Zeta Function Regularization

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Synopsis

This article provides an overview of zeta-function regularization of diverging sums in QFT. We first justify its use by deriving the equivalence of zeta function regularization and asymptotic smoothing of partial sums as methods of assigning finite values to diverging sums, the latter of which has a stronger intuitive physical interpretation. We then demonstrate the application of this method to model the Casimir effect. Next we follow Hawking's derivation (2) of an expression for quadratic path integrals in terms of a generalized zeta function, and demonstrate the method by computing the partition function for a scalar field in 3 dimensions. We finish by deriving a useful relationship between the zeta function and the heat equation, and outlining an example application of this relationship to the zeta function for a scalar field in curved spacetime.

I. MATHEMATICAL BACKGROUND

A. The Generalized Zeta Function

Consider the following infinite sum:

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1}$$

where a_n is some sequence of complex numbers. For most cases relevant to physics, there exists a k such that this sum converges for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > k$. Because it is a power series, the sum must be a holomorphic function of s in this region. This allows us to define the "generalized zeta function" $\zeta : \mathbb{C} \to \mathbb{C}$ for the sequence a_n as the analytic continuation of the sum to the entire complex plane. For the sequence a_n consisting entirely of 1s, this reduces to the more familiar Riemann zeta function.

Alternatively, given a linear operator A, we can define the zeta function ζ_A to be

$$\zeta_A(s) = \operatorname{tr} e^{-A} \tag{2}$$

For operators with a discrete spectrum, this is related to the sum of all eigenvalues $\sum_n \lambda_n^{-s}$. For continuous spectra, we replace the sum with an integral over all eigenstates.

B. Smoothed Asymptotics

We want to consider sums of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \eta\left(\frac{n}{N}\right) \tag{3}$$

where $\eta : \mathbb{R} \to [0, 1]$ is a smooth function satisfying $\eta(1) = 0$, and decaying fast enough such that the sequence converges for all $s \in \mathbb{C}$ save for a finite number of points. In the use cases we care about, N will be some very large number typically representing a UV cutoff. This sum is a Dirichlet series with an extra "smoothing factor" that counteracts the effect of higher order terms. Such a sum appears in the physical examples we consider in the next section. We will follow the method of (1) to find an approximation for this expression in terms of a zeta function. We first choose some very large number A and define a new function f_A:

$$f_A(x) = e^{Ax}\eta(e^x) \tag{4}$$

Then computing its Fourier transform

$$\tilde{f}_A(t) = \int e^{ixt} f_A(x)\mathrm{d}x = \frac{1}{2\pi}\int e^{Ax+ixt}\eta(e^x)\mathrm{d}x = -\frac{1}{2\pi(A+it)}\int_0^\infty y^{A+it}\eta'(y)\mathrm{d}y \tag{5}$$

For readability we take the integral above and wrap it in its own function F:

$$\tilde{f}_A(t) = -\frac{1}{2\pi(A+it)}F(A+it) \tag{6}$$

And then rewriting

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}\eta(\frac{n}{N}) = N^A\sum_{n=1}^{\infty}\frac{1}{n^{s+A}}f_A(\log\frac{n}{N}) \tag{7}$$

$$= N^A\sum_{n=1}^{\infty}\frac{a_n}{n^{s+A}}\int \tilde{f}_A(t)\left(\frac{n}{N}\right)^{-it}\mathrm{d}t \tag{8}$$

$$= \int \tilde{f}_A(t)N^{A+it}\sum_{n=1}^{\infty}\frac{a_n}{n^{s+A+it}}\mathrm{d}t \tag{9}$$

Assuming A is large enough, the sum is equivalent to the generalized zeta function for the sequence a_n:

$$= \int_{-\infty}^{\infty}\tilde{f}_A(t)N^{A+it}\zeta(s+A+it)\mathrm{d}t \tag{10}$$

Now making the substitution $z = s + A + it$ and substituting what we found for $\tilde{f}_A$, this becomes

$$= -\frac{1}{2\pi i}\int_{s+A-i\infty}^{s+A+i\infty}\zeta(z)\frac{N^{z-s}F(z-s)}{z-s}\mathrm{d}z \tag{11}$$

