Exercise 16.7

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June 20, 2021

4.

$$\begin{aligned}\iint_S f(x,y,z)dS &= \iint_S g(\sqrt{x^2+y^2+z^2})dS \\ &= \iint_S g(2)dS \\ &= -5\iint_S dS = -5\times 16\pi = -80\pi\end{aligned}$$

6.

$$\overrightarrow{r_u} = \langle \cos v, \sin v, 1\rangle$$

$$\overrightarrow{r_v} = \langle -u\sin v, u\cos v, 0\rangle$$

$$\overrightarrow{r_u}\times\overrightarrow{r_v} = \langle -u\cos v, -u\sin v, u\rangle$$

$$|\overrightarrow{r_u}\times\overrightarrow{r_v}| = \sqrt{u^2+u^2} = \sqrt{2}u$$

$$\begin{aligned}\iint_S xyzdS &= \int_0^1\int_0^{\frac{\pi}{2}} \sqrt{2}u^4\sin v\cos vdvdu \\ &= \sqrt{2}\int_0^1\int_0^{\frac{\pi}{2}} u^4\sin vd\sin vdu \\ &= \sqrt{2}\int_0^1 u^4(\frac{\sin^2 v}{2})\Big|_0^{\frac{\pi}{2}}du \\ &= \frac{\sqrt{2}}{2}\int_0^1 u^4du \\ &= \frac{\sqrt{2}}{2}\times(\frac{u^5}{5})\Big|_0^1 \\ &= \frac{\sqrt{2}}{10}\end{aligned}$$

10.

$\because z = 4 - 2x - 2y$

$\therefore \frac{\partial z}{\partial x} = -2, \frac{\partial z}{\partial y} = -2$

$$\begin{aligned}
\iint_S xz dS &= \iint_{D_{xy}} x(4-2x-2y)\sqrt{(-2)^2+(-2)^2+1} dA \\
&= \int_0^2 \int_0^{2-x} 3x(4-2x-2y) dy dx \\
&= \int_0^2 (12xy - 6x^2y - 3xy^2)\Big|_0^{2-x} dx \\
&= \int_0^2 [12x(2-x) - 6x^2(2-x) - 3x(2-x)^2] dx \\
&= \int_0^2 (3x^3 - 12x^2 + 12x) dx \\
&= (\frac{3x^4}{4} - 4x^3 + 6x^2)\Big|_0^2 \\
&= 12 - 4 \times 8 + 6 \times 4 = 4
\end{aligned}$$

14.

$\because x = y + 2z^2$

$\therefore \frac{\partial x}{\partial y} = 1, \frac{\partial x}{\partial z} = 4z$

$$\begin{aligned}
\iint_S z dS &= \iint_{D_{yz}} z\sqrt{1^2 + (4z)^2 + 1} dA \\
&= \int_0^1 \int_0^1 z\sqrt{16z^2+2} dy dz \\
&= \int_0^1 z\sqrt{16z^2+2} dz \\
&= \frac{1}{32}\int_0^1 \sqrt{16z^2+2} d(16z^2+2) \\
&= \frac{1}{32} \times \frac{2}{3}(16z^2+2)^{\frac{3}{2}}\Big|_0^1 \\
&= \frac{1}{48}(162\sqrt{2} - 2\sqrt{2}) \\
&= \frac{10\sqrt{2}}{3}
\end{aligned}$$

15.

$$\begin{aligned}\iint_S y dS &= \iint_D (x^2+z^2)\sqrt{4x^2+4z^2+1}dA\\ &= \int_0^{2\pi}\int_0^2 r^2\sqrt{4r^2+1}rdrd\theta\end{aligned}$$

Let $t = 4r^2+1$, $r^2 = \frac{t-1}{4}$ then $dt = 8rdr$, so

$$\begin{aligned}\int_0^{2\pi}\int_0^2 r^2\sqrt{4r^2+1}rdrd\theta &= \int_0^{2\pi}\int_0^2 \frac{t-1}{4}\sqrt{t}\frac{1}{8}dtd\theta\\ &= \frac{1}{32}\int_0^{2\pi}\int_0^2 (t^{\frac{3}{2}}-t^{\frac{1}{2}})dtd\theta\\ &= \frac{1}{32}\int_0^{2\pi}(\frac{2}{5}t^{\frac{5}{2}}-\frac{2}{3}t^{\frac{3}{2}})\Big|_0^2 d\theta\\ &= \frac{\pi}{8}(\frac{4\sqrt{2}}{5}-\frac{2\sqrt{2}}{3})\end{aligned}$$

18.

