Exercise 11.5

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8. $\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}$

Let $a_n = (-1)^n \frac{n}{\sqrt{n^3+2}}, b_n = \frac{n}{\sqrt{n^3+2}} = \frac{1}{\sqrt{n+\frac{2}{n^2}}}, c_n = n + \frac{2}{n^2}$

Let $f(x) = x + \frac{2}{x^2}, x \geq 1$
$\because$ when $x < 2, f(1) = 1 + 2 = 3 > f(2) = 2 + \frac{1}{2} = \frac{5}{2}$
$\because$ when $x \geq 2, f'(x) = 1 - \frac{4}{x^3} > \frac{1}{2} > 0, f(x)$ is increasing
$\therefore \forall n \in N_+, c_n$ is increasing
$\therefore \forall n \in N_+, b_n$ is decreasing, $b_{n+1} < b_n$
$\because \lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{n^3+2}{n^2} = \lim_{n\to\infty} \frac{3n^2}{2n} = \infty$
$\therefore \lim_{n\to\infty} b_n = 0$
$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}$ is convergent

9. $\sum_{n=1}^{\infty}(-1)^n e^{-n}$

Let $b_n = e^{-n} = \frac{1}{e^n}, a_n = (-1)^n b_n$
$\because e^{n+1} > e^n$ and $\lim_{n\to\infty} e^n = \infty$
$\therefore b_{n+1} < b_n$ and $\lim_{n\to\infty} b_n = 0$
$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^n e^{-n}$ is convergent

12. $\sum_{n=1}^{\infty}(-1)^{n+1} n e^{-n}$

Let $b_n = ne^{-n}, a_n = (-1)^{n+1} b_n$
Let $f(x) = xe^{-x}, x \geq 1$, then $f'(x) = (1-x)e^{-x} \leq 0$, $f(x)$ is decreasing
Also,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$\therefore b_{n+1} < b_n$ and $\lim_{n\to\infty} b_n = 0$
$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^{n+1} n e^{-n}$ is convergent

13. $\sum_{n=1}^{\infty}(-1)^{n-1} e^{\frac{2}{n}}$

Let $b_n = e^{\frac{2}{n}}, a_n = (-1)^{n-1} b_n$

Let $f(x) = e^{\frac{2}{x}}$, obviously $f(x)$ is decreasing, and

$$\lim_{x\to\infty} f(x) = e^{\lim_{x\to\infty} \frac{2}{x}} = 1$$

$\therefore b_{n+1} < b_n$ and $\lim_{n\to\infty} b_n = 0$
$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} e^{\frac{2}{n}}$ is convergent

14. $\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$

Let $b_n = \arctan n, a_n = (-1)^{n-1} b_n$
$\because y = \arctan x$ is increasing and

$$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$$

$\therefore b_{n+1} > b_n$ and $\lim_{n\to\infty} b_n \neq 0$
$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is divergent

17. $\sum_{n=1}^{\infty} (-1)^n \sin(\frac{\pi}{n})$

Let $b_n = \sin(\frac{\pi}{n}), a_n = (-1)^n b_n$
$\because f(x) = \sin(\frac{\pi}{x})$ is decreasing when $x \geq 1$ and

$$\lim_{x\to\infty} f(x) = \sin(\lim_{x\to\infty} \frac{\pi}{x}) = \sin 0 = 0$$

$\therefore b_{n+1} < b_n$ and $\lim_{n\to\infty} b_n = 0$
$\therefore \sum_{n=1}^{\infty} a_n = (-1)^n b_n$ is convergent

18. $\sum_{n=1}^{\infty} (-1)^n \cos(\frac{\pi}{n})$

Let $b_n = \cos(\frac{\pi}{n}), a_n = (-1)^n b_n$, and let $f(x) = \cos(\frac{\pi}{x}), x \geq 1$
$\because x \geq 1 \quad \therefore 0 < \frac{\pi}{x} \leq \pi$
$\because y = \cos x$ is decreasing when $0 < x \leq \pi$, and $y = \frac{\pi}{x}$ is also decreasing
$\therefore f(x) = \cos(\frac{\pi}{x})$ is increasing
$\therefore b_{n+1} > b_n$

$$\because \lim_{n\to\infty} b_n = \cos(\lim_{n\to\infty} \frac{\pi}{n}) = \cos 0 = 1 \neq 0$$

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ is divergent

33. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+p}$

Let $a_n = \frac{(-1)^n}{n+p}, b_n = \frac{1}{n+p}$
$\because$ for all $p \in R$, b_n is decreasing and $\lim_{n\to\infty} b_n = 0$
$\therefore$ for all $p \in R$, $b_{n+1} < b_n$ and $\lim_{n\to\infty} b_n = 0$
$\therefore$ for all $p \in R$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+p}$ is convergent.

34. $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(\ln n)^p}{n}$

Let $a_n = (-1)^{n-1}\frac{(\ln n)^p}{n}, b_n = \frac{(\ln n)^p}{n}, f(x) = \frac{(\ln x)^p}{x}, x \geq 2$

$$f'(x) = \frac{p(\ln x)^{p-1} - (\ln x)^p}{x^2} = \frac{(\ln x)^{p-1}(p - \ln x)}{x^2}$$

$\therefore$ when $x < e^p, p > \ln x, f(x)$ is increasing;
$\therefore$ when $x > e^p, p < \ln x, f(x)$ is decreasing

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{p(\ln n)^{p-1}}{n} = \lim_{n\to\infty}\frac{p(p-1)(\ln n)^{p-2}}{n} = \cdots = \lim_{n\to\infty}\frac{p!}{n} = 0$$

For $p \geq \ln 2$, $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(\ln n)^p}{n}$ is convergent.