Exercise 15.5

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May 4, 2021

6.

The total mass is

$$
\begin{aligned}
m &= \int_0^2 \int_x^{6-2x} x^2 dydx \\
&= \int_0^2 (6x^2 - 3x^3) dydx \\
&= (2x^3 - \frac{3x^4}{4})\Big|_0^2 \\
&= 16 - 12 = 4
\end{aligned}
$$

$$
\begin{aligned}
\int_0^2 \int_x^{6-2x} x^2 ydydx &= \int_0^2 (\frac{x^2y^2}{2})\Big|_x^{6-2x} dx \\
&= \int_0^2 \frac{x^2}{2}(3x^2 - 24x + 36)dx \\
&= (\frac{3}{10}x^5 - 3x^4 + 6x^3)\Big|_0^2 \\
&= \frac{48}{5}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^2 \int_x^{6-2x} x^3 dydx &= \int_0^2 (x^3)(6-3x)dx \\
&= (-\frac{3}{5}x^5 + \frac{3}{2}x^4)\Big|_0^2 \\
&= \frac{24}{5}
\end{aligned}
$$

$$
\overline{y} = \frac{\iint_D \rho(x,y)ydA}{\iint_D \rho(x,y)dA} = \frac{\frac{48}{5}}{4} = \frac{12}{5}
$$

$$
\overline{x} = \frac{\iint_D \rho(x,y)xdA}{\iint_D \rho(x,y)dA} = \frac{\frac{24}{5}}{4} = \frac{6}{5}
$$

$\therefore$ the coordinate of center of mass is $(\frac{6}{5}, \frac{12}{5})$

13.

Let $\rho(x,y) = k\sqrt{x^2+y^2}$, then in polar coordinate,

$$\begin{aligned} m &= \int_0^\pi \int_1^2 krrdrd\theta \\ &= \int_0^\pi k(\frac{r^3}{3})\Big|_1^2 d\theta \\ &= \int_0^\pi \frac{7k}{3} d\theta \\ &= \frac{7k\pi}{3} \end{aligned}$$

$$\begin{aligned} \int_0^\pi \int_1^2 kr \times r\cos\theta \times rdrd\theta &= \int_0^\pi \int_1^2 kr^3 \cos\theta drd\theta \\ &= \int_0^\pi k\cos\theta(\frac{x^4}{4})\Big|_1^2 d\theta \\ &= \int_0^\pi \frac{15k\cos\theta}{4} d\theta \\ &= \frac{15k}{4}(\sin\theta)\Big|_0^\pi = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} \int_0^\pi \int_1^2 kr \times r\sin\theta \times rdrd\theta &= \int_0^\pi \frac{15k\sin\theta}{4} d\theta \\ &= \frac{15k}{4}(-\cos\theta)\Big|_0^\pi \\ &= \frac{15k}{2} \end{aligned}$$

$$\overline{y} = \frac{\iint_D \rho(x,y)ydA}{\iint_D \rho(x,y)dA} = \frac{\frac{15k}{2}}{\frac{7k\pi}{3}} = \frac{45}{14\pi}$$

$$\overline{x} = \frac{\iint_D \rho(x,y)xdA}{\iint_D \rho(x,y)dA} = 0$$

$\therefore$ the coordinate of center of mass is $(0, \frac{45}{14\pi})$

17.

$$\begin{aligned}
I_x &= \iint_D y^2 \rho(x,y) dA \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} ky^3 dydx \\
&= \int_{-1}^{1} k(\frac{y^4}{4})\Big|_0^{1-x^2} dydx \\
&= \int_{-1}^{1} \frac{k}{4}(1-x^2)^2 dx \\
&= \frac{k}{2}\int_{0}^{1} (1-x^2)^2 dx \\
&= \frac{k}{2}(\frac{x^5}{5} - \frac{2x^3}{3} + x)\Big|_0^1 \\
&= \frac{4k}{15}
\end{aligned}$$

$$\begin{aligned}
I_y &= \iint_D x^2 \rho(x,y) dA \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} kx^2 ydydx \\
&= \int_{-1}^{1} kx^2(\frac{y^2}{2})\Big|_0^{1-x^2} dx \\
&= \int_{-1}^{1} \frac{kx^2}{2}(x^4 - 2x^2 + 1)dx \\
&= \frac{k}{2}\int_{-1}^{1} (x^6 - 2x^4 + x^2)dx \\
&= k(\frac{x^7}{7} - \frac{2}{5}x^5 + \frac{x^3}{3})\Big|_0^1 \\
&= \frac{8k}{105}
\end{aligned}$$

$$
\begin{aligned}
I_0 &= \iint_D (x^2+y^2)\rho(x,y)dA \\
&= \int_{-1}^{1}\int_{0}^{1-x^2} ky(x^2+y^2)dydx \\
&= \int_{-1}^{1}\int_{0}^{1-x^2} \frac{k}{2}(x^2+y^2)d(x^2+y^2)dx \\
&= \int_{-1}^{1} \frac{k}{2}[\frac{(x^2+y^2)^2}{2}]\Big|_0^{1-x^2} dx \\
&= \int_{-1}^{1} \frac{k}{2}(\frac{x^4-x^2+1}{2}-\frac{x^4}{2})dx \\
&= \frac{k}{2}\int_{-1}^{1} \frac{-x^2+1}{2}dx \\
&= \frac{k}{2}(-\frac{x^3}{6}+\frac{x}{2})\Big|_{-1}^{1} \\
&= \frac{k}{3}
\end{aligned}
$$