Let the surface of the cylinder on yOz plane, on the side face, on $x+y=5$ be S_1, S_2, S_3, respectively.

$$\iint_{S_1} xzdS = 0$$

Let $y = 3\cos\theta, z = 3\sin\theta, x = x$, then

$$\overrightarrow{r_\theta} = \langle 0, -3\sin\theta, 3\cos\theta\rangle, \overrightarrow{r_x} = \langle 1,0,0\rangle$$

$$\begin{aligned}\iint_{S_2} xzdS &= \iint_D xz|\overrightarrow{r_\theta}\times\overrightarrow{r_x}|dA\\ &= 3\iint_D xzdA\\ &= 3\int_0^{2\pi}\int_0^{5-3\cos\theta} 3x\sin\theta dxd\theta\\ &= 9\int_0^{2\pi}\sin\theta(\frac{x^2}{2})\Big|_0^{5-3\cos\theta} d\theta\\ &= \frac{9}{2}\int_0^{2\pi}(25\sin\theta+9\sin\theta\cos^2\theta-30\sin\theta\cos\theta)d\theta\\ &= 0\end{aligned}$$

$$\iint_{S_3} xzdS = \iint_D (5-y)z\sqrt{1+(\frac{\partial x}{\partial y})^2+(\frac{\partial x}{\partial z})^2}dA$$
$$= \int_0^{2\pi}\int_0^3 (5-r\cos\theta)(r\sin\theta)\sqrt{1+1+0}rdrd\theta$$
$$= \sqrt{2}\int_0^{2\pi}\sin\theta\int_0^3(5r^2-r^3\cos\theta)drd\theta$$
$$= \sqrt{2}\int_0^{2\pi}(\frac{5}{3}r^3-\frac{r^4\cos\theta}{4})\Big|_0^3 d\theta$$
$$= \sqrt{2}\int_0^{2\pi}(45-\frac{81}{4}\cos\theta)d\theta$$
$$= \sqrt{2}(45\theta-\frac{81}{4}\sin\theta)\Big|_0^{2\pi}$$
$$= 90\sqrt{2}\pi$$
$$\therefore \iint_S xzdS = \iint_{S_1} xzdS + \iint_{S_2} xzdS + \iint_{S_3} xzdS = 90\sqrt{2}\pi$$

24.

$$F(x,y,z) = \langle -x, -y, z^3\rangle$$

$$z = \sqrt{x^2+y^2}, (x,y) \in [1,9]$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\overrightarrow{n} = \frac{\frac{\partial z}{\partial x}\overrightarrow{i}+\frac{\partial z}{\partial y}\overrightarrow{j}-\overrightarrow{k}}{\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}} = \frac{\frac{x}{\sqrt{x^2+y^2}}\overrightarrow{i}+\frac{y}{\sqrt{x^2+y^2}}\overrightarrow{j}-\overrightarrow{k}}{\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}}$$

$$
\begin{aligned}
\iint_S \overrightarrow{F}\cdot\overrightarrow{n}\,dS &= \iint_D \frac{-\frac{x^2}{\sqrt{x^2+y^2}}-\frac{y^2}{\sqrt{x^2+y^2}}-z^3}{\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}}\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}dA \\
&= \iint_D -\sqrt{x^2+y^2}-(x^2+y^2)dA \\
&= \int_0^{2\pi}\int_1^3(-r-r^2)rdrd\theta \\
&= \int_0^{2\pi}(-\frac{r^3}{3}-\frac{r^4}{4})\Big|_1^3 d\theta \\
&= \int_0^{2\pi}(-9-\frac{81}{4}+\frac{1}{3}+\frac{1}{4})ad\theta \\
&= 2\pi\times(-9-20+\frac{1}{3}) \\
&= -\frac{172\pi}{3}
\end{aligned}
$$

25.

$$\overrightarrow{F}(x,y,z)=\langle x,-z,y\rangle$$

Let $x=\sin\varphi\cos\theta, y=\sin\varphi\sin\theta, z=\cos\varphi$

$$\overrightarrow{r_\varphi}=\langle\cos\varphi\cos\theta,\cos\varphi\sin\theta,-\sin\varphi\rangle,\overrightarrow{r_\theta}=\langle-\sin\varphi\sin\theta,\sin\varphi\cos\theta,0\rangle$$

$$\overrightarrow{n}=\frac{\overrightarrow{r_\varphi}\times\overrightarrow{r_\theta}}{|\overrightarrow{r_\varphi}\times\overrightarrow{r_\theta}|}=\frac{\langle\sin^2\varphi\cos\theta,\sin^2\varphi\sin\theta,\sin\varphi\cos\varphi\rangle}{|\overrightarrow{r_\varphi}\times\overrightarrow{r_\theta}|}$$

$$
\begin{aligned}
\iint_S \overrightarrow{F}\cdot\overrightarrow{n}\,dS &= \iint_D(\sin^3\varphi\cos^2\theta+\sin^3\varphi\sin^2\theta+\sin^2\varphi\cos^2\varphi)dA \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(\sin^3\varphi+\sin^2\varphi\cos^2\varphi)d\varphi d\theta \\
&= \frac{\pi}{2}\int_0^{2\pi}(\cos^2\varphi-1)d\cos\varphi+\frac{\pi}{8}\int_0^{\frac{\pi}{2}}\sin^2(2x)dx \\
&= \frac{\pi}{2}[\frac{\cos^3 x}{3}-\varphi]\Big|_0^{2\pi}+\frac{\pi}{8}\int_0^{\frac{\pi}{2}}\frac{1-\cos 4x}{2}dx \\
&= -\pi^2+\frac{\pi}{16}(\frac{\pi}{2}-\frac{\sin 4x}{4}\Big|_0^{\frac{\pi}{2}}) \\
&= -\pi^2+\frac{\pi^2}{32}=-\frac{31}{32}\pi^2
\end{aligned}
$$

30.

$$\overrightarrow{F} = \langle x, y, 5 \rangle$$

Let the surface of the cylinder in plane $y = 0$, in side face, in $x + y = 2$ be S_1, S_2, S_3, respectively.

