Assignment 3

Question 1 - Consider the following English sentences.

- ***All employees are people.***
- ***The SSN of an employee is the SSN of a person.***

1.1 Convert the sentences into first-order predicate logic. Be extremely careful about quantification; because not everything in the universe is a person, or an employee, or a ssn, you will need both kinds of quantification. Use the following lexicon:
Predicates

person (X) -- X is a person.

employee (X) -- X is an employee

ssn(S,X) — S is the SSN of X.

Ans 1.1 - *All employees are people.*
$\forall\, x\ employee(x)\ \Rightarrow\ person(x)$

The SSN of an employee is the SSN of a person.
$\forall x\ \forall s\ employee(x) \wedge ssn(s,x)\ \Rightarrow\ \exists p\ person(p) \wedge ssn(s,p)$

1.2 Following the steps on text book to convert the logic statements into CNF.
Ans 1.2 - *All employees are people* $\forall\, x\ employee(x)\ \Rightarrow\ person(x)$

1. Eliminate Implication using a ⇒ b ≡ $\neg\, a \vee b$
   $\forall\, x\ \neg\, employee(x) \vee person(x)$

2. Move ¬ inwards – Nothing to do here as it is already inwards
   $\forall\, x\ \neg\, employee(x) \vee person(x)$

3. Standardize variables – They are already in standard form
   $\forall\, x\ \neg\, employee(x) \vee person(x)$

4. Skolemize – There is not existential quantifier to skolemize
   $\forall\, x\ \neg\, employee(x) \vee person(x)$

5. Drop universal quantifier
   $\neg\, employee(x) \vee person(x)$

6. Distribute ⋁ over ⋀ - There is no ⋀ in equation
   $\neg\, \mathbf{employee}(\mathbf{x}) \vee \mathbf{person}(\mathbf{x})$

Second - *The SSN of an employee is the SSN of a person.*
$\forall x\ \forall s\ employee(x) \wedge ssn(s,x)\ \Rightarrow\ \exists p\ person(p) \wedge ssn(s,p)$

1. Eliminate Implication
   $\forall x\ \forall s\ \neg\, (employee(x) \wedge ssn(s,x)) \vee (\exists p\ person(p) \wedge ssn(s,p))$

2. Move ¬ inwards
   $\forall x\ \forall s\ (\neg\, employee(x) \vee \neg\, ssn(s,x)) \vee (\exists p\ person(p) \wedge ssn(s,p))$

3. Standardize variables – They are already in standard form
$\forall x\ \forall s\ (\neg\ employee(x) \lor \neg\ ssn(s,x)) \lor (\exists p\ person(p) \land ssn(s,p))$

4. Skolemize – Remove the existential quantifier in front of person with skolem function f(s) because s is a universal quantifier and covers the scope of existential quantifier p
$\forall$x $\forall$s $(\neg\ employee(x) \lor \neg\ ssn(s,x)) \lor (person(f(s)) \land ssn(s,f(s)))$

5. Drop universal quantifier
$(\neg\ employee(x) \lor \neg\ ssn(s,x)) \lor (person(f(s)) \land ssn(s,f(s)))$

6. Distribute $\lor$ over $\land$

$[\neg\ \boldsymbol{employee(x)} \lor \neg\ \boldsymbol{ssn(s,x)} \lor \boldsymbol{person(f(s))}] \land [\neg\ \boldsymbol{employee(x)} \lor \neg\ \boldsymbol{ssn(s,x)} \lor \boldsymbol{ssn(s,f(s))}]$

1.3 **Using FOPL resolution prove that the second of the original sentences follows from the first. Number your clauses, and indicate explicitly step-by-step what resolves together, under what substitution.**

Ans 1.3 For FOPL resolution that KB $\vDash \alpha$ we can prove that KB $\land \neg\alpha$ as unsatisfiable. By unsatisfiable we mean that we can prove that this statement turns out to be False always.

α = $(\neg\ employee(x) \lor \neg\ ssn(s,x) \lor person(f(s))) \land (\neg\ employee(x) \lor \neg\ ssn(s,x) \lor ssn(s,f(s)))$

Let us convert this α into $\neg\ \alpha$
$\neg[\ (\neg\ employee(x) \lor \neg\ ssn(s,x) \lor person(\ f(s)\ )) \land (\neg\ employee(x) \lor \neg\ ssn(s,x) \lor ssn(s,f(s)))\ ]$

Let's take $\neg$ inside and apply using De Morgan Law
$[\ \neg(\neg\ employee(x)) \land \neg(\neg\ ssn(s,x)) \land (\neg\ person(\ f(s)\ )\ ) \lor \neg(\neg\ employee(x)) \land \neg(\neg\ ssn(s,x)) \land \neg(ssn(s,f(s)))\ ]$

