Exercise 17.2

Wang Yue from CS Elite Class

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6.

The auxiliary equation is $\lambda^2 - 4\lambda + 4 = 0$ with a double root 2.

For the equation $y'' - 4y' + 4y = 0$, $y_c = C_1 e^{2x} + C_2 x e^{2x}$

For the equation $y'' - 4y' + 4y = xe^{0x}$, let $y_{p_1}(x) = Ax + B$.

Then $y'_{p_1}(x) = A$, $y''_{p_1}(x) = 0$, so

$$-4A + 4(Ax + B) = 4Ax + 4B - 4A = 1x + 0$$

$\therefore A = B = \frac{1}{4}$

$\therefore y_{p_1}(x) = \frac{1}{4}x + \frac{1}{4}$

For the equation $y'' - 4y' + 4y = -\sin x$, let $y_{p_2}(x) = C\sin x + D\cos x$.

Then $y'_{p_2}(x) = C\cos x - D\sin x, y''_{p_2}(x) = -C\sin x - D\cos x$, so

$$(-C\sin x - D\cos x) - 4(C\cos x - D\sin x) + 4(C\sin x + D\cos x) = -\sin x$$

$$(4D + 3C)\sin x + (-4C + 3D)\cos x = -1\sin x + 0\cos x$$

Finally, $C = -\frac{3}{25}, D = \frac{4}{3}D = -\frac{4}{25}$

$\therefore y_{p_2}(x) = -\frac{3}{25}\sin x - \frac{4}{25}\cos x$

$\therefore$ the general solution is

$$\begin{aligned} y &= y_c + y_{p_1} + y_{p_2} \\ &= C_1 e^{2x} + C_2 x e^{2x} + \frac{1}{4}x + \frac{1}{4} - \frac{3}{25}\sin x - \frac{4}{25}\cos x \end{aligned}$$

where C_1 and C_2 are arbitrary constants.

7.

The auxiliary equation $\lambda^2 + 1 = 0$, with complex roots $\lambda = \pm i$.

For the equation $y'' + y = 0$, $y_c = C_1\cos x + C_2\sin x$

For the equation $y'' + y = e^x$, let $y_{p_1}(x) = Ae^x$.

Then $y'_{p_1}(x) = y''_{p_1}(x) = Ae^x$, so

$$Ae^x + Ae^x = e^x$$

$\because$ Obviously, $A = \frac{1}{2}$
$\therefore y_{p_1}(x) = \frac{1}{2}e^x$
For the equation $y'' + y = x^3 e^{0x}$, let $y_{p_2}(x) = Ax^3 + Bx^2 + Cx + D$.
Then $y'_{p_2}(x) = 3Ax^2 + 2Bx + C, y''_{p_2}(x) = 6Ax + 2B$, so

$$Ax^3 + Bx^2 + (6A + C)x + 2B + D = x^3$$

$\therefore A = 1, C = -6, B = D = 0$
$\therefore y_{p_2}(x) = x^3 - 6x$
$\therefore$ the general solution is

$$\begin{aligned} y &= y_c + y_{p_1} + y_{p_2} \\ &= C_1 \cos x + C_2 \sin x + \frac{1}{2}e^x + x^3 - 6x \end{aligned}$$

$\because y(0) = C_1 + \frac{1}{2} = 2, y'(0) = C_2 + \frac{1}{2} - 6 = 0$
$\therefore C_1 = \frac{3}{2}, C_2 = \frac{11}{2}$
$\therefore$ the general solution is

$$y = \frac{3}{2}\cos x + \frac{11}{2}\sin x + \frac{1}{2}e^x + 3x^2 - 6x$$

8.

