Exercise 15.4

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20.

Let region $D = \{(x, y)|18 - 2x^2 - 2y^2 \geq 0\} = \{(x, y)|x^2 + y^2 \leq 9\}$, then

$$\begin{aligned}
\iint_D (18 - 2x^2 - 2y^2)dA &= \int_0^{2\pi} \int_0^3 (18 - 2r^2)rdrd\theta \\
&= \int_0^{2\pi} [-\int_0^3 (9 - r^2)d(9 - r^2)]d\theta \\
&= \int_0^{2\pi} -\frac{(9 - r^2)^2}{2}\bigg|_0^3 d\theta \\
&= \int_0^{2\pi} (-0 + \frac{81}{2})d\theta \\
&= 81\pi
\end{aligned}$$

22.

Let the volume of the solid inside the hyperpoloid $x^2 + y^2 + z^2 = 16$ and inside the cylinder $x^2 + y^2 = 4$ be V_0, then

$$\begin{aligned}
V_0 &= \int_0^{2\pi} \int_0^2 \sqrt{16 - r^2}rdrd\theta \\
&= \int_0^{2\pi} \int_0^2 (-\frac{1}{2})\sqrt{16 - r^2}d(16 - r^2)drd\theta \\
&= \int_0^{2\pi} (-\frac{1}{2})\frac{2}{3}(16 - r^2)^{\frac{3}{2}}\bigg|_0^2 d\theta \\
&= \int_0^{2\pi} (-\frac{1}{3})(24\sqrt{3} - 64)d\theta \\
&= \frac{128\pi}{3} - 16\sqrt{3}\pi
\end{aligned}$$

$\therefore V = \frac{4}{3}\pi \times 4^3 - 2V_0 = 32\sqrt{3}\pi$

25.

The boundary D is calculated by

$$x^2+y^2+(\sqrt{x^2+y^2})^2=2(x^2+y^2)=1 \implies x^2+y^2=\frac{1}{2}$$

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\frac{\sqrt{2}}{2}}(\sqrt{1-r^2}-r)rdrd\theta \\
&= \int_0^{2\pi}[\int_0^{\frac{\sqrt{2}}{2}}\sqrt{1-r^2}rdr-\int_0^{\frac{\sqrt{2}}{2}}r^2dr]d\theta \\
&= \int_0^{2\pi}[-\frac{1}{2}\int_0^{\frac{\sqrt{2}}{2}}\sqrt{1-r^2}d(1-r^2)-(\frac{r^3}{3})\Big|_0^{\frac{\sqrt{2}}{2}}]d\theta \\
&= \int_0^{2\pi}[-\frac{1}{2}\times\frac{2}{3}(1-r^2)^{\frac{3}{2}}\Big|_0^{\frac{\sqrt{2}}{2}}-\frac{\sqrt{3}}{18}]d\theta \\
&= \int_0^{2\pi}[-\frac{1}{3}(\frac{\sqrt{2}}{4}-1)-\frac{\sqrt{3}}{18}]d\theta \\
&= \int_0^{2\pi}(\frac{1}{3}-\frac{\sqrt{2}}{12}-\frac{\sqrt{3}}{18})d\theta \\
&= \frac{2\pi}{3}-\frac{\sqrt{2}\pi}{6}-\frac{\sqrt{3}\pi}{9}
\end{aligned}$$

31.

$$\begin{aligned}
\int_0^1\int_y^{\sqrt{2-y^2}}(x+y)dxdy &= \int_0^{\frac{\pi}{4}}\int_0^1 r^2(\sin\theta+\cos\theta)drd\theta \\
&= \int_0^{\frac{\pi}{4}}(\sin\theta+\cos\theta)(\frac{r^3}{3})\Big|_0^1 d\theta \\
&= \frac{1}{3}(sin\theta-\cos\theta)\Big|_0^{\frac{\pi}{4}} \\
&= \frac{(0)-(0-1)}{3}=\frac{1}{3}
\end{aligned}$$

32.