Now use $\neg\ (\neg\ p)\ =\ p$ double negation elimination
$(\ employee(x) \land ssn(s,x) \land \neg\ person(\ f(s)\ )) \lor (\ employee(x) \land\ ssn(s,x) \land \neg ssn(s,f(s)))$

Now with advantage of association I can put this together $(employee(x) \land ssn(s,x))$
$(\ (employee(x) \land ssn(s,x)) \land \neg\ person(\ f(s)\ )) \lor (\ (employee(x) \land\ ssn(s,x)) \land \neg ssn(s,f(s)))$

Now applying law distributivity of $\land\ over \lor$
$employee(x) \land ssn(s,x) \land (\ \neg\ person(\ f(s)\ ) \lor \neg ssn(s,f(s))\ )$

$\neg\ \boldsymbol{\alpha} =\ employee(x) \land ssn(s,x) \land (\ \neg\ person(\ f(s)\ ) \lor \neg ssn(s,f(s))\ )$

Now we will have to prove that KB $\land\ \neg\ \alpha$ is satisfiable.
KB $\land\ \neg\ \alpha$:

$(\neg employee(x) \lor person(x))$
$employee(x)$
$ssn(s,x)$
$(\ \neg\ person(\ f(s)\ ) \lor \neg ssn(s,f(s))\ )$

Using above all the terms in KB $\land\ \neg\ \alpha$ we will prove satisfiability:

$(\neg employee(x) \lor person(x))$ $(\neg\ person(\ f(s)\ ) \lor \neg ssn(s,f(s))\ )$

Substitute f(s)/x

$(\neg\ employee(x) \lor person(x))$ $(\ \neg\ person(x) \lor \neg ssn(s,x)\ )$

Resolving person predicate

employee(x) $(\ \neg\ employee(x) \lor \neg ssn(s,x))$

Resolving employee predicate we get

ssn(s, x) $\neg ssn(s,x)$

Resolving ssn predicate we get empty clause

It is Satisfiable. Hence, we can say that second of the original sentence follows from the first.

Question 2 - Convert $\exists \boldsymbol{x}\ \forall \boldsymbol{y}\ ([\boldsymbol{P}(\boldsymbol{x},\boldsymbol{y}) \Rightarrow \forall \boldsymbol{z}\ \boldsymbol{Q}(\boldsymbol{y},\boldsymbol{z})]) \ \Rightarrow\ [\exists \boldsymbol{u}\ \exists \boldsymbol{v}\ (\boldsymbol{P}(\boldsymbol{x},\boldsymbol{v}) \wedge \boldsymbol{R}(\boldsymbol{y},\boldsymbol{v}))]$ to CNF, show exactly the conversion steps as those listed in the textbook.
Answer-

1. Eliminate Implication
   $\exists x\ \forall y\ \neg([P(x,y) \Rightarrow \forall z\ Q(y,z)]) \vee [\exists u\ \exists v\ (P(x,v) \wedge R(y,v))]$

   Now apply implication to $([P(x,y) \Rightarrow \forall z\ Q(y,z)])$
   $\exists x\ \forall y\ (\neg[\neg P(x,y) \vee \forall z\ Q(y,z)]) \vee [\exists u\ \exists v\ (P(x,v) \wedge R(y,v))]$

2. Move $\neg$ inwards
   $\exists x\ \forall y\ ([\neg(\neg P(x,y)) \wedge \neg\ (\forall z\ Q(y,z))]) \vee [\exists u\ \exists v\ (P(x,v) \wedge R(y,v))]$

   Apply $\neg\ (\neg\ p)\ =\ p$ and $\neg\ (\forall\ x\ q(x)) =\ \exists \boldsymbol{x} \neg\ q(x)$
   $\exists x\ \forall y\ ([P(x,y) \wedge \neg\ (\forall z\ Q(y,z))]) \vee [\exists u\ \exists v\ (P(x,v) \wedge R(y,v))]$

3. Standardize variables – They are already in standard form
   $\exists x\ \forall y\ ([P(x,y) \wedge \neg\ (\forall z\ Q(y,z))]) \vee [\exists u\ \exists v\ (P(x,v) \wedge R(y,v))]$

4. Skolemize – Remove the existential quantifiers $\exists \boldsymbol{x}$ by a constant X and $\exists \boldsymbol{v}$ by skolem function F(y), we use 'y' as parameter of skolem function because y is a universal quantifier covering scope of existential quantifier v
   $\forall y\ ([P(X,y) \wedge \neg\ (\forall z\ Q(y,z))]) \vee [(P(X,F(y)) \wedge R(y,F(y)))]$

5. Drop universal quantifier
   $([P(X,y) \wedge \neg\ Q(y,z)]) \vee [(P(X,F(y)) \wedge R(y,F(y)))]$

6. Distribute $\vee$ over $\wedge$ -
   $[\ (\ P(X,y) \wedge \neg\ Q(y,z)\ )\ \vee P(X,F(y))\ ]\ \wedge [\ (\ P(X,y) \wedge \neg\ Q(y,z)\ )\ \vee R(y,F(y))\ ]$