The integrand has simple poles at $z = 1$ and $z = s$. We can close the contour such that it encircles both poles and apply the residue theorem to obtain

$$= -\frac{1}{2\pi i}\int_{s-B-i\infty}^{s-B+i\infty}\zeta(z)\frac{N^{z-s}F(z-s)}{z-s}\mathrm{d}z + \frac{N^{1-s}F(1-s)}{1-s} - \zeta(s)F(0) \tag{12}$$

We note that we have taken advantage of the analytic continuation of ζ here; while the initial contour was in a region of the complex plane where ζ was given by our initial convergent sum, we had to make use of its analytic continuation to evaluate the residues. Evaluating the integral and plugging in F gives the result

$$\sum_{n=1}^{\infty}\frac{1}{n^s}\eta\left(\frac{n}{N}\right) = \zeta(s) + N^{1-s}\int_0^\infty y^{-s}\eta(y)\mathrm{d}y + O(N^{-B}) \tag{13}$$

$$= \zeta(s) + \int_0^N \frac{1}{y^s}\eta\left(\frac{y}{N}\right)\mathrm{d}y + O(N^{-B}) \tag{14}$$

where to get the second equality we assume that N is very large. We can make B arbitrarily large to discard the last term, leaving us with the zeta function plus a term that depends linearly on N. This is a very useful expansion; if we assume we've constructed our theory well enough that it does not depend on the energy cutoff N, then we know the second term must cancel out in our final result. The process of zeta function regularization is essentially just ignoring this term, as the only remaining term that does not depend on N is the ζ function.

II. THE 1D CASIMIR EFFECT

A. Solution via Zeta Regularization

Consider a complex massless scalar field ϕ in a setup between two parallel conducting plates separated by a small distance a. The eigenstates of the field between the plates must satisfy

$$\phi_n = e^{-in\pi t/a} \sin\Big(\frac{n\pi}{a} z\Big) \tag{15}$$

Now we can compute the vacuum energy of the field between the plates by summing over all the modes:

$$E = \sum_{n=1}^{\infty} \frac{\pi n}{2a} \tag{16}$$

Clearly this sum diverges, and this is where zeta regularization save us. We can replace the diverging sum with the Riemann zeta function $\zeta(-1) = -1/12$, giving the result

$$E = -\frac{\pi}{24a} \tag{17}$$

The force on the plates is the derivative of energy with respect to a:

$$F = -\frac{\mathrm{d}E}{\mathrm{d}a} = -\frac{\pi}{24a^2} \tag{18}$$

which is our final result.

B. Solution via Asymptotic Cutoff

The above solution seems fairly miraculous; we found a diverging sum for energy, and just replaced the sum with a more or less arbitrary negative value with no justification. The reason we still get the correct result is subtle; we actually ignored several important phenomena in our derivation of the expression for energy, and zeta function regularization automatically accounted for those phenomena. To demonstrate this more explicitly, we will follow the method of (3) using asymptotic regulation to derive the same result as above.

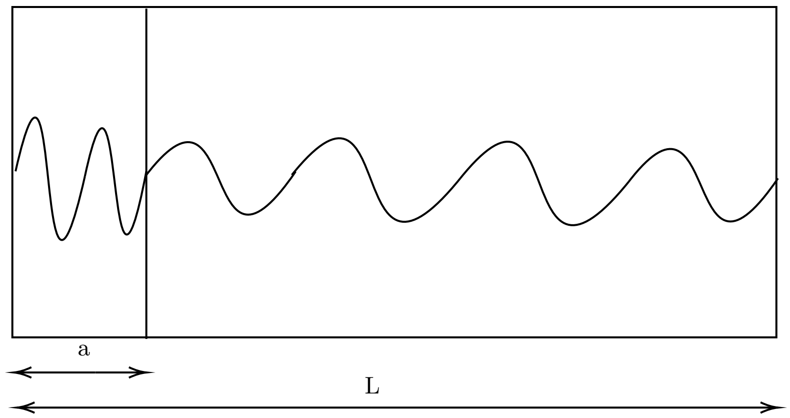
Our first mistake was assuming that our theory is valid up to arbitrarily high energy levels. In QFT we generally assume that our theory is only valid up to some finite UV energy cutoff, which we denote Λ. We can model this energy cutoff by claiming the energy is actually given by the sum

$$E = \frac{\pi}{2a} \sum_{n=1}^{\infty} n\eta\left(\frac{n}{a\Lambda}\right) \tag{19}$$

where η is a smooth real function satisfying $\eta(0) = 1$ and decreasing faster than $O(1/x)$. This function models our arbitrary cutoff at high momenta; the $\eta(0) = 1$ requirement ensures it won't significantly affect the low energies we care about, and the $O(1/x)$ requirement guarantees that the sum converges by reducing the influence of high momenta. More general setups will have additional requirements such that η doesn't break any symmetries of the setup, but for this simple 1D model we don't need to worry about those.

