2)}) \\ &= \arctan(\lim_{x\to 2} \frac{x+2}{3x}) \\ &= \arctan(\frac{2}{3})\end{aligned}$$

46. Find the values of a and b that make f continuous everywhere.

To make $f(x)$ continuous everywhere, we must satisfy:

$$\begin{cases}\lim\limits_{x\to 2^-} f(x) = f(2) \\ \lim\limits_{x\to 3^-} f(x) = f(3)\end{cases}$$

$\because \lim_{x\to 2^-} f(x) = \lim x \to 2^- \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2^-}(x+2) = 4$
$\because \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(ax^2 - bx + 3) = 9a - 3b + 3$
$\therefore$ we get the equations:

$$\begin{cases} 4a - 2b + 3 = 4 \\ 6 - a + b = 9a - 3b + 3 \end{cases}$$

The solution is $\begin{cases} a = \dfrac{1}{2} \\ b = \dfrac{1}{2} \end{cases}$

53. $e^x = 3 - 2x, \quad (0, 1)$

Let $f(x) = e^x + 2x - 3$

$\because f(0) = 1 + 0 - 3 = -2 < 0, f(1) = e + 2 - 3 = e - 1 > 0$

$\because f(x)$ is continuous, $f(0)f(1) < 0$

$\therefore$ by the Intermediate Value Theorem, $\exists \xi \in (0, 1), s.t.$

$$f(\xi) = 0 \iff e^{\xi} = 3 - 2\xi$$

54. $\sin x = x^2 - x, \quad (1, 2)$

Let $f(x) = x^2 - x - \sin x$

$\because f(1) = 1 - 1 - \sin 1 < 0, f(2) = 4 - 2 - \sin 2 > 2 - 1 > 0$

$\because f(x)$ is continuous and $f(1)f(2) < 0$

$\therefore$ by the Intermediate Value Theorem, $\exists \xi \in (1, 2), s.t.$

$$f(\xi) = 0 \iff \sin \xi = \xi^2 - \xi$$

61. Prove that cosine is a continuous function.

Proof. We will prove it by the contradiction.

Suppose that cosine is a discontinuous function, so $\exists x_0 \in R, s.t.$

$$\lim_{x \to x_0} \cos x \neq \cos x_0$$

But $\forall \epsilon > 0, \exists \delta = \epsilon$.

When $\delta > 0$, if $0 < |x - x_0| < \delta$, then

$$\begin{aligned} |\cos x - \cos x_0| &= |\cos(\frac{x + x_0}{2} + \frac{x - x_0}{2}) - \cos(\frac{x + x_0}{2} - \frac{x - x_0}{2})| \\ &= | - 2 \sin \frac{x + x_0}{2} \frac{x - x_0}{2}| \\ &= 2|\sin \frac{x + x_0}{2} \frac{x - x_0}{2}| \\ &\leq 2|\sin \frac{x - x_0}{2}| \\ &< 2|\frac{x - x_0}{2}| = |x - x_0| = \delta = \epsilon \end{aligned}$$

The calculation above indicates that $\lim_x x \to x_0 \cos x = \cos x_0$, which contradicts with the hypothesis.

So we can prove that the cosine function always satisfies $\lim_x x \to x_0 \cos x = \cos x_0$, and it's continuous.

$\square$

67. Show that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & \textbf{if } x \neq 0 \\ 0 & \textbf{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$

$\because$ Obviously, $y = \frac{1}{x}, y = \sin x$ are continuous functions,

$\therefore y = \sin(1/x)$ is continuous function.

$\because y = x^4$ is also continuous function,

$\therefore f(x) = x^4 \sin(1/x), \quad x \neq 0$ is continuous function on $\{x \in R | x \neq 0\}$

$\because x^4$ is an infinitesimal as $x \to 0$, and $\sin(1/x)$ is bounded on $[-1, 1]$ as $x \to 0$,

$\therefore$

$$\lim_{x \to 0} x^4 \sin(1/x) = 0 = f(0)$$

$\therefore f(x)$ is continuous on $(-\infty, \infty)$

68.(a)

Proof. $\because$ Obviously,

$$F(x) = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$therefore$ When $x \neq 0$, $f(x)$ is continous on $(-\infty, 0)$ and $(0, \infty)$

$\because \lim_{x \to 0^-} F(x) = \lim_{x \to 0^+} F(x) = 0 = F(0)$

$\therefore F(x)$ is continuous everywhere. $\square$

86.(b)

Proof. If $f(x) \geq 0$, $|f(x)| = f(x)$ is also continuous.

If $f(x) \leq 0$, $|f(x)| = -f(x)$ is also continuous.

If $\exists x_0, s.t. f(x_0) = 0$, then we discuss about a common situation: $|f(x)|$ on $(a, b), x_0 \in (a, b)$

Without loss of generality, let $f(x) > 0$ as $x \in (a, x_0)$, $f(x) < 0$ as $x \in (x_0, b)$, $f(x) = 0$ as $x = x_0$

If we can prove in such a common situation $|f(x)|$ is continuous, then we can conclude that $|f(x)|$ is continuous in the domain of $f(x)$.

Obviously, when $x \in (a, x_0), |f(x)| = f(x)$ is continous, so is $x \in (x_0, b)$.

$\because \lim_{x \to x_0^-} |f(x)| = \lim_{x \to x_0^+} |f(x)| = 0 = f(x_0)$

$\therefore |f(x)|$ is continuous on (a, b)
$\therefore |f(x)|$ is continuous. $\square$

68.(c)

No. A counterexample is $f(x) = \begin{cases} x-2 & \text{if } x < 0 \\ x+2 & \text{if } x \geq 0 \end{cases}$

In this case, when $x < 0$, $|f(x)| = |x-2| = 2-x$, $|f(x)| = |x|+2$, which is obviously continuous.

However, the original function $f(x)$ is not continuous at $x = 0$, for

$$\lim_{x \to 0^-} f(x) = -2, \lim_{x \to 0^+} f(x) = 2$$