Exercise 7_1

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Exercise 7.3

39.

(a) Use trigonometric substitution to verify that

$$\int_0^x \sqrt{a^2-t^2}dt = \frac{1}{2}a^2\sin^{-1}(x/a) + \frac{1}{2}x\sqrt{a^2-x^2}$$

Proof. Let $t = a\sin\theta$, then $\theta = \arcsin\frac{t}{a}$, $\cos\theta = \frac{\sqrt{a^2-t^2}}{a}$, so

$$\begin{aligned}
\int \sqrt{a^2-t^2}dt &= \int \sqrt{a^2-a^2\sin^2\theta}a\cos\theta d\theta \\
&= \int a^2\cos^2\theta d\theta \\
&= \int \frac{a^2}{2} + \frac{a^2\cos 2\theta}{2}d\theta \\
&= \frac{a^2}{2}\theta + \frac{a^2}{4}\sin 2\theta + C \\
&= \frac{a^2}{2}\sin^{-1}\frac{t}{a} \times (2\frac{t}{a}\frac{\sqrt{a^2-t^2}}{a}) + C \\
&= \frac{a^2\sin^{-1}\frac{t}{a}}{2} + \frac{t\sqrt{a^2-t^2}}{2} + C
\end{aligned}$$

$$\begin{aligned}
\therefore \int_0^x \sqrt{a^2-t^2}dt &= (\frac{a^2\sin^{-1}\frac{t}{a}}{2} + \frac{t\sqrt{a^2-t^2}}{2})\Big|_0^x \\
&= \frac{1}{2}a^2\sin^{-1}\frac{x}{a} + \frac{1}{2}x\sqrt{a^2-x^2}
\end{aligned}$$

□

(b) Use the figure to give trigonometric interpretations of both terms on the right side of the equation in part (a).

Let the area of the sector be S_1 and the area of the triangular be S_2.

By the figure, we have:

$$\begin{cases} \theta & = \arcsin \frac{x}{a} \\ S_1 & = \frac{1}{2} x \sqrt{a^2 - x^2} \\ s_2 & = \frac{1}{2} \theta a^2 \end{cases}$$

$$\therefore S = S_1 + S_2 = \frac{1}{2} a^2 \sin^{-1}(x/a) + \frac{1}{2} x \sqrt{a^2 - x^2} = \int_0^x \sqrt{a^2 - t^2} dt$$

Exercise 7.1

7. $\int (x^2 + 2x) \cos x dx$

$$\begin{aligned} \int (x^2 + 2x) \cos x dx &= \int (x^2 + 2x) d \sin x \\ &= (x^2 + 2x) \sin x - \int \sin x d(x^2 + 2x) \\ &= (x^2 + 2x) \sin x - \int (2x + 2) \sin x dx \\ &= (x^2 + 2x) \sin x + \int (2x + 2) d \cos x \\ &= (x^2 + 2x) \sin x + (2x + 2) \cos x - \int \cos x d(2x + 2) \\ &= (x^2 + 2x) \sin x + (2x + 2) \cos x - \int 2 \cos x dx \\ &= (x^2 + 2x - 2) \sin x + (2x + 2) \cos x + C \end{aligned}$$

9. $\int \ln \sqrt[3]{x} dx$

$$\begin{aligned} \int \ln \sqrt[3]{x} dx &= \frac{1}{3} \int \ln x dx \\ &= \frac{1}{3} \times \frac{1}{x} + C \\ &= \frac{1}{3x} + C \end{aligned}$$

10. $\int \sin^{-1} x dx$

$$\begin{aligned}\int \sin^{-1} x dx &= x \sin^{-1} x - \int x d \sin^{-1} x \\ &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx \\ &= x \arcsin x - \int \frac{dx^2}{2\sqrt{1-x^2}} \\ &= x \arcsin x + \sqrt{1-x^2} + C\end{aligned}$$

15. $\int (\ln x)^2 dx$

$$\begin{aligned}\int (\ln x)^2 dx &= x(\ln x)^2 - \int x d(\ln x)^2 \\ &= x(\ln x)^2 - \int 2 \ln x dx \\ &= x(\ln x)^2 - 2(x \ln x - x) + C \\ &= x(\ln x)^2 - 2x \ln x + 2x + C\end{aligned}$$

