A Numerical Method for Processing GPR Data

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Author Note

This senior thesis is submitted as partial fulfillment of the graduation requirements of Kettering University needed to obtain a Bachelor of Science in Mathematics. The conclusions and opinions expressed in this thesis are from myself and do not necessarily represent the position of Kettering University or anyone else affiliated with this culminating undergraduate experience.

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Abstract

One of the main challenges in processing GPR data is the accurate evaluation of time intervals between signals that are sent and reflected. Small errors in the length of these intervals generate large errors in the thickness of the layers because of the high propagation speed of electromagnetic waves. In this thesis, a new approach for numerical calculation of the time interval between two signals is proposed. Instead of finding the distances between the maxima points of signals, we calculate the distances between the centroids of the signals. The numerical experiments conducted in this thesis show that this new approach provides considerably higher accuracy in finding the layer thickness.

Keywords: Ground Penetrating Radar, Laplace Transform, Fourier Transform, Complex Frequencies, Analytic Continuation, Electromagnetic Waves.

Table of Contents

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The objective for ground penetrating radar (GPR) is to create a two dimensional cross sectional image of the electrical and magnetic characteristics within a volume by transmitting electromagnetic waves from the surface. One method to calculate the permittivity, permeability, and layer thickness beneath a surface is called "Layer Stripping". This paper improves upon the method of layer stripping in the complex frequency domain by proposing a way to more accurately find the time between reflected wavelets for each layer.

A GPR system typically consists of a transmitter located slightly above the surface and a receiver located on the surface. A central unit provides an electromagnetic pulse from the transmitter, and the receiver listens for the horizontal components of the electromagnetic field that is partially back-scattered from the boundary between two materials. This information is recorded and used to determine the electrical and magnetic properties of each layer, as well as the layers thickness.

Layer stripping implies both direct and inverse methods. In the direct method, one knows the properties of each layer and calculates the electric field along with its normal derivative of a wave that travels through it. In the inverse method, one knows the electric field and its normal derivative of the top layer, and then calculates the properties for the next layer. The GPR system records the electric field and its normal derivative at the surface. The inversion method is applied to the data to get the properties, as well as the thickness, of the layer. The direct method is then applied to get the electric field and its normal derivative at the bottom of the layer, which is also the top of the layer beneath it. These steps are repeated for the rest of the layers.

This paper applies the layer stripping method in the complex frequency domain, which allows one to damp the effect that the lower layers have in the inversion calculation. This is because the imaginary component of the frequency gives the effect of a lossy medium, in which the energy quickly decays.

In the following sections, this paper proposes a new method to calculate the layer thickness and supports that proposition with numerical results. GPR Theory will be introduced to derive the equations used in the method.

Conclusions and Recommendations

This papers novelty is the proposed method to increase accuracy of the calculation for layer thickness, which can then be applied to ground penetrating radars in practice. Two methods for calculating the time interval between signals, and therefore layer thickness, were discussed. One method, which takes the time interval between signal maxima to compute the layer thickness proved to be the less accurate of the two methods. Denote this method by Method 1.

Both methods considered the case where the conductivity was zero, and the case where the conductivity was greater than zero. When the conductivity was assumed to be zero, one could reconstruct it perfectly, so no error was propagated. In this best case scenario, using Method 1 the measurements of layer thickness had a relatively low percent error, however, when considering the absolute error, the layer thickness for any given layer could be up to of the order of one decimeter off from where it should be.

The second proposed method, denoted by Method 2, takes the time interval between the centroids of the wavelet. The centroid of the wavelet was calculated by integrating the wavelet and finding the time, t, at which there was an equal area under the wavelet both to the left and right of t.. This method proved to be more successful in measuring layer thickness. In the scenario where the conductivity was assumed to be zero, the absolute error was off by of the order of a centimeter. This provided considerably better measurements in the best case scenario, however, one also must consider what happens when error is introduced from the conductivity.

The conductivity reconstruction algorithm needs to be improved upon, since it only measured the first layer accurately. Despite this problem, a worst case scenario was described where, for all layers except the first layer, the percent error in reconstructed conductivity was 100 percent. Even in the worst case scenario, the method of finding centroids of the wavelet to calculate layer thickness proved to reconstruct the layer thickness with lower percent error than the method of finding maxima to calculate layer thickness. Additionally, when Method 2 was used with a 10 percent error in conductivity, it yielded much better results when compared to when Method 1 was used with no error in conductivity

GPR Theory

Consider Maxwells Equations

$$\nabla \times \boldsymbol{E} = -\mu\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}, \qquad \nabla \times \boldsymbol{H} = \epsilon\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \sigma \boldsymbol{E} + \boldsymbol{j}, \tag{1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic fields. σ is conductivity, ϵ is the relative permittivity, μ is the relative permeability, $\mu_0 = 1.257 \cdot 10^{-6} \frac{h}{m}$, $\epsilon_0 = 8.854 \cdot 10^{-12} \frac{f}{m}$, and $\boldsymbol{j}$ is external source.

Assume that the z-axis is perpendicular to the surface and directed inwards such that positive z is below the surface. Above the surface, let $\epsilon = 1$ and $\sigma = 0$. For all z, the relative magnetic permeability, μ, is assumed to equal one. This assumption is reasonable for non-magnetic materials. The wave number in the x direction, k_{x}, is assumed to be zero because the wave is directed perpendicular to the surface.

