Exercise 16.4

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1.

$P(x,y) = x - y, Q(x,y) = x + y$

Method 1: $\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$

$$\begin{aligned} \oint_C (x-y)dx + (x+y)dy &= \int_0^{2\pi} 4(\cos t - \sin t)(-\sin t) + 4(\cos t + \sin t)(\cos t)dt \\ &= 4\int_0^{2\pi} \cos^2 t + \sin t\cos t - \sin t\cos t + \sin^2 t dt \\ &= 8\pi \end{aligned}$$

Method2:

$$\frac{\partial Q}{\partial x} = 1, \frac{\partial P}{\partial y} = -1$$

$$\oint_C (x-y)dx + (x+y)dy = \iint_D [1 - (-1)]dA$$

where $D = \{(x,y)|x^2 + y^2 \leq 4\}$

$\therefore \iint_D 2dA = 2 \times 4\pi = 8\pi$

3.

$P(x,y) = xy, Q(x,y) = x^2y^3$

Method 1: {

Let C_1, C_2, C_3 be the path from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(1,2)$ and from $(1,2)$ to $(0,0)$, respectively. Then

$$\int_{C_1} xydx + x^2y^3dy = 0$$
$$\int_{C_2} xydx + x^2y^3dy = \int_0^2 t^3dt = 4$$
$$\int_{C_3} xydx + x^2y^3dy = \int_1^0 2t^2 + 16t^5dt = -\frac{10}{3}$$
$$\oint_C xydx + x^2y^3dy = 0 + 4 - \frac{10}{3} = \frac{2}{3}$$

Method 2:

$$\frac{\partial Q}{\partial x} = 2xy^3, \frac{\partial P}{\partial y} = x$$

$$\begin{aligned}\oint_C xydx + x^2y^3dy &= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx \\ &= \int_0^1 x(\frac{y^4}{2} - y)\Big|_0^{2x} dx \\ &= \int_0^1 (8x^5 - 2x^2)dx \\ &= (\frac{4x^6}{3} - \frac{2x^3}{3})\Big|_0^1 \\ &= \frac{2}{3}\end{aligned}$$

7.

$P(x,y) = y + e^{\sqrt{x}}, Q(x,y) = 2x + \cos y^2$

$$\frac{\partial Q}{\partial x} = 2, \frac{\partial P}{\partial y} = 1$$

$$\begin{aligned}\therefore \int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy &= \int_0^1 \int_{x^2}^{\sqrt{x}} (2-1)dydx \\ &= \int_0^1 (x^{\frac{1}{2}} - x^2)dx \\ &= (\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3})\Big|_0^1 \\ &= \frac{1}{3}\end{aligned}$$

8.

$P(x,y) = y^4, Q(x,y) = 2xy^3$

$$\frac{\partial Q}{\partial x} = 2y^3, \frac{\partial P}{\partial y} = 4y^3$$

$$\therefore \int_C y^4 dx + 2xy^3 dy = \int_D (2y^3 - 4y^3) dA$$

where $D = \{(x,y) | \frac{x^2}{2} + y^2 \leq 1\}$

Let $\begin{cases} x = \sqrt{2}\cos t \\ y = \sin t \end{cases}$, then

$$\begin{aligned}
\int_D (-2y^3) dA &= -2 \int_0^{2\pi} \int_0^1 \sin^3 t \sqrt{2} r dr dt \\
&= -2\sqrt{2} \int_0^{2\pi} \sin^3 t (\frac{r^2}{2}) \Big|_0^1 dt \\
&= \sqrt{2} \int_0^{2\pi} (1 - \cos^2 t) d\cos t \\
&= \sqrt{2} (\cos t - \frac{\cos^3 t}{3}) \Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

12.

$P(x,y) = e^{-x} + y^2, Q(x,y) = e^{-y} + x^2$

$$\frac{\partial Q}{\partial x} = 2x, \frac{\partial P}{\partial y} = 2y$$

$$\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} = -\iint_D (2x - 2y) dA$$

where $D = \{(x,y) | -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \cos x\}$

$$\begin{aligned}
\oint_C \overrightarrow{F} \cdot d\overrightarrow{r} &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos x} (y-x)dydx \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\frac{y^2}{2} - xy)\Big|_0^{\cos x} dx \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\frac{\cos^2 x}{2} - x\cos x)dx \\
&= 4\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{2} dx \\
&= 2\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} dx \\
&= \int_0^{\frac{\pi}{2}} (x + \frac{1}{2}\sin 2x)\Big|_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}
\end{aligned}$$

13.

