Exercise 16.9

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7.

$$\operatorname{div}\overrightarrow{F} = \langle 3y^2 + 0 + 3z^2 = 3y^2 + 3z^2$$

$$\begin{aligned}
\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} &= \iiint_E (3y^2 + 3z^2)\rangle dV \\
&= \int_0^{2\pi} \int_0^1 \int_{-1}^2 3r^2 dx r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 9r^3 dr d\theta \\
&= \int_0^{2\pi} 9\frac{r^4}{4}\bigg|_0^1 d\theta \\
&= \frac{9}{4} \times 2\pi = \frac{9}{2}\pi
\end{aligned}$$

8.

$$\operatorname{div}\overrightarrow{F} = 3x^2 + 3y^2 + 3z^2$$

$$\begin{aligned}
\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} &= \iiint_E 3(x^2 + y^2 + z^2) dV \\
&= 3\int_0^{\pi} \int_0^{2\pi} \int_0^2 \rho^2 \rho^2 \sin\varphi d\rho d\theta d\varphi \\
&= 3\int_0^{\pi} \int_0^{2\pi} \sin\varphi (\frac{\rho^5}{5})\bigg|_0^2 d\theta d\varphi \\
&= \frac{96}{5} \int_0^{\pi} 2\pi \sin\varphi d\varphi \\
&= \frac{384\pi}{5}
\end{aligned}$$

10.

$$\operatorname{div}\overrightarrow{F} = 0 + 1 + x = x + 1$$

$$\begin{aligned}
\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} &= \iiint_E (x+1)dV \\
&= \int_0^b \int_0^c \int_0^a (x+1)dxdzdy \\
&= \int_0^b \int_0^c (\frac{x^2}{2} + x)\Big|_0^a dzdy \\
&= bc(\frac{a^2}{2} + a)
\end{aligned}$$

13.

$$\overrightarrow{F} = \langle x\sqrt{x^2+y^2+z^2}, y\sqrt{x^2+y^2+z^2}, z\sqrt{x^2+y^2+z^2}\rangle$$

$$\operatorname{div}\overrightarrow{F} = \sqrt{x^2+y^2+z^2}(x\frac{x}{\sqrt{x^2+y^2+z^2}} + y\frac{y}{\sqrt{x^2+y^2+z^2}} + z\frac{z}{\sqrt{x^2+y^2+z^2}}) = x^2+y^2+z^2$$

$$\begin{aligned}
\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} &= \iiint_E (x^2+y^2+z^2)dV \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^R \rho^2\rho^2 \sin\varphi d\rho d\varphi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{R^5}{5} \sin\varphi d\varphi d\theta \\
&= \int_0^{2\pi} \frac{R^5}{5}(-\cos\varphi)\Big|_0^{\pi} d\theta \\
&= \int_0^{2\pi} \frac{2}{5}R^5 d\theta = \frac{4}{5}R^5\pi
\end{aligned}$$

24.

Let $\overrightarrow{F} = \langle P, Q, R\rangle$

$$\overrightarrow{n} = \frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}} = \langle x, y, z\rangle$$

$$\overrightarrow{F} \cdot \overrightarrow{n} = Px + Qy + Rz = 2x + 2y + z^2$$

$$\therefore P = 2, Q = 2, R = z, \overrightarrow{F} = \langle 2, 2, z\rangle$$

$$\operatorname{div}\overrightarrow{F} = 1$$

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_E dV = \frac{4}{3}\pi$$

25.

Proof. $\because \overrightarrow{a}$ is a constant vector
$\therefore \operatorname{div}\overrightarrow{a} = 0$
$\therefore \iint_S \overrightarrow{a} \cdot \overrightarrow{n}\, dS = \iiint_E \operatorname{div}\overrightarrow{a}\, dV = 0$ □

26.

Proof.

$$\operatorname{div}\overrightarrow{F} = 1 + 1 + 1 = 3$$

$$\frac{1}{3}\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \frac{1}{3}\iiint_E \operatorname{div}\overrightarrow{F} dV = \iiint_E dV = V(E)$$

□

27.

Proof. By using the Divergence Theorem, we have

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_E \operatorname{div}\overrightarrow{F} dV$$

$$\therefore \iint_S \operatorname{curl}\overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_E \operatorname{div}(\operatorname{curl}\overrightarrow{F}) dV = 0$$

□

28.

Proof.

$$\operatorname{div}(D_n f) = \nabla \cdot (\overrightarrow{n} \cdot \nabla f) = \nabla^2 f$$

$$\iint_D {}_n f dS = \iiint_E \operatorname{div}(D_n f) dV = \iiint_E^2 f dV$$

□

29.

Proof.

$$\begin{aligned}\operatorname{div}(f\nabla g) &= \frac{\partial}{\partial x}(f\nabla g) + \frac{\partial}{\partial y}(f\nabla g) + \cdots \\ &= \frac{\partial f}{\partial x}(\nabla g) + \frac{\partial \nabla g}{\partial x} f + \frac{\partial f}{\partial y}(\nabla g) + \frac{\partial \nabla g}{\partial y} f + \cdots \\ &= f\nabla^2 g + \nabla f \nabla g\end{aligned}$$

$$\therefore \iint_S (f\nabla g)\cdot \overrightarrow{n}\, dS = \iiint_E \operatorname{div}(f\nabla g) dV = \iiint_E (f\nabla^2 g + \nabla f \nabla g) dV$$

□

30.

Linearity of diveregence:

$$\operatorname{div}(f+g) = \operatorname{div} f + \operatorname{div} g$$

Proof.

$$\begin{aligned}\operatorname{div}(f\nabla g - g\nabla f) &= \operatorname{div}(f\nabla g) - \operatorname{div}(g\nabla f) \\ &= (\nabla f\cdot\nabla g + f\nabla^2 g) - (\nabla g\cdot\nabla f + g\nabla^2 f) \\ &= f\nabla^2 g - g\nabla^2 f\end{aligned}$$

$$\iint_S (fg - g\nabla f)\cdot \overrightarrow{n}\, dS = \iiint_E \operatorname{div}(f\nabla g - g\nabla f) dV = \iiint_E (f\nabla^2 g - g\nabla^2 f) dV$$

□