Exercise 15.10

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June 2, 2021

18.

Substitude $\begin{cases} x = \sqrt{2}u - \sqrt{\frac{2}{3}}v \\ y = \sqrt{2}u + \sqrt{\frac{2}{3}}v \end{cases}$ into $x^2 - xy + y^2 = 2$, we have

$$2(2u^2 + \frac{2}{3}v^2) - (2u^2 - \frac{2}{3}v^2) = 2u^2 + 2v^2 = 2 \iff u^2 + v^2 = 1$$

Let $S = \{(u, v)|u^2 + v^2 \leq 1\}$, then

$$\begin{aligned}
\iint_R (x^2 - xy + y^2)dA &= \iint_S (\sqrt{2}u - \sqrt{\frac{2}{3}}v)^2 - (2u^2 - \frac{2}{3}v^2) + (\sqrt{2}u + \sqrt{\frac{2}{3}}v)^2 |\frac{\partial(x, y)}{\partial(u, v)}|dudv \\
&= \iint_S (2u^2 + 2v^2)\begin{vmatrix} \sqrt{2} & -\sqrt{\frac{2}{3}} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{vmatrix} dudv \\
&= \frac{8\sqrt{3}}{3} \int_0^{2\pi} \int_0^1 r^2 r dr d\theta \\
&= \frac{8\sqrt{3}}{3} \int_0^{2\pi} (\frac{r^4}{4})\Big|_0^1 d\theta \\
&= \frac{8\sqrt{3}}{3} \times \frac{\pi}{2} = \frac{4\sqrt{3}\pi}{3}
\end{aligned}$$

19.

Substitue $\begin{cases} x = \frac{u}{v} \\ y = v \end{cases}$ into boundary of R, we get

$y = x, y = 3x \implies v^2 = u, v^2 = 3u, xy = 1, xy = 3 \implies u = 1, u = 3$

Let $S = \{(u, v)|1 \leq u \leq 3, u \leq v^2 \leq 3u\}$

$$
\begin{aligned}
\iint_R xy\,dA &= \iint_S u\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv \\
&= \iint_S \frac{u}{v}dudv \\
&= \int_1^3 \int_{\sqrt{u}}^{3\sqrt{u}} \frac{u}{v}dvdu \\
&= \int_1^3 u(-\frac{1}{v^2})\Big|_{\sqrt{u}}^{3\sqrt{u}} du \\
&= \int_1^3 u(\frac{1}{u}-\frac{1}{9u})du \\
&= \frac{16}{9}
\end{aligned}
$$

24.

Let $\begin{cases} x = \frac{u+v}{2} \\ y = \frac{v-u}{2} \end{cases}$, then the boundary of R becomes $u = 0, u = 2, v = 0, v = 3$

Let $S = \{(u,v)|0 \le u \le 2, 0 \le v \le 3\}$

$$
\begin{aligned}
\iint_R (x+y)e^{x^2-y^2}dA &= \iint_S ve^{uv}\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}dudv \\
&= \frac{1}{2}\iint_S ve^{uv}dudv \\
&= \frac{1}{2}\int_0^3 \int_0^2 ve^{uv}dudv \\
&= \frac{1}{2}\int_0^3 (e^{uv})\Big|_{u=0}^{u=2} dv \\
&= \frac{1}{2}\int_0^3 (e^{2v}-1)dv \\
&= \frac{1}{2}(\frac{1}{2}e^{2v}-v)\Big|_0^3 \\
&= \frac{1}{2}(\frac{e^6-1}{2}-3) \\
&= \frac{e^6-7}{4}
\end{aligned}
$$

26.

Let $x = \frac{r\cos\theta}{3}, y = \frac{r\sin\theta}{2}$, then

$$
\begin{aligned}
\iint_R \sin(9x^2+4y^2)dA &= \int_0^{2\pi}\int_0^1 \sin(r^2)\frac{r}{6}drd\theta \\
&= \frac{1}{6}\int_0^{2\pi}\int_0^1 \frac{1}{2}\sin(r^2)dr^2d\theta \\
&= \frac{1}{12}\int_0^{2\pi}(-\cos r^2)\Big|_{r=0}^{r=1}d\theta \\
&= \frac{1}{12}\times 2\pi\times(1-\cos 1) = \frac{(1-\cos 1)\pi}{6}
\end{aligned}
$$

28.

Proof. Let $x=\frac{u+v}{2}, y=\frac{u-v}{2}$
$\because 0\le x+y\le 1 \quad \therefore 0\le u\le 1$

$$
\begin{aligned}
\iint_R f(x,y)dA &= \iint_R f(u)\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}dudv \\
&= \frac{1}{2}\iint_R f(u)dudv \\
&= \int_0^1\int_{-u}^{u}\frac{1}{2}f(u)dvdu \\
&= \int_0^1 \frac{1}{2}\times(2u)f(u)du \\
&= \int_0^1 f(u)du
\end{aligned}
$$

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