Matrix Models and Fermionic String Theory

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Synopsis

We provide an introduction to the relationship between matrix models and string theories. We begin with a review of the $1/N$ expansion for matrix path integrals, and interpret of the expansion as a sum over triangulations of 2D surfaces. We then repeat the calculation for a 1D $N \times N$ matrix path integral to show the emergence of string theory with a 1 dimensional target space. Lastly, we reformulate the large-N limit of the 1D matrix Hamiltonian as a theory of N noninteracting Dirac fermions, and prove its integrability.

I. PURE 2D QUANTUM GRAVITY FROM 0D MATRIX MODELS

We begin by demonstrating how pure 2D quantum gravity arises from a matrix path integral, following the procedure of (1; 2). The partition function for a free 0d matrix model is

$$\mathcal{Z} = \int \mathrm{d}M \, \exp\left[-N \operatorname{Tr} \frac{M^2}{2}\right] \tag{1}$$

We are in zero dimensions, so the path integral is just an ordinary integral in N^2 variables. Here $\mathrm{d}M = \prod_i \mathrm{d}M_{ii} \prod_{j>i} \mathrm{d}\,\mathrm{Re}(M_{ij})\mathrm{d}\,\mathrm{Im}(M_{ij})$, which is the Haar measure on $\mathbb{R}^N \times \mathrm{SU}(N)$ and is invariant under unitary transformations $M \to U^\dagger M U$. Gaussian integration gives the two-point correlator as

$$\langle M_{ij} M_{kl} \rangle = \frac{1}{N} \delta_{il} \delta_{kj} \tag{2}$$

For any sequence I of L pairs of integers $1 \le i, j \le N$, we can follow the proof of Wick's theorem from to obtain

$$\left\langle \prod_{(ij) \in I} M_{ij} \right\rangle = \sum_{P \in \text{2-partitions}} \prod_{(ij),(kl) \in P} \langle M_{ij} M_{kl} \rangle \tag{3}$$

where we sum over all ways to partition I into pairs (really, pairs of pairs). Before moving on, it will be useful to clarify some terminology. Given some graph, we can construct a 2D surface called a *ribbon graph* or *fatgraph*, by "fattening" each vertex to a 2D disk, and fattening each edge into a rectangle connecting the disks. A vertex connected to j edges is said to be j*-valent*. With this object, we can visualize expressions graphically by associating each element of each pair of I to a point, and then summing over all ribbon graphs that can be constructed by connecting pairs of points to other pairs of points via rectangles. Using this formalism, one finds that an expression of the form

$$\left\langle \prod_{j=1} \operatorname{Tr}\left(M^j\right)^{n_j} \right\rangle \tag{4}$$

corresponds to a sum over all ribbon graphs with n_j j-valent vertices. An example is provided in Fig 1.

Looking at our propagator, when taking the trace we expect each loop to contribute N, and each edge to contribute an addition $1/N$; therefore each graph contributes $N^{F(\Gamma)-E(\Gamma)}$, where E and F denote the number of edges and faces (equivalently, the number of loops) respectively.

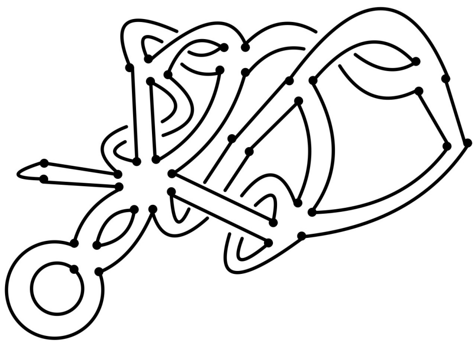

FIG. 1. One particular ribbon graph that appears in the Wick expansion of $\left\langle \mathrm{Tr}(M)\,\mathrm{Tr}\big(M^2\big)^2\,\mathrm{Tr}\big(M^3\big)^2\,\mathrm{Tr}\big(M^4\big)^3\,\mathrm{Tr}\big(M^5\big)\,\mathrm{Tr}\big(M^8\big)\right\rangle$. The graph has one 1-valent vertex, two 2-valent vertices, two 3-valent, three 4-valent, one 5-valent, and one 8-valent vertex.

