Exercise 14.7

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9. $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$

$$f_x(x,y) = 6xy - 12x, f_y(x,y) = 3y^2 + 3x^2 - 12y$$

Let $f_x(x,y) = f_y(x,y) = 0$, then $\begin{cases} x = 0 \\ y = 0 \end{cases}$ or $\begin{cases} x = \pm 2 \\ y = 2 \end{cases}$

$$f_{xx} = 6y - 12, f_{yy} = 6y - 12, f_{xy} = 6x$$

$$D = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6y-12)^2 - (6x)^2 = 36(y-2+x)(y-2-x)$$

When $x = 0, y = 0$, $D = 36 \times (-2)^2 > 0, f_{xx}(0,0) = -12 < 0$
thus $f(0,0)$ is a local maximum
When $x = \pm 2, y = 2$, $D = 36 \times 2 \times (-2) < 0$
thus $f(\pm 2, 2)$ is a saddle point, i.e. not a local maximum or local minimum

15. $f(x,y) = (x^2 + y^2)e^{y^2 - x^2}$

$$f_x(x,y) = 2xe^{y^2-x^2} + (x^2+y^2)e^{y^2-x^2}(-2x) = 2xe^{y^2-x^2}(1 - x^2 - y^2)$$

$$f_y(x,y) = 2ye^{y^2-x^2} + (x^2+y^2)e^{y^2-x^2}2y = 2ye^{y^2-x^2}(1 + x^2 + y^2)$$

Let $f_x(x,y) = f_y(x,y) = 0$, then $\begin{cases} x = 0 \\ y = 0 \end{cases}$ or $\begin{cases} x = \pm 1 \\ y = 0 \end{cases}$

$$\begin{aligned} f_{xx}(x,y) &= 2e^{y^2-x^2}(1 - x^2 - y^2) + 2e^{y^2-x^2}(-2x)(1 - x^2 - y^2) + 2xe^{y^2-x^2}(-2x) \\ &= 2e^{y^2-x^2}[(1 - 2x^2)(1 - x^2 - y^2) - 2x^2] \end{aligned}$$

$$\begin{aligned} f_{yy}(x,y) &= 2e^{y^2-x^2}(1 + x^2 + y^2) + 2ye^{y^2-x^2}(2y)(1 + x^2 + y^2) + 2ye^{y^2-x^2}(2y) \\ &= 2e^{y^2-x^2}[(1 + 2y^2)(1 + x^2 + y^2) + 2y^2] \end{aligned}$$

$$f_{xy}(x,y) = 2xe^{y^2-x^2}(2y)(1-x^2-y^2) + 2xe^{y^2-x^2}(-2y)$$
$$= 4xye^{y^2-x^2}(-x^2-y^2)$$

When $x = 0, y = 0$,

$$D = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = 2 \times 2 - 0 = 4 > 0, \quad f_{xx}(x,y) > 0$$

$\therefore f$ reach its minimum at $(0,0)$
When $x = \pm 1, y = 0$,

$$D = (-4e^{-1}) \times 4e^{-1} - 0 < 0, \quad f_{xx}(x,y) < 0$$

$\therefore (\pm 1, 0)$ are saddle points

33. $f(x,y) = x^4+y^4-4xy+2, \quad D = \{(x,y)|0 \le x \le 3, 0 \le y \le 2\}$

$$f_x(x,y) = 4x^3 - 4y, f_y(x,y) = 4y^3 - 4x$$

Let $f_x(x,y) = f_y(x,y) = 0$, then $x = 1, y = 1$
$f(0,y) = 0 + y^4 - 0 + 2 = x^4 + 2 \in [2, 83]$
$f(x,0) = y^4 + 2 \in [2, 18]$
$f(3,y) = 81+y^4-12y+2 = y(y^3-12)+83\ in[3^{\frac{4}{3}}-12\times 3^{\frac{1}{3}}+83, 83+16-24]$
$f(x,2) = x^4 + 16 - 8x + 2 = x^4 - 8x + 18 \in [2^{\frac{4}{3}} - 2^{\frac{10}{3}} + 18, 81 - 24 + 18]$
$\therefore$ the absolute maximum value is 83, and the absolute minimum value is 2.

