Exercise 11. Fourier Series

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1. Find the Fourier series of the following function

$$f(x) = \begin{cases} x+1, & -\pi \leq x < 0 \\ x^2, & 0 \leq x \leq \pi \end{cases}$$

and find the sum of the Fourier series.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \\
&= \frac{1}{\pi}(\int_{-\pi}^{0}(x+1)\mathrm{d}x + \int_{0}^{\pi} x^2\mathrm{d}x) \\
&= \frac{1}{\pi}[(\frac{x^2}{2}+x)\Big|_{-\pi}^{0} + (\frac{x^3}{3})\Big|_{0}^{\pi}] \\
&= \frac{1}{\pi}(0 - \frac{\pi^2}{2} + \pi + \frac{\pi^3}{3}) \\
&= \frac{\pi^2}{3} - \frac{\pi}{2} + 1
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\mathrm{d}x \\
&= \frac{1}{\pi}(\int_{-\pi}^{0}(x+1)\cos nx\mathrm{d}x + \int_{0}^{\pi} x^2\cos nx\mathrm{d}x) \\
&= \frac{1}{\pi}(\int_{-\pi}^{0}\frac{x+1}{n}\mathrm{d}\sin nx + \int_{0}^{\pi}\frac{x^2}{n}\mathrm{d}\sin nx) \\
&= \frac{1}{\pi}[\frac{(x+1)\sin nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin nx}{n}\mathrm{d}x + \frac{x^2\sin nx}{n}\Big|_{0}^{\pi} - \int_{0}^{\pi}\frac{2x}{n}\sin nx\mathrm{d}x] \\
&= \frac{1}{\pi}[0 + (\frac{\cos nx}{n^2})\Big|_{-\pi}^{0} + 0 + 2\int_{0}^{\pi}\frac{x}{n^2}\mathrm{d}\cos nx] \\
&= \frac{1}{\pi}(0 + \frac{1-(-1)^n}{n^2} + 2(\frac{x\cos nx}{n^2})\Big|_{0}^{\pi} - 2\int_{0}^{\pi}\frac{\cos nx}{n^2}\mathrm{d}x) \\
&= \frac{1}{\pi}(0 + \frac{1-(-1)^n}{n^2} + \frac{2\pi(-1)^n}{n^2} - \frac{2}{n}(\sin nx)\Big|_{0}^{\pi}) \\
&= \frac{1+(-1)^n(2\pi-1)}{n^2\pi}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\mathrm{d}x \\
&= \frac{1}{\pi}(\int_{-\pi}^{0}(x+1)\sin nx\mathrm{d}x + \int_{0}^{\pi} x^2\sin nx\mathrm{d}x) \\
&= -\frac{1}{\pi}(\int_{-\pi}^{0}\frac{x+1}{n}\mathrm{d}\cos nx + \int_{0}^{\pi}\frac{x^2}{n}\mathrm{d}\cos nx) \\
&= -\frac{1}{\pi}[\frac{(x+1)\cos nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\cos nx}{n}\mathrm{d}x + \frac{x^2\cos nx}{n}\Big|_{0}^{\pi} - \frac{1}{n}\int_{0}^{\pi} 2x\cos nx\mathrm{d}x] \\
&= -\frac{1}{\pi}[\frac{1-(1-\pi)(-1)^n}{n} - \frac{\sin nx}{n^2}\Big|_{-\pi}^{0} + \frac{(-1)^n\pi^2}{n} - \frac{2}{n^2}\int_{0}^{\pi} x\mathrm{d}\sin nx] \\
&= -\frac{1}{\pi}[\frac{1+(-1)^n(\pi-1)}{n} - 0 + \frac{(-1)^n\pi^2}{n} + \frac{2}{n^2}\int_{0}^{\pi}\sin nx\mathrm{d}x] \\
&= -\frac{1}{\pi}[\frac{1+(-1)^n(\pi-1)}{n} - 0 + \frac{(-1)^n\pi^2}{n} - \frac{2}{n}(\cos nx)\Big|_{0}^{\pi}] \\
&= -\frac{1}{\pi}[\frac{1+(-1)^n(\pi-1)}{n} - 0 + \frac{(-1)^n\pi^2}{n} - 2\frac{(-1)^n-1}{n}] \\
&= \frac{1+(-1)^n(\pi^2-\pi-1)}{n\pi}
\end{aligned}$$

