Exercise 11.10

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47. $\int x\cos(x^3)\mathbf{d}x$

$\because \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$
$\therefore x\cos(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+1}}{(2n)!}$
$\therefore \int x\cos(x^3)\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n \int \frac{x^{6n+1}}{(2n)!}\mathrm{d}x$
$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!} + C$
where C is a constant.

50. $\int \arctan(x^2)\mathbf{d}x$

$\because \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$
$\therefore \int \arctan(x^2)\mathrm{d}x = \int \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(2n+1)!}\mathrm{d}x$
$= \sum_{n=0}^{\infty} \int \frac{x^{4n+2}}{(2n+1)!}\mathrm{d}x$
$= \sum_{n=0}^{\infty} \frac{x^{4n+3}}{(4n+3)(2n+1)!} + C$

55. $\lim_{x\to 0} \frac{x-\ln(1+x)}{x^2}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n}$

$$\begin{aligned}
\lim_{x\to 0} \frac{x-\ln(1+x)}{x^2} &= \lim_{x\to 0} \frac{x-(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots)}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\cdots}{x^2} \\
&= \lim_{x\to 0}(\frac{1}{2}-\frac{x}{3}+\frac{x^2}{4}-\frac{x^3}{5}+\cdots) \\
&= \frac{1}{2}
\end{aligned}$$

56. $\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$

$e^x = 1 + x + \frac{x^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} &= \lim_{x\to 0} \frac{-\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}}{-\sum_{n=2}^{\infty} \frac{x^n}{n!}} \\ &= \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots}{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots} \\ &= 1\end{aligned}$$

57. $\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

$\sin x = x - \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} &= \lim_{x\to 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x^5} \\ &= \lim_{x\to 0}\left(\frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \frac{x^6}{11!} + \cdots\right) \\ &= \frac{1}{120}\end{aligned}$$

65. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{n5^n}$

More generally, let $f(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \ln(1+x)$
$\therefore \sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{n5^n} = f(\frac{3}{5}) = \ln\frac{8}{5}$

68. $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots$

More generally, let $f(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = e^{-x}$
$\therefore 1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots = f(\ln 2) = e^{-\ln 2} = \frac{1}{2}$

70. $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \cdots$

More generally, let $f(x) = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \arctan x$
$\therefore \frac{1}{1\cdot 2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \cdots = \arctan\frac{1}{2}$

72. If $f(x) = (1+x^3)^{30}$, what is $f^{(58)}(0)$?

$\because (1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$
$\therefore f(x) = (1+x^3)^{30} = \sum_{n=0}^{\infty}\binom{30}{n}x^{3n} = \binom{30}{0}x^0 + \binom{30}{1}x^3 + \binom{30}{2}x^6 + \cdots$
$\therefore f^{(58)}(x) = \binom{30}{20}60 \times 59 \times \cdots \times 3x^2 + \binom{30}{21}63 \times 62 \times \cdots \times 6x^5 + \cdots$
$\therefore f^{(58)}(0) = 0$