Natural log formula with Harmonic Numbers

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1 Introduction

In this paper, the following formula will be derived.

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{x^{n+1}} H_n \tag{1}$$

Where H_n is the nth Harmonic Number

2 Derivation

This formula can be derived by beginning with the following infinite series and rearranging.

$$0 = \sum_{n=1}^{\infty} \frac{1}{x^n} \left(\frac{1}{n+1} - \frac{1}{n+1} \right) \tag{2}$$

From the definition of harmonic numbers,

$$\frac{1}{n+1} = H_{n+1} - H_n$$

Substituting this into (2) gives the following

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \frac{1}{x^n} \left(\frac{1}{n+1} - (H_{n+1} - H_n) \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{x^n} \left(\frac{1}{n+1} - H_{n+1} + H_n \right) \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{x^n (n+1)} - \frac{H_{n+1}}{x^n} + \frac{H_n}{x^n} \right) \\ 0 &= \sum_{n=1}^{\infty} \frac{1}{x^n (n+1)} - \sum_{n=1}^{\infty} \frac{H_{n+1}}{x^n} + \sum_{n=1}^{\infty} \frac{H_n}{x^n} \end{aligned}$$

Add $1 - H_1$ (which equates to 0) to the right hand side.

$$\begin{aligned} 0 &= 1 - H_1 + \sum_{n=1}^{\infty} \frac{1}{x^n (n+1)} - \sum_{n=1}^{\infty} \frac{H_{n+1}}{x^n} + \sum_{n=1}^{\infty} \frac{H_n}{x^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{x^n (n+1)} + 1 - H_1 - \sum_{n=1}^{\infty} \frac{H_{n+1}}{x^n} + \sum_{n=1}^{\infty} \frac{H_n}{x^n} \\ 0 &= \left(\sum_{n=1}^{\infty} \frac{1}{x^n (n+1)} + 1 \right) - \left(H_1 + \sum_{n=1}^{\infty} \frac{H_{n+1}}{x^n} \right) + \sum_{n=1}^{\infty} \frac{H_n}{x^n} \end{aligned}$$

$$\left(\sum_{n=1}^{\infty}\frac{1}{x^{n}\left(n+1\right)}+1\right)=\left(H_{1}+\sum_{n=1}^{\infty}\frac{H_{n+1}}{x^{n}}\right)-\sum_{n=1}^{\infty}\frac{H_{n}}{x^{n}} \tag{3}$$

Because $\frac{1}{x^{0}\left(0+1\right)}=1$, it is evident that

$$\sum_{n=1}^{\infty}\frac{1}{x^{n}\left(n+1\right)}+1=\sum_{n=0}^{\infty}\frac{1}{x^{n}\left(n+1\right)}$$

Take out a factor of x on the right hand side

$$\sum_{n=1}^{\infty}\frac{1}{x^{n}\left(n+1\right)}+1=x\sum_{n=0}^{\infty}\frac{1}{x^{n+1}\left(n+1\right)}$$

Substituting n for $n-1$ in the right hand side gives

$$\sum_{n=1}^{\infty}\frac{1}{x^{n}\left(n+1\right)}+1=x\sum_{n=1}^{\infty}\frac{1}{nx^{n}} \tag{4}$$

Similarly, because $\frac{H_{0+1}}{x^{0}}=1$, it is evident that

$$H_{1}+\sum_{n=1}^{\infty}\frac{H_{n+1}}{x^{n}}=\sum_{n=0}^{\infty}\frac{H_{n+1}}{x^{n}}$$

Take out a factor of x on the right hand side

$$H_{1}+\sum_{n=1}^{\infty}\frac{H_{n+1}}{x^{n}}=x\sum_{n=0}^{\infty}\frac{H_{n+1}}{x^{n+1}}$$

Substituting n for $n-1$ in the right hand side gives

$$H_{1}+\sum_{n=1}^{\infty}\frac{H_{n+1}}{x^{n}}=x\sum_{n=1}^{\infty}\frac{H_{n}}{x^{n}} \tag{5}$$

Substitute (4) and (5) into (3).

$$x\sum_{n=1}^{\infty}\frac{1}{nx^{n}}=x\sum_{n=1}^{\infty}\frac{H_{n}}{x^{n}}-\sum_{n=1}^{\infty}\frac{H_{n}}{x^{n}}$$

Factor out $\sum_{n=1}^{\infty}\frac{H_{n}}{x^{n}}$ on the right hand side.

$$x\sum_{n=1}^{\infty}\frac{1}{nx^{n}}=(x-1)\sum_{n=1}^{\infty}\frac{H_{n}}{x^{n}}$$

$$\therefore\ \frac{x}{x-1}\sum_{n=1}^{\infty}\frac{1}{nx^{n}}=\sum_{n=1}^{\infty}\frac{H_{n}}{x^{n}} \tag{6}$$

The Mercator series states the following:

$$\ln\left(1+x\right)=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{n}x^{n}$$

Substitute x for $-\dfrac{1}{x}$.

$$\ln\left(1-\frac{1}{x}\right)=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{n}\left(-\frac{1}{x}\right)^{n}$$

$$\ln\left(\frac{x-1}{x}\right)=\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}\left(-1\right)^{n}}{nx^{n}}$$

$$-\ln\left(\frac{x-1}{x}\right)=-\sum_{n=1}^{\infty}\frac{-\left(-1\right)^{2n}}{nx^{n}}$$

$$\ln\left(\frac{x}{x-1}\right)=\sum_{n=1}^{\infty}\frac{1}{nx^{n}} \tag{7}$$

Substituting (7) into (6) gives

$$\frac{x}{x-1}\ln\left(\frac{x}{x-1}\right)=\sum_{n=1}^{\infty}\frac{H_n}{x^n}$$

$$\ln\left(\frac{x}{x-1}\right)=\frac{x-1}{x}\sum_{n=1}^{\infty}\frac{H_n}{x^n}$$

$$\ln\left(\frac{x}{x-1}\right)=\left(x-1\right)\sum_{n=1}^{\infty}\frac{H_n}{x^{n+1}}$$

Substitute x for $\dfrac{x}{x-1}$. Because $\dfrac{\frac{x}{x-1}}{\frac{x}{x-1}-1}=x$,

$$\begin{aligned}\ln\left(x\right)&=\left(\frac{x}{x-1}-1\right)\sum_{n=1}^{\infty}\frac{H_n}{\left(\frac{x}{x-1}\right)^{n+1}}\\&=\left(\frac{1}{x-1}\right)\sum_{n=1}^{\infty}\frac{H_n}{\left(\frac{x}{x-1}\right)^{n+1}}\\&=\sum_{n=1}^{\infty}\frac{H_n}{\left(x-1\right)\left(x-1\right)^{-n-1}x^{n+1}}\\\ln\left(x\right)&=\sum_{n=1}^{\infty}\frac{\left(x-1\right)^{n}}{x^{n+1}}H_n\end{aligned}$$

Q.E.D.