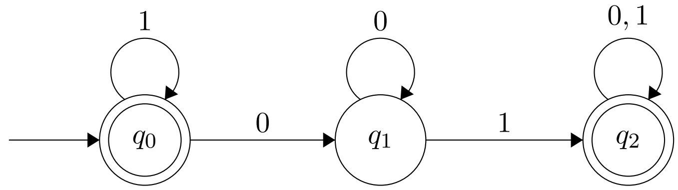
A5.1 Consider DFA D over alphabet 0,1 depicted in the transition diagram of attached file DFA.png.



(i) Identify D's set of states.

Solution. $Q = \{q_0, q_1, q_2\}$ ■

(ii) Identify D's transition function.

Solution. δ is described as:

	0	1
q_0	q_1	q_0
q_1	q_1	q_2
q_2	q_2	q_2

■

(iii) Identify D's initial state.

Solution. q_0 is the initial state. ■

(iv) Identify D's set of final states.

Solution. $F = \{q_0, q_2\}$ ■

A5.2 Suppose that you have constructed an NFA N over alphabet 0,1 that is equivalent to DFA D.

(i) Identify N's set of states.

Solution. $Q = \{q_0, q_1, q_2\}$ ■

(ii) Identify N's transition function.

Solution. δ is described as:

	0	1	ϵ
q_0	q_1	q_0	$\emptyset$
q_1	q_1	q_2	$\emptyset$
q_2	q_2	q_2	$\emptyset$

■

(iii) Identify N's initial state.

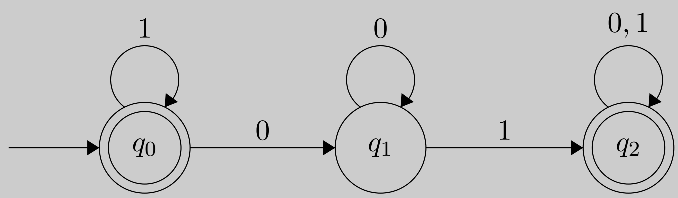
Solution. q_0 is the initial state. ■

(iv) Identify N's set of final states.

Solution. $F = \{q_0, q_2\}$ ■

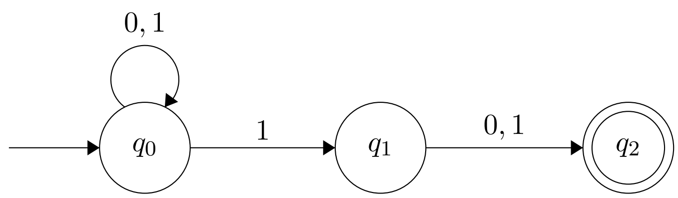
A5.3 Depict the NFA N thus constructed in a transition diagram.

Solution.



■

A5.4 Consider NFA M over alphabet 0,1 depicted in the transition diagram of attached file NFA.png.



(i) Identify M's set of states.

Solution. $Q = \{q_0, q_1, q_2\}$ ■

(ii) Identify M's transition function.

Solution. δ is described as:

	0	1	ϵ
q_0	q_0	$\{q_0, q_1\}$	$\emptyset$
q_1	q_2	q_2	$\emptyset$
q_2	$\emptyset$	$\emptyset$	$\emptyset$

■

(iii) Identify M's initial state.

Solution. q_0 is the initial state. ■

(iv) Identify M's set of final states.

Solution. $F = \{q_2\}$ ■

A5.5 Suppose that you have constructed an DFA C over alphabet 0,1 that is equivalent to NFA M.

(i) Identify C's set of states.

Solution. $Q = \{q_0, \{q_0q_1\}, \{q_0q_2\}, \{q_0q_1q_2\}\}$ ■

(ii) Identify C's transition function.

Solution. δ is described as:

	0	1
q_0	q_0	$\{q_0q_1\}$
$\{q_0q_1\}$	$\{q_0q_2\}$	$\{q_0q_1q_2\}$
$\{q_0q_2\}$	q_0	$\{q_0q_1\}$
$\{q_0q_1q_2\}$	$\{q_0q_2\}$	$\{q_0q_1q_2\}$

■

(iii) Identify C's initial state.

