Exercise 11.9

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2.

$\because \sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$

$\therefore \lim_{n\to\infty} |\frac{b_{n+1}}{b_n}| = \frac{1}{2}$

$\therefore \lim_{n\to\infty} |\frac{\frac{b_{n+1}}{n+2}}{\frac{b_n}{n+1}}| = \lim_{n\to\infty} \frac{n+1}{n+2} |\frac{b_{n+1}}{b_n}| = \lim_{n\to\infty} |\frac{b_{n+1}}{b_n}| = \frac{1}{2}$

$\therefore \sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ also converges for $|x| < 2$

In fact,

$$\because \int \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

Due to integration to a power series does not change its radius of convergence, $\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ also converges for $|x| < 2$.

13.

(a)

$$\begin{aligned}
\because \int_0^x f(t)\mathrm{d}t &= \int_0^x \frac{1}{(1+t)^2}\mathrm{d}t \\
&= -\frac{1}{1+x} \\
&= -\sum_{n=0}^{\infty} (-x)^n \\
&= \sum_{n=0}^{\infty} (-1)^{n+1} x^n
\end{aligned}$$

$$\therefore \frac{1}{(1+x)^2} = f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x f(t)\mathrm{d}t = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$$

$$\because \rho = \lim_{n\to\infty} |\frac{(-1)^{n+2}(n+1)}{(-1)^{n+1} n}| = 1$$

$\therefore$ the radius of convergence is $R = \frac{1}{\rho} = 1$.

(b)

$$\begin{aligned}
\because \int_0^x f(t)\mathrm{d}t &= \int_0^x \frac{1}{(1+t)^3}\mathrm{d}t \\
&= -\frac{1}{2}\frac{1}{(1+x)^2} \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n}nx^{n-1}
\end{aligned}$$

$$\therefore \frac{1}{(1+x)^3} = f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\mathrm{d}t = \sum_{n=2}^{\infty}(-1)^n\frac{n(n-1)}{2}x^{n-2}$$

(c)

$$\begin{aligned}
f(x) &= \frac{x^2}{(1+x)^3} \\
&= x^2\sum_{n=2}^{\infty}(-1)^n\frac{n(n-1)}{2}x^{n-2} \\
&= \sum_{n=2}^{\infty}(-1)^n\frac{n(n-1)}{2}x^{n}
\end{aligned}$$

17. $f(x) = \frac{x}{(1+4x)^2}$

$$\begin{aligned}
\frac{1}{(1+x)^2} &= -\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-(-x)} \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty}(-x)^n \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}
\end{aligned}$$

$$\begin{aligned}
f(x) = x\frac{1}{(1+4x)^2} &= x\sum_{n=1}^{\infty}(-1)^{n+1}n(4x)^{n-1} \\
&= \sum_{n=1}^{\infty}(-4)^{n-1}nx^n
\end{aligned}$$

$$\because \rho = \lim_{n\to\infty}|\frac{(-4)^n(n+1)}{(-4)^{n-1}n}| = \lim_{n\to\infty}\frac{4(n+1)}{n} = 4$$

$\therefore$ the radius of convergence is $R = \frac{1}{\rho} = \frac{1}{4}$

18. $f(x) = (\frac{x}{2-x})^3$

$$\begin{aligned}
\frac{1}{(1-x)^3} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{(1-x)^2} \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x^2}\frac{1}{1-x} \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x^2}\sum_{n=0}^{\infty} x^n \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{\infty} nx^{n-1} \\
&= \sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^{n-2}
\end{aligned}$$

$$\begin{aligned}
f(x) = x^3\frac{1}{(2-x)^3} &= \frac{x^3}{8}\frac{1}{(1-\frac{x}{2})^3} \\
&= \frac{x^3}{8}\sum_{n=2}^{\infty}\frac{n(n-1)}{2^{n-1}}x^{n-2} \\
&= \sum_{n=2}^{\infty}\frac{n(n-1)}{2^{n+2}}x^{n+1}
\end{aligned}$$

$$\because \rho = \lim_{n\to\infty}\left|\frac{\frac{n(n+1)}{2^{n+3}}}{\frac{n(n-1)}{2^{n+2}}}\right| = \lim_{n\to\infty}\frac{n+1}{2(n-1)} = \frac{1}{2}$$

$\therefore$ the radius of convergence is $R = \frac{1}{\rho} = 2$

40.

(a)

$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}x^n = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty}x^n = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{1}{(1-x)^2}$$

(b)

$$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}$$

$\sum_{n=1}^{\infty}\frac{n}{2^n}$ is the special case of $\sum_{n=1}^{\infty} nx^n$ when $x = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty}\frac{n}{2^n} = \frac{\frac{1}{2}}{(\frac{1}{2})^2} = 2$$

(c) (i)

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{\infty} nx^{n-1} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}$$

$$\sum_{n=2}^{\infty} n(n-1)x^{n} = x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2x^2}{(1-x)^3}$$

(ii) This series is the special case of $\sum_{n=2}^{\infty} n(n-1)x^n$ when $x = \frac{1}{2}$.

$$\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = \frac{2 \times \frac{1}{4}}{(\frac{1}{2})^3} = 4$$

(iii)

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^2}{2^n} &= \sum_{n=1}^{\infty}(\frac{n^2-n}{2^n} + \frac{n}{2^n}) \\ &= \sum_{n=2}^{\infty} \frac{n^2-n)}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n} \\ &= 4 - 2 = 2\end{aligned}$$