Acceleration in Special Relativity

Prakhar Bansal

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1 Proper acceleration

Definition. *The particle's* ***proper acceleration,*** α*, is defined as its acceleration measured in its instantaneous rest frame.*

Suppose at time t the particle has velocity v.Then it is instantaneously at rest in the frame Σ' moving with velocity v relative to Σ.Then $\alpha = \frac{dv'}{dt'}$ Now we relate acceleration observed in Σ, $(\frac{dv}{dt})$, to α

$$\begin{aligned} \alpha &= \frac{dv'}{dt'} \\ &= \frac{dv'}{dv} \times \frac{dv}{dt} \times \frac{dt}{dt'} \end{aligned} \tag{1}$$

Now we have

$$v' = \frac{v-u}{1-\frac{v.u}{c^2}} \tag{2}$$

where u is the velocity of reference frame Σ' in Σ.

$$\begin{aligned} dv' &= \frac{(1-\frac{v.u}{c^2})dv + (v-u)\frac{u}{c^2}dv}{(1-\frac{v.u}{c^2})^2} \\ &= dv(\frac{1-\frac{u^2}{c^2}}{(1-\frac{v.u}{c^2})^2}) \end{aligned}$$

at the time of observation u $\approx$ v

$$\begin{aligned} \frac{dv'}{dv} &= \frac{1}{1-\frac{v^2}{c^2}} \\ &= \gamma^2 \end{aligned} \tag{3}$$

Now

$$\begin{aligned} t &= \gamma(t' + \frac{v.x'}{c^2}) \\ \frac{dt}{dt'} &= \gamma \end{aligned} \tag{4}$$

Using 3 and 4 we get

$$\frac{dv}{dt} = \gamma^{-3}\alpha \tag{5}$$

2 Constant proper acceleration

Suppose a particle experiences constant proper acceleration, $\alpha = \alpha_0$.Suppose it starts out at rest in Σ at t=0.We determine the velcity v(t) of the particle (as measured in Σ) Rearranging 4 and integrating we get

$$\int_0^v dv \frac{1}{(1-\frac{v^2}{c^2})^{3/2}} = \int_0^t \alpha_0 dt$$

Using substitution $v = c\sin\theta$ the integral becomes quite simplified and final expression becomes

$$v(t) = \frac{\alpha_0 t}{\sqrt{1+\frac{\alpha_0 t}{c^2}}} \tag{6}$$

We can write a similar equation for a velocity v'(t')as

$$v'(t') = \alpha_0 t' \tag{7}$$

(Since the proper acceleration α_0 is constant)

3 Hyperbolic Space Travel

Using 6 we determine the expression for x(t).Here we take $\alpha = g = 10m/s^2$ as it gives a standard result.

Fact. An acceleration equal to "g" for one year can impart a velocity equal to c i.e.

$$c = g.t \tag{8}$$

where t is no. of seconds in a year

To make our equations reflect more **practical** situations we convert distances into light years and time into years.

$$\begin{aligned} X &= \frac{x}{c.t} \\ &= \frac{x}{c.\frac{c}{g}} \\ &= \frac{x.g}{c^2} \end{aligned} \tag{9}$$

where X is in light years and x is in m.
Similarly for time in seconds(t) and time in years(T) we obtain relation

$$T = \frac{t.g}{c} \tag{10}$$

v(t)in 6 is $\frac{dx}{dt}$ which is $c \times \frac{dX}{dT}$.So final expression becomes

$$\frac{dX}{dT} = \frac{T}{\sqrt{1+T^2}}$$

$$\int_0^X dX = \int_0^T \frac{T}{\sqrt{1+T^2}} dT$$

which gives us

$$X(T) = \sqrt{1+T^2} - 1 \tag{11}$$

We now have an expression for the distance travelled by a body (say **A Spaceship**) in a time T if accelerated by a **constant proper acceleration** equal to g.

3.1 Proper time interval in case of accelerated motion

Suppose there is a spaceship in which astronauts are sent from one galaxy to the other and which travels at a constant proper acceleration g.Now if we are given the task to calculate the time(T)(in the rest frame in which space ship was released (Σ)) it takes for the spaceship to reach from one galaxy to the other (given the distance X between them) we can easily do that by using the eq 11.
Now suppose we are asked to find the time calculated in astronauts' refernce frame Σ in the same journey.
We can't use the simple time-dilation equation here because the velocity of refernce frame Σ' is continuously changing here. This is where the invariant equation of SR comes into picture.
We can use that equation to determine dτ (by using eq 11) and integrate it to obtain proper time interval τ in astronauts' refernce frame.
The invariant equation

$$\boxed{d\tau^2 = dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2}} \tag{12}$$

If we rewrite the equation in terms of T and X and express τ also in years then we can get rid of the factor of c and finally get

$$d\tau^2 = dT^2 - dX^2 \tag{13}$$

Using eq 11

$$dX = \frac{T}{\sqrt{1+T^2}}dT$$

Substituting in eq13

$$\begin{aligned}
d\tau^2 &= \frac{dT^2}{1+T^2} \\
\int_0^\tau d\tau &= \int_0^T \frac{dT}{\sqrt{1+T^2}} \\
\tau &= \ln\left(T + \sqrt{1+T^2}\right) \\
\implies T &= \frac{e^\tau - e^{-\tau}}{2}
\end{aligned}$$

$$\boxed{T = \sinh\tau} \tag{14}$$