Exercise 14.4

Wang Yue from CS Elite Class

April 11, 2021

21. Find the linear approximation of $f(x,y,z)=\sqrt{x^2+y^2+z^2}$ at $(3,2,6)$. Approximate $\sqrt{(3.02)^2+(1.97)^2+(5.99)^2}$.

$$f_x(x,y,z)=\frac{x}{\sqrt{x^2+y^2+z^2}}$$
$$f_y(x,y,z)=\frac{y}{\sqrt{x^2+y^2+z^2}}$$
$$f_z(x,y,z)=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

Therefore, the linear approximation of f is

$$\begin{aligned}&f(3,2,6)+f_x(3,2,6)(x-3)+f_y(3,2,6)(y-2)+f_z(3,2,6)(z-6)\\&=7+\frac{3(x-3)+2(y-2)+6(z-6)}{7}\end{aligned}$$

When $x=3.02, y=1.97, z=5.99,\ f(x,y,z)\approx 7+\frac{0.06-0.06-0.06}{7}\approx 6.99$

25. $z=e^{-2x}\cos 2\pi t$

$$\frac{\partial z}{\partial x}=-2e^{-2x}\cos 2\pi t,\quad \frac{\partial z}{\partial t}=-2\pi e^{-2x}\sin 2\pi t$$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial r}dr=-2e^{-2x}\cos 2\pi t dx-2\pi e^{-2x}\sin 2\pi t dt$$

28. $T=\frac{v}{1+uvw}$

$$\frac{\partial T}{\partial u}=\frac{-v^2w}{(1+uvw)^2},\quad \frac{\partial T}{\partial w}=\frac{-v^2u}{(1+uvw)^2}$$

$$\frac{\partial T}{\partial v}=\frac{1(1+uvw)-v(uw)}{(1+uvw)^2}=\frac{1}{(1+uvw)^2}$$

$$dT=\frac{\partial T}{\partial u}du+\frac{\partial T}{\partial v}dv+\frac{\partial T}{\partial w}dw=\frac{-v^2wdu+dv-v^2udw}{(1+uvw)^2}$$

30. $L = xze^{-y^2-z^2}$

$$\frac{\partial L}{\partial x} = ze^{-y^2-z^2}, \quad \frac{\partial L}{\partial y} = -2xyze^{-y^2-z^2}, \quad \frac{\partial L}{\partial z} = x(1-2z^2)e^{-y^2-z^2}$$

$$dL = \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial y}dy + \frac{\partial L}{\partial z}dz = \frac{zdx - 2xyzdy + x(1-2z^2)dz}{e^{y^2+z^2}}$$

42.

First, find the tangent lines of $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$.

$$\frac{d\overrightarrow{r_1}}{dt} = \langle 3, -2t, 2t-4 \rangle$$

$\because \overrightarrow{r_1}(0) = (2,1,3), \quad \therefore \overrightarrow{r_1'}(0) = (3,0,-4)$

$$\frac{d\overrightarrow{r_2}}{dt} = \langle 2u, 6u^2, 2 \rangle$$

$\because \overrightarrow{r_2}(1) = (2,1,3), \quad \therefore \overrightarrow{r_2'}(1) = (2,6,2)$

$$\therefore \overrightarrow{n} = \overrightarrow{r_1}(0) \times \overrightarrow{r_2}(1) = (-16, 2, 10)$$

$\therefore$ the equation of the tangent plane at P is $-16(x-2)+2(y-1)+10(z-3) = 0$, or

$$8x - y + 5z = 30$$

45.

$\because$ by the definition of linear approximation,

$$f(a+\Delta x, b+\Delta y) = f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y + o(\sqrt{(\Delta x)^2 + (\Delta y)^2})$$

$$\therefore \lim_{(\Delta x, \Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b)$$

$\because f$ is differentiable,
$\therefore f$ is continuous at (a,b).

46.

(a)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} f(0+\Delta x, 0+\Delta y) = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x\Delta y}{(\Delta x)^2+(\Delta y)^2}$$

Let $\Delta y = k\Delta x$, then

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{k(\Delta x)^2}{(1+k^2)(\Delta x)^2} = \frac{k}{1+k^2} \neq 0 \text{ when } k \neq 0$$

$\therefore \lim_{(\Delta x,\Delta y)\to(0,0)} f(\Delta x, \Delta y)$ does not exist.

$\therefore f$ is not continuous at $(0,0)$.

$\therefore$ by the result of T45, f is not differentiable.

(b) $\because f$ is not differentiable at $(0,0)$

$\therefore f_x$ and f_y are not continuous at $(0,0)$