17. $\int e^{2\theta} \sin 3\theta d\theta$

$$\begin{aligned}\int e^{2\theta} \sin 3\theta d\theta &= \frac{1}{2} \int \sin 3\theta de^{2\theta} \\ &= \frac{1}{2}(e^{2\theta} \sin 3\theta - \int e^{2\theta} d \sin 3\theta) \\ &= \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{2} \int e^{2\theta} \cos 3\theta d\theta \\ &= \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{4} \int \cos 3\theta de^{2\theta} \\ &= \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{4} e^{2\theta} \cos 3\theta + \frac{3}{4} \int e^{2\theta} d \cos 3\theta \\ &= \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{4} e^{2\theta} \cos 3\theta - \frac{9}{4} \int e^{2\theta} \sin 3\theta d\theta\end{aligned}$$

$$\therefore \int e^{2\theta} \sin 3\theta d\theta = \frac{4}{13} e^{2\theta} (\frac{1}{2} \sin 3\theta - \frac{3}{4} \cos 3\theta) = e^{2\theta} (\frac{2}{13} \sin 3\theta - \frac{3}{13} \cos 3\theta)$$

21. $\int \frac{xe^{2x}}{(1+2x)^2}dx$

26. $\int_4^9 \frac{\ln y}{\sqrt{y}}dy$

$$\begin{aligned}
\int_4^9 \frac{\ln y}{\sqrt{y}}dy &= 2\int_4^9 \ln y d\sqrt{y} \\
&= 2(\ln y\sqrt{y})\Big|_4^9 - 2\int_4^9 \sqrt{y} d\ln y \\
&= 2(6\ln 3 - 4\ln 2) - 2\int_4^9 \frac{\sqrt{y}}{y}dy \\
&= 12\ln 3 - 8\ln 2 - 4(\sqrt{y})\Big|_4^9 \\
&= 12\ln 3 - 8\ln 2 - 4
\end{aligned}$$

30. $\int_1^{\sqrt{3}} \arctan\frac{1}{x}dx$

Let $u = \frac{1}{x}, x = \frac{1}{u}$, then

$$\begin{aligned}
\int_1^{\sqrt{3}} \arctan\frac{1}{x}dx &= \int_1^{\frac{\sqrt{3}}{3}} \arctan u d\frac{1}{u} \\
&= \int_1^{\frac{\sqrt{3}}{3}} -\frac{\arctan u}{u}du \\
&= -\int_1^{\frac{\sqrt{3}}{3}} \frac{1}{u}d\frac{1}{1+u^2} \\
&= -(\frac{1}{u(1+u^2)})\Big|_1^{\frac{\sqrt{3}}{3}} + \int_1^{\frac{\sqrt{3}}{3}} \frac{1}{1+u^2}d\frac{1}{u} \\
&= -(\frac{9}{4\sqrt{3}} - \frac{1}{2}) + \int_1^{\sqrt{3}} \frac{x^2+1-1}{x^2+1}dx \\
&= \frac{1}{2} - \frac{3\sqrt{3}}{4} + \int_1^{\sqrt{3}} (1 - \frac{1}{1+x^2})dx \\
&= \frac{1}{2} - \frac{3\sqrt{3}}{4} + (x - \arctan x)\Big|_1^{\sqrt{3}} \\
&= \frac{1}{2} - \frac{3\sqrt{3}}{4} + \sqrt{3} - \frac{\pi}{3} - 1 + \frac{\pi}{4} \\
&= \frac{\sqrt{3}}{4} - \frac{\pi}{12} - \frac{1}{2}
\end{aligned}$$

37. $\int \cos x\sqrt{x}dx$

Let $u = \sqrt{x}, x = u^2$, then

$$\begin{aligned}
\int \cos\sqrt{x}dx &= \int 2u\cos u du \\
&= 2\int u d\sin u \\
&= 2u\sin u - 2\int \sin u du \\
&= 2u\sin u + 2\cos u + C \\
&= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C
\end{aligned}$$

38. $\int t^3 e^{t^2} dt$

$$\begin{aligned}
\int t^3 e^{t^2} dt &= \frac{1}{2}\int t^2 e^{t^2} dt^2 \\
&= \frac{1}{2}(t^2 - 1)e^{t^2} + C
\end{aligned}$$

70.

(a) Use integration by parts to show that

$$\int f(x)dx = xf(x) - \int xf'(x)dx$$

Proof. $\because \int f(x)dg(x) = f(x)g(x) - \int g(x)df(x)$

$\therefore$ let $g(x) = x$, then

$$\int f(x)dx = xf(x) - \int xf'(x)dx$$

□

(b) If f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy$$

Proof. By part (a), we know

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_a^b xf'(x)dx$$

which is equivalent to

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_a^b xdf(x)$$

$\because x = g(f(x))$
$\therefore$ let $y = f(x)$, then

$$\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy$$

$\square$

(c) I just omit the graph.

(d) Use part(b) to evaluate $\int_1^e \ln x dx$

$$\int_1^e \ln x dx = e - 0 - \int_0^1 e^y dy = e - 0 - (e - 1) = 1$$