Our second mistake was that we completely ignored how the vacuum state of the field *outside* the plates changed with respect to a. Assuming our expreimental setup is contained within a large box of length L, the vacuum energy outside the plates is

$$E_{out} = \frac{\pi}{2} \sum_{n} \frac{n}{L-a} \eta\left(\frac{n}{(L-a)\Lambda}\right) \tag{20}$$



Because the box is large, we can take the limit as $L \to \infty$, which turns the sum into a continuous integral and gives the result

$$E_{out} = \frac{\pi}{2}\int \frac{n}{L-a}\eta\left(\frac{n}{(L-a)\Lambda}\right)\mathrm{d}n = \frac{\pi}{2}(L-a)\Lambda^2\int x\eta(x)\mathrm{d}x \tag{21}$$

We only care about terms that depend on a, so we can express the energy as

$$E_{out} = -\frac{\pi}{2}a\Lambda^2\int x\eta(x)\mathrm{d}x \tag{22}$$

We now have a completely accurate physical model of the setup. The correct value for total energy is obtained by summing the inside and outside terms:

$$E_{tot} = \frac{\pi}{2a}\sum_{n=1}^{\infty} nf\left(\frac{n}{a\Lambda}\right) - a\frac{\pi}{2}\Lambda^2\int x\eta(x)\mathrm{d}x \tag{23}$$

$$= \frac{\pi}{2a}\left[\sum_{n=1}^{\infty} n\eta\left(\frac{n}{a\Lambda}\right) - \int n\eta\left(\frac{n}{a\Lambda}\right)\mathrm{d}n\right] \tag{24}$$

The factor in brackets exactly matches the expression (14) for the zeta function with $s = -1$; therefore the energy is

$$E = \frac{\pi}{2a}\zeta(-1) = -\frac{\pi}{24a} \tag{25}$$

exactly as we derived before. So zeta function regularization acts as a shortcut to this result; it models our arbitrary cutoff function, and also subtracts out the vacuum energy outside the plates.

III. PATH INTEGRALS

We now consider a path integral of the form

$$Z = \int \mathcal{D}\{\phi\}e^{-\frac{1}{2}\int \mathrm{d}^4x\ \phi A\phi} \tag{26}$$

for some linear operator A. This can be interpreted as the partition function of a scalar field with action $\int \phi A\phi$. Expanding A in terms of an orthonormal eigenvector basis ϕ_n so that $A\phi_n = \lambda_n\phi_n$, and assuming the eigenfunctions form a complete basis, we can expand ϕ in terms of the eigenfunctions as

$$\phi = \sum_n a_n\phi_n \tag{27}$$

This allows us to express the integration measure as

$$\mathcal{D}\{\phi\} = \prod_n \mu \mathrm{d}a_n \tag{28}$$

where μ is some normalization constant. Plugging this all in, the path integral is now

$$Z = \int \prod_n \mu \mathrm{d}a_n \, \exp\left[-\frac{1}{2}\int \mathrm{d}^4x \sum_n \lambda_n a_n^2 \phi_n\right] \tag{29}$$

$$= \mu \prod_n \int \mathrm{d}a_n \, \exp\left[-\frac{1}{2}\int \mathrm{d}^4x \lambda_n a_n^2 \phi_n\right] \tag{30}$$

$$= \frac{\mu}{2}\prod_n \sqrt{\frac{\pi}{\lambda_n}} \tag{31}$$

$$= \sqrt{\frac{1}{\det\left(\frac{4A}{\mu^2\pi}\right)}} \tag{32}$$

In general, λ_n will be unbounded so the determinant will diverge. However we can still assign it a meaningful value by introducing the generalized zeta function ζ as the analytic continutation of the sum

$$\zeta(s) = \sum_n \lambda_n^{-s} \tag{33}$$

The derivative at $s = 0$ is:

$$\zeta'(0) = \lim_{h\to 0}\sum_n \frac{\lambda_n^h - 1}{h} = \lim_{h\log\lambda\to 0}\sum_n \frac{e^{h\log\lambda_n} - 1}{h\log\lambda_n}\log\lambda_n = -\sum_n \log(\lambda_n) \tag{34}$$

