# Exercise 14.2

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**6.** $\lim_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y)$

Let $u = 1\times(-1) = -1, v = 1-1 = 0$, then

$$\lim_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y) = \lim_{(u,v)\to(-1,1)} e^{-u}\cos v = \lim_{u\to -1} e^{-u} \times \lim_{v\to 1}\cos v = e$$

**11.** $\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4+y^4}$

Let $y = kx$, then

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4+y^4} &= \lim_{(x,y)\to(0,0)} \frac{k^2x^2\sin^2 x}{(1+k^4)x^4} \\ &= \frac{k^2}{1+k^4} \times \lim_{(x,y)\to(0,0)} (\frac{\sin x}{x})^2 \\ &= \frac{k^2}{1+k^4}\end{aligned}$$

If $k = 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4+y^4} = \frac{0}{1} = 0$$

If $k = 1$, then

$$\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4+y^4} = \frac{1}{2}$$

$\therefore$ the limit does not exist.

**12.** $\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$

Let $x-1 = ky$, then

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2} = \lim_{(x,y)\to(1,0)} \frac{ky^2}{(k^2+1)y^2} = \frac{k}{k^2+1}$$

$\because \lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$ is not a constant
$\therefore$ the limit does not exist.

**13.** $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} \times \lim_{y\to 0} y$$

$\because \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ is bounded in $(0,1)$, and $\lim_y y = 0$
$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

**16.** $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2}$

$\forall \epsilon > 0, \exists \delta = \sqrt{\epsilon}$, such that if $0 < \sqrt{x^2+y^2} < \delta$, then

$$\frac{x^2 \sin^2 y}{x^2+2y^2} < \frac{x^2 y^2}{x^2+y^2} < \frac{(x^2+y^2)^2}{(x^2+y^2)} = \delta^2 = \epsilon$$

$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2}$ exists.

**17.** $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}+1)}{(x^2+y^2+1)-1} \\ &= \lim_{(x,y)\to(0,0)} (\sqrt{x^2+y^2+1}+1) \\ &= 1+1 = 2\end{aligned}$$

**32.** $H(x,y) = \frac{e^x+e^y}{e^{xy}-1}$

When $x \neq 0$ and $y \neq 0$, $e^{xy}-1 \neq 0$, $H(x,y)$ is continuous since it is a rational function.

When $x = 0$ or $y = 0$, let $y = x$, then we get

$$\lim_{(x,y)\to(0,0)} \frac{e^x+e^y}{e^{xy}-1} = \lim_{x\to 0} \frac{2e^x}{e^{x^2}-1} = \infty$$

$\therefore \lim_{(x,y)\to(0,0)} \frac{e^x+e^y}{e^{xy}-1}$ does not exist.
$\therefore$ the function is continuous at $\{(x,y)|x \neq 0 \text{ and } y \neq 0\}$

**33.** $G(x,y) = \ln(x^2+y^2-4)$

The domain of $G$ is $\{(x,y)|x^2+y^2 > 4\}$
$\because$ the inner function $z = x^2+y^2-4$ is continuous in $D(G)$
$\therefore G(x,y)$ is continuous in $\{(x,y)|x^2+y^2 > 4\}$

**37.** $f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \textbf{if } (x,y) \neq (0,0) \\ 1 & \textbf{if } (x,y) = (0,0) \end{cases}$

When $(x,y) \neq (0,0)$, $f(x,y)$ is continuous since it is a rational function.

When $(x,y) = (0,0)$, we can prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0 \neq 1$.

$\forall \epsilon > 0, \exists \delta = \sqrt[3]{\epsilon}$, such that if $0 < \sqrt{x^2+y^2} < \delta$,

$$|f(x,y) - 0| = \frac{x^2y^3}{2x^2+y^2} < \frac{x^2}{x^2+y^2}y^3 < (\sqrt{x^2+y^2})^3 = \delta^3 = \epsilon$$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$

$\therefore f$ is not continuous at $(0,0)$.

$\therefore f(x,y)$ is continuous in $\{(x,y) | x \neq 0 \text{ and } y \neq 0\}$

**38.** $f(x,y) = \begin{cases} \frac{xy}{x^2+xy+y^2} & \textbf{if}(x,y) \neq (0,0) \\ 0 & \textbf{if}(x,y) = (0,0) \end{cases}$

When $(x,y) \neq (0,0)$, $f(x,y)$ is continuous since it is a rational function.

When $(x,y) = (0,0)$, we can prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Let $y = x$, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{x^2}{3x^2} = \frac{1}{3}$$

Let $y = -x$, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{-x^2}{2x^2 - x^2} = -1$$

$\therefore \lim_{(x,y)\to(0,0)}$ does not exist, thus not continuous at $(0,0)$.

$\therefore f(x,y)$ is continuous in $\{(x,y) | x \neq 0 \text{ and } y \neq 0\}$

**40.** $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

Taking the substitution $r^2 = x^2 + y^2$, then the limit is equivalent to

$$\lim_{r\to 0^+} r^2 \ln r^2 = \lim_{r\to 0^+} \frac{2\ln r}{r^{-2}} = \lim_{r\to 0^+} \frac{2}{r(-2r^{-3})} = \lim_{r\to 0^+} -r^2 = 0$$

**41.** $\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$

Taking the substitution $r^2 = x^2 + y^2$, then the limit is equivalent to

$$\lim_{r\to 0^+} \frac{e^{-r^2}-1}{r^2} = \lim_{r\to 0^+} \frac{-2re^{-r^2}}{2r} = \lim_{r\to 0^+} -e^{-r^2} = -1$$