Exercise 14.x

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1. (2) $x = a\cos\alpha\cos t, y = a\sin\alpha\cos t, z = a\sin t, t = t_0$

$$\because x^2 + y^2 + z^2 = a^2(\cos^2\alpha + \sin^2\alpha)\cos^2 t + a^2\sin^2 t = a^2$$

$\therefore$ the curve is a the shell of the sphere, with radius $|a|$.
Let $\overrightarrow{r}(t) = \langle a\cos\alpha\cos t, a\sin\alpha\cos t, a\sin t\rangle$, then

$$\overrightarrow{r}'(t) = \langle -a\cos\alpha\sin t, -a\sin\alpha\sin t, a\cos t\rangle$$

$$\therefore \overrightarrow{r}'(t_0) = \langle -a\cos\alpha\sin t_0, -a\sin\alpha\sin t_0, a\cos t_0\rangle$$

1. (4) $z = f(x, y), \frac{x-x_0}{\cos\alpha} = \frac{y-y_0}{\sin\alpha}, f$ is differentiable, $M_0(x_0, y_0, z_0)$

Let $z = \varphi(t), \varphi(t) = f(x_0 + t\cos\alpha, y_0 + t\sin\alpha)$

$$\overrightarrow{r}(t) = \langle x_0 + t\cos\alpha, y_0 + t\sin\alpha, f(x_0 + t\cos\alpha, y_0 + t\sin\alpha)\rangle$$

Taking differentiation to $\varphi(t)$ with respect to t, we have

$$\overrightarrow{r}'(t) = \langle \cos\alpha, \sin\alpha, f_x\cos\alpha + f_y\sin\alpha\rangle$$

4.

Let $F(x, y, z) = x^2 + y^2 + z^2 - 3x, G(x, y, z) = 2x - 3y + 5z - 4$, then

$$\nabla F = \langle 2x - 3, 2y, 2z\rangle, \quad \nabla G = \langle 2, -3, 5\rangle$$

When $x = 1, y = 1, z = 1, \nabla F = \langle -1, 2, 2\rangle$
$\therefore$ the tangent vector at $(1, 1, 1)$ is

$$\nabla F \times \nabla G = \langle 16, -9, -1\rangle$$

$\therefore$ the equation of the tangent line is $\frac{x-1}{16} = \frac{y-1}{-9} = \frac{z-1}{-1}$.
$\therefore$ the equation of the normal plane is $16(x - 1) - 9(y - 1) - (z - 1) = 0$

7. (2) $z = e^y + x + x^2 + 6, \quad P_0(1, 0, 9)$

Let $f(x, y) = e^y + x + x^2 + 6, F(x, y, z) = f(x, y) - z$, then

$$f_x(x, y) = 2x + 1, \quad f_y(x, y) = e^y$$

$$\therefore \nabla F(x, y, z) = \langle 2x + 1, e^y, -1\rangle$$

When $x = 1, y = 0, z = 9, \nabla F = \langle 3, 1, -1\rangle$
$\therefore$ the equation of the tangent plane at P_0 is $3(x - 1) + (y - 0) - (z - 9) = 0$
$\therefore$ the equation of the normal line at P_0 is

$$\frac{x - 1}{3} = \frac{y}{1} = \frac{z - 9}{-1}$$

7. (7) $z = y + \ln \frac{x}{z}, \quad P_0(1, 1, 1)$

Let $f(x, y) = y + \ln \frac{x}{z}, F(x, y, z) = f(x, y) - z$, then
Let $f(x, y, z) = y + \ln \frac{x}{z} - z$, then

$$f_x = \frac{1}{\frac{x}{z}} \frac{1}{z} = \frac{1}{x}, \quad f_y = 1, \quad f_z = -\frac{1}{\frac{x}{z}} \frac{x}{z^2} - 1 = -\frac{1}{z} - 1$$

$$\therefore \nabla f(x, y, z) = \langle \frac{1}{x}, 1, -\frac{1}{z} - 1\rangle$$

When $x = 1, y = 1, z = 1$, $\nabla f = \langle 1, 1, -2\rangle$
$\therefore$ the equation of the tangent plane at P_0 is $(x - 1) + (y - 1) - 2(z - 1) = 0$
$\therefore$ the equation of the normal line at P_0 is

$$\frac{x - 1}{1} = \frac{y - 1}{1} = \frac{z - 1}{-2}$$

8.

Let $F(x, y, z) = 2x^2 + 3y^2 + z^2 - 9$, then $F_x = 4x, F_y = 6y, F_z = 2z$

$$\therefore \nabla F(x, y, z) = \langle 4x, 6y, 2z\rangle$$

Let $\overrightarrow{n} = \langle 2, -3, 2\rangle$, and let $\nabla F \parallel \overrightarrow{n}$, then we have

$$\frac{4x}{2} = \frac{6y}{-3} = \frac{2z}{2}$$

$$\therefore y = -x, z = 2x$$

$\therefore 2x^2 + 3x^2 + 4x^2 = 9x^2 = 9$, then $x = \pm 1$
$\therefore$ the normal vector is $(4, -6, 4)$ or $(-4, 6, -4)$
$\therefore$ the equation of the tangent plane is

$$4(x - 1) - 6(y + 1) + 4(z - 2) = 0 \text{ or } -4(x - 1) + 6(y + 1) - 4(z - 2) = 0$$

$$\text{i.e. } 2x - 3y + 2z = \pm 9$$

B

1.

Proof. Let $\overrightarrow{r}(t) = \begin{cases} x = ae^t \cos t \\ y = ae^t \sin t \\ z = ae^t \end{cases}$, then $\overrightarrow{r'}(t) = \begin{cases} x = ae^t(\cos t - \sin t) \\ y = ae^t(\sin t + \cos t) \\ z = ae^t \end{cases}$

The direction vector of generatrix is $\overrightarrow{s}(t) = \begin{cases} x = ae^t \cos t \\ y = ae^t \sin t \\ z = ae^t \end{cases}$

Let θ be the angle between $\overrightarrow{r}$ and $\overrightarrow{s}$, then

$$\cos\theta = \frac{\overrightarrow{r} \cdot \overrightarrow{s}}{|\overrightarrow{r}||\overrightarrow{s}|} = \frac{\sqrt{6}}{3}$$

$\therefore \theta$ is a constant

□

3.

Let $\overrightarrow{n_1} = (1, -1, -1), \overrightarrow{n_2} = (1, -1, -\frac{1}{2})$, then

$$\overrightarrow{n} = \overrightarrow{n_1} \times \overrightarrow{n_2} = (-\frac{1}{2}, \frac{1}{2}, 0)$$

Let $F(x, y, z) = x^2 + y^2 + z^2 - x$, then

$$F_x = 2x - 1, \quad F_y = 2y, \quad F_z = 2z$$

$$\therefore \nabla F(x, y, z) = \langle 2x - 1, 2y, 2z \rangle$$

$$\therefore y = -x + \frac{1}{2}, \quad z = 0$$

$$\therefore F(x, y, z) = x^2 + x^2 + \frac{1}{4} - x + 0 - x = 0$$

$$\therefore x = \frac{1 \pm \frac{\sqrt{2}}{2}}{2} = \frac{2 \pm \sqrt{2}}{4}, y = \frac{\pm\sqrt{2}}{4}, z = 0$$

$\therefore$ the equation of the tangent plane is

$$x + y = \frac{1 \pm \sqrt{2}}{2}$$