# Exercise 16.3

Wang Yue from CS Elite Class

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### 7.

Let $P(x,y) = ye^x + \sin y, Q(x,y) = e^x + x\cos y$, then

$$\frac{\partial P}{\partial y} = e^x + \cos y, \frac{\partial Q}{\partial x} = e^x + \cos y$$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
$\because$ the domain of $\overrightarrow{F}$ is in $\mathbb{R}^2$
$\therefore \overrightarrow{F}$ is conservative
Let $\overrightarrow{F} = \nabla f$, then

$$f(x,y) = \int P(x,y)dx + \varphi(y) = ye^x + x\sin y + \varphi(y)$$

$$Q(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\int P(x,y)dx) + \varphi'(y)$$

$$\frac{\partial}{\partial y}(ye^x + x\sin y) + \varphi'(y) = e^x + x\cos y + \varphi'(y) = e^x + x\cos y$$

$\therefore \varphi'(y) = 0, \varphi(y) = C$

$$\therefore f(x,y) = ye^x + x\sin y + C$$

### 8.

Let $P(x,y) = 2xy + y^{-2}, Q(x,y) = x^2 - 2xy^{-3}$, where $y > 0$, then

$$\frac{\partial P}{\partial y} = 2x - 2y^{-3}, \frac{\partial Q}{\partial x} = 2x - 2y^{-3}$$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
$\because \frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous throughout the domain of $\overrightarrow{F}$
$\therefore \overrightarrow{F}$ is conservative

$$f(x,y) = \int P(x,y)dx + \varphi(y) = x^2y + xy^{-2} + \varphi(y)$$

$$Q(x,y) = \frac{\partial f}{\partial y} = x^2 - 2xy^{-3} + \varphi'(y) = x^2 - 2xy^{-3}$$

$\therefore \varphi'(y) = 0, \varphi(y) = C$

$$\therefore f(x,y) = x^2 y + xy^{-2} + C$$

## 9.

Let $P(x,y) = \ln y + 2xy^3, Q(x,y) = 3x^2y^2 + \frac{x}{y}$ where $y > 0$, then

$$\frac{\partial P}{\partial y} = \frac{1}{y} + 6xy^2 = \frac{\partial Q}{\partial x} = 6xy^2 + \frac{1}{y}$$

$\because \frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ are continuous throughout the domain of $\overrightarrow{F}$
$\therefore \overrightarrow{F}$ is conservative

$$f(x,y) = \int P(x,y)dx + \varphi(y) = x \ln y + x^2y^3 + \varphi(y)$$

$$Q(x,y) = \frac{\partial f}{\partial y} = \frac{x}{y} + 3x^2y^2 + \varphi'(y) = 3x^2y^2 + \frac{x}{y}$$

$\therefore \varphi'(y) = 0, \varphi(y) = C$

$$\therefore f(x,y) = x \ln y + x^2y^3 + C$$

## 13.

Let $P(x,y) = xy^2, Q(x,y) = x^2y$

$$\frac{\partial P}{\partial y} = 2xy = \frac{\partial Q}{\partial x} = 2xy$$

$\therefore \overrightarrow{F}$ is conservative, i.e., $\exists f, s.t. \overrightarrow{F} = \nabla f$

$$f(x,y) = \int P(x,y)dx + \varphi(y) = \frac{1}{2}x^2y^2 + \varphi(y)$$

$$Q(x,y) = \frac{\partial f}{\partial y} = x^2y + \varphi'(y) = x^2y$$

$\therefore \varphi'(y) = 0, \varphi(y) = C$

$$\therefore f(x,y) = \frac{1}{2}x^2y^2$$

$$\therefore \int_C \nabla f \cdot d\overrightarrow{r} = f(\overrightarrow{r}(b)) - f(\overrightarrow{r}(a)) = f(2,1) - f(0,1) = 2$$

## 17.

Obviously, $\overrightarrow{F}$ is conservative.