Now we want to compute the following expectation for a general polynomial action:

$$\mathcal{Z}(g_1, g_2, ...) = \left\langle \exp\left[N \sum_{j=1}^{\infty} g_j \frac{\mathrm{Tr}\, M^j}{j}\right]\right\rangle \tag{5}$$

Expanding the exponential as a Taylor series and computing the nth power of the resulting series, we get

$$\mathcal{Z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left(\sum_{j=1}^{\infty} N g_j \frac{\mathrm{Tr}\, M^j}{j}\right)^n \right\rangle = \sum_{(n_k)} \frac{1}{(\sum_k n_k)!} \prod_{k=1} \left(\frac{N g_k}{k}\right)^{n_k} \left\langle \prod_{j=1} \mathrm{Tr}\big(M^j\big)^{n_j} \right\rangle \tag{6}$$

where we sum over all sets of nonnegative integers (n_k). Now we can apply our earlier result

$$\left\langle \prod_{j=1} \mathrm{Tr}\big(M^j\big)^{n_j} \right\rangle = \sum_{\Gamma} N^{F(\Gamma)-E(\Gamma)} \tag{7}$$

where the sum is over all ribbon graphs with n_j j-valent vertices, and we can also simplify

$$\prod_k N^{n_k} = N^{V(\Gamma)} \qquad\qquad \left(\sum_k n_k\right)! \prod_k k^{n_k} = |\mathrm{Aut}(\Gamma)| \tag{8}$$

where we have pulled a fact from graph theory about the order of the automorphism group of a graph (7). Then we obtain

$$\mathcal{Z} = \sum_{\Gamma} \frac{N^{V(\Gamma)+F(\Gamma)-E(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \prod_{k=1} g_k^{n_k} \tag{9}$$

where now the sum is over all ribbon graphs. If we formally expand the logarithm of each side, the sum is restricted only to connected ribbon graphs. Notably, on a connected graph, the exponent on N in Eq (9) is exactly the Euler characteristic $\chi = F - E + V$, which tells us the topology of the lowest-genus surface on which this graph can be embedded. So our new expression is

$$F = \log \mathcal{Z} = \sum_{\Gamma} \frac{N^{\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \prod_{k=1} g_k^{n_k} \tag{10}$$

with the sum over all connected ribbon graphs. Now we'll examine the special case where $g_k = 0$ for all $k \neq 3$. Then Eq 10 reduces to

$$F = \sum_{\Gamma} \frac{g_3^{V(\Gamma)} N^{\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \tag{11}$$

Now we can define the *dual* Γ^* of a graph Γ to be the graph obtained by associating a vertex to each face of Γ, and an edge between any two adjacent faces. Notably under this map, n-valent vertices will map onto n-sided faces. An example is depicted in Fig 2. Now we can re-express Eq 11 as a sum over dual graphs:

$$F = \sum_{\Gamma^*} \frac{g_3^{F(\Gamma^*)} N^{\chi(\Gamma^*)}}{|\mathrm{Aut}(\Gamma^*)|} \tag{12}$$

where we have made use of the fact that $V(\Gamma) = F(\Gamma^*)$ and $\chi(\Gamma) = \chi(\Gamma^*)$.

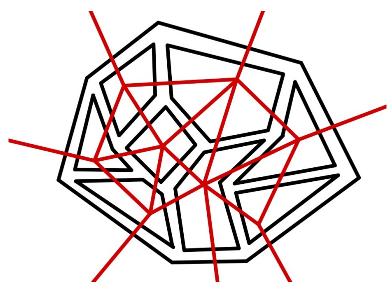

FIG. 2. Ribbon graph (black) and its dual (red). Note that since the ribbon graph only has trivalent vertices, all the faces of the dual have three sides. We can imagine the outward-pointing red lines all join together at a single point offscreen, so that the dual graph triangulates a sphere.

