Exercise 15.7

Wang Yue from CS Elite Class

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10.

$$
\begin{aligned}
\iiint_E e^{\frac{z}{y}}\,dV &= \int_0^1 \int_y^1 \int_0^{xy} e^{\frac{z}{y}}\,dzdxdy \\
&= \int_0^1 \int_y^1 (ye^{\frac{z}{y}})\Big|_{z=0}^{z=xy} dxdy \\
&= \int_0^1 \int_y^1 y(e^x - 1)dxdy \\
&= \int_0^1 \int_0^x y(e^x - 1)dydx \\
&= \int_0^1 (e^x - 1)\frac{y^2}{2}\Big|_0^x dx \\
&= \frac{1}{2}\int_0^1 (e^x - 1)x^2 dx \\
&= \frac{1}{2}\int_0^1 x^2 e^x dx - \frac{1}{2}\int_0^1 x^2 dx \\
&= \frac{1}{2}\int_0^1 x^2 de^x - \frac{1}{6} \\
&= \frac{1}{2}(x^2 e^x\Big|_0^1 - \int_0^1 e^x dx^2) - \frac{1}{6} \\
&= \frac{1}{2}(e - 2\int_0^1 xde^x) - \frac{1}{6} \\
&= \frac{1}{2}(e - 2(xe^x\Big|_0^1 - \int_0^1 e^x dx)) - \frac{1}{6} \\
&= \frac{1}{2}(e - 2) - \frac{1}{6} \\
&= \frac{3e - 7}{6}
\end{aligned}
$$

11.

$$\begin{aligned}
\iiint_E \frac{z}{x^2+z^2}dV &= \int_1^4\int_y^4\int_0^z \frac{z}{x^2+z^2}dxdzdy \\
&= \int_0^1\int_y^4\int_0^z \frac{\frac{1}{z}}{(\frac{x}{z})^2+1}dxdzdy \\
&= \int_0^1\int_y^4\int_0^z \frac{1}{(\frac{x}{z})^2+1}d(\frac{x}{z})dzdy \\
&= \int_0^1\int_y^4 \arctan(\frac{x}{z})\Big|_{x=0}^{x=z}dzdy \\
&= \int_0^1\int_y^4 \frac{\pi}{4}dzdy \\
&= \frac{\pi}{4}\times\frac{3\times 3}{2} = \frac{9\pi}{8}
\end{aligned}$$

17.

$$\begin{aligned}
\iiint_E xdV &= \int_0^4\int_0^{2\pi}\int_0^{\frac{\sqrt{x}}{2}} xrdrd\theta dx \\
&= \int_0^4\int_0^{2\pi} x(\frac{r^2}{2})\Big|_{r=0}^{r=\frac{\sqrt{x}}{2}}d\theta dx \\
&= \int_0^4 2\pi x(\frac{x}{8})dx \\
&= \frac{\pi}{4}\int_0^4 x^2dx \\
&= \frac{\pi}{4}(\frac{x^3}{3})\Big|_0^4 \\
&= \frac{16\pi}{3}
\end{aligned}$$

20.

$$\because \left\{\begin{array}{l} y = x^2+z^2 \\ y = 8-x^2-z^2 \end{array}\right. \implies \left\{\begin{array}{l} y = 4 \\ x^2+z^2 = 4 \end{array}\right.$$

$V = 2V_1$

$$
\begin{aligned}
V_1 &= \int_0^4 \int_0^{2\pi} \int_0^2 r dr d\theta dy \\
&= \int_0^4 \int_0^{2\pi} \frac{r^2}{2}\Big|_{r=0}^{r=2} d\theta dy \\
&= 2\int_0^4 \int_0^{2\pi} d\theta dy \\
&= 4\pi \int_0^4 dy \\
&= 16\pi
\end{aligned}
$$

$\therefore$ the volume of the solid is $V = 2V_1 = 32\pi$

22.

