Exercise 14.5

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6. $w = \ln\sqrt{x^2+y^2+z^2}, \quad x = \sin t, \quad y = \cos t, \quad z = \tan t$

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= \frac{2x}{2(x^2+y^2+z^2)}\cos t - \frac{2y}{2(x^2+y^2+z^2)}\sin t + \frac{2z}{2(x^2+y^2+z^2)}\frac{1}{\cos^2 t} \\
&= \frac{x\cos t - y\sin t + z\sec^2 t}{x^2+y^2+z^2} \\
&= \frac{x\cos t - y\sin t + z\sec^2 t}{\sec^2 t}
\end{aligned}$$

11. $z = e^r\cos\theta, \quad r = st, \quad \theta = \sqrt{s^2+t^2}$

$$\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} \\
&= e^r t\cos\theta - e^r\sin\theta\frac{s}{\sqrt{s^2+t^2}} \\
&= \frac{te^r\sqrt{s^2+t^2}\cos\theta - se^r\sin\theta}{\sqrt{s^2+t^2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} \\
&= se^r\cos\theta - e^r\sin\theta\frac{t}{\sqrt{s^2+t^2}} \\
&= \frac{se^r\sqrt{s^2+t^2}\cos\theta - te^r\sin\theta}{\sqrt{s^2+t^2}}
\end{aligned}$$

29. $\tan^{-1}(x^2y) = x + xy^2$

Let $F(x,y) = x + xy^2 - \tan^{-1}(x^2y)$, then $F(x,y) = 0$.

$$\frac{\partial F}{\partial x} = 1 + y^2 - \frac{2xy}{1+x^4y^2}$$

$$\frac{\partial F}{\partial y} = 2xy - \frac{x^2}{1+x^4y^2}$$

$$\therefore \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{(1+y^2)(1+x^4y^2)-2xy}{1+x^4y^2}}{\frac{2xy(1+x^4y^2)-x^2}{1+x^4y^2}} = \frac{1+x^4y^2+y^2+x^4y^4-2xy}{x^2-2xy-2x^5y^3}$$

34. $yz + x\ln y = z^2$

Let $F(x,y,z) = yz + x\ln y - z^2$, then $F(x,y,z) = 0$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\ln y}{y-2z} = \frac{\ln y}{2z-y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z+\frac{x}{y}}{y-2z} = \frac{yz+x}{y(2z-y)}$$

52. If $z = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find

(a)

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\\ &= f_x\cos\theta + f_y\sin\theta\end{aligned}$$

(b)

$$\begin{aligned}\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}\\ &= -rf_x\sin\theta + rf_y\cos\theta\end{aligned}$$

(c)

$$\begin{aligned}\frac{\partial^2 z}{\partial r\partial\theta} &= \frac{\partial}{\partial r}(-rf_x\sin\theta + rf_y\cos\theta)\\ &= -\sin\theta\frac{\partial}{\partial r}(rf_x) + \cos\theta\frac{\partial}{\partial r}(rf_y)\\ &= -\sin\theta(f_x + r\frac{\partial^2 z}{\partial r\partial x}) + \cos\theta(f_y + \frac{\partial^2 z}{\partial r\partial y})\end{aligned}$$

54. Suppose $z = f(x,y)$, where $x = g(s,t)$ and $y = h(s,t)$.

(a) Show that

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}(\frac{\partial x}{\partial t})^2 + 2\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}(\frac{\partial y}{\partial t})^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

Proof.

$$\frac{\partial z}{\partial t} = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2} &= (f_{xx}\frac{\partial x}{\partial t} + f_{xy}\frac{\partial y}{\partial t})\frac{\partial x}{\partial t} + f_x\frac{\partial^2 x}{\partial t^2} + (f_{yx}\frac{\partial x}{\partial t} + f_{yy}\frac{\partial y}{\partial t})\frac{\partial y}{\partial t} + f_y\frac{\partial^2 y}{\partial t^2} \\
&= f_{xx}(\frac{\partial x}{\partial t})^2 + f_{xy}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + f_x\frac{\partial^2 x}{\partial t^2} + f_{yx}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + f_{yy}(\frac{\partial y}{\partial t})^2 + f_y\frac{\partial^2 y}{\partial t^2} \\
&= \frac{\partial^2 z}{\partial x^2}(\frac{\partial x}{\partial t})^2 + 2\frac{\partial^2 z}{\partial x\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 z}{\partial y^2}(\frac{\partial y}{\partial t})^2 + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}
\end{aligned}$$

□

(b) Find a similar formula for $\frac{\partial^2 z}{\partial s\partial t}$.

$$\frac{\partial z}{\partial t} = f_x \frac{\partial x}{\partial t} + f_y \frac{\partial y}{\partial t}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial s\partial t} &= (f_{xx}\frac{\partial x}{\partial s} + f_{xy}\frac{\partial y}{\partial s})\frac{\partial x}{\partial t} + f_x\frac{\partial^2 x}{\partial s\partial t} + (f_{yx}\frac{\partial x}{\partial s} + f_{yy}\frac{\partial y}{\partial s})\frac{\partial y}{\partial t} + f_y\frac{\partial^2 y}{\partial s\partial t} \\
&= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x\partial y}(\frac{\partial x}{\partial t}\frac{\partial y}{\partial s} + \frac{\partial x}{\partial s}\frac{\partial y}{\partial t}) + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\partial t}
\end{aligned}$$

55. A function f is called homogeneous of degree n if it satisfies the equation $f(tx, ty) = t^n f(x, y)$ for all t, where n is a positive integer and f has continuous second-order partial derivatives.

(b) Show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

Proof. Taking differentiation to $f(tx, ty) = t^n f(x, y)$ with respect to t, we have

$$f_x(tx, ty)x + f_y(tx, ty)y = nt^{n-1}f(x, y)$$

When $t = 1$, we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

□

58.

Proof. Taking partial derivative to $F(x, y, z) = 0$ with respect to x, then

$$F_x + F_z \frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

Taking partial derivative to $F(x, y, z) = 0$ with respect to y, then

$$F_x \frac{\partial x}{\partial y} + F_y = 0 \implies \frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$$

Taking partial derivative to $F(x, y, z) = 0$ with respect to z, then

$$F_y \frac{\partial y}{\partial z} + F_z = 0 \implies \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$$

$$\therefore \frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1$$

□

59.

Proof. Taking differential to $F(x, y) = 0$ on both sides, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

$\therefore$taking differentiation to $\frac{dy}{dx}$, we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{\partial}{\partial x}(-\frac{F_x}{F_y}) + \frac{\partial}{\partial y}(-\frac{F_x}{F_y})\frac{dy}{dx} \\
&= -\frac{F_{xx}F_y - F_xF_{yx}}{F_y^2} - \frac{F_{xy}F_y - F_xF_{yy}}{F_y^2}(-\frac{F_x}{F_y}) \\
&= -\frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{F_y^3}
\end{aligned}$$

□