[A.P.]. Pedersen

0.1 Examine the following formal descriptions of sets so that you understand which members they contain. Write a short informal English description of each set.

$$a.\ \{1, 3, 5, 7, \dots\}$$

Solution. A set of positive odd numbers. ■

$$b.\ \{\dots, -4, -2, 0, 2, 4, \dots\}$$

Solution. A set of even integer numbers. ■

$$c.\ \{n \mid n = 2m \text{ for some } m \text{ in } \mathbb{N}\}$$

Solution. A set of even natural numbers. ■

$$d.\ \{n \mid n = 2m \text{ for some } m \text{ in } \mathbb{N}, \text{ and } n = 3k \text{ for some } k \text{ in } \mathbb{N}\}$$

Solution. A set where n is divisible by both 2 and 3, therefore, it is a set where n is a multiple of 6. ■

$$e.\ \{w \mid w \text{ is a string of 0s and 1s and } w \text{ equals the reverse of } w\}$$

Solution. Since w is a string of 0s and 1s, the next iteration of the string is the reverse, so sets of w could be 0110 or 1001. ■

f. $\{n \mid n \text{ is an integer and } n = n + 1\}$

Solution. An empty set since there does not exist an integer n such that $n = n + 1$. ■

0.2 Write formal descriptions of the following sets.

a. The set containing the numbers 1, 10, and 100

Solution. $\{1, 10, 100\}$ ■

b. The set containing all integers that are greater than 5

Solution. $\{n \in \mathbb{Z} \mid n > 5\}$ ■

c. The set containing all natural numbers that are less than 5

Solution. $\{n \in \mathbb{N} \mid n < 5\}$ ■

d. The set containing the string aba

Solution. {aba} ■

e. The set containing the empty string

Solution. {} or ϵ ■

f. The set containing nothing at all

Solution. $\emptyset$ ■

0.3 Let A be the set {x,y,z} and B be the set {x,y}.

a. Is A a subset of B?

Solution. A is not a subset of B because z $\notin B$. ■

b. Is B a subset of A?

Solution. B is a subset of A because every element of B is in A. ■

c. What is $A \cup B$?

Solution. {x,y,z} ■

d. What is $A \cap B$?

Solution. {x,y} ■

e. What is $A \times B$?

Solution. {(x,x), (x,y), (y,x), (y,y), (z,x), (z,y)} ■

f. What is the power set of B?

Solution. $\{\emptyset, \{x\}, \{y\}, \{x,y\}\}$ ■

0.4 If A has a elements and B has b elements, how many elements are in $A \times B$? Explain your answer.

Solution. There are ab elements because for every a element, there will be a b element within it. Each a element starting from the first would contain each possible b element, similar to multiplying. ■

0.5 If C is a set with c elements, how many elements are in the power set of C? Explain your answer.

Solution. There are 2^c elements because the power set is a collection of all subsets including the empty set and set C itself. By observation, suppose we have a set with one element: {a}, then our power set would be $\{\emptyset, \{a\}\}$, which there are 2 elements. Another observation is suppose we have a set with two elements: {a,b}, then our power set would be $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, which there are 4 elements. Suppose we have another set with 3 elements: {a,b,c}, then our power set would be $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$, which there are 8 elements. By observation, we can see that as the number of elements increase, there is a 2^c elements correlation. ■

0.6 Let X be the set $\{1, 2, 3, 4, 5\}$ and Y be the set $\{6, 7, 8, 9, 10\}$. The unary function $f\colon X \to Y$ and the binary function $g : X \times Y \to Y$ are described in the following tables.

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

g	6	7	8	9	10
1	10	10	10	10	10
2	7	8	9	10	6
3	7	7	8	8	9
4	9	8	7	6	10
5	6	6	6	6	6

a. What is the value of $f(2)$?

Solution. $f(2) = 7$ ■

b. What are the range and domain of f?

Solution. The range of f is {6,7} and the domain is X or {1,2,3,4,5}. ■

c. What is the value of $g(2, 10)$?

Solution. $g(2, 10) = 6$ ■

d. What are the range and domain of g?

Solution. The range of g is Y or {6,7,8,9,10} and the domain of g is {{1,6}, {1,7}, {1,8}, {1,9}, {1,10}, {2,6}, {2,7}, {2,8}, {2,9}, {2,10}, {3,6}, {3,7}, {3,8}, {3,9}, {3,10}, {4,6}, {4,7}, {4,8}, {4,9}, {4,10}, {5,6}, {5,7}, {5,8}, {5,9}, {5,10}}. ■

e. What is the value of $g(4, f(4))$?

Solution. $g(4, f(4)) = 8$ ■

0.7 For each part, give a relation that satisfies the condition.

a. Reflexive and symmetric but not transitive

Solution. Let $A = \{a, b, c\}$ and $B = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, b)(c, c)\}$. B is reflexive because every element of A is in B and B is symmetric because every element in B has its symmetrical counterpart. B is not transitive because (a, b) and (b, c) exist in B but (a, c) does not exist in B. ■

b. Reflexive and transitive but not symmetric

Solution. Let $A = \{a, b, c\}$ and $B = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$. B is reflexive because every element of A is in B and B is transitive because (a, b) and (b, c) exist, and so does (a, c). B is not symmetric because B contains no symmetrical counterparts, such as $(a, b), (a, c), (b, c)$ exists but $(b, a), (c, a), (c, b)$ does not exist in B. ■

c. Symmetric and transitive but not reflexive

Solution. Let $A = \{a, b, c\}$ and $B = \{(a, a), (a, b), (b, a), (b, b)\}$. B is symmetric because for (a, b), there exists (b, a) and B is transitive because for (a, b) and (b, a), there exists (a, a). B is not reflexive because there does not exist (c, c) in B. ■

A2 Let X be a nonempty set. (i). Rigorously state the mathematical conditions a family F of subsets of X satisfies precisely when F is a partition of X. (ii). Given partitions of F and G of X, say the partition of F is *finer* than partition G if every element of G is the set-theoretic union of some subfamily of F. Prove or disprove the following statement: For every nonempty subset D of the set of real numbers, there is a partition F of D such that no partition of D is finer than F.

Solution. (i) A family F of subsets of X is a partition of X if and only if: (1) $\emptyset \notin F$: the family F does not contain empty sets, (2) $\bigcup_{A \in P} A = X$: the union of the sets in F is equal to X, and (3) $\forall A, B \in F \to A \cap B = \emptyset$: the intersection of any two distinct sets in F is empty. ■

Solution. (ii) For any nonempty subset D of the set of real numbers: $D = \{a_1, a_2, a_3, \ldots, a_n\}$ where $a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}$. All possible partitions of D contain every possible non-empty subset of D, so let $P(D) - \emptyset$ be the power set of D which will contain every element of all possible partitions of D. Every element of $P(D) - \emptyset$ is a set-theoretic union of the single element subsets: $\{a_1, a_2, a_3, \ldots, a_n\} = \{a_1\} \cup \{a_2\} \cup \{a_3\} \cup \cdots \cup \{a_n\}$. Thus, partition F of D where $F = \{\{a_1\}, \{a_2\}, \{a_3\}, \ldots, \{a_n\}\}$ will be finer than any other partition of D because the elements in F can be used to form any subset of D, which means the following statement is true. ■

Source to Solve A2: `https://en.wikipedia.org/wiki/Partition_of_a_set`