Exercise 16.2

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3.

$$\begin{cases} x = 4\cos\theta \\ y = 4\sin\theta \end{cases}$$

$$\begin{aligned}
\int_C xy^4 ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos\theta)(4\sin\theta)^4 \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^5 \cos\theta \sin^4\theta \sqrt{16\sin^2\theta + 16\cos^2\theta} d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^6 \sin^4\theta d\sin\theta \\
&= 4^6 (\frac{\sin^5\theta}{5})\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{2^{13}}{5}
\end{aligned}$$

5.

$$\begin{cases} x = y^2 \\ y = y \end{cases}$$

$$\begin{aligned}
\int_C (x^2y^3 - \sqrt{x}) dy &= \int_1^2 (y^7 - y) dy \\
&= (\frac{y^8}{8} - \frac{y^2}{2})\Big|_1^2 \\
&= \frac{243}{8}
\end{aligned}$$

6.

$$\begin{cases} x = y^3 \\ y = y \end{cases}$$

$$\begin{aligned}\int_C e^x dx &= \int_{-1}^{1} e^{y^3} 3y^2 dy \\ &= \int_{-1}^{1} e^{y^3} dy^3 \\ &= e^{y^3}\Big|_{y=-1}^{y=1} \\ &= e - \frac{1}{e}\end{aligned}$$

8.

Let $C_1 = \{(x,y)|x^2+y^2=4\}, C_2 = \{(x,y)|y = \frac{x}{4}+2\}$

$$\begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases}$$

$$\begin{aligned}\int_{C_1} x^2dx + y^2dy &= \int_0^{\frac{\pi}{2}} 4\cos^2\theta(-2\sin\theta) + 4\sin^2\theta(2\cos\theta)d\theta \\ &= \int_0^{\frac{\pi}{2}} -8\sin\theta\cos^2\theta d\theta + \int_0^{\frac{\pi}{2}} 8\cos\theta\sin^2\theta d\theta \\ &= 8\int_0^{\frac{\pi}{2}} \cos^2\theta d\cos\theta + 8\int_0^{\frac{\pi}{2}} \sin^2\theta d\sin\theta \\ &= 8(\frac{\sin^3\theta}{3})\Big|_0^{\frac{\pi}{2}} + 8(\frac{\cos^3\theta}{3})\Big|_0^{\frac{\pi}{2}} \\ &= \frac{8}{3} - \frac{8}{3} = 0\end{aligned}$$

$$\begin{cases} x = x \\ y = \frac{x}{4}+2 \end{cases}$$

$$\begin{aligned}\int_{C_2} x^2dx + y^2dy &= \int_0^4 x^2 + \frac{1}{4}(\frac{x^2}{16} + x + 4)dx \\ &= \int_0^4 \frac{65}{64}x^2 + \frac{x}{4} + 1dx \\ &= (\frac{65}{64\times 3}x^3 + \frac{x^2}{8} + x)\Big|_0^4 \\ &= \frac{65}{3} + 2 + 4 = \frac{83}{3}\end{aligned}$$

12.

$$\begin{aligned}
\int_C (x^2+y^2+z^2)ds &= \int_0^{2\pi} (t^2+\cos^2 2t+\sin^2 2t)\sqrt{1+(-2\sin 2t)^2+(2\cos 2t)^2}dt \\
&= \int_0^{2\pi} (t^2+1)\sqrt{1+4}dt \\
&= \sqrt{5}(\frac{t^3}{3}+t)\Big|_0^{2\pi} \\
&= \frac{8\sqrt{5}\pi^3}{3}+2\sqrt{5}\pi
\end{aligned}$$

24.

$\overrightarrow{F} = (y\sin z, z\sin x, x\sin y)$, $\overrightarrow{r}(t) = (\cos t, \sin t, \sin 5t)$, $\overrightarrow{r}'(t) = (-\sin t, \cos t, 5\cos 5t)$

$$\overrightarrow{F}(\overrightarrow{r}(t)) = (\sin t\sin\sin 5t, \sin 5t\sin\cos t, \cos t\sin\sin t)$$

$$\begin{aligned}
\int_C \overrightarrow{F}\cdot d\overrightarrow{r} &= \int_0^{\pi} \overrightarrow{F}(\overrightarrow{r}(t))\cdot\overrightarrow{r}'(t)dt \\
&= \int_0^{\pi} (-\sin^2 t\sin\sin 5t+\sin 5t\cos t\sin\cos t+5\cos t\cos 5t\sin\sin t)dt \\
&\approx -0.1363
\end{aligned}$$

26.

$$\begin{aligned}
\int_C ze^{-xy}ds &= \int_0^1 e^{-t-t^3}\sqrt{1+4t^2+e^{-2t}}dt \\
&\approx 0.8208
\end{aligned}$$

28.

$\overrightarrow{r}(t) = (t, 1+t^2)$, $\overrightarrow{r}'(t) = (1, 2t)$

$$\overrightarrow{F}(\overrightarrow{r}(t)) = (\frac{t}{\sqrt{t^4+3t^2+1}}, \frac{1+t^2}{\sqrt{t^4+3t^2+1}})$$

$$\begin{aligned}
\int_C \overrightarrow{F}\cdot d\overrightarrow{r} &= \int_{-1}^{1} \frac{3t+2t^3}{\sqrt{t^4+3t^2+1}}dt \\
&= 0
\end{aligned}$$

32(a).

$\overrightarrow{r}(t) = (2\cos t, 2\sin t)$, $\overrightarrow{r}'(t) = (-2\sin t, 2\cos t)$

$$\overrightarrow{F}(\overrightarrow{r}(t)) = (4\cos^2 t, 4\sin t\cos t)$$

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_0^{2\pi} -8\sin t\cos^2 t + 8\sin t\cos^2 t dt = 0$$

49.

Proof.

$$\begin{aligned}\int_C \overrightarrow{v} \cdot d\overrightarrow{r} &= \int_a^b \overrightarrow{v} \cdot \overrightarrow{r}'(t)dt \\ &= \overrightarrow{v} \cdot \int_a^b \overrightarrow{r}'(t)dt \\ &= \overrightarrow{v} \cdot (\overrightarrow{b} - \overrightarrow{a})\end{aligned}$$

$\square$

50.

Proof.

$$\begin{aligned}\int_C \overrightarrow{r} \cdot d\overrightarrow{r} &= \int_a^b \overrightarrow{r} \cdot \overrightarrow{r}'dt \\ &= (\frac{\overrightarrow{r}^2}{2})\Big|_a^b \\ &= \frac{1}{2}(|\overrightarrow{r}(b)|^2 - |\overrightarrow{r}(a)|^2)\end{aligned}$$

$\square$