Exercise 11.6

Wang Yue from CS Elite Class

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6. $\sum_{n=0}^{\infty}\frac{(-3)^n}{(2n+1)!}$

We check whether $\sum_{n=0}^{\infty}\frac{3^n}{(2n+1)!}$ is convergent or not.

By the ratio test, we have:

$$\frac{\frac{3^{n+1}}{(2n+3)!}}{\frac{3^n}{(2n+1)!}}=\frac{3}{(2n+2)(2n+3)}\leq\frac{3}{2\times 3}=\frac{1}{2}<1$$

$\therefore \sum_{n=0}^{\infty}\frac{3^n}{(2n+1)!}$ is convergent
$\therefore \sum_{n=0}^{\infty}\frac{(-3)^n}{(2n+1)!}$ is absolutely convergent

7. $\sum_{k=1}^{\infty}k(\frac{2}{3})^k$

$\because$ By the limit ratio test, we have:

$$\lim_{k\to\infty}\frac{(k+1)(\frac{2}{3})^{k+1}}{k(\frac{2}{3})^k}=\lim_{k\to\infty}\frac{2}{3}\frac{k+1}{k}=\frac{2}{3}<1$$

$\therefore \sum_{k=1}^{\infty}k(\frac{2}{3})^k$ is convergent
$\because$ each term of $\sum_{k=1}^{\infty}k(\frac{2}{3})^k$ is positive term
$\therefore \sum_{k=1}^{\infty}k(\frac{2}{3})^k$ is absolutely convergent

10. $\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+2}}$

Let $a_n=(-1)^n\frac{n}{\sqrt{n^3+2}}, b_n=\frac{n}{\sqrt{n^3+2}}=\frac{1}{\sqrt{n+\frac{2}{n^2}}}, c_n=n+\frac{2}{n^2}$

Let $f(x)=x+\frac{2}{x^2}, x\geq 1$
$\because$ when $x<2, f(1)=1+2=3>f(2)=2+\frac{1}{2}=\frac{5}{2}$
$\because$ when $x\geq 2, f'(x)=1-\frac{4}{x^3}>\frac{1}{2}>0, f(x)$ is increasing
$\therefore \forall n\in N_+, c_n$ is increasing
$\therefore \forall n\in N_+, b_n$ is decreasing, $b_{n+1}<b_n$
$\because \lim_{n\to\infty}c_n=\lim_{n\to\infty}\frac{n^3+2}{n^2}=\lim_{n\to\infty}\frac{3n^2}{2n}=\infty$
$\therefore \lim_{n\to\infty}b_n=0$
$\therefore \sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+2}}$ is absolutely convergent

11. $\sum_{n=1}^{\infty} \frac{(-1)^n e^{\frac{1}{n}}}{n^3}$

Let $a_n = \frac{(-1)^n e^{\frac{1}{n}}}{n^3}, b_n = \frac{e^{\frac{1}{n}}}{n^3}$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{n^3 e^{\frac{1}{n+1}}}{(n+1)^3 e^{\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^3 e^{\frac{1}{n(n+1)}}} = 1$$

$\therefore$ TODO is inconclusive.
$\because$ when n increases, $e^{\frac{1}{n}}$ is decreasing, n^3 is increasing
$\therefore b_n$ is decreasing, $b_{n+1} < b_n$

$$\because \lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{n^3} = \frac{0}{\infty} = 0$$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n e^{\frac{1}{n}}}{n^3}$ is absolutely convergent

13. $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

$a_n = \frac{10^n}{(n+1)4^{2n+1}}$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{10(n+1)}{16(n+2)} = \lim_{n\to\infty} \frac{10}{16} = \frac{5}{8} < 1$$

$\therefore \sum_{n=1}^{\infty} a_n$ is absolutely convergent

17. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

Let $a_n = \frac{(-1)^n}{\ln n}, b_n = \frac{1}{\ln n}, n \geq 2$
Obviously, we can know b_n is decrasing, $b_{n+1} < b_n$, when $n \geq 2$

$$\because \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0$$

$\therefore \sum_{n=2}^{\infty} a_n$ is convergent
$\because \ln n < n$

$$\therefore \sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$$

$\because \sum_{n=2}^{\infty} \frac{1}{n}$ is divergent
$\therefore \sum_{n=2}^{\infty} b_n$ is also divergent
$\therefore$ it is conditionally convergent.

21. $\sum_{n=1}^{\infty}(\frac{n^2+1}{2n^2+1})^n$

Let $a_n = (\frac{n^2+1}{2n^2+1})^n$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \lim_{n\to\infty} \frac{2n}{4n} = \frac{1}{2} < 1$$

$\therefore \sum_{n=1}^{\infty}(\frac{n^2+1}{2n^2+1})^n$ is absolutely convergent and therefore convergent

23. $\sum_{n=1}^{\infty}(1+\frac{1}{n})^{n^2}$

Let $a_n = (1+\frac{1}{n})^{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty}(1+\frac{1}{n})^n = e > 1$$

$\therefore \sum_{n=1}^{\infty}(1+\frac{1}{n})^{n^2}$ is divergent

32. A series $\sum a_n$ is defined by the equations

$$a_1 = 1 \qquad a_{n+1} = \frac{2+\cos n}{\sqrt{n}} a_n$$

Determine whether $\sum a_n$ converges or diverges.

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{2+\cos n}{\sqrt{n}} = 0$$

$\therefore \sum a_n$ is absolutely convergent
$\therefore \sum a_n$ is convergent

34. $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n b_1 b_2 \cdots b_n}$

Let $a_n = \frac{(-1)^n n!}{n^n b_1 b_2 \cdots b_n}$

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{n^n}{(n+1)^n b_{n+1}} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} \lim_{n\to\infty} \frac{1}{b_{n+1}} = \frac{1}{\frac{e}{2}} = \frac{2}{e} < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n b_1 b_2 \cdots b_n}$ is absolutely convergent
$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n b_1 b_2 \cdots b_n}$ is convergent

38.

(a) *Proof.*

$$\begin{aligned}
\because \{r_n\} &\text{ is decreasing} \\
\therefore R_n &= a_{n+1} + a_{n+2} + a_{n+3} + \cdots \\
&\le a_{n+1} + r_{n+1}a_{n+1} + r_{n+1}^2 a_{n+1} + \cdots \\
&= \sum_{k=1}^{\infty} a_{n+1} r_{n+1}^{k-1} \\
&= \frac{a_{n+1}}{1 - r_{n+1}}
\end{aligned}$$

□

(b) *Proof.*

$$\begin{aligned}
\because \{r_n\} &\text{ is increasing} \\
\therefore R_n &= a_{n+1} + a_{n+2} + a_{n+3} + \cdots \\
&\le a_{n+1} + a_{n+1} \lim_{n\to\infty} r_n + a_{n+1}(\lim_{n\to\infty} r_n)^2 + \cdots \\
&= \sum_{k=1}^{\infty} a_{n+1} L^{k-1} \\
&= \frac{a_{n+1}}{1 - L}
\end{aligned}$$

□