Exercise 11.8

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3. $\sum_{n=1}^{\infty}(-1)^n nx^n$

Let $c_n = (-1)^n n \neq 0$, and

$$\rho = \lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

Therefore, the radius of $\sum_{n=1}^{\infty}(-1)^n nx^n$ is $R = \frac{1}{\rho} = 1$
When $x = -1$,

$$\sum_{n=1}^{\infty}(-1)^n nx^n = \sum_{n=1}^{\infty}(-1)^n n(-1)^n = \sum_{n=1}^{\infty} n$$

which is divergent.

When $x = 1$, the series

$$\sum_{n=1}^{\infty}(-1)^n nx^n = \sum_{n=1}^{\infty}(-1)^n n$$

which is also divergent.

$\therefore$ the interval of convergence of the series is $(-1, 1)$.

8. $\sum_{n=1}^{\infty} n^n x^n$

Let $c_n = n^n$, then

$$\rho = \lim_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} n = \infty$$

Therefore, the radius of $\sum_{n=1}^{\infty} n^n x^n$ is $R = 0$.
Also, the interval of convergence of the series is $\{0\}$.

9. $\sum_{n=1}^{\infty}(-1)^n \frac{n^2 x^n}{2^n}$

Let $c_n = (-1)^n \frac{n^2}{2^n}$, then

$$\rho = \lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \lim_{n\to\infty} \frac{(n+1)}{2n} = \frac{1}{2}$$

Therefore, the radius of $\sum_{n=1}^{\infty}(-1)^n \frac{n^2x^n}{2^n}$ is $R = \frac{1}{\rho} = 2$
When $x = 2$,

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2x^n}{2^n} = \sum_{n=1}^{\infty}(-1)^n n^2$$

$\because \sum_{n=1}^{\infty} n^2$ is not decreasing and $\lim_{n\to\infty} n^2 = \infty$
$\therefore \sum_{n=1}^{\infty}(-1)^n n^2$ is obviously divergent
When $x = -2$,

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2x^n}{2^n} = \sum_{n=1}^{\infty} n^2$$

which is obviously divergent
$\therefore$ the interval of convergence of the series is $(-2, 2)$.

13. $\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{4^n \ln n}$

Let $c_n = (-1)^n \frac{1}{4^n \ln n}$, then

$$\lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| = \lim_{n\to\infty} \frac{\ln n}{4\ln(n+1)} = \frac{1}{4}$$

Therefore, the radius of $\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{4^n \ln n}$ is $R = \frac{1}{\rho} = 4$
When $x = 4$,

$$\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}$$

Denote $\frac{1}{\ln n}$ to be b_n.
$\because b_n$ is decreasing, $b_{n+1} < b_n$, and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0$$

$\therefore \sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}$ is convergent
When $x = -4$,

$$\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

$\because \ln n < n \quad \therefore \frac{1}{\ln n} > \frac{1}{n}$

$$\therefore \sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$$

$\because \sum_{n=2}^{\infty} \frac{1}{n}$ is divergent
$\therefore \sum_{n=2}^{\infty} \frac{1}{\ln n}$ is also divergent
$\therefore$ the interval of convergence of the series is $(-4, 4]$

20. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

First we convert the series to standard form:

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(x-\frac{1}{2})^n}{10^n\sqrt{n}}$$

Let $c_n = \frac{1}{10^n\sqrt{n}}$, then

$$\rho = \lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| = \lim_{n\to\infty} \frac{\sqrt{n}}{10\sqrt{n+1}} = \frac{1}{10}$$

Therefore, the radius of $\sum_{n=1}^{\infty} \frac{(x-\frac{1}{2})^n}{10^n\sqrt{n}}$ is $R = \frac{1}{\rho} = 10$
When $x = \frac{1}{2} + 10 = \frac{21}{2}$,

$$\sum_{n=1}^{\infty} \frac{(x-\frac{1}{2})^n}{10^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$\because \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series, whose $p = \frac{1}{2} < 1$
$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent
When $x = \frac{1}{2} - 10 = -\frac{19}{2}$,

$$\sum_{n=1}^{\infty} \frac{(x-\frac{1}{2})^n}{10^n\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$$

$\because \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$
$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ is convergent
$\therefore$ the interval of convergence of the series is $[-\frac{19}{2}, \frac{21}{2})$

30.

$\because \sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$
$\therefore R \in [4, 6)$, where R is the radius of $\sum_{n=0}^{\infty} c_n x^n$

$$\therefore \lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| = \frac{1}{R} \in [\frac{1}{6}, \frac{1}{4})$$

(a) convergent

$\because \lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| < \frac{1}{4} < 1$

$\therefore \sum_{n=0}^{\infty} c_n$ is absolutely convergent, and therefore convergent

(b) divergent

$$\because \lim_{n\to\infty} |\frac{8^{n+1}c_{n+1}}{8^n c_n}| = \lim_{n\to\infty} |\frac{8c_{n+1}}{c_n}| \geq 8 \times \frac{1}{6} > 1$$

$\therefore \sum_{n=0}^{\infty} c_n 8^n$ is divergent

(c) convergent

$$\because \lim_{n\to\infty} |\frac{c_{n+1}(-3)^{n+1}}{c_n(-3)^n}| = \lim_{n\to\infty} 3|\frac{c_{n+1}}{c_n}| < 3 \times \frac{1}{4} < 1$$

$\therefore \sum_{n=0}^{\infty} c_n(-3)^n$ is convergent

(d) divergent

$$\because \lim_{n\to\infty} |\frac{(-1)^{n+1}c_{n+1}9^{n+1}}{(-1)^n c_n 9^n}| = \lim_{n\to\infty} 9|\frac{c_{n+1}}{c_n}| \geq 9 \times \frac{1}{6} > 1$$

$\therefore \sum_{n=0}^{\infty}(-1)^n c_n 9^n$ is divergent