The auxiliary equation $\lambda^2 - 4\lambda = 0$, with roots $\lambda = 4$ and $\lambda = 0$
For the equation $y'' - 4y = 0$, $y_c = C_1 + C_2 e^{4x}$
For the equation $y'' - 4y = e^x \cos x$, let $y_p(x) = Ae^x \cos x + Be^x \sin x$
Then $y'_p(x) = Ae^x(\cos x - \sin x) + Be^x(\sin x + \cos x)$
Then $y''_p(x) = Ae^x(\cos x - \sin x - \sin x - \cos x) + Be^x(\sin x + \cos x + \cos x - \sin x)$
Then $y''_p(x) = -2Ae^x \sin x + 2Be^x \cos x$, so

$$(-2A - 4B)e^x \sin x + (2B - 4A)e^x \cos x = e^x \cos x$$

Since $-2A - 4B = 0, 2B - 4A = 1,\ A = -\frac{1}{5}, B = \frac{1}{10}$
Therefore, $y_p(x) = -\frac{1}{5}e^x \cos x + \frac{1}{10}e^x \sin x$
$\therefore$ the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 + C_2 e^{4x} - \frac{1}{5}e^x \cos x + \frac{1}{10}e^x \sin x \end{aligned}$$

$\because y(0) = C_1 + C_2 - \frac{1}{5} = 2, y'(0) = 4C_2 - \frac{1}{5}(1-0) + \frac{1}{10}(0+1) = 4C_2 - \frac{1}{10} = 2$
$\therefore C_2 = \frac{21}{40}, C_1 = \frac{67}{40}$
$\therefore$ the general solution is

$$y = \frac{67}{40} + \frac{21}{40}e^{4x} - \frac{1}{5}e^x \cos x + \frac{1}{10}e^x \sin x$$

9.

The auxiliary equation is $\lambda^2 - \lambda = 0$, with roots $\lambda = 0$ and $\lambda = 1$

For the equation $y'' - y' = 0$, $y_c = C_1 + C_2 e^x$

For the equation $y'' - y' = xe^x$, let $y_p(x) = (Ax^2 + Bx)e^x$

Then $y_p'(x) = e^x(Ax^2 + Bx + 2Ax + B)$

Then $y_p''(x) = e^x(Ax^2 + Bx + 2Ax + B + 2Ax + B + 2A)$, so

$$e^x(2Ax + B + 2A) = xe^x$$

Since $2A = 1, B + 2A = 0$, $A = \frac{1}{2}, B = -1$

$\therefore$ the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 + C_2 e^x + (\frac{1}{2}x^2 - x)e^x \end{aligned}$$

$\because y(0) = C_1 + C_2 = 2, y'(0) = C_2 + (-1) = 1$

$\therefore C_2 = 2, C_1 = 0$

$\therefore$ the general solution is

$$y = 2e^x + (\frac{1}{2}x^2 - x)e^x$$

16.

The auxiliary equation is $\lambda^2 + 3\lambda - 4 = 0$, with roots $\lambda = -4$ and $\lambda = 1$

For equation $y'' + 3y' - 4y = 0$, $y_c = C_1 e^x + C_2 e^{-4x}$

For equation $y'' + 3y' - 4y = x^3 e^x$, let $y_{p_1}(x) = x(Ax^3 + Bx^2 + Cx + D)e^x$

For equation $y'' + 3y' - 4y = xe^x$, let $y_{p_2}(x) = x(Ex + F)e^x$

$\therefore$ the trial solution is

$$\begin{aligned} y &= y_c + y_{p_1} + y_{p_2} \\ &= C_1 e^x + C_2 e^{-4x} + (Ax^3 + Bx^2 + Cx + Ex + D + F)xe^x \end{aligned}$$

17.

The auxiliary equation is $\lambda^2 + 2\lambda + 10 = 0$, with $\Delta = \sqrt{-36}$

Therefore, the solution is $\lambda = \frac{-2+6i}{2} = -1 + 3i$ or $\lambda = \frac{-2-6i}{2} = -1 - 3i$

For the equation $y'' + 2y' + 10y = 0$, $y_c = e^{-x}(C_1 \cos 3x + C_2 \sin 3x)$

For the equation $y'' + 2y' + 10y = x^2 e^{-x} \cos 3x$,

let $y_p = e^{-x}[(Ax^2 + Bx + C)\cos 3x + (Dx^2 + Ex + F)\sin 3x]$

$\because \cos 3x, \sin 3x$ are solutions to y_c

$\therefore$ the trial solution is

$$\begin{aligned} y &= y_c + xy_p \\ &= \{[C_1 + x(Ax^2 + Bx + C)]\cos 3x + [C_2 + x(Dx^2 + Ex + F)]\sin 3x]\}e^{-x} \end{aligned}$$

18.

