Exercise 15.8 & 15.9

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15.8

18.

$D = \{(x, y, z)|x^2 + y^2 \leq z, 0 \leq z \leq 4\}$

$$\begin{aligned}
\iiint_E zdV &= \int_0^{2\pi} \int_0^4 \int_0^{\sqrt{z}} zrdrdzd\theta \\
&= \int_0^{2\pi} \int_0^4 z(\frac{r^2}{2})\Big|_0^{\sqrt{z}} dzd\theta \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^4 z^2 dzd\theta \\
&= \frac{1}{2} \int_0^{2\pi} \frac{z^3}{3}\Big|_0^4 d\theta \\
&= \frac{1}{6} \int_0^{2\pi} \frac{64}{3} d\theta \\
&= \frac{64\pi}{9}
\end{aligned}$$

19.

$D = \{(x, y, z)|0 < x > 2, 0 < y < 2, 0 < z < 4 - x^2 - y^2\}$

$$
\begin{aligned}
\iiint_E (x+y+z)dV &= \int_0^{2\pi}\int_0^4\int_0^{\sqrt{4-z}} (r\cos\theta + r\sin\theta + z) r dr dz d\theta \\
&= \int_0^{2\pi}\int_0^4 (\cos\theta+\sin\theta)(\frac{r^3}{3})\Big|_0^{\sqrt{4-z}} + z(\frac{r^2}{2})\Big|_0^{\sqrt{4-z}} dz d\theta \\
&= -\int_0^{2\pi}\int_0^4 \frac{(4-z)^{\frac{3}{2}}}{3}(\cos\theta+\sin\theta) + \frac{z(4-z)}{2} d(4-z) d\theta \\
&= -\int_0^{2\pi} \frac{\cos\theta+\sin\theta}{3}\frac{2}{5}(4-z)^{\frac{5}{2}}\Big|_0^4 d\theta + \int_0^{2\pi}\int_0^4 (2z - \frac{z^2}{2}) dz d\theta \\
&= \int_0^{2\pi} \frac{\cos\theta+\sin\theta}{3}\frac{64}{5} d\theta + \int_0^{2\pi} (z^2 - \frac{z^3}{6})\Big|_0^4 d\theta \\
&= \frac{64}{15}(\sin\theta - \cos\theta)\Big|_0^{2\pi} + \int_0^{2\pi} (16 - \frac{32}{3}) d\theta \\
&= \frac{32\pi}{3}
\end{aligned}
$$

20.

$$
\begin{aligned}
\iiint_E x dV &= \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta + 5} x r dz dr d\theta \\
&= \int_0^{2\pi}\int_2^3 r^2\cos\theta(r\cos\theta + r\sin\theta + 5) dr d\theta \\
&= \int_0^{2\pi}\int_2^3 (r^3\cos^2\theta + r^3\sin\theta\cos\theta + 5r^2\cos\theta) dr d\theta \\
&= \int_0^{2\pi} (\frac{r^4(\cos^2\theta + \sin\theta\cos\theta)}{4} + \frac{5r^3\cos\theta}{3})\Big|_2^3 d\theta \\
&= \int_0^{2\pi} \frac{65\cos^2\theta + 65\sin\theta\cos\theta}{4} + \frac{95\cos\theta}{3} d\theta \\
&= \int_0^{2\pi} \frac{65 + 65\cos 2\theta}{8} d\theta + \int_0^{2\pi} \frac{65\sin\theta}{4} d\sin\theta \\
&= \frac{65\pi}{4} + \frac{65}{4}(\frac{\sin^2\theta}{2})\Big|_0^{2\pi} = \frac{65\pi}{4}
\end{aligned}
$$

21.

$$\begin{aligned}
\iiint_E x^2 dV &= \int_0^{2\pi}\int_0^1\int_0^{2r}(r^3\cos^2\theta)dzdrd\theta \\
&= \int_0^{2\pi}\int_0^1 (2r^4\cos^2\theta)drd\theta \\
&= 2\int_0^{2\pi}\cos^2\theta(\frac{r^5}{5})\Big|_0^1 d\theta \\
&= \frac{2}{5}\int_0^{2\pi}\cos^2\theta d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)d(2\theta) \\
&= \frac{1}{5}\frac{1}{2}(2\theta+\sin 2\theta)\Big|_0^{2\pi} \\
&= \frac{2\pi}{5}
\end{aligned}$$

24.

Let $z = 2 - z^2$, then $z = -2$ (abandoned) or $z = 1$

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^1\int_0^z rdrdzd\theta \\
&= \int_0^{2\pi}\int_0^1 (\frac{r^2}{2})\Big|_0^z dzd\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^1 z^2 dzd\theta \\
&= \frac{1}{2}\int_0^{2\pi}\frac{z^3}{3}\Big|_0^1 d\theta \\
&= \frac{\pi}{3}
\end{aligned}$$

29.

$$\begin{aligned}
\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\,dzdxdy &= \int_0^{2\pi}\int_0^2\int_r^2 (zr^2\cos\theta)dzdrd\theta \\
&= \int_0^{2\pi}\int_0^2 r^2\cos\theta(\frac{z^2}{2})\Big|_r^2 drd\theta \\
&= \int_0^{2\pi}\int_0^2 (2r^2-\frac{r^4}{2})\cos\theta drd\theta \\
&= \int_0^{2\pi}\int_0^2 \cos\theta(\frac{2r^3}{3}-\frac{r^5}{10})\Big|_0^2 d\theta \\
&= \frac{32}{15}\int_0^{2\pi}\cos\theta d\theta \\
&= 0
\end{aligned}$$

