Exercise 15.4

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20.

Let region $D = \{(x, y)|18 - 2x^2 - 2y^2 \geq 0\} = \{(x, y)|x^2 + y^2 \leq 9\}$, then

$$\begin{aligned}\iint_D (18 - 2x^2 - 2y^2)dA &= \int_0^{2\pi} \int_0^3 (18 - 2r^2) r dr d\theta \\ &= \int_0^{2\pi} [-\int_0^3 (9 - r^2) d(9 - r^2)] d\theta \\ &= \int_0^{2\pi} -\frac{(9 - r^2)^2}{2}\bigg|_0^3 d\theta \\ &= \int_0^{2\pi} (-0 + \frac{81}{2}) d\theta \\ &= 81\pi\end{aligned}$$

22.

Let the volume of the solid inside the hyperpoloid $x^2 + y^2 + z^2 = 16$ and inside the cylinder $x^2 + y^2 = 4$ be V_0, then

$$\begin{aligned}V_0 &= \int_0^{2\pi} \int_0^2 \sqrt{16 - r^2} r dr d\theta \\ &= \int_0^{2\pi} \int_0^2 (-\frac{1}{2}) \sqrt{16 - r^2} d(16 - r^2) dr d\theta \\ &= \int_0^{2\pi} (-\frac{1}{2}) \frac{2}{3} (16 - r^2)^{\frac{3}{2}} \bigg|_0^2 d\theta \\ &= \int_0^{2\pi} (-\frac{1}{3})(24\sqrt{3} - 64) d\theta \\ &= \frac{128\pi}{3} - 16\sqrt{3}\pi\end{aligned}$$

$\therefore V = \frac{4}{3}\pi \times 4^3 - 2V_0 = 32\sqrt{3}\pi$

25.

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^1 (\sqrt{1-r^2}-r)rdrd\theta \\
&= TODO
\end{aligned}
$$

31.

$$
\begin{aligned}
\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)dydx &= \int_0^{\pi}\int_0^3 r\sin r^2 drd\theta \\
&= \int_0^{\pi}\frac{1}{2}\int_0^3 \sin r^2 dr^2 d\theta \\
&= \int_0^{\pi}\frac{1}{2}(-\cos r^2)\Big|_0^3 d\theta \\
&= \frac{1}{2}\int_0^{\pi}(1-\cos 9)d\theta \\
&= \frac{(1-\cos 9)\pi}{2}
\end{aligned}
$$

32.

$$
\begin{aligned}
\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}dydx &= \int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2 drd\theta \\
&= \int_0^{\frac{\pi}{2}}(\frac{r^3}{3})\Big|_0^{2\cos\theta} d\theta \\
&= \int_0^{\frac{\pi}{2}}\frac{8\cos^3\theta}{3}d\theta \\
&= \frac{8}{3}\int_0^{\frac{\pi}{2}}(1-\sin^2\theta)d\sin\theta \\
&= \frac{8}{3}(\sin\theta-\frac{\sin^3\theta}{3})\Big|_0^{\frac{\pi}{2}} \\
&= \frac{8}{3}\times\frac{2}{3} = \frac{16}{9}
\end{aligned}
$$

34.

$$\iint_D xy\sqrt{1+x^2+y^2}dA = \int_0^{\frac{\pi}{2}}\int_0^1 r^2\sin\theta\cos\theta\sqrt{1+r^2}rdrd\theta$$
$$= \int_0^1\int_0^{\frac{\pi}{2}} r^2\sin\theta\cos\theta\sqrt{1+r^2}rd\theta dr$$
$$= \int_0^1\int_0^{\frac{\pi}{2}} r^3\sqrt{1+r^2}\sin\theta d\sin\theta dr$$
$$= \int_0^1 r^3\sqrt{1+r^2}(\frac{\sin^2\theta}{2})\Big|_0^{\frac{\pi}{2}} dr$$
$$= \int_0^1 \frac{r^3\sqrt{1+r^2}}{2}dr$$
$$\approx 0.1609$$

39.

$$\int_{\frac{\sqrt{2}}{2}}^1\int_{\sqrt{1-x^2}}^x xydydx + \int_1^{\sqrt{2}}\int_0^x xydydx + \int_{\sqrt{2}}^2\int_0^{\sqrt{4-x^2}} xydydx$$
$$= \int_0^{\frac{\pi}{4}}\int_1^2 r\cos\theta r\sin\theta rdrd\theta$$
$$= \int_1^2\int_0^{\frac{\pi}{4}} r^3\sin\theta d\sin\theta dr$$
$$= \int_1^2 r^3(\frac{\sin^2\theta}{2})\Big|_0^{\frac{\pi}{4}} dr$$
$$= \int_1^2 \frac{r^3}{4}dr$$
$$= (\frac{r^4}{16})\Big|_1^2 = \frac{15}{16}$$

40.

(a) *Proof.*

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dA &= \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}dA \\ &= \lim_{a\to\infty}\int_0^{2\pi}\int_0^{a} e^{-r^2}rdrd\theta \\ &= \lim_{a\to\infty}\int_0^{2\pi}(-\frac{1}{2})e^{-r^2}d(-r^2)d\theta \\ &= \lim_{a\to\infty}\int_0^{2\pi}(-\frac{1}{2})(e^{-r^2})\Big|_0^a d\theta \\ &= \lim_{a\to\infty}\int_0^{2\pi}\frac{1-e^{-a^2}}{2}d\theta \\ &= \lim_{a\to\infty}(1-e^{-a^2})\pi = \pi\end{aligned}$$

□

(b) *Proof.*

TODO

□

(c) *Proof.* Let $I = \int_{-\infty}^{\infty} e^{-x^2}dx = \int_{-\infty}^{\infty} e^{-y^2}dy$, then$I^2 = \pi$

$\because e^{-x^2} > 0 \therefore I > 0$

$$\therefore I = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

□

(d) *Proof.* Let $t = \sqrt{2}x$, then $x = \frac{t}{\sqrt{2}}$, then

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}d\frac{t}{\sqrt{2}} = \sqrt{\pi}$$

which is equivalent to

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}dt = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx = \sqrt{2\pi}$$

□

41.

(a)

$$
\begin{aligned}
\int_0^\infty x^2 e^{-x^2} dx &= \int_0^\infty (-\frac{x}{2}) de^{-x^2} \\
&= (-\frac{x}{2}e^{-x^2})\Big|_0^\infty - \int_0^\infty e^{-x^2} d(-\frac{x}{2}) \\
&= \frac{1}{2}\int_0^\infty e^{-x^2} dx \\
&= \frac{1}{4}\int_{-\infty}^\infty e^{-x^2} dx \\
&= \frac{\sqrt{\pi}}{4}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\int_0^\infty \sqrt{x} e^{-x} dx &= \int_0^\infty (-\sqrt{x}) de^{-x} \\
&= (-\sqrt{x}e^{-x})\Big|_0^\infty - \int_0^\infty e^{-x} d(-\sqrt{x}) \\
&= \int_0^\infty e^{-x} d\sqrt{x}
\end{aligned}
$$

Let $x = t^2$, then

$$
\int_0^\infty \sqrt{x} e^{-x} dx = \int_0^\infty e^{-t^2} dt = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}
$$