So we now have a sum over graphs built out of triangles. If we "fill in" the face of each triangle, we end up with a triangulation of some surface with Euler characteristic χ. Assuming each triangular face has unit area, then the total area of the triangulation will be equal to $F(\Gamma^*)$. Therefore, it intuitively makes sense that we could rewrite our sum over graphs as an integral over Riemann surfaces Σ of Euler characteristic χ:

$$F \sim \int \mathcal{D}\Sigma\, g_3^{\mathrm{Area}(\Sigma)} N^{\chi(\Sigma)} \tag{13}$$

We can re-express this in terms of an integral over the metric g and Ricci scalar R, since for any surface, the area is given by $\int \sqrt{g}$ and the Euler characteristic by $\int R\sqrt{g}$. Therefore let $\Lambda = \log g_3$ and $\alpha = \log N$, and we get

$$\int \mathcal{D}g\, \exp\left[\int \mathrm{d}^2\sigma\, (\Lambda + \alpha R)\sqrt{g}\right] \tag{14}$$

And term in the exponential is exactly the Einstein-Hilbert action with a cosmological constant. The expansion of our matrix model in powers of N corresponds to a sum over surface topologies for this theory, in line with string theory. In the large N limit, the contribution from the spherical topology dominates.

More general coupling choices g_k correspond to different "polygonulations" of the surface. For example, if we set all g_k to zero except for g_4, we would get a sum over squarulations of the surface.

II. C=1 CFT FROM 1D MATRIX MODELS

Following (4; 3) we consider the partition function

$$\mathcal{Z} = \int \mathcal{D}M(t)\ \exp\left[-N \int \mathrm{d}t\ \mathrm{Tr}\left[\frac{1}{2}\dot{M}^2 + U(M)\right]\right] \tag{15}$$

which is a 1-dimensional version of our original matrix theory. The corresponding free propagator is

$$\langle M_{ij}(t_1) M_{k\ell}(t_2)\rangle = \frac{1}{N}\delta_{i\ell}\delta_{jk} e^{-\sqrt{2}|t_1-t_2|} \tag{16}$$

Following all the same steps as before, and assuming only trivalent graphs for simplicity, we end up with

$$F = \sum_{\Gamma} \frac{g^{V(\Gamma)} N^{\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \int \prod_i \mathrm{d}t_i \prod_{<jk>} e^{-\sqrt{2}|t_j-t_k|} \tag{17}$$

which is like the 0d expression, with an additional integral over all propagators between vertices. Now when we take the dual of the graph, we must Fourier transform the propagator as

$$\exp\Big(-\sqrt{2}|t_j-t_k|\Big) \to \exp\left(-\frac{1}{4}\sum_{<jk>}(p_i-p_j)^2\right) \tag{18}$$

The second-order Taylor approximation of this exponential will look like $\frac{1}{2} - \frac{1}{4}(p_i - p_j)^2$. Making this simplification gives

$$F = \sum_{\Gamma^*} \frac{g^{F(\Gamma^*)} N^{\chi(\Gamma^*)}}{|\mathrm{Aut}(\Gamma^*)|} \int \prod_i \mathrm{d}p_i e^{-\frac{1}{4}\sum_{<jk>}(p_i-p_j)^2} \tag{19}$$

Next, we observe that $\sum_{<jk>}(p_i - p_j)^2$ is really a discretized form of $\int \mathrm{d}^2\sigma\sqrt{g}\partial_\sigma X \partial^\sigma X$ (we have changed variables from p to X to make the following equation more recognizable as the Polyakov action). Therefore, in the continuum limit where the sum over graphs becomes an integral over surfaces, we obtain

$$\int \mathcal{D}X\mathcal{D}g\ \exp\left[\int \mathrm{d}^2\sigma\ (\Lambda + \alpha R - \frac{1}{4}\partial_\sigma X \partial^\sigma X)\sqrt{g}\right] \tag{20}$$

which is the Einstein-Hilbert action for 2d gravity coupled to a 1D matter field, or equivalently, the Polyakov action for $c = 1$ string theory with a cosmological constant on the worldsheet.

Our second order simplification of the Green's function may be invalid for higher-genus surfaces; some further commentary on this simplification, as well as analysis of the expansion on the original lattice Γ rather than its dual, is provided in (4).

