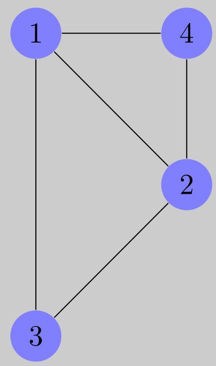
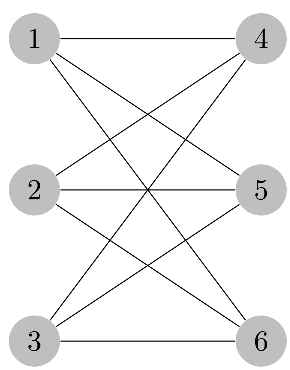
0.8 Consider the undirected graph $G = (V, E)$ where V, the set of nodes, is {1,2,3,4} and E, the set of edges, is $\{\{1,2\},\{2,3\},\{1,3\},\{2,4\},\{1,4\}\}$. Draw the graph G. What are the degrees of each node? Indicate a path from node 3 to node 4 on your drawing of G.

Solution.



deg(1) = 3
deg(2) = 3
deg(3) = 2
deg(4) = 2
One path from node 3 to node 4 would be to go from node 3 to node 2, then go from node 2 to node 4. ■

0.9 Write a formal description of the following graph.



Solution. An undirected graph $G = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6\}$ and
$E = \{\{1,4\},\{1,5\},\{1,6\},\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\}\}$ ■

0.11 Let $S(n) = 1+2+\cdots+n$ be the sum of the first n natural numbers and let $C(n) = 1^3+2^3+\cdots+n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n, to arrive at the curious conclusion that $C(n) = S^2(n)$ for every n.

$$a.\ S(n) = \frac{1}{2}n(n+1).$$

$$b.\ C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$$

Solution. **a.** Prove that $S(n) = \frac{1}{2}n(n+1)$.
Base Case: Let $n = 1$. Then $S(n) = 1$ by its own definition and

$$\begin{aligned} S(1) &= \frac{1}{2}(1)(1+1) \\ &= \frac{1}{2}(1)(2) \\ &= 1 \end{aligned}$$

Induction Step: Assume $k = n + 1$. We will then have

$$\begin{aligned} S(k) &= 1 + 2 + \cdots + n + k \\ S(n+1) &= 1 + 2 + \cdots + n + (n+1) \\ S(k) &= \frac{1}{2}k(k+1) \\ S(n+1) &= \frac{1}{2}(n+1)((n+1)+1) \\ &= \frac{1}{2}(n+1)(n+2) \end{aligned}$$

The formula $S(n) = \frac{1}{2}n(n+1)$ holds for both n and $n+1$.

b. Prove that $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$.
Base Case: Let $n = 1$. Then $C(n) = 1$ by its own definition and

$$\begin{aligned} C(1) &= \frac{1}{4}((1)^4 + 2(1)^3 + (1)^2) = \frac{1}{4}1^2(1+1)^2 \\ &= \frac{1}{4}(1)(4) \\ &= 1 \end{aligned}$$

Induction Step: Assume $k = n + 1$. We will then have

$$\begin{aligned} C(k) &= 1^3 + 2^3 + \cdots + n^3 + k^3 \\ C(n+1) &= 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 \\ C(k) &= \frac{1}{4}(k^4 + 2k^3 + k^2) = \frac{1}{4}k^2(k+1)^2 \\ C(n+1) &= \frac{1}{4}((n+1)^4 + 2(n+1)^3 + (n+1)^2) = \frac{1}{4}(n+1)^2((n+1)+1)^2 \\ &= \frac{1}{4}(n+1)^2(n+2)^2 \end{aligned}$$

The formula $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$ holds for both n and $n+1$.

