

Exercise 14.2

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6. $\lim_{(x,y) \rightarrow (1,-1)} e^{-xy} \cos(x+y)$

11. $\lim_{(x,y) \rightarrow (0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4}$

Let $y = kx$, then

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4} &= \lim_{(x,y) \rightarrow (0,0)} \frac{k^2 x^2 \sin^2 x}{(1 + k^4)x^4} \\ &= \frac{k^2}{1 + k^4} \times \lim_{(x,y) \rightarrow (0,0)} \left(\frac{\sin x}{x}\right)^2 \\ &= \frac{k^2}{1 + k^4} \end{aligned}$$

If $k = 0$, then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4} = \frac{0}{1} = 0$$

If $k = 1$, then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4} = \frac{1}{2}$$

\therefore the limit does not exist.

12. $\lim_{(x,y) \rightarrow (1,0)} \frac{xy-y}{(x-1)^2+y^2}$

Let $x - 1 = ky$, then

$$\lim_{(x,y) \rightarrow (1,0)} \frac{xy-y}{(x-1)^2+y^2} = \lim_{(x,y) \rightarrow (1,0)} \frac{ky^2}{(k^2+1)y^2} = \frac{k}{k^2+1}$$

$\therefore \lim_{(x,y) \rightarrow (1,0)} \frac{xy-y}{(x-1)^2+y^2}$ is not a constant

\therefore the limit does not exist.

13. $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{x}{\sqrt{x^2+y^2}} \times \lim_{y \rightarrow 0} y$$

$\because \lim_{(x,y) \rightarrow (0,0)} \frac{x}{\sqrt{x^2+y^2}}$ is bounded in $(0, 1)$, and $\lim_y y = 0$

$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

16. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2}$

$\forall \epsilon > 0, \exists \delta = \sqrt{\epsilon}$, such that if $0 < \sqrt{x^2+y^2} < \delta$, then

$$\frac{x^2 \sin^2 y}{x^2+2y^2} < \frac{x^2 y^2}{x^2+y^2} < \frac{(x^2+y^2)^2}{(x^2+y^2)} = \delta^2 = \epsilon$$

$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2}$ exists.

17. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{(x^2+y^2+1)-1} \\ &= \lim_{(x,y) \rightarrow (0,0)} (\sqrt{x^2+y^2+1}+1) \\ &= 1+1=2 \end{aligned}$$

32. $H(x, y) = \frac{e^x+e^y}{e^{xy}-1}$

When $x \neq 0$ and $y \neq 0$, $e^{xy} - 1 \neq 0$, $H(x, y)$ is continuous since it is a rational function.

When $x = 0$ or $y = 0$, let $y = x$, then we get

$$\lim_{(x,y) \rightarrow (0,0)} \frac{e^x+e^y}{e^{xy}-1} = \lim_{x \rightarrow 0} \frac{2e^x}{e^{x^2}-1} = \infty$$

$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{e^x+e^y}{e^{xy}-1}$ does not exist.

\therefore the function is continuous at $\{(x, y) | x \neq 0 \text{ and } y \neq 0\}$

33. $G(x, y) = \ln(x^2+y^2-4)$

The domain of G is $\{(x, y) | x^2+y^2 > 4\}$

\because the inner function $z = x^2+y^2-4$ is continuous in $D(G)$

$\therefore G(x, y)$ is continuous in $\{(x, y) | x^2+y^2 > 4\}$

$$\mathbf{37.} \quad f(x, y) = \begin{cases} \frac{x^2 y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

When $(x, y) \neq (0, 0)$, $f(x, y)$ is continuous since it is a rational function.

When $(x, y) = (0, 0)$, we can prove that $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0 \neq 1$.

$\forall \epsilon > 0, \exists \delta = \sqrt[3]{\epsilon}$, such that if $0 < \sqrt{x^2 + y^2} < \delta$,

$$|f(x, y) - 0| = \frac{x^2 y^3}{2x^2 + y^2} < \frac{x^2}{x^2 + y^2} y^3 < (\sqrt{x^2 + y^2})^3 = \delta^3 = \epsilon$$

$\therefore \lim_{(x, y) \rightarrow (0, 0)} f(x, y) \neq f(0, 0)$

$\therefore f$ is not continuous at $(0, 0)$.

$\therefore f(x, y)$ is continuous in $\{(x, y) | x \neq 0 \text{ and } y \neq 0\}$

$$\mathbf{38.} \quad f(x, y) = \begin{cases} \frac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

When $(x, y) \neq (0, 0)$, $f(x, y)$ is continuous since it is a rational function.

When $(x, y) = (0, 0)$, we can prove that $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist.

Let $y = x$, then

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \frac{x^2}{3x^2} = \frac{1}{3}$$

Let $y = -x$, then

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \frac{-x^2}{2x^2 - x^2} = -1$$

$\therefore \lim_{(x, y) \rightarrow (0, 0)}$ does not exist, thus not continuous at $(0, 0)$.

$\therefore f(x, y)$ is continuous in $\{(x, y) | x \neq 0 \text{ and } y \neq 0\}$

$$\mathbf{40.} \quad \lim_{(x, y) \rightarrow (0, 0)} (x^2 + y^2) \ln(x^2 + y^2)$$

Taking the substitution $r^2 = x^2 + y^2$, then the limit is equivalent to

$$\lim_{r \rightarrow 0^+} r^2 \ln r^2 = \lim_{r \rightarrow 0^+} \frac{2 \ln r}{r^{-2}} = \lim_{r \rightarrow 0^+} \frac{2}{r(-2r^{-3})} = \lim_{r \rightarrow 0^+} -r^2 = 0$$

$$\mathbf{41.} \quad \lim_{(x, y) \rightarrow (0, 0)} \frac{e^{-x^2 - y^2} - 1}{x^2 + y^2}$$

Taking the substitution $r^2 = x^2 + y^2$, then the limit is equivalent to

$$\lim_{r \rightarrow 0^+} \frac{e^{-r^2} - 1}{r^2} = \lim_{r \rightarrow 0^+} \frac{-2re^{-r^2}}{2r} = \lim_{r \rightarrow 0^+} -e^{-r^2} = -1$$