$P(x,y) = y - \cos y, Q(x,y) = x\sin y$

$$\frac{\partial Q}{\partial x} = \sin y, \frac{\partial P}{\partial y} = 1 + \sin y$$

Let $D = \{(x,y)|(x-3)^2 + (y+4)^2 \leq 4\}$, then

$$\begin{aligned}
\int_C \overrightarrow{F} \cdot d\overrightarrow{r} &= -\iint_D (\sin y - 1 - \sin y)dA \\
&= \iint_D dA = 4\pi
\end{aligned}$$

19.

$$\begin{aligned}
A = \oint_C xdy &= \int_{2\pi}^0 (t - \sin t)\sin t dt \\
&= -\int_{2\pi}^0 td\cos t - \int_{2\pi}^0 \frac{1-\cos 2t}{2} dt \\
&= -(t\cos t)\Big|_{2\pi}^0 + \int_{2\pi}^0 \cos t dt - (\frac{t}{2} - \frac{\sin 2t}{4})\Big|_{2\pi}^0 \\
&= 2\pi + 0 - (-\pi) = 3\pi
\end{aligned}$$

27.

Let C' be a counterclockwise-oriented circle with center the origin and radius a, where

$$P(x,y)=\frac{2xy}{(x^2+y^2)^2}, Q(x,y)=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x}=\frac{-2x(x^2+y^2)^2-(y^2-x^2)2(x^2+y^2)2x}{(x^2+y^2)^4}=\frac{-2x(x^2+y^2)+4x(x^2-y^2)}{(x^2+y^2)^3}=\frac{2x^3-6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial P}{\partial y}=\frac{2x(x^2+y^2)^2-4xy(x^2+y^2)2y}{(x^2+y^2)^4}=\frac{2x^3-6xy^2}{(x^2+y^2)^3}$$

Let D be the region bounded by C and C', then

$$\int_C Pdx+Qdy+\int_{-C'} Pdx+Qdy=\iint_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dA=0$$

$$\therefore \int_C Pdx+Qdy=\int_{C'} Pdx+Qdy$$

Let a parametric equation of C' be $\begin{cases} x=a\cos t \\ y=a\sin t \end{cases}$, then

$$\begin{aligned}
\therefore \int_C \overrightarrow{F}\cdot d\overrightarrow{r} &= \int_{C'} \overrightarrow{F}\cdot d\overrightarrow{r} \\
&= \int_0^{2\pi} \overrightarrow{F}(\overrightarrow{r}(t))\cdot \overrightarrow{r}'(t)dt \\
&= \int_0^{2\pi} \frac{2a^2\sin t\cos t}{a^4}(-a\sin t)+\frac{a^2(\sin^2 t-\cos^2 t)}{a^4}(a\cos t)dt \\
&= \int_0^{2\pi} -\frac{2}{a}\sin^2 t\cos t+\frac{1}{a}(2\sin^2 t-1)\cos tdt \\
&= \int_0^{2\pi} -\frac{1}{a}\cos tdt \\
&= -\frac{1}{a}(\sin t)\Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

31.

Proof. $\because \begin{cases} x=g(u,v) \\ y=h(u,v) \end{cases}$

$$\begin{aligned}
\therefore \iint_R dxdy &= \frac{1}{2}\int_{\partial R} xdy-ydx \\
&= \frac{1}{2}\int_{\partial S} g(u,v)(\frac{\partial h}{\partial u}du+\frac{\partial h}{\partial v}dv)-h(u,v)(\frac{\partial g}{\partial u}du+\frac{\partial g}{\partial v}dv) \\
&= \frac{1}{2}\int_{\partial S}[g(u,v)\frac{\partial h}{\partial u}-h(u,v)\frac{\partial g}{\partial u}]du+[g(u,v)\frac{\partial h}{\partial v}-h(u,v)\frac{\partial g}{\partial v}]dv
\end{aligned}$$

Let $P(u,v) = g(u,v)\frac{\partial h}{\partial u} - h(u,v)\frac{\partial g}{\partial u}, Q(u,v) = g(u,v)\frac{\partial h}{\partial v} - h(u,v)\frac{\partial g}{\partial v}$, then

$$\frac{\partial Q}{\partial u} = \frac{\partial g}{\partial u}\frac{\partial h}{\partial v} + g(u,v)\frac{\partial^2 h}{\partial u \partial v} - \frac{\partial h}{\partial u}\frac{\partial g}{\partial v} - h(u,v)\frac{\partial^2 g}{\partial u \partial v}$$

$$\frac{\partial P}{\partial v} = \frac{\partial g}{\partial v}\frac{\partial h}{\partial u} + g(u,v)\frac{\partial^2 h}{\partial v \partial u} - \frac{\partial h}{\partial v}\frac{\partial g}{\partial u} - h(u,v)\frac{\partial^2 g}{\partial v \partial u}$$

$$\begin{aligned}
\frac{1}{2}\int_{\partial S} P(u,v)du + Q(u,v)dv &= \frac{1}{2}\int_S (\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v})dudv \\
&= \frac{1}{2}\int_S (2\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u})dudv \\
&= \int_S \frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}dudv \\
&= \int_S \left|\frac{\partial(g,h)}{\partial(u,v)}\right| dudv \\
&= \int_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv
\end{aligned}$$

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