Exercise 15.6

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6.

Let $f(x,y) = 4 - x^2 - y^2$ and $D = \{(x,y)|x^2+y^2 \le 4\}$, then
$f_x(x,y) = -2x, f_y(x,y) = -2y$

$$\begin{aligned}
\iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1} dA &= \iint_D \sqrt{4x^2+4y^2+1} dA \\
&= \int_0^{2\pi} \int_0^2 \sqrt{4r^2+1} r dr d\theta \\
&= \int_0^{2\pi} \int_0^2 \frac{1}{8} \sqrt{4r^2+1} d(4r^2+1) d\theta \\
&= \int_0^{2\pi} \frac{1}{8} \times \frac{2}{3} (4r^2+1)^{\frac{3}{2}} \Big|_0^2 d\theta \\
&= \int_0^{2\pi} \frac{1}{12} (17^{\frac{3}{2}} - 1) d\theta \\
&= \frac{(17^{\frac{3}{2}} - 1)\pi}{6}
\end{aligned}$$

11.

Let $f(x,y) = \sqrt{a^2 - x^2 - y^2}$ and $D = \{(x,y)|x^2+y^2 \le ax\}$, then

$$f_x(x,y) = \frac{-x}{\sqrt{a^2-x^2-y^2}}, f_y(x,y) = \frac{-y}{\sqrt{a^2-x^2-y^2}}$$

$$\iint_D \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}dA = \iint_D \sqrt{\frac{x^2+y^2+a^2-x^2-y^2}{a^2-x^2-y^2}}dA$$
$$= \int_{-\pi}^{\pi}\int_0^{a\cos\theta}\sqrt{\frac{a^2}{a^2-r^2}}rdrd\theta$$
$$= a\int_{-\pi}^{\pi}\int_0^{a\cos\theta}(-\frac{1}{2})(a^2-r^2)^{-\frac{1}{2}}d(a^2-r^2)d\theta$$
$$= a\int_{-\pi}^{\pi} -(a^2-r^2)^{\frac{1}{2}}\Big|_{r=0}^{r=a\cos\theta} d\theta$$
$$= a\int_{-\pi}^{\pi} a(1-\sin\theta)d\theta$$
$$= a^2(\theta+\cos\theta)\Big|_{-\pi}^{\pi}$$
$$= a^2(2\pi-2)$$

12.

$$\because \begin{cases} x^2+y^2+z^2=4z \\ z=x^2+y^2 \end{cases} \implies x^2+y^2=3$$

$\therefore D=\{(x,y)|x^2+y^2=3\}$
$\because x^2+y^2+z^2=4z$
$\therefore z^2-4z+4=4-x^2-y^2=(z-2)^2$
$\therefore z-2=\sqrt{4-x^2-y^2}, z=\sqrt{4-x^2-y^2}+2$

$$\iint_D \sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}dA = \int_0^{2\pi}\int_0^{\sqrt{3}}\sqrt{1+(\frac{-r\cos\theta}{\sqrt{4-r^2}})^2+(\frac{-r\sin\theta}{\sqrt{4-r^2}})^2}rdrd\theta$$
$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\sqrt{\frac{4-r^2+r^2}{4-r^2}}rdrd\theta$$
$$= \int_0^{2\pi}\int_0^{\sqrt{3}}2\sqrt{\frac{1}{4-r^2}}rdrd\theta$$
$$= -\int_0^{2\pi}\int_0^{\sqrt{3}}(4-r^2)^{-\frac{1}{2}}d(4-r^2)drd\theta$$
$$= -\int_0^{2\pi}2(4-r^2)^{\frac{1}{2}}\Big|_{r=0}^{r=\sqrt{3}}d\theta$$
$$= -\int_0^{2\pi}2(1-2)d\theta$$
$$= 4\pi$$

22.

The area of the top half of the sphere is

$$\lim_{t\to a^-}\iint_D \sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}dA$$

where $\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{t^2-x^2-y^2}}$, $\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{t^2-x^2-y^2}}$, $D=\{(x,y)|x^2+y^2\leq t^2\}$

$$\begin{aligned}
\lim_{t\to a^-}\iint_D \sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}dA &= \lim_{t\to a^-}\int_0^{2\pi}\int_0^t \sqrt{1+(\frac{-r\cos\theta}{\sqrt{t^2-r^2}})^2+(\frac{-r\sin\theta}{\sqrt{t^2-r^2}})^2}rdrd\theta \\
&= \lim_{t\to a^-}\int_0^{2\pi}\int_0^t \sqrt{\frac{t^2-r^2+r^2}{t^2-r^2}}rdrd\theta \\
&= \lim_{t\to a^-}\int_0^{2\pi}(-\frac{1}{2})t(t^2-r^2)^{-\frac{1}{2}}d(t^2-r^2)d\theta \\
&= \lim_{t\to a^-}\int_0^{2\pi} -t(t^2-r^2)^{\frac{1}{2}}\Big|_{r=0}^{r=t}d\theta \\
&= \lim_{t\to a^-}\int_0^{2\pi} t^2d\theta \\
&= \lim_{t\to a^-} 2\pi t^2 \\
&= 2\pi a^2
\end{aligned}$$

$\therefore$ the area of a sphere of radius r is $2\times 2\pi a^2=4\pi a^2$

23.

$D=\{(x,y)|x^2+y^2=\leq 25\}$

$$
\begin{aligned}
\iint_D \sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}dA &= \iint_D \sqrt{1+(\frac{\partial y}{\partial x})^2+(\frac{\partial y}{\partial z})^2}dA \\
&= \iint_D \sqrt{1+(2x)^2+(2z)^2}dxdz \\
&= \int_0^{2\pi}\int_0^5 \sqrt{1+4r^2}rdrd\theta \\
&= \frac{1}{8}\int_0^{2\pi}\int_0^5 \sqrt{1+4r^2}d(1+4r^2)d\theta \\
&= \frac{1}{8}\int_0^{2\pi} \frac{2}{3}(1+4r^2)^{\frac{3}{2}}\Big|_{r=0}^{r=5} d\theta \\
&= \frac{1}{12}\int_0^{2\pi} (101)^{\frac{3}{2}} - 1d\theta \\
&= \frac{101^{\frac{3}{2}}-1}{6}\pi
\end{aligned}
$$

24.

Solving $\begin{cases} x^2+z^2=1 \\ y^2+z^2=1 \end{cases}$, intersection points are as follows:

$(1,-1,0),(1,1,0),(-1,1,0),(-1,-1,0),(0,0,1),(0,0,-1)$

$V = 8V_1$

$$
\begin{aligned}
V_1 &= \int_0^1 dz \int_0^{\sqrt{1-z^2}} dx \int_0^{\sqrt{1-z^2}} dy \\
&= \int_0^1 (\sqrt{1-z^2})^2 dz \\
&= (z-\frac{1}{3}z^3)\Big|_0^1 \\
&= 1-\frac{1}{3}-0 = \frac{2}{3}
\end{aligned}
$$

$$\therefore V = 8V_1 = \frac{16}{3}$$