Exercise 14.3

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24. $w = \frac{e^v}{u+v^2}$

$$\frac{\partial w}{\partial u} = -\frac{e^v}{(u+v^2)^2}$$

$$\frac{\partial w}{\partial v} = \frac{e^v(u+v^2-2v)}{(u+v^2)^2}$$

26. $u(r,\theta) = \sin(r\cos\theta)$

$$\frac{\partial u}{\partial r} = \cos(r\cos\theta)\cos\theta$$

$$\frac{\partial u}{\partial \theta} = -r\sin\theta\cos(r\cos\theta)$$

29. $F(x,y) = \int_y^x \cos(e^t)dt$

$$\frac{\partial F}{\partial x} = \cos(e^x)$$

$$\frac{\partial F}{\partial y} = -\cos(e^y)$$

36. $u = x^{\frac{y}{z}}$

$$\frac{\partial u}{\partial x} = \frac{y}{z}x^{\frac{y}{z}-1}$$

$$\frac{\partial u}{\partial y} = \frac{x^{\frac{y}{z}}\ln x}{z}$$

$$\frac{\partial u}{\partial z} = -\frac{yx^{\frac{y}{z}}\ln x}{z^2}$$

39. $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

$$\frac{\partial u}{\partial x_i} = \frac{2x_i}{2\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}$$

where $i = 1, 2, \cdots, n$.

41. $f(x, y) = \ln(x + \sqrt{x^2 + y^2});\quad f_x(3, 4)$

$$f_x(x, y) = \frac{1 + \frac{x}{\sqrt{x^2+y^2}}}{x + \sqrt{x^2 + y^2}}$$

$$f_x(3, 4) = \frac{1 + \frac{3}{5}}{3 + 5} = \frac{1}{5}$$

42. $f(x, y) = \arctan(\frac{y}{x});\quad f_x(2, 3)$

$$f_x(x, y) = \frac{1}{1 + (\frac{y}{x})^2}(-\frac{y}{x^2})$$

$$f_x(2, 3) = \frac{1}{1 + (\frac{3}{2})^2}(-\frac{3}{4}) = -\frac{3}{4} \times \frac{4}{13} = -\frac{3}{13}$$

49. $e^z = xyz$

Differentiating implicitly with respect to x and treating y as constant:

$$e^z \frac{\partial z}{\partial x} = yz + yx\frac{\partial z}{\partial x}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}$$

Differentiating implicitly with respect to y and treating x as constant:

$$e^z \frac{\partial z}{\partial y} = xz + xy\frac{\partial z}{\partial y}$$

$$\therefore \frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$$

50. $yz + x\ln y = z^2$

To compute $\frac{\partial z}{\partial x}$, differentiate implicitly with respect to x:

$$y\frac{\partial z}{\partial x} + \ln y = 2z\frac{\partial z}{\partial x}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{\ln y}{y - 2z}$$

To compute $\frac{\partial z}{\partial y}$, differentiate implicitly with respect to y:

$$z + y\frac{\partial z}{\partial y} + \frac{x}{y} = 2z\frac{\partial z}{\partial y}$$

$$\therefore \frac{\partial z}{\partial y} = \frac{zy + x}{2yz - y^2}$$

52.

(a)

$$\frac{\partial z}{\partial x} = g(y)f'(x)$$

$$\frac{\partial z}{\partial y} = f(x)g'(y)$$

(b)

$$\frac{\partial z}{\partial x} = \frac{f'(\frac{x}{y})}{y}$$

$$\frac{\partial z}{\partial y} = -\frac{xf'(\frac{x}{y})}{y^2}$$

55. $w = \sqrt{u^2 + v^2}$

$$\frac{\partial w}{\partial u} = \frac{2u}{2\sqrt{u^2+v^2}} = \frac{u}{\sqrt{u^2+v^2}}$$

$$\frac{\partial w}{\partial v} = \frac{2v}{2\sqrt{u^2+v^2}} = \frac{v}{\sqrt{u^2+v^2}}$$

$$\frac{\partial}{\partial u}(\frac{\partial w}{\partial u}) = \frac{\sqrt{u^2+v^2} - \frac{u^2}{\sqrt{u^2+v^2}}}{u^2+v^2} = \frac{\frac{v^2}{\sqrt{u^2+v^2}}}{u^2+v^2} = \frac{v^2}{(u^2+v^2)^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial v}(\frac{\partial w}{\partial u}) = \frac{-u\frac{v}{\sqrt{u^2+v^2}}}{u^2+v^2} = \frac{-uv}{(u^2+v^2)^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial u}(\frac{\partial w}{\partial v}) = \frac{-v\frac{2u}{2\sqrt{u^2+v^2}}}{u^2+v^2} = \frac{-uv}{(u^2+v^2)^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial v}(\frac{\partial w}{\partial v}) = \frac{\sqrt{u^2+v^2} - \frac{v^2}{\sqrt{u^2+v^2}}}{u^2+v^2} = \frac{u^2}{(u^2+v^2)^{\frac{3}{2}}}$$

