MATH3401 Assignment 6

Gareth Booth, 44353027

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Equations used:

1. $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ (Geometric Series)
2. $(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$ (Binomial coefficient definition)
3. $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ (Pascal's Rule)
4. $\binom{n}{n} = \binom{n}{0} = 1, n \geq 0$
5. $\lfloor (m-1)/2 \rfloor = \lfloor m/2 \rfloor, m$ odd
6. $\lfloor (m+1)/2 \rfloor = \lfloor m/2 \rfloor + 1, m$ odd
7. $\lfloor m/2 \rfloor = 1 + \lfloor (m-1)/2 \rfloor = \lfloor (m+1)/2 \rfloor, m$ even

We are given S.

$$\begin{aligned}
S &= \frac{1}{1-z-z^2} \\
&= \frac{1}{1-(z+z^2)} \\
&= \sum_{n=0}^{\infty} \left(z+z^2\right)^n \qquad (1) \\
&= \sum_{n=0}^{\infty} z^n (1+z)^n \\
&= \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k} z^k \qquad (2) \\
S &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} z^{n+k}
\end{aligned}$$

We investigate the coefficients of z^l (in the outer sum) for each l. Clearly, z^l terms will only appear when $n \leq l$, since $k \geq 0$. So for z^l, we must investigate the appearances of z^l terms for $n \leq l$.

We see that every appearance of z^l in these partial sums must be of the form $\binom{n}{k} z^l$ for $l = k + n$, and clearly $k \leq n$. So the coefficients of z^l will be the sum of these binomial coefficients satisfying $l = k + n$ and $k \leq n$. Therefore,

$$S = \sum_{l=0}^{\infty} c_l z^l$$

for

$$c_l = \sum_{\substack{n,k \in \mathbb{N}_0 \\ l=n+k \\ k \leq n}} \binom{n}{k}$$

We seek to evaluate c_l. First, we wish to change this expression into a precise summation. Apply $l = n + k$.

$$c_l = \sum_{\substack{n,k \in \mathbb{N}_0 \\ l=n+k \\ k \leq n}} \binom{l-k}{k}$$

Note that we may write $l = n + k$ for integers n, k with $k \leq n$ exactly when $k \leq \left\lfloor \frac{l}{2} \right\rfloor$. (e.g. for $5 = 3 + 2$, 2 is the largest possible k, for $6 = 3 + 3$, 3 is the largest possible k, this pattern continues. Also note the cases $l = 0$ and $l = 1$ are still valid here.) We may therefore write the sum as

$$c_l = \sum_{k=0}^{\lfloor l/2 \rfloor} \binom{l-k}{k}$$

We claim that $c_l = F_{l+1}$, where F_l is the lth Fibonacci number, i.e c_l is a sequence satisfying $c_0 = 1$, $c_1 = 1$ and $c_m = c_{m-1} + c_{m-2}$ for $m \geq 2$.

Proof by induction.

Base cases.

$$c_0 = \sum_{k=0}^{\lfloor 0/2 \rfloor} \binom{0-k}{k} = \sum_{k=0}^{0} \binom{0-k}{k} = \binom{0}{0} = 1 \qquad (4)$$

$$c_1 = \sum_{k=0}^{\lfloor 0/2 \rfloor} \binom{0-k}{k} = \sum_{k=0}^{0} \binom{0-k}{k} = \binom{0}{0} = 1 \qquad (4)$$

Assume $l = m$ and $l = m - 1$ are true. Show $l = m + 1$ holds.

$$c_{m+1} = c_m + c_{m-1}$$

Perform induction step.

$$c_{m+1} = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k} + \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-1-k}{k}$$

Case m odd.

$$RHS = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k} + \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-1-k}{k} \qquad (5)$$

Change summations, sub $k \to k + 1$ in 1st sum. This is valid because $\sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k} = \sum_{k=0}^{\lfloor m/2 \rfloor + 1} \binom{m-k}{k}$, and so the new sum will contain the same terms as the original.

$$
\begin{aligned}
RHS &= \sum_{k+1=0}^{\lfloor m/2 \rfloor} \binom{m-(k+1)}{k+1} + \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-1-k}{k} \\
&= \sum_{k=-1}^{\lfloor m/2 \rfloor} \binom{m-1-k}{k+1} + \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-1-k}{k} \\
&= \binom{m}{0} + \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-1-k}{k+1} + \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-1-k}{k} \\
&= \binom{m+1}{0} + \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-1-k}{k+1} + \binom{m-1-k}{k} \qquad (4) \\
&= \binom{m+1}{0} + \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k+1} \qquad (3) \\
&= \binom{m+1}{0} + \sum_{k-1=0}^{\lfloor m/2 \rfloor+1} \binom{m-(k-1)}{(k-1)+1} \qquad \text{(sub } k \to k-1\text{, keeping no. of terms the same)} \\
&= \binom{m+1}{0} + \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} \binom{m+1-k}{k} \qquad (6) \\
&= \sum_{k=0}^{\lfloor (m+1)/2 \rfloor} \binom{m+1-k}{k}
\end{aligned}
$$

$RHS = c_{m+1} = LHS$

Case m even.

$$
RHS = \sum_{k=0}^{1+\lfloor (m-1)/2 \rfloor} \binom{m-k}{k} + \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-1-k}{k} \qquad (7)
$$

As before, sub $k \to k+1$ in 1st sum. This time we must decrease the sum range, as $\sum_{k=0}^{\lfloor (m-1)/2 \rfloor+1} \binom{m-k}{k} \neq \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-k}{k}$.

$$
\begin{aligned}
RHS &= \sum_{k=-1}^{\lfloor (m-1)/2 \rfloor} \binom{m-1-k}{k+1} + \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-1-k}{k} \\
&= \binom{m}{0} + \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-1-k}{k+1} + \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-1-k}{k} \\
&= \binom{m+1}{0} + \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-1-k}{k+1} + \binom{m-1-k}{k} \qquad (4) \\
&= \binom{m+1}{0} + \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-k}{k+1} \qquad (3) \\
&= \binom{m+1}{0} + \sum_{k=1}^{\lfloor (m-1)/2 \rfloor+1} \binom{m+1-k}{k} \qquad \text{(sub } k \to k-1\text{, keeping no. of terms the same)} \\
&= \sum_{k=0}^{\lfloor (m+1)/2 \rfloor} \binom{m+1-k}{k} \qquad (7)
\end{aligned}
$$

$RHS = c_{m+1} = LHS$

By the induction hypothesis, the statement is true and so $c_l = F_{l+1}$, as required. $\square$