(the first step is the limit definition of the derivative; the second step is just some algebra; the third is an application of L'Hopital's rule). Now we may write the determinant in terms of ζ:

$$\det\left(\frac{4A}{\mu^2\pi}\right) = \prod_n \frac{4\lambda_n}{\mu^2\pi} = \exp\left[\sum_n \log\frac{4}{\mu^2\pi} + \sum_n \log\lambda_n\right] = \exp\left[\zeta(0)\log\frac{4}{\mu^2\pi} - \zeta'(0)\right] \tag{35}$$

And therefore the partition function is

$$Z = \exp\left[\frac{1}{2}\zeta(0)\log\frac{\mu^2\pi}{4} + \frac{1}{2}\zeta'(0)\right] \tag{36}$$

IV. SCALAR FIELD IN A BOX

Consider a complex, massless scalar field in a box of volume V. The Euclidean-space action is

$$E = \int \dot{\phi}^2 + (\boldsymbol{\nabla}\phi)^2 \mathrm{d}^4x = \int \mathrm{d}^4x \, \phi(\partial_t^2 + \boldsymbol{\nabla}^2)\phi \tag{37}$$

where the second expression is obtained via integration by parts. Therefore the partition function for this field is

$$Z = \int \mathcal{D}\{\phi\} \exp\Big[-\int \mathrm{d}^4x\, \phi(\partial_t^2 + \boldsymbol{\nabla}^2)\phi\Big]$$

where the sum is over all fields that are periodic in β. This path integral relies on our ability to compute the determinant of the ∂^2 operator. The eigenstates will be plane waves with period β and eigenvalues

$$\lambda_n = \frac{4\pi^2 n^2}{\beta^2} + k^2 \tag{38}$$

for all $k \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then we can compute the generalized zeta function by summing over all n and integrating over all k:

$$\zeta(s) = \frac{2V}{(2\pi)^3} \sum_{n=-\infty}^{\infty} \int \mathrm{d}^3k \left(\frac{4\pi^2 n^2}{\beta^2} + k^2\right)^{-s} \tag{39}$$

$$= \frac{4\pi V}{(2\pi)^3} \left[\int \mathrm{d}k\, k^{2-s} + 2\sum_{n=1}^{\infty} \int \mathrm{d}k \left(\frac{4\pi^2 n^2}{\beta^2} + k^2\right)^{-s} k^2\right] \tag{40}$$

The first integrand here may approach infinity at $k = 0$; this is an infrared divergence and may be ignored. We can integrate the second term by parts to obtain

$$\zeta(s) = -\frac{8\pi V}{(2\pi)^3} \sum_{n=1}^{\infty} \int \frac{\left(\frac{4\pi^2 n^2}{\beta^2} + k^2\right)^{-s+1}}{2-2s} \mathrm{d}k \tag{41}$$

$$= -\frac{8\pi V}{(2\pi)^3} \sum_{n=1}^{\infty} \int \frac{\left(\frac{4\pi^2 n^2}{\beta^2} + \frac{4\pi^2 n^2}{\beta^2}\sinh^2 y\right)^{-s+1}}{2-2s} \mathrm{d}\left(\frac{2\pi n}{\beta}\sinh y\right) \tag{42}$$

$$= -\frac{8\pi V}{(2\pi)^3} \frac{(2\pi)^{-2s+3}}{(2-2s)\beta^{-2s+3}} \sum_{n=1}^{\infty} n^{-2s+3} \int (\cosh y)^{-2s+3} \mathrm{d}y \tag{43}$$

$$\zeta(s) = \frac{8\pi V}{(2\pi)^{2s}} \frac{\beta^{3-2s}}{2-2s} \zeta_R(2s-3) \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{3}{2})}{2\Gamma(s-1)} \tag{44}$$

where ζ_R is the standard Riemann zeta function. From this we can compute

$$\zeta(0) = 0 \tag{45}$$

$$\zeta'(0) = 2\pi V \beta^{-1} \zeta_R(-3)\Gamma(\frac{1}{2})\Gamma(-\frac{3}{2}) = \frac{\pi^2}{45} V T^3 \tag{46}$$