$D = \{(x, y, z) | x^2 + z^2 \leq 4\}$

$$
\begin{aligned}
V &= \iint_D \int_{-1}^{4-z} dy dx dz \\
&= \int_0^{2\pi} \int_0^2 \int_{-1}^{4-2\sin\theta} dy r dr d\theta \\
&= \int_0^{2\pi} \int_0^2 (5 - 2\sin\theta) r dr d\theta \\
&= \int_0^{2\pi} (5 - 2\sin\theta)(\frac{r^2}{2})\Big|_{r=0}^{r=2} d\theta \\
&= 2\int_0^{2\pi} (5 - 2\sin\theta) d\theta \\
&= 2(5\theta + 2\cos\theta)\Big|_0^{2\pi} \\
&= 2(10\pi) = 20\pi
\end{aligned}
$$

30.

$$\int_0^3 \int_{-2}^2 \int_0^{\sqrt{9-z^2}} dy dx dz$$

$$\int_0^3 \int_{-2}^2 \int_0^{\sqrt{9-y^2}} dz dx dy$$

$$\int_{-2}^2 \int_0^3 \int_0^{\sqrt{9-z^2}} dy dz dx$$

$$\int_{-2}^{2}\int_{0}^{3}\int_{0}^{\sqrt{9-y^2}} dzdydx$$

$$\int_{0}^{3}\int_{0}^{\sqrt{9-z^2}}\int_{-2}^{2} dxdydz$$

$$\int_{0}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{-2}^{2} dxdzdy$$

34.

$$\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{1-x^2} f(x,y,z)dzdydx$$

$$\int_{0}^{1}\int_{0}^{1-y}\int_{0}^{1-x^2} f(x,y,z)dzdxdy$$

$$\int_{0}^{1}\int_{0}^{-y^2+2y}\int_{0}^{1-y} f(x,y,z)dxdzdy + \int_{0}^{1}\int_{-y^2+2y}^{1}\int_{0}^{\sqrt{1-z}} f(x,y,z)dxdzdy$$

$$\int_{0}^{1}\int_{0}^{1-x^2}\int_{0}^{1-x} f(x,y,z)dydxdz$$

$$\int_{0}^{1}\int_{0}^{\sqrt{1-z}}\int_{0}^{1}\int_{0}^{\sqrt{1-z}}\int_{0}^{\sqrt{1-z}} f(x,y,z)dxdydz+\int_{0}^{1}\int_{\sqrt{1-z}}^{1}\int_{0}^{1-y} f(x,y,z)dxdydz$$

37.

$$\iiint_C (4+5x^2yz^2)dV = 4\iiint_C dV + \iiint_C (5x^2yz^2)dV$$

$\because C$ is symmetric w.r.t. xoz plane, $5x^2yz^2$ is odd function w.r.t. y

$\therefore \iiint_C (5x^2yz^2)dV = 0$

$$\therefore \iiint_C (4+5x^2yz^2)dV = 4\iiint_C dV = 4(16\pi) = 64\pi$$

40.

$D = \{(x, y, z)|x \geq 0, -1 \leq y \leq 1, z \geq 0, z \leq \min(1 - y^2, 1 - x)\}$

$$\begin{aligned}
M &= \int_0^1 \int_0^{1-x} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \rho(x, y, z) dydzdx \\
&= 4 \int_0^1 \int_0^{1-x} 2\sqrt{1-z} dzdx \\
&= -8 \int_0^1 \int_0^{1-x} (1-z)^{\frac{1}{2}} d(1-z) dx \\
&= -8 \int_0^1 \frac{2}{3}(1-z)^{\frac{3}{2}} \Big|_0^{1-x} dx \\
&= -\frac{16}{3} \int_0^1 (x^{\frac{3}{2}} - 1) dx \\
&= \frac{16}{3}(\frac{2}{5}x^{\frac{5}{2}} - x)\Big|_1^0 \\
&= \frac{16}{3} \times \frac{3}{5} = \frac{16}{5}
\end{aligned}$$