III. THE FERMIONIC THEORY

Following (5; 6), we consider the partition function for a 1d matrix model

$$\mathcal{Z} = \int \mathcal{D}\Phi(t)\ \exp\left[-\beta \int \mathrm{d}t\, \mathrm{Tr}\left[\dot{\Phi}^2 + U(\Phi)\right]\right] \tag{21}$$

where the potential $\tilde{U}(\Phi) = \sum_{i=1}^{N} U(\lambda_i)$ is a symmetric function of the eigenvalues λ_i of Φ. We note the action has a global SU(N) symmetry under the unitary transformation

$\Phi \to U^\dagger \Phi U$. Our goal is to show that this reduces to a theory of noninteracting Dirac fermions in the large $N \sim \beta$ limit. We can diagonalize $\Phi = \Omega^\dagger \Lambda \Omega$, and would like to compute the Hamiltonian for this theory in terms of the eigenvalues $\lambda_i \equiv \Lambda_{ii}$. We first expand $\mathrm{Tr}\,\dot{\Phi}^2$:[1]

$$\mathrm{Tr}\,\dot{\Phi}^2 = \mathrm{Tr}\Big[(\dot{\Omega}^\dagger \Lambda \Omega + \Omega^\dagger \Lambda \dot{\Omega})^2 + (\Omega^\dagger \dot{\Lambda} \Omega)^2 \tag{22}$$

$$+\Omega^\dagger \dot{\Lambda} \Omega (\dot{\Omega}^\dagger \Lambda \Omega + \Omega^\dagger \Lambda \dot{\Omega}) + (\dot{\Omega}^\dagger \Lambda \Omega + \Omega^\dagger \Lambda \dot{\Omega}) \Omega^\dagger \dot{\Lambda} \Omega\Big] \tag{23}$$

Using cyclicity of the trace and that $\dot{\Omega}^\dagger \Omega + \Omega^\dagger \dot{\Omega} = 0$, we find that the last two terms cancel and the first two terms simplify to

$$\mathrm{Tr}\,\dot{\Phi}^2 = \mathrm{Tr}\Big[\dot{\Lambda}^2 + 2\Lambda^2 + 2(\dot{\Omega}^\dagger \Omega \Lambda)^2\Big] \tag{24}$$

Noting that

$$\mathrm{Tr}\Big[\dot{\Omega}^\dagger \Omega, \Lambda\Big]^2 = \mathrm{Tr}\Big(2\dot{\Omega}^\dagger \Omega \Lambda \dot{\Omega}^\dagger \Omega \Lambda - 2\dot{\Omega}^\dagger \Omega \Lambda^2 \dot{\Omega}^\dagger \Omega\Big) = \mathrm{Tr}\Big(2\dot{\Omega}^\dagger \Omega \Lambda \dot{\Omega}^\dagger \Omega \Lambda + 2\Lambda^2\Big) \tag{25}$$

We may rewrite this as

$$\mathrm{Tr}\,\dot{\Phi}^2 = \mathrm{Tr}\Big[\dot{\Lambda}^2 + [\dot{\Omega}^\dagger \Omega, \Lambda]^2\Big] \tag{26}$$

Using $[A, \Lambda]_{ik} = A_{ik}\Lambda_{kk} - \Lambda_{ii}A_{ik} = A_{ik}(\lambda_k - \lambda_i)$, we find

$$\mathrm{Tr}\Big([\dot{\Omega}^\dagger \Omega, \Lambda]^2\Big) = \sum_{ij} [\dot{\Omega}^\dagger \Omega, \Lambda]_{ij} [\dot{\Omega}^\dagger \Omega, \Lambda]_{ji} = -\sum_{ij} (\dot{\Omega}^\dagger \Omega)_{ij} (\dot{\Omega}^\dagger \Omega)_{ji} (\lambda_j - \lambda_i)^2 \tag{27}$$

$\dot{\Omega}\Omega^\dagger$ is traceless and antihermitian, so we may decompose it in terms of $\mathfrak{su}(N)$ generators as

$$\dot{\Omega}\Omega^\dagger = \frac{i}{\sqrt{2}} \sum_{i<j} \dot{\alpha}_{ij} T_{ij} + \dot{\beta}_{ij} \tilde{T}_{ij} + \sum_{i=1}^{N-1} \dot{\gamma}_i H_i \tag{28}$$

where T_{ij} is the matrix with a 1 at positions (i, j) and (j, i), and zeroes everywhere else, T_{ij} is the matrix with i at position (i, j), $-i$ at position (j, i), and zeroes everywhere else, and H_i are the diagonal generators of the Cartan subalgebra. Plugging this back into the previous expression gives the complete Lagrangian

$$L = \frac{1}{2} \sum_i \dot{\lambda}_i^2 + \frac{1}{2} \sum_{i<j} (\dot{\alpha}_{ij}^2 + \dot{\beta}_{ij}^2)(\lambda_j - \lambda_i)^2 + \sum_i U(\lambda_i) \tag{29}$$