Which gives the partition function

$$Z = \exp\left(\frac{\pi^2 V T^3}{90}\right) \tag{47}$$

From here we may compute the energy, entropy, and pressure of the field at a given temperature:

$$E = -\frac{\partial}{\partial\beta} \log Z = \frac{\pi^2}{30} V T^4 \tag{48}$$

$$S = \beta E + \log Z = \frac{2\pi^2}{45} V T^3 \tag{49}$$

$$P = \beta^{-1} - \frac{\partial}{\partial V} \log Z = \frac{\pi^2}{90} T^4 \tag{50}$$

V. THE HEAT EQUATION

Generally we will not know the exact eigenvalues of the operator A, so explicitly computing ζ_A will not be possible. This is especially true in curved spacetimes. We can however obtain some information about ζ_A by studying the heat equation

$$\frac{\mathrm{d}}{\mathrm{d}t}F(x,y,t) + AF(x,y,t) = 0 \tag{51}$$

Assuming A is a 4-dimensional operator, then x and y represent points on a 4D manifold M with metric g, and F is a map $F : M^2 \times \mathbb{R} \to \mathbb{R}$. We assuming A acts on F by holding y and t constant and only caring about terms that depend on x. One solution to this equation is

$$F(x,y,t) = \sum_n e^{-\lambda_n t}\phi_n(x)\phi_n(y) \tag{52}$$

which is easy to see noting that A acts only on $\phi_n(x)$ and not on the y or t dependent terms. We can try to make this look more similar to ζ_A by defining a new function

$$Y(t) \equiv \int F(x,x,t)\sqrt{g}\,\mathrm{d}^4x = \sum_n e^{-\lambda_n t}\int \phi_n(x)^2\sqrt{g}\mathrm{d}^4x = \sum_n e^{-\lambda_n t} \tag{53}$$

Where in the last step we used that the eigenfunctions λ_n are normalized. $Y(t)$ and $\zeta_A(s)$ are related by an operation called a Mellin transform:

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}Y(t)\mathrm{d}t = \frac{1}{\Gamma(s)}\sum_n \int_0^\infty t^{s-1}e^{-\lambda_n t}\mathrm{d}t \tag{54}$$

$$= \frac{1}{\Gamma(s)}\sum_n \frac{1}{\lambda_n^s}\int_0^\infty (\lambda_n t)^{s-1}e^{-\lambda_n t}\,\mathrm{d}(\lambda_n t) \tag{55}$$

$$= \frac{1}{\Gamma(s)}\sum_n \frac{1}{\lambda_n^s}\Gamma(s) \tag{56}$$

$$= \zeta_A(s) \tag{57}$$

Therefore we have reduced the problem of computing ζ_A to that of solving the heat equation for F. This means that whatever analytical approximations or numerical methods we have for solving the heat equation, we may also use to compute path integrals.

A. Massless Scalar Field in Curved Spacetime

We now present an approximations of $Y(t)$ for a scalar field in curved spacetime. The proof of this equation is far beyond the scope of what I studied here, I mainly want to demonstrate a use case for this method in general relativity. For most second-order differential operators on a compact manifold, the following approximation is valid:

$$Y(t) \sim \sum_n t^{n-2}\int b_n\sqrt{g}\mathrm{d}^4x \tag{58}$$

where b_n denotes a sequence of terms with $b_n \sim O(R^n)$. In the specific case of a scalar field coupled to the Ricci scalar R, which is goverened by $A = -\Box^2 + \xi R$ for some constant ξ (usually 0 or 1/6), the first few terms in this sequence are

$$b_0 = \frac{1}{16\pi^2} \tag{59}$$

$$b_1 = \frac{1}{16\pi^2}(\frac{1}{6} - R) \tag{60}$$

$$b_2 = \frac{1}{2880\pi^2}\left[R^{abcd}R_{abcd} - R^{ab}R_{ab} + 30(1-6\xi)^2R^2 + (6-30\xi)\Box R\right] \tag{61}$$

So this sequence together with $Y(t)$ may be used to compute ζ_A, which in turn can be used to find the partition function for this field.

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