$\therefore$ the Fourier series of $f(x)$ is

$$(\frac{\pi^2}{6}-\frac{\pi}{4}+\frac{1}{2})+\sum_{n=1}^{\infty}(\frac{1+(-1)^n(2\pi-1)}{n^2\pi}\cos nx+\frac{1+(-1)^n(\pi^2-\pi-1)}{n\pi}\sin nx)$$

Denote the sum function to be $s(x)$, then we have

$$s(x)=\begin{cases} x+1 & \text{if } -\pi<x<0 \\ \frac{1}{2} & \text{if } x=0 \\ x^2 & \text{if } 0<x<\pi \\ \frac{\pi^2+\pi+1}{2} & \text{if } x=\pi \end{cases}$$

2. Find the sine series of $f(x)=\frac{x^2}{2}(0\leq x\leq\pi)$

Let $F(x)=\begin{cases} f(x), & 0\leq x\leq\pi \\ -f(-x), & -\pi\leq x<0 \end{cases}$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}F(x)\mathrm{d}x=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}F(x)\cos nx\mathrm{d}x=0$$

$$\begin{aligned} b_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}F(x)\sin nx\mathrm{d}x \\ &=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\mathrm{d}x \\ &=\frac{1}{\pi}\int_0^{\pi}x^2\sin nx\mathrm{d}x \\ &=-\frac{1}{n\pi}\int_0^{\pi}x^2\mathrm{d}\cos nx \\ &=-\frac{1}{n\pi}[(x^2\cos nx)\Big|_0^{\pi}-\int_0^{\pi}2x\cos nx\mathrm{d}x] \\ &=-\frac{1}{n\pi}[(-1)^n\pi^2-\frac{2}{n}\int_0^{\pi}x\mathrm{d}\sin nx] \\ &=-\frac{1}{n\pi}[(-1)^n\pi^2-\frac{2}{n}(0-\int_0^{\pi}\sin nx\mathrm{d}x)] \\ &=-\frac{1}{n\pi}[(-1)^n\pi^2-\frac{2}{n^2}(\cos nx)\Big|_0^{\pi}] \\ &=\frac{1}{n\pi}[2\frac{(-1)^n-1}{n^2}-(-1)^n\pi^2] \\ &=\frac{(-1)^n n^2(2-\pi^2)-2}{n^3\pi} \end{aligned}$$

$\therefore$ the sine series of $f(x)=\frac{x^2}{2}(0\leq x\leq\pi)$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2(2-\pi^2)-2}{n^3\pi} \sin nt$$

3. Find the Fourier series of $f(x) = x(1 < x < 3)$, and prove the equality

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$$

Suppose $f(x)$ has the period $T = 2$, then:

when $-1 < x < 1, f(x) = x + 2$, when $-3 < x < -1, f(x) = x + 4$

$a_0 = 0$

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\mathrm{d}x \\ &= \frac{1}{2}\int_{-2}^{-1}(x+4)\cos\frac{n\pi x}{2}\mathrm{d}x + \frac{1}{2}\int_{-1}^{1}(x+2)\cos\frac{n\pi x}{2}\mathrm{d}x + \frac{1}{2}\int_{1}^{2} x\cos\frac{n\pi x}{2}\mathrm{d}x \\ &= \frac{-2}{n^2\pi^2}\cos n\pi \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\mathrm{d}x \\ &= \frac{1}{2}\int_{-2}^{-1}(x+4)\sin\frac{n\pi x}{2}\mathrm{d}x + \frac{1}{2}\int_{-1}^{1}(x+2)\sin\frac{n\pi x}{2}\mathrm{d}x + \frac{1}{2}\int_{1}^{2} x\sin\frac{n\pi x}{2}\mathrm{d}x \\ &= \frac{8}{n\pi}\cos n\pi \end{aligned}$$

$\therefore$ the Fourier series of $f(x)$ in $[-2, 2]$ is

$$\sum_{n=1}^{\infty}[\frac{-2}{n^2\pi^2}\cos n\pi \cos nx + \frac{8}{n\pi}\cos n\pi \sin nx]$$

When $x = \frac{3\pi}{2}$, $\frac{3\pi}{2} = 6\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$, which is equivalent to

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$$