Notably, this Lagrangian does not depend on the Cartan subalgebra coefficients γ_i, so our states should have weight zero. The corresponding Hamiltonian for this Lagrangian is

$$H = -\frac{1}{2\beta^2 \Delta(\lambda)} \sum_i \frac{\mathrm{d}^2}{\mathrm{d}\lambda_i^2} \Delta(\lambda) + \sum_i U(\lambda_i) + \sum_{i<j} \frac{\Pi_{ij}^2 + \tilde{\Pi}_{ij}^2}{(\lambda_j - \lambda_i)^2} \tag{30}$$

where Π_{ij} and $\tilde{\Pi}_{ij}$ are the canonical conjugate momenta for α_{ij} and β_{ij} respectively, and $\Delta(\lambda) \equiv \prod_{i<j}(\lambda_i - \lambda_j)$. This Hamiltonian looks complicated, but we can make a drastic simplifcation. We are ultimately interested in computing the ground state in the large

[1] This procedure is similar to the one used for the Calogero equation on homework 1.

$N \sim \beta$ limit. The ground state should have SU(N) symmetry, so we may restrict our attention to SU(N) singlet states $\psi(\lambda)$. Then the third term in 30 will act trivially on ψ. Additionally, since SU(N) contains matrices which permute the eigenvalues in Λ, our states $\psi(\lambda_1, ..., \lambda_N)$ must be symmetric in their inputs. This allows us to define a corresponding antisymmetrized state $\chi(\lambda) = \Delta(\lambda)\psi(\lambda)$. Since antisymmetry forces $\sum_i \frac{\mathrm{d}\Delta}{\mathrm{d}\lambda_i} = 0$, we can compute

$$\sum_i \frac{\mathrm{d}^2}{\mathrm{d}\lambda_i^2}\Delta(\lambda)\chi(\lambda) = \sum_i \frac{\mathrm{d}^2\Delta}{\mathrm{d}\lambda_i^2}\chi + \frac{\mathrm{d}\Delta}{\mathrm{d}\lambda_i}\frac{\mathrm{d}\chi}{\mathrm{d}\lambda_i} + \Delta\frac{\mathrm{d}^2\chi}{\mathrm{d}\lambda_i^2} = 0 + 0 + \Delta\left[\sum_i \frac{\mathrm{d}^2}{\mathrm{d}\lambda_i^2}\right]\chi \tag{31}$$

Therefore, when acting on these antisymmetric χ states, the Hamiltonian reduces to

$$H = \sum_i -\frac{1}{2\beta^2}\sum_i \frac{\mathrm{d}^2}{\mathrm{d}\lambda_i^2} + U(\lambda_i) \tag{32}$$

This Hamiltonian represents N non-interacting particles moving through some potential in 1+1 dimensions. By the antisymmetry of χ, we know these particles are fermions. We also note that the spatial dimension of this theory has emerged from the space of eigenvalues of the matrix, while the time dimension comes directly from the time dimension in the original matrix theory. Moving on, we can second quantize the Hamiltonian by definining field operators as

$$\Psi(\lambda, t) = \sum_{i=1}^{N} e^{-it}\chi_i(\lambda)a(\lambda_i) \tag{33}$$

with $a(\lambda_i)$ the Schrödinger-picture annhilation operators, and χ_i the single-particle wavefunction derived from χ. Then the Hamiltonian becomes

$$H = \int \mathrm{d}\lambda \, \frac{1}{2\beta^2}\frac{\partial\Psi^\dagger}{\partial\lambda}\frac{\partial\Psi}{\partial\lambda} + U(\lambda)\Psi^\dagger\Psi - \mu_F(\Psi^\dagger\Psi - N) \tag{34}$$

