Exercise 12.5

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18.

Let $A(10, 3, 1), B(5, 6, -3)$, then $\overrightarrow{AB} = (-5, 3, -4)$

Suppose P is an arbitrary point in the line segment AB, then

$$\overrightarrow{OP} = \overrightarrow{OA} + t\overrightarrow{AB} = (10 - 5t, 3 + 3t, 1 - 4t)$$

where $0 \leq t \leq 1$.

$\therefore$ parametric equations for the line segment is

$$x = 10 - 5t \quad y = 3 + 3t \quad z = 1 - 4t \quad 0 \leq t \leq 1$$

26.

$\because$ the plane is perpendicular to the line $x = 3t, y = 2 - t, z = 3 + 4t$

$\therefore \overrightarrow{n} = (3, -1, 4)$

$\because a = 3, b = -1, c = 4, x_0 = 2, y_0 = 0, z_0 = 1$

$\therefore$ an euqation of the plane is $3(x - 2) - (y - 0) + 4(z - 1) = 0$, or

$$3x - y + 4z = 10$$

27.

$\because$the normal vector of the plane $5x - y - z = 6$ is $\overrightarrow{n} = (5, -1, -1)$.

$\therefore a = 5, b = -1, c = -1, x_0 = 1, y_0 = -1, z_0 = -1$

$\therefore$ an equation of the plane is $5(x - 1) - (y + 1) - (z + 1) = 0$, or

$$5x - y - z = 7$$

34.

Let $P(3t, 1 + t, 2 - t)$, then we know P can be an arbitrary point in the line $x = 3t, y = 1 + t, z = 2 - t$.

Let $P_0(1, 2, 3)$, then $\overrightarrow{P_0P} = (3t - 1, t - 1, -t - 1)$

Suppose the normal vector $\overrightarrow{n} = (a, b, c)$, then

$$\overrightarrow{n} \cdot \overrightarrow{P_0P} = (3t-1)a + (t-1)b - (t+1)c = 0$$

When $t = 1$, we can get $2a - 2c = 0 \Rightarrow a = c$
When $t = -1$, we can get $-4a - 2b = 0 \Rightarrow b = -2a$
Suppose $a = 1$, then we know one of the normal vector is $\overrightarrow{n} = (1, -2, 1)$
$\therefore a = 1, b = -2, c = 1, x_0 = 1, y_0 = 2, z_0 = 3$
$\therefore$ an equation of the plane is $(x-1) - 2(y-2) + (z-3) = 0$, or

$$x - 2y + z = 0$$

40.

The plane that passes through the line of intersection of the plane$x - z = 1$ and $y + 2z = 3$ can be represented as

$$(x - z - 1) + \lambda(y + 2z - 3) = 0$$

which is equivalent to

$$x + \lambda y + (2\lambda - 1)z - 3\lambda - 1 = 0$$

$\therefore$ the normal vector of the plane is $\overrightarrow{m} = (1, \lambda, 2\lambda - 1)$
$\because$ the normal vector of $x + y - 2z = 1$ is $\overrightarrow{n} = (1, 1, -2)$
$\because$ these two planes are perpendicular to each other
$\therefore \overrightarrow{m}$ is perpendicular to $\overrightarrow{n}$
$\therefore \overrightarrow{m} \cdot \overrightarrow{n} = 1 + \lambda - 4\lambda + 2 = 0$
$\therefore$ we can solve that $\lambda = 1$
$\therefore$ an equation of the plane is $x - z - 1 + y + 2z - 3 = 0$, or

$$x + y + z = 4$$

49.

$$\because \begin{cases} x + y + z = 1 \\ x + z = 0 \end{cases} \Rightarrow \begin{cases} x + z = 0 \\ y = -1 \end{cases}$$

Let $P_0(0, -1, 0)$, $\overrightarrow{v} = (1, 0, 1)$
$\therefore$ the direction numbers of $\begin{cases} x + z = 0 \\ y = -1 \end{cases}$ is 1, 0 and 1.

50.

The normal vector of $x + y + z = 0$ is $\overrightarrow{n} = (1, 1, 1)$
Similarly, the normal vector of $x + 2y + 3z = 1$ is $\overrightarrow{m} = (1, 2, 3)$
$\therefore$ if θ is the angle between the planes, then

$$\cos\theta = \frac{\overrightarrow{n} \cdot \overrightarrow{m}}{|\overrightarrow{n}||\overrightarrow{m}|} = \frac{1 + 2 + 3}{\sqrt{3}\sqrt{14}} = \frac{2\sqrt{3}}{\sqrt{14}}$$

$$\therefore \theta = \arccos(\frac{2\sqrt{3}}{\sqrt{14}}) \approx 22.2^\circ$$

61.

Let $A(1, 0, -2), B(3, 4, 0)$, then we can get their midpoint $M(2, 2, -1)$.

Plus, we know $\overrightarrow{AB} = (2, 4, 2)$.

The plane in which every point is equidistant from A and B has normal vector $\overrightarrow{AB}$ and passes the point M. That is $2(x-2) + 4(y-2) + 2(z+1) = 0$. That is

$$x + 2y + z = 5$$

75.

Proof. Let $A(x_1, y_1, z_1)$ be an arbitrary point in the plane $ax + by + cz + d_1 = 0$

Then we can compute the distance between A and $ax + by + cz + d_2 = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$$

□

79.

Let $A(2, 0, -1), B(1, -1, 1), C(4, 1, 3)$

Then L_1 can be represented as

$$t\overrightarrow{OA} = (2t, 0, -t)$$

Then L_2 can be represented as

$$\overrightarrow{OB} + t\overrightarrow{BC} = (1, -1, 1) + (3t, 2t, 2t) = (3t + 1, 2t - 1, 2t + 1)$$

Let $\overrightarrow{v_1} = \overrightarrow{OA} = (2, 0, -1), \overrightarrow{v_2} = \overrightarrow{BC} = (3, 2, 2)$

Then $\overrightarrow{v_1} \times \overrightarrow{v_2} = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 3 & 2 & 2 \end{vmatrix} = 2\overrightarrow{i} - 7\overrightarrow{j} + 4\overrightarrow{k}$

Since L_1 and L_2 are skew, they can be viewed as lying on two parallel planes P_1 and P_2.

An equation of P_1 is $2x - 7y + 4k = 0$

An equation of P_2 is $2(x-1) - 7(y+1) + 4(z-1) = 0$ or $2x - 7y + 4z = 13$

Therefore, the distance of L_1 and L_2 is equal to the distance from $O(0, 0, 0)$ to $2x - 7y + 4z = 13$. That is

$$D = \frac{|0 - 0 + 0 - 13|}{\sqrt{2^2 + (-7)^2 + 4^2}} = \frac{13}{\sqrt{69}} \approx 1.565$$

81.

Proof. $\because ax + by + cz + d = 0$ can be rewritten as

$$a(x + \frac{d}{a}) + b(y - 0) + c(z - 0) = 0$$

which can be expressed as $(a, b, c) \cdot (x + \frac{d}{a}, y, z) = 0$

Let $\overrightarrow{n} = (a, b, c)$, $\overrightarrow{r} = (x, y, z)$, $\overrightarrow{r_0} = (-\frac{d}{a}, 0, 0)$, then $(a, b, c) \cdot (x + \frac{d}{a}, y, z) = 0$ can be rewritten as

$$\overrightarrow{n} \cdot (\overrightarrow{r} - \overrightarrow{r_0}) = 0$$

which is called the vector equation of the plane.

$\therefore ax + by + cz + d = 0$ represents a plane and (a, b, c) is normal vector. $\square$