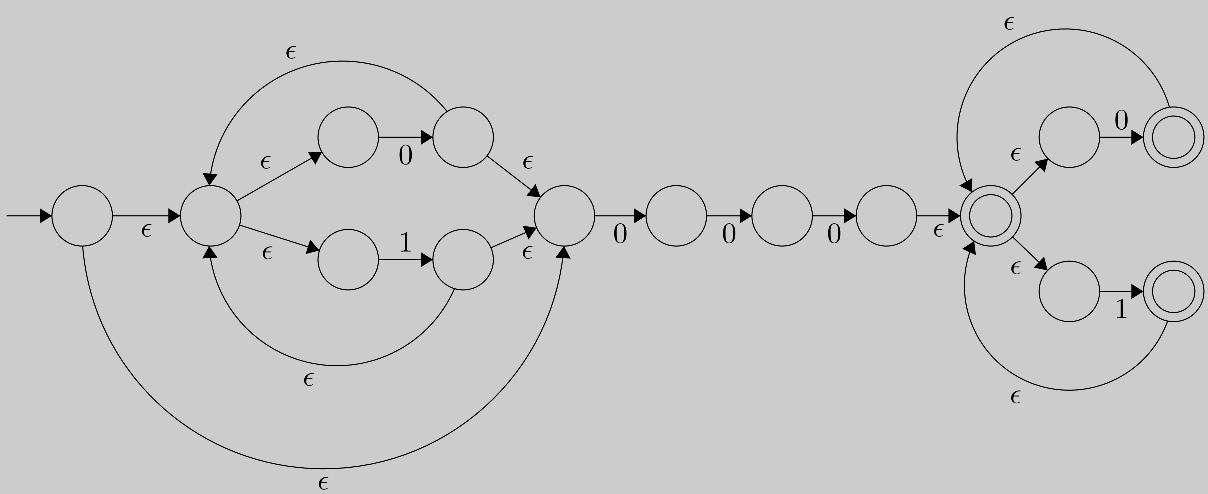
1.19 Use the procedure described in Lemma 1.55 to convert the following regular expressions to non deterministic finite automata.

a. $(0 \cup 1)^* 000 (0 \cup 1)^*$

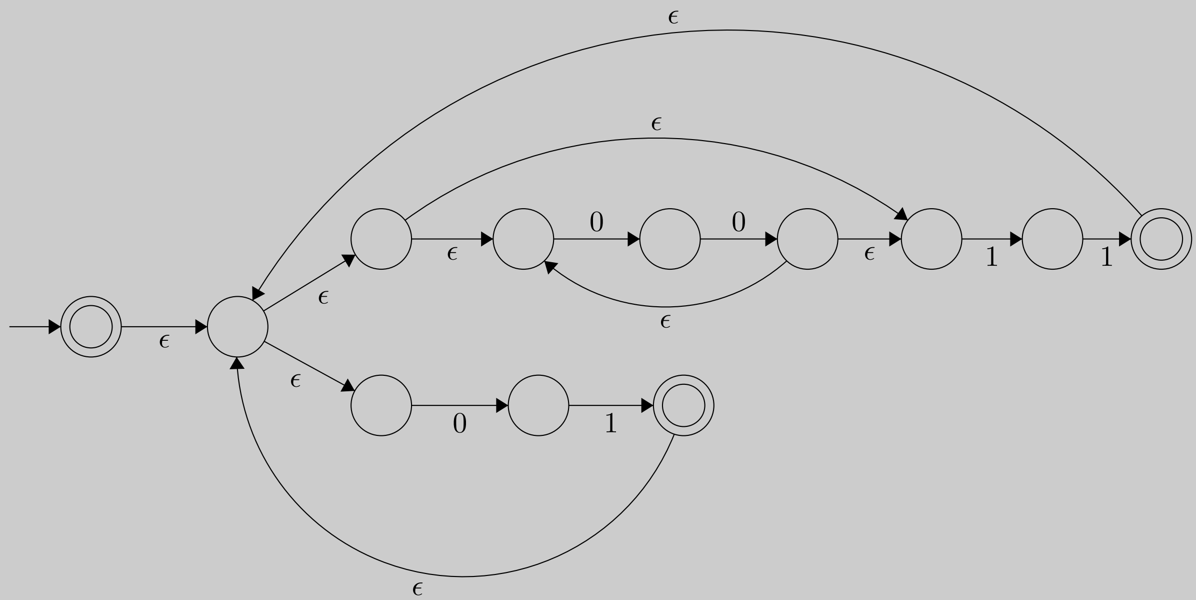
Solution.