Here the final term acts as a Lagrange multiplier that fixes the number of fermions to be N, with μ_F set to the Fermi level of the N-fermion system. Next we decompose Ψ into a new set of fermionic variables Ψ_L and Ψ_R. We define them as

$$\Psi(\lambda, t) \equiv \frac{\mathrm{e}^{-i\mu_\mathrm{F}t}}{\sqrt{2v(\lambda)}}\Big[\exp\left(-i\beta\int^\lambda \mathrm{d}\lambda' v\left(\lambda'\right) + i\pi/4\right)\Psi_\mathrm{L}(\lambda, t) \tag{35}$$

$$+\exp\left(i\beta\int^\lambda \mathrm{d}\lambda' v\left(\lambda'\right) - i\pi/4\right)\Psi_\mathrm{R}(\lambda, t)\Big] \tag{36}$$

where $v(\lambda) = \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}$ is the classical velocity at the Fermi level, so

$$\frac{1}{2}v^2 = \mu_F - U(\lambda) \tag{37}$$

We want to rewrite the Hamiltonian in terms of these components. The calculation is long, but an important intermediate step is the derivative

$$\frac{\partial\Psi}{\partial\lambda} = (-\frac{1}{\sqrt{2}v}\frac{\partial v}{\partial\lambda} - i\beta v\gamma_0)\Psi + \frac{\mathrm{e}^{-i\mu_\mathrm{F}t}}{\sqrt{2v}}\Big[\exp\left(-i\beta\int^\lambda \mathrm{d}\lambda' v\left(\lambda'\right) + i\pi/4\right)\frac{\partial\Psi_\mathrm{L}}{\partial\lambda} \tag{38}$$

$$+\exp\left(i\beta\int^\lambda \mathrm{d}\lambda' v\left(\lambda'\right) - i\pi/4\right)\frac{\partial\Psi_\mathrm{R}}{\partial\lambda}\Big] \tag{39}$$

We can re-express this in terms of $v' \equiv \frac{\mathrm{d}v}{\mathrm{d}\tau}$ to get

$$\frac{\partial \Psi}{\partial \lambda} = (-\frac{v'}{\sqrt{2}v^2} - i\beta v\gamma_0)\Psi + \frac{\mathrm{e}^{-i\mu_{\mathrm{F}}t}}{\sqrt{2v^3}}\Big[\exp\left(-i\beta\int^{\lambda}\mathrm{d}\lambda' v\left(\lambda'\right) + i\pi/4\right)\frac{\partial \Psi_{\mathrm{L}}}{\partial \tau} \tag{40}$$

$$+\exp\left(i\beta\int^{\lambda}\mathrm{d}\lambda' v\left(\lambda'\right) - i\pi/4\right)\frac{\partial \Psi_{\mathrm{R}}}{\partial \tau}\Big] \tag{41}$$

In this step, we have reinterpreted the classical time τ to be our new spatial coordinate, rather than λ; the time coordinate in the second-quantized theory is still t. When we plug these derivatives back into the Hamiltonian, we get many terms with factors of $e^{i\beta\int v}$, which will vanish in the large $\beta \sim N$ limit under the stationary phase approximation. After a long calculation and dropping these rapidly oscillating terms, the resulting Hamiltonian decomposes as

$$H = \frac{1}{2\beta}(H_L + H_R) \tag{42}$$

where

$$H_{L,R} = \int_0^{T/2}\mathrm{d}\tau\,\frac{1}{2\beta v^2}\frac{\partial \Psi^\dagger_{L,R}}{\partial \tau}\frac{\partial \Psi_{L,R}}{\partial \tau} + \frac{1}{4\beta}\left(\frac{v''}{v^3} - \frac{5}{2}\left(\frac{v'}{v^2}\right)^2\right)\Psi^\dagger_{L,R}\Psi_{L,R} \mp i\Psi^\dagger_{L,R}\frac{\partial \Psi_{L,R}}{\partial \tau} \tag{43}$$

and $T/2$ is the period of classical motion. The odd-looking factor involving v'' arises from integrating $\partial_\lambda\Psi^\dagger\partial_\lambda\Psi \to \Psi^\dagger\partial_\lambda^2\Psi$ by parts. For convenience we can write $\Psi \equiv (\Psi_L, \Psi_R)$, and the complete Hamiltonian becomes

$$H = \int_0^{T/2}\mathrm{d}\tau\,\frac{1}{2\beta v^2}\frac{\partial \Psi^\dagger}{\partial \tau}\frac{\partial \Psi}{\partial \tau} + \frac{1}{4\beta}\left(\frac{v''}{v^3} - \frac{5}{2}\left(\frac{v'}{v^2}\right)^2\right)\Psi^\dagger\Psi - i\Psi^\dagger\gamma_0\frac{\partial \Psi}{\partial \tau} \tag{44}$$