$$\begin{aligned}
M_{yz} &= \int_0^1 \int_0^{1-x} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} x\rho(x, y, z) dydzdx \\
&= 4 \int_0^1 \int_0^{1-x} 2x\sqrt{1-z} dzdx \\
&= -8 \int_0^1 \int_0^{1-x} x\sqrt{1-z} d(1-z) dx \\
&= -8 \int_0^1 \frac{2}{3}x(1-z)^{\frac{3}{2}} \Big|_{z=0}^{z=1-x} dx \\
&= -\frac{16}{3} \int_0^1 (x^{\frac{5}{2}} - x) dx \\
&= \frac{16}{3}(\frac{2}{7}x^{\frac{7}{2}} - \frac{1}{2}x^2)\Big|_1^0 \\
&= \frac{16}{3} \times \frac{3}{14} = \frac{8}{7}
\end{aligned}$$

$$\begin{aligned}
M_{xz} &= \int_0^1 \int_0^{1-x} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} y\rho(x, y, z) dydzdx \\
&= 4 \int_0^1 \int_0^{1-x} \frac{y^2}{2}\Big|_{-\sqrt{1-x}}^{\sqrt{1-x}} dzdx \\
&= 0
\end{aligned}$$

$$\begin{aligned}
M_{xy} &= \int_0^1 \int_0^{1-x} \int_{-\sqrt{1-z}}^{\sqrt{1-z}} z\rho(x,y,z)dydzdx \\
&= 8\int_0^1 \int_0^{1-x} z\sqrt{1-z}dzdx \\
&= 8\int_0^1 \int_0^{1-x} (1-z-1)\sqrt{1-z}d(1-z)dx \\
&= 8\int_0^1 (\int_0^{1-x} (1-z)^{\frac{3}{2}}d(1-z) - \int_0^{1-x}(1-z)d(1-z))dx \\
&= 8\int_0^1 (\frac{2}{5}(1-z)^{\frac{5}{2}}\Big|_0^{1-x} - \frac{2}{3}(1-z)^{\frac{3}{2}}\Big|_0^{1-x})dx \\
&= 8\int_0^1 (\frac{2}{5}(x^{\frac{5}{2}}-1) - \frac{2}{3}(x^{\frac{3}{2}}-1))dx \\
&= 8\int_0^1 (\frac{4}{15} + \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}})dx \\
&= 8(\frac{4}{15}x + \frac{4}{35}x^{\frac{7}{2}} - \frac{4}{15}x^{\frac{5}{2}})\Big|_0^1 \\
&= \frac{32}{35}
\end{aligned}$$

$$\overline{x} = \frac{M_{yz}}{M} = \frac{8}{7}\times\frac{5}{16} = \frac{7}{10}$$

$$\overline{y} = \frac{M_{xz}}{M} = 0$$

$$\overline{z} = \frac{M_{xy}}{M} = \frac{32}{35}\times\frac{5}{16} = \frac{2}{7}$$

$\therefore$ the coordinate of center of mass is $(\frac{7}{10}, 0, \frac{2}{7})$.

46. (Assume that the solid has constant density k)

Let the radius of the cone be t, then the moment inertia about the z-axis is

$$\begin{aligned}
\iiint_E (x^2+y^2)\rho(x,y,z)dV &= k\int_0^{2\pi}\int_0^t\int_r^h dzrdrd\theta \\
&= k\int_0^{2\pi}\int_0^t (hr-r^2)drd\theta \\
&= k\int_0^{2\pi} (\frac{hr^2}{2} - \frac{r^3}{3})\Big|_{r=0}^{r=t} d\theta \\
&= k\int_0^{2\pi} (\frac{ht^2}{2} - \frac{t^3}{3})d\theta \\
&= 2\pi k(\frac{ht^2}{2} - \frac{t^3}{3})
\end{aligned}$$