■

b. $(((00)^*(11)) \cup 01)^*$

Solution.



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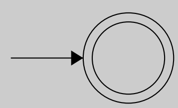
c. $\emptyset^*$

Solution.



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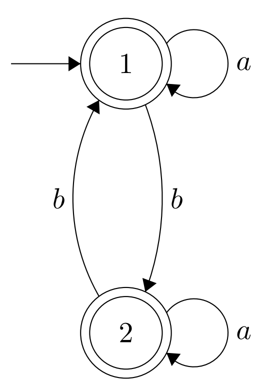
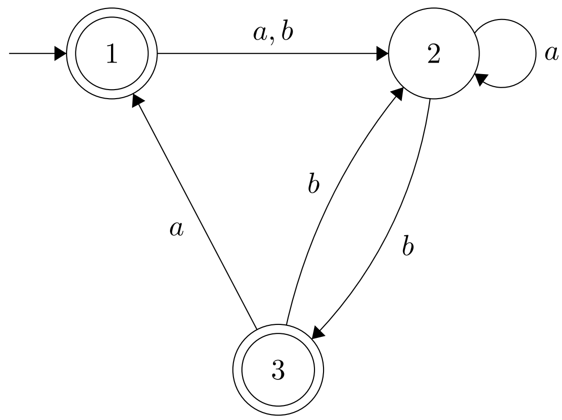
1.21 Use the procedure described in Lemma 1.60 to convert the following finite automata to regular expressions.



Solution. $(a \cup ba^*b)^*ba^*$ ■

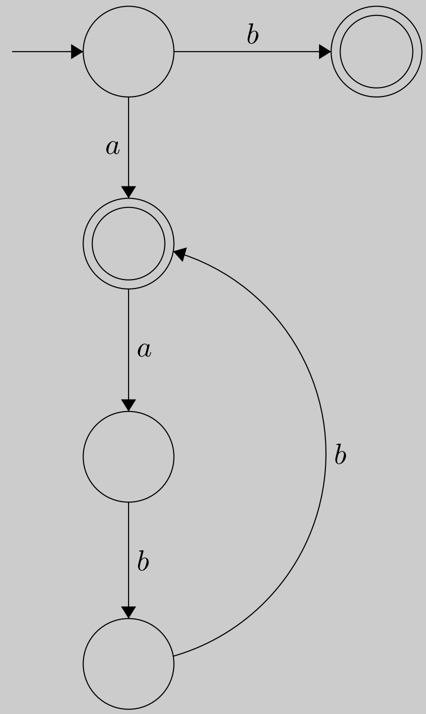


Solution. $((a \cup b)a^*b(ba^*b)^*a)^*\epsilon \cup (a \cup b)a^*b(ba^*b)^*$ ■

1.28 Convert the following regular expressions to NFAs using the procedure given in Theorem 1.54. In all parts, $\sum = \{a, b\}$.

a. $a(abb)^* \cup b$

Solution.



