Exercise 16.8

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11(a)

$$\text{curl}\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & xy^2 & z^2 \end{vmatrix} = \langle 0, x^2, y^2 \rangle$$

$$z = 1 - x - y, \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = -1$$

$$\begin{aligned} \int_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \iint_S \text{curl}\overrightarrow{F} \cdot d\overrightarrow{S} = \iint_D (x^2 + y^2) dA \\ &= \int_0^{2\pi} \int_0^3 r^3 dr d\theta \\ &= \int_0^{2\pi} \frac{81}{4} d\theta = \frac{81}{2}\pi \end{aligned}$$

12(a)

$$\text{curl } \overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & \frac{1}{3}x^3 & xy \end{vmatrix} = \langle x, -y, 0 \rangle$$

$$z = y^2 - x^2, \frac{\partial z}{\partial x} = -2x, \frac{\partial z}{\partial y} = 2y$$

$$\begin{aligned} \int_C \overrightarrow{F} \cdot \overrightarrow{r} &= \iint_S \text{curl}\overrightarrow{F} \cdot d\overrightarrow{S} = \iint_D (2x^2 + 2y^2) dA \\ &= \int_0^{2\pi} \int_0^1 2r^3 dr d\theta \\ &= \int_0^{2\pi} 2\frac{r^4}{4}\Big|_0^1 d\theta = \pi \end{aligned}$$

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$$\text{curl}\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2yz & y & 3x \end{vmatrix} = \langle 0, -2y-3, 2z \rangle$$

$$z = 5 - x^2 - y^2, z \in [1, 5], \frac{\partial z}{\partial x} = -2x, \frac{\partial z}{\partial y} = -2y$$

$$\begin{aligned}
\int_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \iint_D (-2y(2y+3) + 2z) dA \\
&= \iint_D (-6y^2 - 2x^2 - 6y + 10) dA \\
&= \int_0^{2\pi} \int_0^2 (-6r^3 \sin^2\theta - 2r^3 \cos^2\theta - 6r^2 \sin\theta + 10r) dr d\theta \\
&= \int_0^{2\pi} (-24 \sin^2\theta - 8\cos^2\theta - 16 \sin\theta + 20) d\theta \\
&= \int_0^{2\pi} (12 - 16 \sin^2\theta - 16 \sin\theta) d\theta \\
&= \int_0^{2\pi} (12 - 8 + 8\cos 2\theta - 16 \sin\theta) d\theta \\
&= [4 + 4 \sin 2\theta + 16 \cos\theta] \Big|_0^{2\pi} \\
&= 8\pi
\end{aligned}$$

Let $\overrightarrow{r}(t) = \langle 2\cos t, 2\sin t, 1 \rangle$, then $\overrightarrow{r}'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$

$$\overrightarrow{F}(\overrightarrow{r}(t)) = \langle -4\sin t, 2\sin t, 6\cos t \rangle$$

$$\begin{aligned}
\int_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \int_0^{2\pi} \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t) dt \\
&= \int_0^{2\pi} 8\sin^2 t + 4 \sin t \cos t dt \\
&= 8 \int_0^{2\pi} \frac{1 - \cos 2t}{2} dt + 4 \int_0^{2\pi} \sin t d \sin t \\
&= 8\pi
\end{aligned}$$

$\therefore$ the Stoke's Theorem is true in this example.

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Proof. Use Stoke's Theorem.

$$\overrightarrow{F} = \langle z, -2x, 3y \rangle$$

$$\text{curl}\overrightarrow{F} = \langle 3, 1, -2 \rangle$$

$$z = 1 - x - y, \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = -1$$

$$\int_D \overrightarrow{F} \cdot d\overrightarrow{t} = \iint_S \text{curl}\overrightarrow{F} \cdot d\overrightarrow{S} = \iint_S (3 + 1 - 2) dS = 2 \iint_S dS$$

which means it only depends on the area of the region C.

□

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$$\text{curl}\overrightarrow{F} = \langle -2z, -3x^2, -1 \rangle$$

$\because \overrightarrow{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$
$\therefore z = 2xy$
$\therefore \frac{\partial z}{\partial x} = 2y, \frac{\partial z}{\partial y} = 2x$

$$\begin{aligned}
\int_C (y + \sin x)dx + (z^2 + \cos y)dy + x^3 dz &= \iint_S (8xy^2 + 6x^3 - 1) dA \\
&= \int_0^{2\pi} \int_0^1 (8r^4 \cos\theta \sin^2\theta + 6r^4 \cos^3\theta - r) dr d\theta \\
&= \int_0^{2\pi} \frac{8}{5} \sin^2\theta \cos\theta + \frac{6}{5} \cos^3\theta - 1 d\theta \\
&= \frac{8}{5} \int_0^{2\pi} \sin^2\theta d\sin\theta + \frac{6}{5} \int_0^{2\pi} (1 - \sin^2\theta) d\sin\theta - 2\pi \\
&= -2\pi
\end{aligned}$$

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Proof. By using the Divergence Theorem, we have

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_E \text{div}\overrightarrow{F} dV$$

$$\therefore \iint_S \text{curl}\overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_E \text{div}(\text{curl}\overrightarrow{F}) dV = 0$$

□

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$$\operatorname{curl}(\overrightarrow{F}+\overrightarrow{G})=\operatorname{curl}\overrightarrow{F}+\operatorname{curl}\overrightarrow{G}$$

$$\operatorname{curl} f\overrightarrow{F}=f\operatorname{curl}\overrightarrow{F}+(\nabla f)\times\overrightarrow{F}$$

1.

$$\operatorname{curl}(f\nabla g)=f\operatorname{curl}\nabla g+(\nabla f)\times(\nabla g)$$

$\because \operatorname{curl}\nabla g=0$

$\therefore \operatorname{curl}(f\nabla g)=(\nabla f)\times(\nabla g)$

$\therefore$ by the Stoke's Theorem,

$$\int_C (f\nabla g)\cdot d\overrightarrow{r}=\iint_S(\nabla f\times\nabla g)\cdot d\overrightarrow{S}$$

2.

$$\operatorname{curl}(f\nabla f)=f\operatorname{curl}\nabla f+(\nabla f)\times(\nabla f)=0+0=0$$

$\therefore$ by the Stoke's Theorem,

$$\int_C (f\nabla f)\cdot dr=\iint_S 0\cdot d\overrightarrow{S}=0$$

3.

$$\begin{aligned}\operatorname{curl}(f\nabla g+g\nabla f)&=\operatorname{curl}(f\nabla g)+\operatorname{curl}(g\nabla f)\\&=(\nabla f)\times(\nabla g)+(\nabla g)\times(\nabla f)=0\end{aligned}$$