56. $v = \frac{xy}{x-y}$

$$\frac{\partial v}{\partial x} = \frac{y(x-y) - xy}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{x(x-y) - xy(-1)}{(x-y)^2} = \frac{x^2}{(x-y)^2}$$

$$\frac{\partial}{\partial x}(\frac{\partial v}{\partial x}) = \frac{-(-y^2)2(x-y)}{(x-y)^4} = \frac{2y^2}{(x-y)^3}$$

$$\frac{\partial}{\partial y}(\frac{\partial v}{\partial y}) = \frac{2x^2(x-y)}{(x-y)^4} = \frac{2x^2}{(x-y)^3}$$

$$\frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) = \frac{2x(x-y)^2 - 2x^2(x-y)}{(x-y)^4} = \frac{2x(x-y) - 2x^2}{(x-y)^3} = \frac{-2xy}{(x-y)^3}$$

$$\frac{\partial}{\partial y}(\frac{\partial v}{\partial x}) = \frac{-2y(x-y)^2 - 2y^2(x-y)}{(x-y)^4} = \frac{-2xy}{(x-y)^3}$$

67. $u = e^{r\theta}\sin\theta; \quad \frac{\partial^3 u}{\partial r^2 \partial\theta}$

$$\frac{\partial u}{\partial \theta} = e^{r\theta} r \sin\theta + e^{r\theta}\cos\theta$$

$$\frac{\partial^2 u}{\partial r \partial\theta} = \sin\theta(e^{r\theta} + r\theta e^{r\theta}) + e^{r\theta}\theta\cos\theta = (1+r\theta)e^{r\theta}\sin\theta + e^{r\theta}\cos\theta$$

$$\frac{\partial^3 u}{\partial r^2 \partial\theta} = e^{r\theta}\sin\theta(2\theta + r\theta^2) + e^{r\theta}\theta\cos\theta = e^{r\theta}\theta[(2+r\theta)\sin\theta + \cos\theta]$$

71. If $f(x,y,z) = xy^2z^3 + \arcsin(x\sqrt{z})$, find f_{xzy}.

$$f_x(x,y,z) = y^2z^3 + \frac{\sqrt{z}}{\sqrt{1-zx^2}}$$

$$f_{xz}(x,y,z) = 3y^2z^2 + \frac{\frac{\sqrt{1-zx^2}}{2\sqrt{z}} + \sqrt{z}\frac{zx}{\sqrt{1-zx^2}}}{1-zx^2} = 3y^2z^2 + \frac{1-zx^2+2xz^2}{2\sqrt{z}(1-zx^2)^{\frac{3}{2}}}$$

$$f_{xzy}(x,y,z) = 6z^2y$$

76.

(d) $u = \ln \sqrt{x^2 + y^2}$

$$u_x = \frac{1}{\sqrt{x^2 + y^2}} \frac{1}{2\sqrt{x^2 + y^2}} 2x = \frac{x}{x^2 + y^2}$$

$$u_{xx} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

By the property of symmetry, similarly we have:

$$u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\therefore u_{xx} + u_{yy} = \frac{y^2 - x^2 + x^2 - y^2}{(x^2 + y^2)^2} = 0$$

which indicates that u is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.

77.

Proof. $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$

$$u_x = \frac{-\frac{2x}{2\sqrt{x^2+y^2+z^2}}}{x^2 + y^2 + z^2} = \frac{-x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$u_{xx} = -\frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - x\frac{3}{2}\sqrt{x^2 + y^2 + z^2} \cdot 2x}{(x^2 + y^2 + z^2)^3} = -\frac{x^2 + y^2 + z^2 - 3x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

Similarly, we can get:

$$u_{yy} = -\frac{x^2 + y^2 + z^2 - 3y^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

$$u_{zz} = -\frac{x^2 + y^2 + z^2 - 3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

$$\therefore u_{xx} + u_{yy} + u_{zz} = -\frac{3(x^2 + y^2 + z^2) - 3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = 0$$

which indicates u is a solution of the three-dimensional Laplace equation. □

79.

Proof. $u(x,t) = f(x+at) + g(x-at)$

$$u_x = f'(x+at) + g'(x-at)$$

$$u_{xx} = f''(x+at) + g''(x-at)$$

$$u_t = af'(x+at) - ag'(x-at)$$

$$u_{tt} = a^2 f''(x+at) + a^2 g''(x-at)$$

$$\therefore u_{tt} = a^2[f''(x+at) + g''(x-at)] = a^2 u_{xx}$$

$\therefore u(x,t)$ is a solution of the wave equation in Exercise 78. $\square$

101.

(a)

(b) When $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{(3yx^2 - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{(x^3 - 3xy^2)(x^2+y^2) - (x^3y - xy^3)(2y)}{(x^2+y^2)^2} = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

(c)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{x\to x_0} \frac{\frac{0}{x^2} - 0}{x - 0} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{y\to y_0} \frac{\frac{0}{y^2} - 0}{y - 0} = 0$$

(d)

$$\begin{aligned} f_{xy}(0,0) &= \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{0-h^5}{(0+h^2)^2} - 0}{h} = -1 \\ f_{yx}(0,0) &= \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h^5}{(h^2+0)^2} - 0}{h} = 1 \end{aligned}$$

(e) No. Because f is not continuous at $(0,0)$.

$$f_{xy}(x,y) = \frac{\partial}{\partial y}(\frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}(\frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

which illustrates that Clairaut's Theorem is still correct.