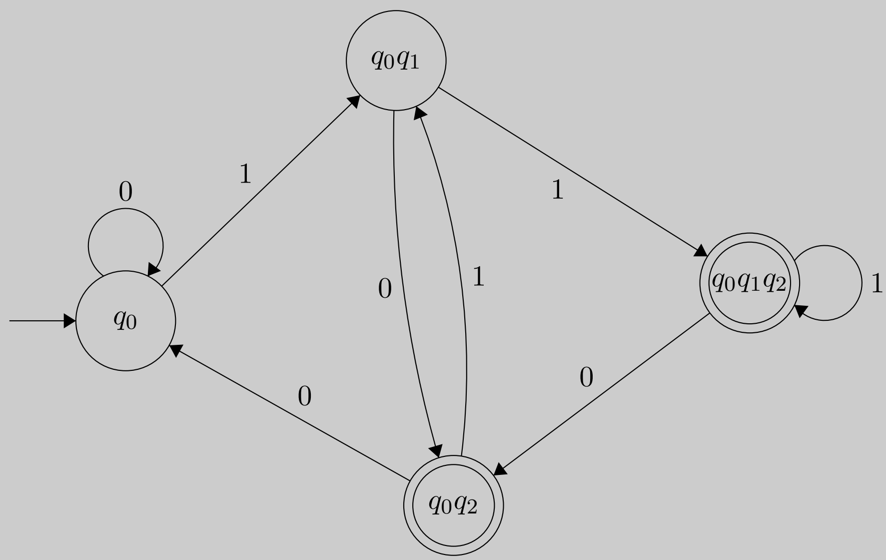
Solution. q_0 is the initial state. ■

(iv) Identify C's set of final states.

Solution. $F = \{\{q_0q_2\}, \{q_0q_1q_2\}\}$ ■

A5.6 Depict the DFA C thus constructed in a transition diagram.

Solution.



■

A5.7 State a rigorous mathematical definition for a given algebraic structure to be called a *monoid.* Is a vector space a monoid? Explain your answer in detail. Show that the set of all strings over a given alphabet, when equipped with the binary operation of string concatenation, forms a monoid.

Solution. Given a structure G and an operation $*$ to be used on G, a monoid is a category in which the structure satisfies the conditions:
Associative: Let $\forall x, y, z \in G$, then $(x * y) * z = x * (y * z)$.
Identity: Given an identity element $e \in G$ and $\forall y \in G, y * e = e * y = y$.

A vector space is a monoid because the basic properties of a vector space fulfill the requirements for it to be a monoid. Given an vector set V, there exists a zero vector: $0 \in V$, such that

$v * 0 = v$ for all $v \in V$. A vector space also has the associative property as well given $u, v, w \in V$, such that $(u + v) + w = u + (v + w)$.

Assume we have a given alphabet $A = \{a_1, a_2, a_3, \ldots, a_n\}$ and a set of all strings $S = \{s_1, s_2, s_3, \ldots, s_n\}$ over the given A. Using the binary operation of string concatenation, we can see that it forms a monoid through its properties. It satisfies the identity property because the identity of a set of all strings is $\emptyset$, and concatenating any element s in S with $\emptyset$ gives you the element s. It also satisfies the associative property because of how string concatenation operates. Assume we take 3 random elements from $S : s_a, s_b, s_c$. Performing $(s_a \circ s_b) \circ s_c = s_a s_b \circ s_c = s_a s_b s_c$ and performing $s_a \circ (s_b \circ s_c) = s_a \circ s_b s_c = s_a s_b s_c$, which satisfies the associative property. ■

A5.8 State a rigorous mathematical definition for a given mapping from a given algebraic structure into another given algebraic structure to be called a *monoid homomorphism.* Is a linear map from one vector space into another space a monoid homomorphism? Explain your answer in detail.

Solution. Given a monoid A with its operation $*$ and identity i_A, and another monoid B with its operation $\diamond$ and identity i_B, the mapping $f : A \to B$ is such that $f(A_x) \diamond f(A_y) = f(A_x * A_y)$ where $A_x, A_y \in A$.

A linear map from one vector space into another space is a monoid homomorphism because we are preserving all the group structures. The operation preserving is also a property of linear mapping, which we are preserving as well in the homomorphism. ■

A5.9 Prove or disprove: There is a string homomorphism transforming each binary representation of a given number to its hexadecimal representation.

Solution. Assume we have a string set $B = \{0000, 0001, 0010, \ldots, 1111\}$ that is the binary representation of decimal numbers from 0 to 15, and another string set $H = \{0, 1, 2, \ldots, D, E, F\}$ that is the hexadecimal conversion based on the binary representation from 0 to 15 in decimal. When we take a binary representation of a number, we are essentially taking a concatenated version of the elements from B and splitting it into elements that are in B, e.g. binary representation of decimal number 217 would be 11011001, which is the concatenation of 1101 and 1001. To convert to hexadecimal, we take those separate elements from B and convert or map them to H using our function f. Thus, we would have $f(B_e) = H_e$, and in the example given, $f(1101) = D$ and $f(1001) = 9$, and to get our hexadecimal result, D and 9 would be concatenated to get $D9$. By the definition of homomorphism, $f(B_x \circ B_y) = f(B_x) \circ f(B_y)$, using our example:
$f(1101 \circ 1001) = f(1101) \circ f(1001)$
$f(11011001) = D \circ 9$
$f(11011001) = D9$
We can say that the claim is correct as there is a string homomorphism when transforming a binary representation of a number to its hexadecimal representation. ■

Sources used for this assignment:
`http://people.csail.mit.edu/madhu/ST12/scribe/lect03.pdf`

https://en.wikipedia.org/wiki/Vector_space
https://www.youtube.com/watch?v=hJCv5KDMAFI
https://en.wikipedia.org/wiki/Homomorphism
https://en.wikipedia.org/wiki/Linear_map