$$
\begin{aligned}
\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}dydx &= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 dr d\theta \\
&= \int_0^{\frac{\pi}{2}} (\frac{r^3}{3})\Big|_0^{2\cos\theta} d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{8\cos^3\theta}{3} d\theta \\
&= \frac{8}{3}\int_0^{\frac{\pi}{2}} (1-\sin^2\theta) d\sin\theta \\
&= \frac{8}{3}(\sin\theta - \frac{\sin^3\theta}{3})\Big|_0^{\frac{\pi}{2}} \\
&= \frac{8}{3}\times\frac{2}{3} = \frac{16}{9}
\end{aligned}
$$

34.

$$
\begin{aligned}
\iint_D xy\sqrt{1+x^2+y^2}dA &= \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \sin\theta\cos\theta\sqrt{1+r^2} r dr d\theta \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} r^2 \sin\theta\cos\theta\sqrt{1+r^2} r d\theta dr \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} r^3\sqrt{1+r^2}\sin\theta d\sin\theta dr \\
&= \int_0^1 r^3\sqrt{1+r^2}(\frac{\sin^2\theta}{2})\Big|_0^{\frac{\pi}{2}} dr \\
&= \int_0^1 \frac{r^3\sqrt{1+r^2}}{2} dr \\
&\approx 0.1609
\end{aligned}
$$

39.

$$\int_{\frac{\sqrt{2}}{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xydydx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xydydx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xydydx$$
$$= \int_0^{\frac{\pi}{4}}\int_1^2 r\cos\theta r\sin\theta rdrd\theta$$
$$= \int_1^2\int_0^{\frac{\pi}{4}} r^3\sin\theta d\sin\theta dr$$
$$= \int_1^2 r^3(\frac{\sin^2\theta}{2})\Big|_0^{\frac{\pi}{4}} dr$$
$$= \int_1^2 \frac{r^3}{4}dr$$
$$= (\frac{r^4}{16})\Big|_1^2 = \frac{15}{16}$$

40.

(a) *Proof.*

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dA &= \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}dA \\ &= \lim_{a\to\infty}\int_0^{2\pi}\int_0^a e^{-r^2}rdrd\theta \\ &= \lim_{a\to\infty}\int_0^{2\pi}(-\frac{1}{2})e^{-r^2}d(-r^2)d\theta \\ &= \lim_{a\to\infty}\int_0^{2\pi}(-\frac{1}{2})(e^{-r^2})\Big|_0^a d\theta \\ &= \lim_{a\to\infty}\int_0^{2\pi}\frac{1-e^{-a^2}}{2}d\theta \\ &= \lim_{a\to\infty}(1-e^{-a^2})\pi = \pi\end{aligned}$$

□

(b) *Proof.*

TODO

□

(c) *Proof.* Let $I = \int_{-\infty}^{\infty} e^{-x^2}dx = \int_{-\infty}^{\infty} e^{-y^2}dy$, then$I^2 = \pi$

$\because e^{-x^2} > 0 \therefore I > 0$

$$\therefore I = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

$\square$

(d) *Proof.* Let $t = \sqrt{2}x$, then $x = \frac{t}{\sqrt{2}}$, then

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} d\frac{t}{\sqrt{2}} = \sqrt{\pi}$$

which is equivalent to

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

$\square$

41.

(a)

$$\begin{aligned}
\int_0^{\infty} x^2 e^{-x^2} dx &= \int_0^{\infty} (-\frac{x}{2}) de^{-x^2} \\
&= (-\frac{x}{2}e^{-x^2})\Big|_0^{\infty} - \int_0^{\infty} e^{-x^2} d(-\frac{x}{2}) \\
&= \frac{1}{2}\int_0^{\infty} e^{-x^2} dx \\
&= \frac{1}{4}\int_{-\infty}^{\infty} e^{-x^2} dx \\
&= \frac{\sqrt{\pi}}{4}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^{\infty} \sqrt{x}e^{-x} dx &= \int_0^{\infty} (-\sqrt{x}) de^{-x} \\
&= (-\sqrt{x}e^{-x})\Big|_0^{\infty} - \int_0^{\infty} e^{-x} d(-\sqrt{x}) \\
&= \int_0^{\infty} e^{-x} d\sqrt{x}
\end{aligned}$$

Let $x = t^2$, then

$$\int_0^{\infty} \sqrt{x}e^{-x} dx = \int_0^{\infty} e^{-t^2} dt = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$