The auxiliary equation $\lambda^2 + 4 = 0$, with complex roots $\lambda = \pm 2i$

For the equation $y'' + 4y = 0$, $y_c = C_1 \cos 2x + C_2 \sin 2x$

For the equation $y'' + 4y = e^{3x}$, let $y_{p_1} = Ae^{3x}$

For the equation $y''+4y = x \sin 2x$, let $y_{p_2} = (Bx+C) \sin 2x+(Dx+E) \cos 2x$

$\because \sin 2x, \cos 2x$ are solutions to y_c

$\therefore$ the trial solution is

$$\begin{aligned} y &= y_c + y_{p_1} + xy_{p_2} \\ &= Ae^{3x} + (Bx^2 + Cx + C_2) \sin 2x + (Dx^2 + Ex + C_1) \cos 2x \end{aligned}$$

23.

The auxiliary equation is $\lambda^2 + 1 = 0$, with complex roots $\lambda = \pm i$

For the equation $y'' + y = 0$, $y_c = C_1 \cos x + C_2 \sin x$

Let $y_p(x) = u_1(x) \cos x + u_2(x) \sin x$

$y_p'(x) = u_1'(x) \cos x + u_2'(x) \sin x - u_1(x) \sin x + u_2(x) \cos x$

Impose the constraint that $u_1'(x) \cos x + u_2'(x) \sin x = 0$, then

$y_p''(x) = -u_1'(x) \sin x + u_2'(x) \cos x - u_1(x) \cos x - u_2(x) \sin x$

$$-u_1'(x) \sin x + u_2'(x) \cos x = \sec^2 x$$

$$u_1'(x) = \frac{\begin{vmatrix} 0 & \sin x \\ \sec^2 x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = -\sin x \sec^2 x = -\tan x \sec x$$

$$u_1(x) = -\sec x + C_3$$

$$u_2'(x) = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^2 x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \cos x \sec^2 x = \sec x = \frac{1}{\cos x}$$

$$u_2(x) = \ln(\sec x + \tan x) + C_4$$

$$\therefore y_p(x) = (-\sec x + C_3) \cos x + (\ln(\sec x + \tan x) + C_4) \sin x$$

$\therefore$ the solution is

$$y = (-\sec x + C_3 + C_1) \cos x + (\ln(\sec x + \tan x) + C_4 + C_2) \sin x$$

25.

The auxiliary equation is $\lambda^2 - 3\lambda + 2 = 0$, with roots $\lambda = 1$ and $\lambda = 2$

For the equation $y'' - 3y' + 2y = 0$, $y_c = C_1e^x + C_2e^{2x}$

Let $y_p(x) = u_1(x)e^x + u_2(x)e^{2x}$

Then $y_p'(x) = u_1'(x)e^x + u_2'(x)e^{2x} + u_1(x)e^x + 2u_2(x)e^{2x}$

Impose the constraint that $u_1'(x)e^x + u_2'(x)e^{2x} = 0$

Then $y_p''(x) = u_1'(x)e^x + 2u_2'(x)e^{2x} + u_1(x)e^x + 4u_2(x)e^{2x}$

$$u_1'(x)e^x+2u_2'(x)e^{2x}+u_1(x)e^x+4u_2(x)e^{2x}-3u_1(x)e^x-6u_2(x)e^{2x}+2u_1(x)e^x+2u_2(x)e^{2x} = \frac{1}{1+e^{-x}}$$

$$u_1'(x)e^x + 2u_2'(x)e^{2x} = \frac{1}{1+e^{-x}}$$

$$\because u_1'(x)e^x + u_2'(x)e^{2x} = 0$$

$$u_1'(x) = \frac{-e^{-x}}{1+e^{-x}}, u_2'(x) = \frac{e^{-2x}}{1+e^{-x}}$$

$$u_1(x) = \ln(1+e^{-x}), u_2(x) = -e^{-x} + \ln(1+e^{-x})$$

$\therefore$ the general solution is

$$y = C_1e^x + C_2e^{2x} + \ln(1+e^{-x)e^x+(-e^{-x}+\ln(1+e^{-x}))e^{2x}}$$