For surface S_1, $\overrightarrow{n} = \langle 0, 1, 0 \rangle$

$$\iint_{S_1} \overrightarrow{F} \cdot \overrightarrow{n}\, dS = \iint_{S_1} y dS = 0$$

For surface S_2, let $x = \cos\theta, z = \sin\theta, y = y$, then $\overrightarrow{n} = \langle \cos\theta, 0, \sin\theta \rangle$

$$\begin{aligned}
\iint_{S_2} \overrightarrow{F} \cdot \overrightarrow{n}\, dS &= \int_0^{2\pi} \int_0^{2-\cos\theta} (\cos^2\theta + 5\sin\theta) dy d\theta \\
&= \int_0^{2\pi} (5\sin\theta + \cos^2\theta)(2 - \cos\theta) d\theta \\
&= \int_0^{2\pi} 10\sin\theta - 5\sin\theta\cos\theta + 2\cos^2\theta - \cos^3\theta d\theta \\
&= 2\pi
\end{aligned}$$

For surface S_3, let $x = u\cos\theta, y = 2 - u\cos\theta, z = u\sin\theta$

$$\overrightarrow{r_u} = \langle \cos\theta, -\cos\theta, \sin\theta \rangle, \overrightarrow{r_\theta} = \langle -u\sin\theta, u\sin\theta, u\cos\theta \rangle$$

$$n = -\frac{\overrightarrow{r_u} \times \overrightarrow{r_\theta}}{|\overrightarrow{r_u} \times \overrightarrow{r_\theta}|} = \frac{\langle u, u, 0 \rangle}{|\overrightarrow{r_u} \times \overrightarrow{r_\theta}|}$$

$$\begin{aligned}
\iint_{S_3} \overrightarrow{F} \cdot \overrightarrow{n}\, dS &= \iint_D (u^2\cos\theta + 2u - u^2\cos\theta) dA \\
&= 2\iint_D u dA \\
&= \int_0^{2\pi} \int_0^1 2u du d\theta \\
&= \int_0^{2\pi} (u^2)\Big|_0^1 d\theta \\
&= 2\pi
\end{aligned}$$

$\therefore$ the general solution is

$$y = 0 + 2\pi + 2\pi = 4\pi$$

31.

Let $x = v, y = \cos u, z = \sin u, u \in [0, \pi], v \in [0, 2]$

$$\overrightarrow{F} = \langle v^2, \cos^2 u, \sin^2 u \rangle$$

$$\overrightarrow{r_u} = \langle 0, -\sin u, \cos u \rangle, \overrightarrow{r_v} = \langle 1, 0, 0 \rangle$$

$$\overrightarrow{n} = \frac{\overrightarrow{r_u} \times \overrightarrow{r_v}}{|\overrightarrow{r_u} \times \overrightarrow{r_v}|} = \frac{\langle 0, \cos u, \sin u \rangle}{|\overrightarrow{r_u} \times \overrightarrow{r_v}|}$$

$$\begin{aligned}\iint_S \overrightarrow{F} \cdot \overrightarrow{n} dS &= \iint_D \cos^3 u + \sin^2 u \cos u dA \\ &= \int_0^{\pi} \int_0^2 \cos u dv du \\ &= \int_0^{\pi} 2 \cos u du \\ &= 0\end{aligned}$$

37.

$$\overrightarrow{n} = \frac{\langle \frac{\partial h}{\partial x}, -1, \frac{\partial h}{\partial z} \rangle}{|\overrightarrow{\langle \frac{\partial h}{\partial x}, -1, \frac{\partial h}{\partial z} \rangle}|}$$

Let $\overrightarrow{F} = \langle P, Q, R \rangle$, then

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iint_D (P\frac{\partial h}{\partial x} - Q + R\frac{\partial h}{\partial z})dA$$

38.

$$\overrightarrow{n} = \frac{\langle 1, -\frac{\partial k}{\partial y}, -\frac{\partial k}{\partial z} \rangle}{|\langle 1, -\frac{\partial k}{\partial y}, -\frac{\partial k}{\partial z} \rangle|}$$

Let $\overrightarrow{F} = \langle P, Q, R \rangle$, then

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iint_D (P - Q\frac{\partial k}{\partial y} - R\frac{\partial k}{\partial z})dA$$