where $\gamma_0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is a 2D gamma matrix acting on the two components of Ψ. This Hamiltonian represents Dirac fermions moving through some potential. There are no interaction terms between fermions, so there is some hope that the theory is exactly solvable. As a sidenote, here we are neglecting to discuss the boundary conditions of this theory. (5) provides a semiclassical analysis deriving that $\Psi_R(\tau = 0) = \Psi_L(\tau = 0)$ and $\Psi_R(\tau = T/2) = \Psi_L(T/2) = 0$, which implies that $\bar{\Psi}\gamma_1\Psi$ vanishes at $\tau \in \{0, T/2\}$; so our fermions are confined to a box of width $T/2$. We compute Hamilton's equation of motion to be

$$i\gamma^\mu\partial_\mu\Psi = K(\tau)\cdot\Psi \tag{45}$$

where $K(\tau)$ is the differential operator

$$K(\tau) = \gamma_0\left[\partial_\tau\frac{1}{2\beta v^2}\partial_\tau - \frac{1}{4\beta}\left(\frac{v''}{v^3} - \frac{5}{2}\left(\frac{v'}{v^2}\right)^2\right)\right] \tag{46}$$

Naively it seems that K should approach zero when we drop off terms of order $1/\beta \sim 1/N$ in the large-N limit, which would simply leave us with the 2D massless Dirac equation. However, we must be more precise about the way we take the limit. As we bring $\beta \to \infty$, we simultaneously scale v in such a way that the string coupling $1/(\beta v^2)$ remains constant. Then these extra terms will be non-negligible, so we are unfortunately left with a more complicated theory. Nonetheless, this theory is still integrable; we have an infinite family of conserved currents labelled by a spatial coordinate a:

$$J_\mu(\tau, t; a) = \bar{\Psi}(\tau + a, t)\gamma_\mu\mathcal{T}\left\{\exp\left[-i\int_\tau^{\tau+a}\gamma_1 K\left(\tau'\right)\cdot d\tau'\right]\right\}\Psi(\tau, t) \tag{47}$$

where $\mathcal{T}$ denotes τ-ordering. We can demonstrate the conservation law first by differentiating according to the product rule:

$$\partial^\mu J_\mu(\tau,t;a) = \partial^\mu \bar{\Psi}(\tau+a,t)\gamma_\mu \mathcal{T}\left\{\exp\left[-i\int_\tau^{\tau+a}\gamma_1 K(\tau')\cdot d\tau'\right]\right\}\Psi(\tau,t) \tag{48}$$

$$+\bar{\Psi}(\tau+a,t)\gamma_\mu \partial^\mu \mathcal{T}\left\{\exp\left[-i\int_\tau^{\tau+a}\gamma_1 K(\tau')\cdot d\tau'\right]\right\}\Psi(\tau,t) \tag{49}$$

$$+\bar{\Psi}(\tau+a,t)\gamma_\mu \mathcal{T}\left\{\exp\left[-i\int_\tau^{\tau+a}\gamma_1 K(\tau')\cdot d\tau'\right]\right\}\partial^\mu\Psi(\tau,t) \tag{50}$$

Plugging in the equation of motion gives the first term as

$$(i\overline{\Psi}(\tau+a)K(\tau+a))\mathcal{T}\left\{\exp\left[-i\int_\tau^{\tau+a}\gamma_1 K(\tau')\cdot d\tau'\right]\right\}\Psi(\tau,t) \tag{51}$$

and the third term as

$$\bar{\Psi}(\tau+a,t)\mathcal{T}\left\{\exp\left[-i\int_\tau^{\tau+a}\gamma_1 K(\tau')\cdot d\tau'\right]\right\}(-iK(\tau)\Psi(\tau,t)) \tag{52}$$

On the middle term we can simplify $\partial_\mu \to \partial_\tau$, and computing the derivative yields

$$\bar{\Psi}(\tau+a,t)(-i\gamma_1^2(K(\tau+a)-K(\tau)))\mathcal{T}\left\{\exp\left[-i\int_\tau^{\tau+a}\gamma_1 K(\tau')\cdot d\tau'\right]\right\}\Psi(\tau,t) \tag{53}$$

Substituting $\gamma_1^2 = 1$, this precisely cancels out the other two terms; therefore $\partial^\mu J_\mu = 0$ for any choice of a, as desired.

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