Exercise 14.6

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7.

$$f_x(x,y) = 2\cos(2x+3y), \quad f_y(x,y) = 3\cos(2x+3y)$$

$$\begin{aligned} D_{\overrightarrow{u}} f(x,y) &= \frac{\sqrt{3}}{2} f_x(x,y) - \frac{1}{2} f_y(x,y) \\ &= \sqrt{3}\cos(2x+3y) - \frac{3}{2}\cos(2x+3y) \\ &= (\sqrt{3} - \frac{3}{2})\cos(2x+3y) \end{aligned}$$

$\therefore D_{\overleftrightarrow{u}} f(-6,4) = \sqrt{3} - \frac{3}{2}$

8.

$$f_x(x,y) = -\frac{y^2}{x^2}, \quad f_y(x,y) = \frac{2y}{x}$$

$$\begin{aligned} D_{\overrightarrow{u}} f(x,y) &= \frac{2}{3} f_x(x,y) + \frac{\sqrt{5}}{3} f_y(x,y) \\ &= -\frac{2}{3}\frac{y^2}{x^2} + \frac{2\sqrt{5}}{3}\frac{y}{x} \end{aligned}$$

$\therefore D_{\overrightarrow{u}} f(1,2) = -\frac{8}{3} + \frac{4\sqrt{5}}{3} = \frac{4\sqrt{5}-8}{3}$

11.

$$f_x(x,y) = e^x \sin y, \quad f_y(x,y) = e^x \cos y$$

The unit vector of $\overrightarrow{v}$ is $\overrightarrow{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$.

$$\begin{aligned} D_{\overrightarrow{u}} f(x,y) &= -\frac{3}{5} e^x \sin y + \frac{4}{5} e^x \cos y \\ &= e^x (\frac{4}{5}\cos y - \frac{3}{5}\sin y) \end{aligned}$$

$\therefore D_{\overrightarrow{u}} f(0, \frac{\pi}{3}) = e^0 (\frac{4}{5} \times \frac{1}{2} - \frac{3}{5} \times \frac{\sqrt{3}}{2}) = \frac{4-3\sqrt{3}}{10}$

14.

$$g_r(r,s) = \frac{s}{1+r^2s^2}, \quad g_s(r,s) = \frac{r}{1+r^2s^2}$$

The unit vector of $\overrightarrow{v}$ is $\overrightarrow{u} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$$D_{\overrightarrow{u}} g(r,s) = \frac{s+2r}{(1+r^2s^2)\sqrt{5}}$$

$\therefore D_{\overrightarrow{u}} g(1,2) = \frac{2+2}{(1+4)\sqrt{5}} = \frac{4\sqrt{5}}{25}$

25. $f(x,y,z) = \sqrt{x^2+y^2+z^2}, \quad (3,6,-2)$

$$\nabla f = \langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \rangle$$

$$\therefore \nabla f(3,6,-2) = \langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \rangle$$

The maximum rate of change is $|\nabla f| = \sqrt{\frac{9+36+4}{49}} = 1$

The direction is parallel to the vector $\overrightarrow{u} = \langle \frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \rangle$

39.

$$D_{\overrightarrow{u}} f(x,y) = (3x^2+10xy)\frac{3}{5} + (5x^2+3y^2)\frac{4}{5} = \frac{29}{5}x^2 + 6xy + \frac{12}{5}y^2$$

$$D^2_{\overrightarrow{u}} f(x,y) = (\frac{58}{5}x + 6y)\frac{3}{5} + (6x + \frac{24}{5}y)\frac{4}{5} = \frac{294}{25}x + \frac{186}{25}y$$

$$D^2_{\overrightarrow{u}} f(2,1) = \frac{588}{25} + \frac{186}{25} = \frac{774}{25}$$

40.

(a) *Proof.* $\because D_{\overrightarrow{u}} f = f_x a + f_y b$

$$\therefore D^2_{\overrightarrow{u}} f = (f_{xx}a + f_{xy}b)a + (f_{yx}a + f_{yy}b)b = f_{xx}a^2 + 2f_{xy}ab + f_{yy}b^2$$

□

(b)

$$f_{xx} = \frac{\partial}{\partial x}(e^{2y}) = 0, \quad f_{xy} = \frac{\partial}{\partial y}(e^{2y}) = 2e^{2y}, \quad f_{yy} = \frac{\partial}{\partial y}(2xe^{2y}) = 4xe^{2y}$$

Let the unit vector of $\overrightarrow{v}$ is $\overrightarrow{u} = \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \rangle$, and let $a = \frac{2}{\sqrt{13}}, b = \frac{3}{\sqrt{13}}$

$$\therefore D^2_{\overrightarrow{u}} f(x,y) = 0 + 2(2e^{2y})\frac{6}{13} + 4xe^{2y}\frac{9}{13} = (\frac{24}{13} + \frac{36}{13}x)e^{2y}$$

53.

Proof.

$$z = f(x,y) = c(\frac{x^2}{a^2} + \frac{y^2}{b^2})$$

$$f_x(x,y) = c\frac{2x}{a^2}, \quad f_y(x,y) = c\frac{2y}{b^2}$$

$$\therefore \nabla f = \langle c\frac{2x}{a^2}, c\frac{2y}{b^2}, -1\rangle = \langle \frac{2x}{a^2}, \frac{2y}{b^2}, -\frac{1}{c}\rangle$$

$\therefore$ the tangent plane of f at the point (x_0, y_0, z_0) can be expressed as

$$\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{z - z_0}{c} = 0$$

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} - (\frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2}) - \frac{z - z_0}{c} = 0$$

$$\frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} - \frac{2z_0}{c} - \frac{z - z_0}{c} = 0$$

$$\therefore \frac{2xx_0}{a^2} + \frac{2yy_0}{b^2} = \frac{z + z_0}{c}$$

□

54.

Let $F(x,y,z) = x^2 - y + z^2$, then

$$F_x(x,y,z) = 2x, \quad F_y(x,y,z) = -1, \quad F_z(x,y,z) = 2z$$

$$\therefore \nabla F = \langle 2x, -1, 2z\rangle$$

Let $\overrightarrow{n} = (1,2,3)$, if $\nabla F(x,y,z) = \lambda\overrightarrow{n}$, then we can solve that

$$x = -\frac{1}{4}, \quad y = -1, \quad z = -\frac{3}{4}$$

$\therefore$ the point is $(-\frac{1}{4}, -1, -\frac{3}{4})$.