Exercise 17.1

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June 20, 2021

4.

Solving the auxiliary equation $\lambda^2 - 8\lambda + 12 = (\lambda - 6)(\lambda - 2) = 0$, $\lambda = 6$ or $\lambda = 2$

$\therefore$ the general solution is

$$y = C_1 e^{6x} + C_2 e^{2x}$$

where C_1 and C_2 are arbitrary constants.

5.

Solving the auxiliary equation $9\lambda^2 - 12\lambda + 4 = (3\lambda - 2)^2 = 0, \lambda = \frac{2}{3}$

$\therefore$ the general solution is

$$y = C_1 e^{\frac{2}{3}x} + C_2 x e^{\frac{2}{3}x}$$

where C_1 and C_2 are arbitrary constants.

9.

Solving the auxiliary equation $\lambda^2 - 4\lambda + 13 = 0$,

$\because \Delta = 16 - 4 \times 13 = 16 - 52 = -36$

$\therefore$ the solution to the equation is

$$\lambda = \frac{4 + 6i}{2} = 2 + 3i \text{ or } \lambda = \frac{4 - 6i}{2} = 2 - 3i$$

$\therefore$ the general solution is

$$y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$$

where C_1 and C_2 are arbitrary constants.

21.

Solving the auxiliary equation $\lambda^2 - 6\lambda + 10 = 0$,
$\because \Delta = 36 - 4 \times 10 = -4$
$\therefore$ the solution to the equation is

$$\lambda = \frac{6+2i}{2} = 3+i \text{ or } \lambda = \frac{6-2i}{2} = 3-i$$

$\therefore$ the general solution is

$$y = e^{3x}(C_1 \cos x + C_2 \sin x)$$

$\because y' = e^{3x}(C_2 \cos x - C_1 \sin x), y'(0) = 3$
$\therefore y'(0) = 1 \times (C_2 - 0) = C_2 = 3$
$\because y(0) = 2$
$\therefore y = 1 \times (C_1 - 0) = C_1 = 2$
$\therefore$ the general solution is

$$y = e^{3x}(2\cos x + 3\sin x)$$

34.

Proof. The auxiliary equation is $a\lambda^2 + b\lambda + c = 0$, with a, b, c positive.

1. If $\Delta = \sqrt{b^2 - 4ac} > 0$, the auxiliary equation has two real solution.
   Let the solutions be λ_1 and λ_2, then by the Vieta's Theorem,

$$\lambda_1\lambda_2 = \frac{c}{a} > 0, \lambda_1 + \lambda_2 = -\frac{b}{a} < 0 \implies \lambda_1 < 0, \lambda_2 < 0$$

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$\therefore \lim_{x\to\infty} y(x) = c_1 \lim_{x\to\infty} e^{\lambda_1 x} + c_2 \lim_{x\to\infty} e^{\lambda_2 x} = 0 + 0 = 0$$

2. If $\Delta = \sqrt{b^2 - 4ac} = 0$, the auxiliary equation has a real double root.
   Let the solution be $\lambda_0 = -\frac{b}{2a}$, then

$$y(x) = c_1 e^{\lambda_0 x} + c_2 x e^{\lambda_0 x}$$

$$\therefore \lim_{x\to\infty} y(x) = c_1 \lim_{x\to\infty} e^{\lambda_0 x} + c_2 \lim_{x\to\infty} x e^{\lambda_0 x} = 0 + 0 = 0$$

3. If $\Delta = \sqrt{b^2 - 4ac} < 0$, the auxiliary equation has two complex roots.
   Let $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i$, where $\alpha = -\frac{b}{2a} < 0$

$$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

$$\therefore \lim_{x\to\infty} y(x) = \lim_{x\to\infty} e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \leq (c_1 + c_2) \lim_{x\to\infty} e^{\alpha x} = 0$$

$\square$