Let $P(x,y,z) = yze^{xz}, Q(x,y,z) = e^{xz}, R(x,y,z) = xye^{xz}$

$$f(x,y,z) = \int P(x,y)dx + \varphi(y,z) = ye^{xz} + \varphi(y,z)$$

$$Q(x,y,z) = \frac{\partial f}{\partial y} = e^{xz} + \frac{\partial \varphi}{\partial y} = e^{xz}$$

$$R(x,y,z) = \frac{\partial f}{\partial z} = xye^{xz} + \frac{\partial \varphi}{\partial z} = xye^{xz}$$

$\therefore \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial z} = 0, \therefore \varphi(y,z) = C$

$$\therefore f(x,y,z) = ye^{xz}$$

$$\therefore \int_C \nabla f \cdot d\overrightarrow{r} = f(\overrightarrow{r}(2)) - f(\overrightarrow{r}(0)) = f(5,3,0) - f(1,-1,0) = 3 - (-1) = 4$$

## 19.

Let $P(x,y) = 2xe^{-y}, Q(x,y) = 2y - x^2e^{-y}$, then

$$\frac{\partial P}{\partial y} = -2xe^{-y} = \frac{\partial Q}{\partial x} = -2xe^{-y}$$

$\therefore \overrightarrow{F}$ is conservative

Let $\overrightarrow{F} = \nabla f$, then

$$f(x,y) = \int P(x,y)dx + \varphi(y) = x^2e^{-y} + \varphi(y)$$

$$Q(x,y) = \frac{\partial f}{\partial y} = -x^2e^{-y} + \varphi'(y) = -x^2e^{-y} + 2y$$

$\therefore \varphi'(y) = 2y, \varphi(y) = y^2 + C$

$$\therefore f(x,y) = x^2e^{-y} + y^2 + C$$

$$\therefore \int_C \nabla f \cdot d\overrightarrow{r} = f(2,1) - f(1,0) = 4e^{-1}$$

## 20.

Let $P(x,y) = \sin y, Q(x,y) = x\cos y - \sin y$, then

$$\frac{\partial P}{\partial y} = \cos y = \frac{\partial Q}{\partial x} = \cos y$$

$\therefore \overrightarrow{F}$ is conservative
Let $\overrightarrow{F} = \nabla F$, then

$$f(x,y) = \int P(x,y)dx + \varphi(y) = x\sin y + \varphi(y)$$

$$Q(x,y) = \frac{\partial f}{\partial y} = x\cos y + \varphi'(y) = x\cos y - \sin y$$

$\therefore \varphi'(y) = -\sin y, \varphi(y) = \cos y + C$

$$\therefore f(x,y) = x\sin y + \cos y + C$$

$$\therefore \int_C \nabla f \cdot d\overrightarrow{r} = f(1,\pi) - f(2,0) = -1 - 1 = -2$$

## 29.

$$\because P = \frac{\partial f}{\partial x}, Q = \frac{\partial f}{\partial y}, R = \frac{\partial f}{\partial z}$$

$\because P, Q, R$ all have continuous first-order partial derivatives

$$\therefore \begin{cases} \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \\ \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \\ \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \end{cases}$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},$$

## 30.

Let $P(x,y,z) = y, Q(x,y,z) = x, R(x,y,z) = xyz$, then

$$\because \frac{\partial P}{\partial z} = 0 \neq \frac{\partial R}{\partial x} = yz$$

$\therefore \overrightarrow{F} = P(x,y,z)\overrightarrow{i} + Q(x,y,z)\overrightarrow{j} + R(x,y,z)\overrightarrow{k}$ is not conservative
$\therefore \int_C ydx + xdy + xyzdz$ is not independent of path

**35.**

(a) Let $P(x,y) = -\frac{y}{x^2+y^2}, Q(x,y) = \frac{x}{x^2+y^2}$

$$\frac{\partial P}{\partial y} = -\frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

(b) In region $D = \{(x,y)|x^2+y^2 \leq 1\}$, the parametric equation is $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$

$$\int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_0^{\pi} (-\sin t)^2 + (\cos t)^2 dt = \pi$$

$$\int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{2\pi}^{\pi} (-\sin t)^2 + (\cos t)^2 dt = -\pi$$

This does not contradict with Theorem 6, because $P$ and $Q$ are both undefined in $(0,0)$, which is a point inside the region $D$. Since $P$ and $Q$ do not have continuous first-order derivatives, we cannot get the conclusion of Theorem 6.