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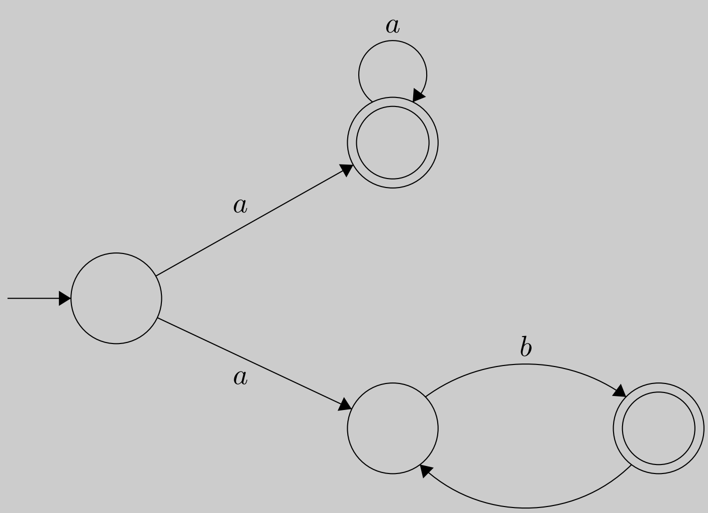
b. $a^+ \cup (ab)^+$

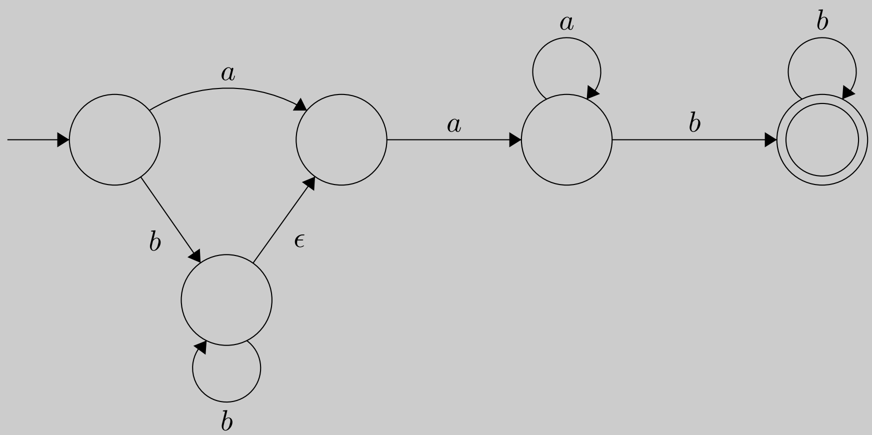
Solution.



■

c. $(a \cup b^+)a^+b^+$

Solution.



■

1.29 Use the pumping lemma to show that the following languages are not regular.

a. $A_1 = \{0^n 1^n 2^n | n \geq 0\}$

Solution. Assume that A_1 is regular and let p be the pumping length given by the pumping lemma, and let s be the string $0^p 1^p 2^p$. Since s is a member of A_1 and is longer than p, the pumping lemma guarantees that s can be split into $s = xyz$, where for any $i \geq 0$, the string xy^iz is in A_1. The string y can only consist of 0, 1, or 2 and because of this, the string $xyyz$ will not have equal numbers of 0, 1, and 2. This contradicts the assumption as $xyyz$ is not a member of A_1. ■

b. $A_2 = \{www | w \in \{a, b\}^*\}$

Solution. Assume A_2 is regular and let p be the pumping length given by the pumping lemma, and let s be the string $ab^pab^pab^p$. With $|xy| \leq p$, y has to be either an a with some b, or y is all b. In this case, pumping up will result in multiple a in the first w of the www that are not repeated in the other two w. In another case, where y contains only b, then pumping up will result in too many b for the first w which are not repeated in the other two w, which contradicts the assumption. ■

c. $A_3 = \{a^{2^n} | n \geq 0\}$(Here, a^{2^n} means a string of $2^n a'$s.)

Solution. Assume A_3 is regular and let p be the pumping length given by the pumping lemma, and let s be the string a^{2^p}. With $|xy| \leq p$, we can get $p < 2^p$ and so $|y| < 2^p$, and $|xyyz| = |xyz| + |y| < 2^p + 2^p = 2^{p+1}$. The second condition requires $|y| > 0$ so $2^p < |xyyz| < 2^{p+1}$, and the length of $xyyz$ cannot be a power of 2, which contradicts the assumption. ■

1.37 Let $C_n = \{x | x \text{ is a binary number that is a multiple of } n\}$. Show that for each $n \geq 1$, the language C_n is regular.

