Exercise 14.8

Wang Yue from CS Elite Class

May 11, 2021

1. Find the point on the curve of intersection of the paraboloid $z = x^2 + y^2$ and the plane $x + y + 2z = 2$ that are closest and farthest from the origin.

Let the distance between a point to the origin be $d = \sqrt{x^2 + y^2 + z^2}$, then for convenience we solve the maximum and minimum of d^2 under constraints.

Construct the Lagrange Function

$$L(x, y, z, \lambda, \mu) = (x^2 + y^2 + z^2) + \lambda(x^2 + y^2 - z^2) + \mu(x + y + 2z - 2)$$

then let

$$\begin{cases} \frac{\partial L}{\partial x} = 2x + 2\lambda x + \mu & = 0 \\ \frac{\partial L}{\partial y} = 2y + 2\lambda y + \mu & = 0 \\ \frac{\partial L}{\partial z} = 2z + 2\lambda z + 2\mu & = 0 \\ \frac{\partial L}{\partial \lambda} = x^2 = y^2 - z^2 & = 0 \\ \frac{\partial L}{\partial \mu} = x + y + 2z - 2 & = 0 \end{cases}$$

then we can get $\begin{cases} x = \frac{1}{2} \\ y = \frac{1}{2} \\ z = -\frac{1}{2} \end{cases}$ or $\begin{cases} x = -1 \\ y = -1 \\ z = 2 \end{cases}$

When $x = y = \frac{1}{2}, z = -\frac{1}{2}$, $d = \sqrt{3 \times \frac{1}{4}} = \frac{\sqrt{3}}{2}$

When $x = y = -1, z = 2$, $d = \sqrt{1 + 1 + 4} = \sqrt{6}$

$\therefore$ the closet distance is $\frac{\sqrt{3}}{2}$, and the farthest distance is $\sqrt{6}$.

2.

Let $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$, then $\nabla F = \langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \rangle$

$\therefore$ the equation of the tangent plane is

$$\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0)$$

Let $x = y = 0$, then we get the intercept in z-axis, i.e.

$$z = (\frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2} + \frac{2z_0^2}{c^2}) \times \frac{c^2}{2z_0} = \frac{c^2}{z_0}$$

And similarly, we can get the intercept $y = \frac{b^2}{y_0}$ in y-axis and $x = \frac{a^2}{x_0}$ in x-axis.
$\therefore$ the volume of the tetrahedron, i.e

$$V = \frac{1}{3}(\frac{1}{2}\frac{a^2}{x_0}\frac{b^2}{y_0})\frac{c^2}{z_0} = \frac{a^2b^2c^2}{6x_0y_0z_0}$$

Construct Lagrange Function $L(x, y, z, \lambda) = \frac{a^2b^2c^2}{6xyz} + \lambda(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1) = 0$, then let

$$\begin{cases} \frac{\partial L}{\partial x} = -\frac{a^2b^2c^2}{6x^2yz} + \frac{2x\lambda}{a^2} = 0 \\ \frac{\partial L}{\partial y} = -\frac{a^2b^2c^2}{6xy^2z} + \frac{2y\lambda}{b^2} = 0 \\ \frac{\partial L}{\partial z} = -\frac{a^2b^2c^2}{6xyz^2} + \frac{2z\lambda}{c^2} = 0 \\ \frac{\partial L}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \end{cases}$$

From $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0$, we can get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{a^2b^2c^2}{12xyz\lambda}$$

$\therefore \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$, which means $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$
$\therefore$ we have the minimum of volume $V_{min} = \frac{\sqrt{3}abc}{2}$

3.

Construct Lagrange Function

$$L(x, y, z, \lambda) = x^2 + \pi y^2 + \frac{\sqrt{3}z^2}{4} + \lambda(4x + 2\pi y + 3z - 2m)$$

then let

$$\begin{cases} \frac{\partial L}{\partial x} = 2x + 4\lambda = 0 \\ \frac{\partial L}{\partial y} = 2\pi y + 2\pi\lambda = 0 \\ \frac{\partial L}{\partial z} = \frac{\sqrt{3}z}{2} + 3\lambda = 0 \\ \frac{\partial L}{\partial \lambda} = 4x + 2\pi y + 3z - 2m = 0 \end{cases}$$

we get

$$\begin{cases} x = -2\lambda \\ y = -\lambda \\ z = -2\sqrt{3}\lambda \end{cases}$$

then $-8\lambda - 2\pi\lambda - 6\sqrt{3}\lambda = 2m$, i.e.

$$\lambda = -\frac{m}{4 + \pi + 3\sqrt{3}}$$

$\therefore x = \frac{2m}{4+\pi+3\sqrt{3}}, y = \frac{m}{4+\pi+3\sqrt{3}}, z = \frac{2\sqrt{3}m}{4+\pi+3\sqrt{3}}$
$\therefore$ the maximum sum of area is

$$x^2 + \pi y^2 + \frac{\sqrt{3}z^2}{4} = \frac{4m^2 + \pi m^2 + 3\sqrt{3}m^2}{(4 + \pi + 3\sqrt{3})^2} = \frac{m^2}{4 + \pi + 3\sqrt{3}}$$

4.

$f(x,y) = x^2y(4-x-y)$
$D = \{(x,y) | x+y \leq 6 \text{ and } x \geq 0 \text{ and } y \geq 0\}$
$f_x(x,y) = y[2x(4-x-y) - x^2] = 8xy - 3x^2y - 2xy^2$
$f_y(x,y) = x^2[(4-x-y) - 1] = 3x^2 - x^3 - x^2y$
Let $f_x(x,y) = f_y(x,y) = 0$ and do not consider boundary, we get $x = 2, y = 1$
$f_{xx}(x,y) = 8y - 6xy - 2y^2$
$f_{yy}(x,y) = -x^2$
$f_{xy}(x,y) = 8x - 3x^2 - 4xy$

$$
\begin{aligned}
D &= f_{xx}f_{yy} - (f_{xy})^2 \\
&= -8x^2y + 6x^3y + 2x^2y^2 - (64x^2 + 9x^4 + 16x^2y^2 - 48x^3 - 64x^2y + 24x^3y) \\
&= 56x^2y - 18x^3y - 14x^2y^2 - 64x^2 - 9x^4 + 48x^3
\end{aligned}
$$

When $x = 2, y = 1$, $D = 56 \times 4 - 18 \times 8 - 14 \times 4 - 64 \times 4 - 9 \times 16 + 48 \times 8 = 8 > 0$
$\because f_{xx}(2,1) = 8 - 12 - 2 = -2 < 0$
$\therefore f(2,1) = 4 \times 1 \times (4-2-1) = 4$ is a local maximum
Fix $x = 0, 0 \leq y \leq 6$, then $f(0,y) = 0$
Fix $y = 0, 0 \leq x \leq 6$, then $f(x,0) = 0$
Fix $x+y = 6$, then $f(x,y) = x^2(6-x)(4-6) = 2x^3 - 12x^2$, where $0 \leq x \leq 6$
Let $g(x) = 2x^3 - 12x^2$, then $g'(x) = 6x^2 - 24x = 6x(x-4)$
When $x = 0, g(x) = 0$. When $x = 4, g(x) = 128 - 12 \times 16 = -64$
$\therefore f(2,1) = 4$ is a local maximum, and there is no local minimum
$\therefore f(2,1) = 4$ is the absolute maximum, and $f(4,2) = -64$ is the absolute minimum.