Prove that $C(n) = S^2(n)$.
Base Case: Let $n = 1$. Then

$$\begin{aligned} C(1) &= S^2(1) \\ \frac{1}{4}(1)^2(1+1)^2 &= (\frac{1}{2}(1)(1+1))^2 \\ \frac{1}{4}(1)(4) &= (\frac{1}{2}(1)(2))^2 \\ 1 &= 1^2 \end{aligned}$$

Induction Step: Assume $k = n + 1$. We will then have

$$\begin{aligned} C(k) &= S^2(k) \\ \frac{1}{4}k^2(k+1)^2 &= (\frac{1}{2}k(k+1))^2 \\ \frac{1}{4}k^2(k+1)^2 &= \frac{1}{4}k^2(k+1)^2 \end{aligned}$$

Therefore, we arrive at the curious conclusion that $C(n) = S^2(n)$ for every n. ■

0.13 Show that every graph with two or more nodes contains two nodes that have equal degrees.

Solution. Suppose we have a graph G with n nodes where $n \geq 2$. This would mean that the degree of every node in G can only range from 0 to $n-1$. If there was a node in G that had a degree of 0, this would mean it is impossible that the same graph G can have a degree of $n-1$, which would mean at least 2 nodes have the same degree. If there wasn't a node with degree of 0, then each node would have at most $n-1$ degree, guaranteeing that there is at least two nodes with equal degrees. This problem also relate to the pigeonhole principle, where there are n pigeons in less than n pigeonholes, then at least one pigeonhole must contain more than one pigeon. ■

0.5 If C is a set with c elements, how many elements are in the power set of C? Use mathematical induction to establish your answer.

Solution. If the cardinality of set C is c, then the cardinality of the power set $\mathcal{P}(C)$ is 2^c.
Base Case: If the set C is empty ($c = 0$), $C = \emptyset$, the cardinality $card(C) = 0$.
Then $\mathcal{P}(C) = \{\emptyset\}$, which contains one element, $\emptyset$.
Therefore, $card(\mathcal{P}(C)) = 2^0 = 1$, which the proof holds.

Induction Step: Let $k = c + 1$ and let $C = \{e_1, e_2, \ldots, e_c\}$ and let $D = \{e_1, e_2, \ldots, e_c, e_k\}$.
Because $D = C \cup \{e_k\}$, every subset of C is also a subset of D. Any subset of D either contains e_k or doesn't contain e_k, and if the subset of D doesn't contain e_k, then it is a subset of C, and there are 2^c subsets.
If a subset of D does contain e_k, then there are 2^c subsets containing e_k.
In total, we would have 2^c subsets containing e_k and 2^c subsets not containing e_k, so we would have $2^c + 2^c = 2^{c+1} = 2^k$. ■

A3.1 Define $f : \mathbb{N} \to \mathbb{Z}$ by setting for each $n \in \mathbb{N}$:

$$a.\ f(n) = \frac{n}{2} \text{ if } n \text{ is even}$$

$$b.\ f(n) = \frac{-(n-1)}{2} \text{if } n \text{ is odd}$$

You can show (and will show) that f is bijective.

Solution. **Base Case:** Let $n = 1 : f(1) = \frac{-(1-1)}{2} = 0$
Let $n = 2 : f(2) = \frac{2}{2} = 1$
So far, the base cases shows that f is bijective.

Induction Step: Let $k = n + 1$. If k is even, then $f(k) = \frac{k}{2} = \frac{n+1}{2}$, which means for every k value that is even, there will be a unique output $f(k) \in \mathbb{Z}$.
If k is odd, then $f(k) = \frac{-(k-1)}{2} = \frac{-((n+1)-1)}{2} = \frac{-n}{2}$, which will also give us a unique negative output $f(k) \in \mathbb{Z}$.
So for each $k \in \mathbb{N}$, there will always be a unique output $f(k) \in \mathbb{Z}$, therefore f is bijective. ■

A3.2 The power set of the natural numbers, $\mathcal{P}(\mathbb{N})$, is uncountable. Show that the set $\mathcal{P}(\mathbb{N})$ is **not** finite.

Solution. For *reductio ad absurdum*, assume that $\mathcal{P}(\mathbb{N})$ is finite. By definition, this means that $\mathcal{P}(\mathbb{N})$ is countable because it is finite. By another definition, there is a injective function $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$, however, this cannot be the case since $\mathcal{P}(\mathbb{N})$ has sets that are not countable, thereby creating the contradiction. ■