Exercise 15.2

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8.

$$\begin{aligned}
\int_1^3 \int_1^5 \frac{\ln y}{xy} dy dx &= \int_1^3 \frac{1}{x} \int_1^5 \ln y d \ln y dx \\
&= \int_1^3 \frac{1}{x} \frac{(\ln y)^2}{2} \Big|_{y=1}^{y=5} dx \\
&= \int_1^3 \frac{(\ln 5)^2}{2x} dx \\
&= \frac{(\ln 5)^2}{2} (\ln 3 - \ln 1) \\
&= \frac{(\ln 5)^2 \ln 3}{2}
\end{aligned}$$

12.

$$\begin{aligned}
\int_0^1 \int_0^1 xy\sqrt{x^2+y^2} dy dx &= \int_0^1 \frac{x}{2} \int_0^1 \sqrt{x^2+y^2} d(x^2+y^2) dx \\
&= \int_0^1 \frac{x}{2} \times \frac{2}{3} (x^2+y^2)^{\frac{3}{2}} \Big|_{y=0}^{y=1} dx \\
&= \int_0^1 [\frac{x}{3}(x^2+1)^{\frac{3}{2}} - \frac{x^4}{3}] dx \\
&= \int_0^1 \frac{1}{6}(x^2+1)^{\frac{3}{2}} d(x^2+1) - \int_0^1 \frac{x^4}{3} dx \\
&= \frac{1}{6} \times \frac{2}{5} (x^2+1)^{\frac{5}{2}} \Big|_0^1 - \frac{x^5}{15} \Big|_0^1 \\
&= \frac{1}{15} \times (4\sqrt{2} - 1) - \frac{1}{15} \\
&= \frac{4\sqrt{2} - 2}{15}
\end{aligned}$$

13.

$$\begin{aligned}\int_0^2\int_0^\pi r\sin^2\theta d\theta dr &= \int_0^2 r\int_0^\pi \frac{1-\cos 2\theta}{2}d\theta dr\\ &= \int_0^2 r(\frac{\theta}{2}-\frac{\sin 2\theta}{4})\Big|_{\theta=0}^{\theta=\pi}dr\\ &= \int_0^2 \frac{\pi}{2}rdr\\ &= \frac{\pi}{2}\frac{r^2}{2}\Big|_0^2\\ &= \pi\end{aligned}$$

15.

$$\begin{aligned}\iint_R \sin(x-y)dA &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin(x-y)dydx\\ &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}-\sin(x-y)d(x-y)dx\\ &= \int_0^{\frac{\pi}{2}}\cos(x-y)\Big|_{y=0}^{y=\frac{\pi}{2}}dx\\ &= \int_0^{\frac{\pi}{2}}[\cos(x-\frac{\pi}{2})-\cos x]dx\\ &= \int_0^{\frac{\pi}{2}}(\sin x-\cos x)dx\\ &= (-\cos x-\sin x)\Big|_0^{\frac{\pi}{2}}\\ &= (-0-1)-(-1-0)=0\end{aligned}$$

17.

$$\begin{aligned}
\iint_R \frac{xy^2}{x^2+1} dA &= \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1} dydx \\
&= \int_0^1 2\int_0^3 \frac{xy^2}{x^2+1} dydx \\
&= \int_0^1 \frac{2x}{x^2+1} (\frac{y^3}{3})\Big|_{y=0}^{y=3} dx \\
&= \int_0^1 \frac{18x}{x^2+1} dx \\
&= \int_0^1 \frac{9}{x^2+1} d(x^2+1) \\
&= 9\ln(x^2+1)\Big|_0^1 \\
&= 9\ln 2
\end{aligned}$$

21.

$$\begin{aligned}
\iint_R ye^{-xy} dA &= \int_0^3 \int_0^2 ye^{-xy} dxdy \\
&= \int_0^3 -e^{-xy}\Big|_{x=0}^{x=2} dy \\
&= \int_0^3 (1-e^{-2y}) dy \\
&= (y+\frac{1}{2}e^{-2y})\Big|_0^3 \\
&= 3+\frac{1}{2}(e^{-6}-1) \\
&= \frac{e^{-6}}{2}+\frac{5}{2}
\end{aligned}$$

26.

$$\begin{aligned}
\int_1^2 \int_{-1}^1 (3y^2 - x^2 + 2)dxdy &= \int_1^2 2\int_0^1 (3y^2 - x^2 + 2)dxdy \\
&= \int_1^2 2(3y^2x + 2x - \frac{x^3}{3})\Big|_{x=0}^{x=1} dy \\
&= \int_1^2 2(3y^2 + 2 - \frac{1}{3})dy \\
&= \int_1^2 (6y^2 + \frac{10}{3})dy \\
&= (2y^3 + \frac{10}{3}y)\Big|_1^2 = \frac{52}{3}
\end{aligned}$$

30.

$$\begin{aligned}
\int_0^4 \int_0^5 (16 - x^2)dydx &= \int_0^4 5(16 - x^2)dx \\
&= (80x - \frac{5}{3}x^3)\Big|_0^4 \\
&= 320 - \frac{320}{3} \\
&= \frac{640}{3}
\end{aligned}$$

40.

(a) Clairaut's Theorem tells us the order of partial derivative does not matter, while Fubini's Theorem indicates that the order of integration does not matter. Therefore, they are similar to each other.

(b) *Proof.* By Fubini's Theorem, we have

$$g(x, y) = \int_a^x \int_c^y f(s, t)dtds = \int_c^y \int_a^x f(s, t)dsdt$$

$$\therefore g_{xy} = \frac{\partial^2}{\partial y \partial x} \int_a^x \int_c^y f(s, t)dtds = \frac{\partial}{\partial y} \int_c^y f(x, t)dt = f(x, y)$$

$$\therefore g_{yx} = \frac{\partial^2}{\partial x \partial y} \int_c^y \int_a^x f(s, t)dsdt = \frac{\partial}{\partial x} \int_a^x f(s, y)ds = f(x, y)$$

$$\therefore g_{xy} = g_{yx} = f(x, y)$$

$\square$