Solution. Let DFA M with n states that recognizes C_n and M has n states which keeps track of the n possible remainders of the binary division simulation. The start state is the only accept state and correspond to 0 remainder. The input string is fed into M starting from the most significant bit, and for each input bit, M doubles the remainder that its current state holds and then adds the input bits. The new state is the sum modulo n and we double the remainder as it corresponds to the left shift of the remainder in the division. If an input string ends at the accept state, the binary number has no remainder on division by n and is a member of C_n. ■

1.51 Let x and y be strings and let L be any language. We say that x and y are ***distinguishable by L*** if some string z exists whereby exactly one of the strings xz and yz is a member of L; otherwise, for every string z, we have $xz \in L$ whenever $yz \in L$ and we say that x and y are ***indistinguishable by L***. If x and y are indistinguishable by L, we write $x \equiv_L y$. Show that $\equiv_L$ is an equivalence relation.

Solution. An equivalence relation must have the three properties: reflexive, symmetric, and transitive. Based on the problem itself, its shown that $\equiv_L$ is reflexive and symmetric. To prove transitivity, let $x \equiv_L y$ and $y \equiv_L u$. First relation means that $xy \in L$ whenever $yz \in L$ for any string z and the second relation means that $yz \in L$ whenever $uz \in L$. Combining this yields to $xz \in L$ whenever $uz \in L$, which results in $x \equiv_L u$, and satisfies the transitivity. ■

1.52 **Myhill-Nerode theorum.** Refer to Problem 1.51. Let L be a language and let X be a set of strings. Say that X is ***pairwise distinguishable by L*** if every two distinct strings in X are distinguishable by L. Define the ***index of L*** to be the maximum number of elements in any set that is pairwise distinguishable by L. The index of L may be finite or infinite.

a. Show that if L is recognized by a DFA with k states, L has index at most k.

Solution. Let M be a k-state DFA that recognizes L, and suppose L has index greater than k. This would assume that some set X with more than k elements is pairwise distinguishable by L and because M has k states, X contains two distinct strings x and y, where $\delta(q_0, x) = \delta(q_0, y)$ based on the pigeonhole principle. $\delta(q_0, x)$ is the state that M is in after starting in the start state q_0 and reading string x. For any string $z \in \sum^*, \delta(q_0, xz) = \delta(q_0, yz)$, which means either both xz and yz are in L or neither are in L, but then x and y are not distinguishable by L, which contradicts the assumption that X is pairwise distinguishable by L. ■

b. Show that if the index of L is a finite number k, it is recognized by a DFA with k states.

Solution. Let $X = \{s_1, s_2, \ldots, s_k\}$ and is pairwise distinguishable by L. Also let DFA $M = (Q, \sum, \delta, q_0, F)$ with k states that recognizes L, and let $Q = \{q_1, q_2, \ldots, q_k\}$ and define $\delta(q_i, a)$ to be q_j, where $s_j \equiv_L s_i a$ for some $s_j \in X$. Let $F = \{q_i | s_i \in L\}$, and let the start state q_0 be the q_i such that $s_i \equiv_L \epsilon$. M is constructed so that for any state $q_i, \{s | \delta(q_0, s) = q_i\} = \{s | s \equiv_L s_i\}$, and therefore, M recognizes L. ■

c. Conclude that L is regular iff it has finite index. Moreover, its index is the size of the smallest DFA recognizing it.

Solution. Let k be the number of states in a DFA recognizing L and L has index at most k. If L has index k, then it is recognized by a DFA with k states and thus is regular. Suppose that L's index is k to show that the index of L is the size of the smallest DFA accepting it. This would mean that there is a k state DFA accepting L and that is the smallest such DFA. ■

Sources used for this assignment:
https://en.wikipedia.org/wiki/Pumping_lemma_for_regular_languages
https://en.wikipedia.org/wiki/Equivalence_relation
https://www.ccs.neu.edu/home/viola/classes/toc-gra-Spring13-scribes-exercises.pdf
http://people.cs.pitt.edu/~xiangxiao/cs1502/recitation11.pdf
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