37. $f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2$

Proof. $f_x(x,y) = -4x(x^2-1) - 2(x^2y - x - 1)(2xy - 1)$
$f_y(x,y) = -2x^2(x^2y - x - 1)$
$f_{xx}(x,y) = -4(x^2-1) - 8x^2 - 2(2xy-1)^2 - 4y(x^2y - x - 1)$
$f_{yy}(x,y) = -2x^2(x^2) = -2x^4$
$f_{xy}(x,y) = -2(x^2)(2xy-1) - 2(x^2y - x - 1)(2x)$
Let $f_x(x,y) = f_y(x,y) = 0$, then $\begin{cases} x = 1 \\ y = 2 \end{cases}$ or $\begin{cases} x = -1 \\ y = 0 \end{cases}$
When $x = 1, y = 2, f_{xx}(x,y) < 0, f_{yy}(x,y) = -2 < 0, f_{xy}(x,y) = -6$
$D = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = 52 - 36 > 0$
$\therefore f$ has a local maximum at $(1,2)$
When $x = -1, y = 0, f_{xx}(x,y) = -10 < 0, f_{yy}(x,y) = -2, f_{xy}(x,y) = 2$
$D = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = 20 - 4 > 0$
$\therefore f$ has a local maximum at $(-1,0)$

$\square$

38.

Proof. $f_x(x,y) = 3e^y - 3x^2 = 3(e^y - x^2)$
$f_y(x,y) = 3xe^y - 3e^{3y} = 3e^y(x - e^{2y})$
Let $f_x(x,y) = f_y(x,y) = 0$, then $x = 1, y = 0$
When $x = 1, y = 0$, $f_{xx}(x,y) = -6 < 0, f_{yy}(x,y) = 3e^y(x - 3e^{2y}) = -6 < 0$
$f_{xy}(x,y) = 3e^y, D = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = 36 - 9 > 0$
$\therefore f$ has a local maximum at $(1,0)$
However, $f(-3,0) = 3(-3) + 27 - 1 = 17 > f(1,0) = 3 - 1 - 1 = 1$
$\therefore f(1,0)$ is not an absolute maximum value

□

42.

Let the coordinate of the point be (x,y,z), then the distance can be expressed as

$$D = \sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + z^2 + xz + 9}$$

Let $g(x,y) = x^2 + y^2 + xy + 9$, then find the minimum value of $g(x,y)$
$g_x(x,y) = 2x + y, g_y(x,y) = 2y + x$
Let $g_x(x,y) = g_y(x,y) = 0$, then $x = 0, y = 0$
$g_{xx}(x,y) = 2 > 0, g_{yy}(x,y) = 2, g_{xy}(x,y) = 1, D = 4 - 1 > 0$
$\therefore g(0,0) = 9$ is the local minimum value, and absolute minimum value of $g(x,y)$
$\therefore D_{min} = \sqrt{9} = 3$

47.

Let the coordinate of the vertex be (x,y,z), then the volume can be expressed as

$$V = xyz = (6 - 2y - 3z)yz$$

where $y \geq 0, z \geq 0$.
Let $g(x,y) = (6 - 2x - 3y)xy$, then find the maximum value of $g(x,y)$
$g_x(x,y) = -4xy + 6y - 3y^2, \quad g_y(x,y) = -6xy + 6x - 2x^2$
Let $g_x(x,y) = g_y(x,y) = 0$ and $x \neq 0, y \neq 0$, then $x = 1, y = \frac{2}{3}$
When $x = 1, y = \frac{2}{3}$, $g_{xx}(x,y) = -4y = -\frac{8}{3} < 0, g_{yy}(x,y) = -6x < 0$
$g_{xy}(x,y) = -4x + 6 - 6y = -2$
$D = g_{xx}(x,y)g_{yy}(x,y) - [g_{xy}(x,y)]^2 = 16 - 4 > 0$
$\therefore g$ attaches its local maximum at $(1, \frac{2}{3})$, and $g(1, \frac{2}{3}) = \frac{4}{3}$
If $x = 0$ or $y = 0$, $g(x,y) = 0 < g(1, \frac{2}{3})$
$\therefore g(1, \frac{2}{3}) = \frac{4}{3}$ is the absolute maximum of $g(x,y)$
$\therefore V_{max} = g(1, \frac{2}{3}) = \frac{4}{3}$