15.9

25.

$$\begin{aligned}
\iiint_E xe^{x^2+y^2+z^2}dV &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 (\rho\sin\varphi\cos\theta e^{\rho^2}\rho^2\sin\varphi)d\rho d\theta d\varphi \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 (e^{\rho^2}\rho^3\sin^2\varphi\cos\theta)d\rho d\theta d\varphi \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^2\varphi\cos\theta\int_0^1(\frac{1}{2}e^{\rho^2}\rho^2)d\rho^2 d\theta d\varphi \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^2\varphi\cos\theta(\rho^2 e^{\rho^2})\Big|_{\rho=0}^{\rho=1} d\theta d\varphi \\
&= \frac{e}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin^2\varphi\cos\theta d\theta d\varphi \\
&= \frac{e}{2}\int_0^{\frac{\pi}{2}}\sin^2\varphi(\sin\theta)\Big|_0^{\frac{\pi}{2}} d\varphi \\
&= \frac{e}{2}\int_0^{\frac{\pi}{2}}\frac{1-\cos 2\varphi}{2}d\varphi \\
&= \frac{e}{2}(\frac{\varphi}{2}+\frac{2\sin 2\varphi}{2})\Big|_0^{\frac{\pi}{2}} \\
&= \frac{e\pi}{8}
\end{aligned}$$

27.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^a \rho^2\sin\varphi d\rho d\theta d\varphi = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_0^{2\pi}\sin\varphi(\frac{\rho^3}{3})\Big|_0^a d\theta d\varphi$$

$$= \frac{2\pi a^2}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\sin\varphi d\varphi$$

$$= \frac{2\pi a^2}{3}(-\cos\varphi)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{2\pi a^2}{3}\times\frac{\sqrt{3}-1}{2}$$

$$= \frac{(\sqrt{3}-1)\pi a^2}{3}$$

30.

Let $x^2+y^2+(\sqrt{x^2+y^2})^2=4$, then $x^2+y^2=z^2=2$, $z=\sqrt{2}$

$$\therefore V = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^2(\rho^2\sin\varphi)d\rho d\theta d\varphi$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\sin\varphi(\frac{\rho^3}{3})\Big|_0^2 d\theta d\varphi$$

$$= \frac{8}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\sin\varphi d\theta d\varphi$$

$$= \frac{16\pi}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin\varphi d\varphi$$

$$= \frac{16\pi}{3}(\cos\varphi)\Big|_{\frac{\pi}{2}}^{\frac{\pi}{4}}$$

$$= \frac{8\sqrt{2}\pi}{3}$$

39.

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xydzdydx &= \int_0^{\frac{\pi}{3}} \int_0^{\frac{\pi}{2}} \int_0^2 (\rho^4 \sin^3\varphi \sin\theta\cos\theta) d\rho d\theta d\varphi \\
&= \int_0^{\frac{\pi}{3}} \int_0^{\frac{\pi}{2}} \sin^3\varphi \sin\theta\cos\theta (\frac{\rho^5}{5})\Big|_0^2 d\theta d\varphi \\
&= \frac{16}{5}\int_0^{\frac{\pi}{3}} \int_0^{\frac{\pi}{2}} \sin^3\varphi \sin\theta\cos\theta d\theta d\varphi \\
&= \frac{16}{5}\int_0^{\frac{\pi}{3}} \int_0^{\frac{\pi}{2}} \sin^3\varphi \sin\theta d\sin\theta d\varphi \\
&= \frac{16}{5}\int_0^{\frac{\pi}{3}} \sin^3\varphi (\frac{\sin^2\theta}{2})\Big|_0^{\frac{\pi}{2}} d\varphi \\
&= \frac{8}{5}\int_0^{\frac{\pi}{3}} (\cos^2\varphi - 1) d\cos\varphi \\
&= \frac{8}{5}(\frac{\cos^3\varphi}{3} - \cos\varphi)\Big|_0^{\frac{\pi}{3}} \\
&= \frac{8}{5}(\frac{1}{24} - \frac{1}{2} - \frac{1}{3} + 1) \\
&= \frac{8}{5} \times \frac{5}{24} = \frac{1}{3}
\end{aligned}$$

41.

The center of the sphere is $(0, 0, 2)$, then we have

$$x^2 + y^2 + (z-2)^2 = 4$$

Change into spherical coordinate, then

$$\rho^2\sin^2\varphi + \rho^2\cos^2\varphi = \rho^2 = 4\rho\cos\varphi \implies \rho = 4\cos\varphi$$

$$
\begin{aligned}
\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}}(x^2+y^2+z^2)^{\frac{3}{2}}dzdydx &= \int_0^{\pi}\int_0^{2\pi}\int_0^{4\cos\varphi}(\rho^3\rho^2\sin\varphi)d\rho d\theta d\varphi \\
&= \int_0^{\pi}\int_0^{2\pi}\sin\varphi(\frac{\rho^6}{6})\Big|_0^{4\cos\varphi}d\theta d\varphi \\
&= \int_0^{\pi}\int_0^{2\pi}\sin\varphi\frac{(4\cos\varphi)^6}{6}d\theta d\varphi \\
&= \frac{4^6\pi}{3}\int_0^{\pi}\sin\varphi(\cos\varphi)^6 d\varphi \\
&= -\frac{4^6\pi}{3}\int_0^{\pi}\cos^6\varphi d\cos\varphi \\
&= -\frac{4^6\pi}{3}(\frac{\cos^7\varphi}{7})\Big|_0^{\pi} \\
&= \frac{2^{13}\pi}{21}
\end{aligned}
$$