MA5_1 Exercise

Wang Yue from CS Elite Class

November 29, 2020

Exercise 5.2

66. Prove $\int_0^{\pi/2} x \sin x dx \leq \frac{\pi^2}{8}$

Proof.

$$\because \int_a^b x dx = (\frac{x^2}{2})\Big|_a^b = \frac{b^2 - a^2}{2}$$

$\because \sin x \leq 1$ when $x \in [0, \frac{\pi}{2}]$

$\therefore$

$$\int_0^{\frac{\pi}{2}} x \sin x dx \leq \int_0^{\frac{\pi}{2}} x dx = \frac{\pi^2}{8}$$

$\square$

68.

(a) If f is continuous on $[a, b]$, show that

$$|\int_a^b f(x) dx| \leq \int_a^b |f(x)| dx$$

Proof. $\because -|f(x)| \leq f(x) \leq |f(x)|$

$\therefore -\int_a^b |f(x)| dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx$, which means that

$$|\int_a^b f(x) dx| \leq \int_a^b |f(x)| dx$$

$\square$

(b) Use the result of part (a) to show that

$$|\int_0^{2\pi} f(x) \sin 2x dx| \leq \int_0^{2\pi} |f(x)| dx$$

Proof. By the conclusion of part (a), we can get:

$$|\int_0^{\frac{\pi}{2}} f(x)\sin 2x dx| \le \int_0^{\frac{\pi}{2}} |f(x)\sin 2x| dx$$

$\because$ when $x \in [0, \frac{\pi}{2}]$, $|f(x)\sin 2x| = |f(x)|\sin 2x \le |f(x)|$

$\therefore$

$$|\int_0^{\frac{\pi}{2}} f(x)\sin 2x dx| \le \int_0^{\frac{\pi}{2}} |f(x)\sin 2x| dx \le \int_0^{\frac{\pi}{2}} |f(x)| dx$$

□

70. Let $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } 0 < x \le 1 \end{cases}$. Show that f is not integrable on $[0, 1]$.

Proof. Suppose that f is integrable on $[0, 1]$, then

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = \frac{1}{n}$

The first term of summation is

$$f(x_1^*)\Delta x = \frac{1}{n} f(x_1^*)$$

Then

□

71. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$

$$\because \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{i}{n})^4 \frac{1}{n}$$

which is the same as the Riemann sum of $f(x) = x^4$ from 0 to 1 when $x_i^* = \frac{1}{n}$

$$\therefore \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \int_0^1 x^4 dx = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5}$$

72. $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2}$

$$\because \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1+(\frac{i}{n})^2} \frac{1}{n}$$

which is the same as the Riemann sum of $g(x) = \frac{1}{1+x^2}$ from 0 to 1 when $x_i^* = \frac{1}{n}$

$$\therefore \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2} = \int_0^1 g(x)dx = \arctan x \Big|_0^1 = \frac{\pi}{4}$$

73. Find $\int_1^2 x^{-2}dx$

Let $x_0 = 1, x_n = 2, \Delta x = \frac{x_n - x_0}{n} = \frac{1}{n}$, $x_i = x_{i-1} + \Delta x$, $x_i^* = \sqrt{x_{i-1}x_i}$, then

$$\begin{aligned}
\int_1^2 \frac{1}{x^2}dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \frac{1}{(x_i^*)^2} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \frac{1}{x_{i-1}x_i} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \sum_{i=1}^{n} (\frac{1}{x_{i-1}} - \frac{1}{x_i}) \\
&= \lim_{n\to\infty} n \times \frac{1}{n} (\frac{1}{1} - \frac{1}{2}) \\
&= \frac{1}{2}
\end{aligned}$$