   Now applying distributivity law again inside brackets

$[\boldsymbol{P}(\boldsymbol{X},\boldsymbol{y}) \vee \boldsymbol{P}(\boldsymbol{X},\boldsymbol{F}(\boldsymbol{y}))\ ] \wedge [\ \neg\ \boldsymbol{Q}(\boldsymbol{y},\boldsymbol{z}) \vee \boldsymbol{P}(\boldsymbol{X},\boldsymbol{F}(\boldsymbol{y}))\ ] \wedge [\ \boldsymbol{P}(\boldsymbol{X},\boldsymbol{y}) \vee \boldsymbol{R}(\boldsymbol{y},\boldsymbol{F}(\boldsymbol{y}))\ ] \wedge [\neg\ \boldsymbol{Q}(\boldsymbol{y},\boldsymbol{z}) \vee \boldsymbol{R}(\boldsymbol{y},\boldsymbol{F}(\boldsymbol{y}))\ ]$

Question 3 - Let *a* be a constant symbol, *f* be a function symbol, *P* be a ternary predicate symbol, and *b, c, y, z* be variables. Use resolution to determine whether the following CNF sentence is satisfiable or unsatisfiable:

$$\boldsymbol{R(a,\ b,\ b) \wedge (\ \neg R(c,\ y,\ z) \vee R(\ f(\ f\ (c)\ ),\ y,\ f(z))\ ) \wedge \neg R(\ f(\ f(a)\ ),\ a,\ f(a))}$$

Answer -

1) Using associativity of $\wedge$ we can write the above equation as :
   $R(a,\ b,\ b) \wedge [\ (\neg R(c,y,z) \vee R(\ f(\ f(c)\ ),y,f(z))\ ) \wedge \neg R(\ f(\ f(a)\ ),\ a,\ f(a)\ )\ ]$

2) Now use law of distributivity $\wedge$ over $\vee$ inside the [] brackets:
   $R(a,\ b,\ b) \wedge [\ (\neg R(c,y,z) \wedge \neg R(f(f(a)),\ a,\ f(a))\ ) \vee (R(f(f(c)),y,f(z)) \wedge \neg R(f(f(a)),\ a,\ f(a))\ )\ ]$

3) We will now substitute **c/a, y/a, z/a** and we get
   $R(a,\ b,\ b) \wedge [(\neg R(a,\ a,\ a) \wedge \neg R(f(f(a)),\ a,\ f(a))\ ) \vee (\ R(f(f(a)),\ a,\ f(a)) \wedge \neg R(f(f(a)),\ a,\ f(a))\ )\ ]$

4) As we observe last term of above equation are equal [$R(f(f(a)), a, f(a))$] but one is negation of the other and we know that $\boldsymbol{\alpha} \wedge \neg \boldsymbol{\alpha}$ will always be False and we can for now write false for that statement.
$R(a, b, b) \wedge [\, (\neg R(a, a, a) \wedge \neg R(f(f(a)), a, f(a))\,) \vee (False)\,]$

5) Now we will use law of distributivity $\vee$ over $\wedge$
$[\, R(a, b, b) \wedge (\neg R(a, a, a) \wedge \neg R(f(f(a)), a, f(a))\,)\,] \vee [\, R(a, b, b) \wedge (False)\,]$

6) Now again if we observe the last term of above equation it comes as False as [anything $\wedge$ *False*] will always will be False and we can write the statement as:
$[\, R(a, b, b) \wedge (\neg R(a, a, a) \wedge \neg R(f(f(a)), a, f(a))\,)\,] \vee [\, False\,]$

7) Now we will use law of associativity in our first term of above equation
$[\, (R(a, b, b) \wedge \neg R(a, a, a)\,) \wedge \neg R(f(f(a)), a, f(a))\,] \vee [\, False\,]$

8) Now we will substitute b/a
$[\, (R(a, a, a) \wedge \neg R(a, a, a)) \wedge \neg R(f(f(a)), a, f(a))\,] \vee [\, False\,]$

9) Now in above equation you see the first terms are same, except the fact that one equation is negation of another which means that this term will always be false, and we can write false for that for ease. Using $\boldsymbol{\alpha} \wedge \neg \boldsymbol{\alpha}$ is always false rule.
$[\, (False) \wedge \neg R(f(f(a)), a, f(a))\,] \vee [\, False\,]$

10) Now in above equations first term we see that we have a False $\wedge \neg R(f(f(a)), a, f(a))$. This term will always be False as we can use $\boldsymbol{False \wedge anthing}$ is always false rule. We will substitute False value of this term as well.
$[\, False\,] \vee [\, False\,]$

11) Above equation is always False and hence we can say that the original equation is unsatisfiable.