Assume $\mathbf{j} = f(t)\delta(x)\delta(z - z_0)\boldsymbol{e}_y$, where $\boldsymbol{e}_y$ is a unit vector in the y direction, the electric field is y polarized, and $z_0 < 0$. Denote by E, the y component of $\boldsymbol{E}$. One gets the following equations from Maxwell's equations:

$$\frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 E}{\partial x^2} - \frac{\partial^2 E}{\partial z^2} + \mu_0\mu\sigma\frac{\partial E}{\partial t} + \mu_0\mu\frac{\partial j}{\partial t} = 0 \tag{2}$$

Denote T to be the time it takes for a signal to be transmitted, f(0) = 0, and supp(f) = [0,T]. Since E does not depend on y and $E(t) = 0$ for $t < 0$, one can apply the Fourier transform to equation 2:

$$E(z, k_x, \omega) = \frac{1}{2\pi}\int_0^\infty \int_0^\infty e^{-i(\omega t + k_x x)} E(z, x, t)\,dt\,dx$$

Next, taking the Fourier transform on both sides of equation 2 one gets k_z.

$$\mathcal{F}(\frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2}-\frac{\partial^2 E}{\partial x^2}-\frac{\partial^2 E}{\partial z^2}+\mu_0\mu\sigma\frac{\partial E}{\partial t})=\mathcal{F}(-\mu_0\mu\frac{\partial j}{\partial t})$$

$$\frac{\partial^2 E}{\partial z^2}+(\omega^2\frac{\mu\epsilon}{c^2}-i\omega\mu\mu_0\sigma-k_x^2)E=h(\omega)\delta(z-z_0) \tag{3}$$

Denote $k_z^2=\omega^2\frac{\mu\epsilon}{c^2}-i\omega\mu\mu_0\sigma-k_x^2$ and $h(\omega)=i\mu\mu_0\omega f(\omega)$.

$$k_z=-\sqrt{\mu\omega^2\frac{\epsilon}{c^2}-i\mu_0\mu\sigma\omega-k_{\mathrm{x}}^2} \tag{4}$$

The square root denotes a positive root if real, and a root with negative imaginary part if complex.

Let

$$I=\frac{\frac{\partial E}{\partial z}}{E}$$

Since

$$\frac{\partial^2 E}{\partial z^2}+k_z^2E=0$$

and

$$I'=(\frac{\frac{\partial E}{\partial z}}{E})'=\frac{E\frac{\partial^2 E}{\partial z^2}-(\frac{\partial E}{\partial z})^2}{E^2}=\frac{\frac{\partial^2 E}{\partial z^2}}{E}-I^2$$

$$\frac{\partial^2 E}{\partial z^2}+k_z^2E=0\to\frac{\frac{\partial^2 E}{\partial z^2}}{E}=-k_z^2$$

One obtains

$$I'+I^2=\frac{\frac{\partial^2 E}{\partial z^2}}{E}=-k_z^2$$

Thus,

$$I'+I^2+k_z^2=0,\quad I(L)=ik_z(L)$$

Direct Problem

Consider a piecewise constant layered medium. Let $0 = z_{\text{initial}} < z_1 < ... < z_n$. For layer j, $z \in (z_{j-1}, z_j)$, $\epsilon(z) = \epsilon(j)$, $\sigma(z) = \sigma(j)$, and $k_z(z) = k_z(j)$. Since f = 0 when $t \leq 0$, the electric field is zero for $t < 0$. This means that the electric field can be continued into the lower half of the complex plane. Let $\omega = \omega_1 + i\omega_2$, where $\omega_1 = \Re(\omega) > 0$, and $\omega_2 = \Im(\omega) \leq 0$. For $z_{j-1} \leq z \leq z_j$

$$\frac{dI}{dz} + I^2 + k_j^2 = 0$$

$$\int \frac{dI}{-(I^2 + k_j^2)} = \int -dz$$

$$z + c = \int \frac{dI}{(I + ik_j)(I - ik_j)}$$

$$\frac{1}{(I + ik_j)(I - ik_j)} = \frac{A}{I + ik_j} + \frac{B}{I - ik_j}$$

$$A = -\frac{1}{2ik_j} \qquad B = \frac{1}{2ik_j}$$

$$\frac{I + ik_j}{I - ik_j} = c_j e^{-2ik_j z}$$

$$c_j = \frac{I + ik_j}{I - ik_j} e^{2ik_j z}$$

Let $l_j = z_j - z_{j-1}$, then from the continuity condition $I_j(z_j) = I_{j+1}(z_j)$, the following back recursion formulas can be used to calculate I of the previous layers.

$$\begin{aligned} I_n &= ik_z(z_n) \\ \alpha &= \frac{I_j - ik_z}{I_j + ik_z} \\ I_{j-1} &= ik_z(z_j)\frac{1 + \alpha e^{2ik_z(z_j)l_j}}{1 - \alpha e^{2ik_z(z_j)l_j}} \end{aligned} \tag{5}$$

E can be found from equation 3 and Sommerfeld's radiation conditions:

$$\lim_{z\to-\infty}\frac{dE}{dz}+ik_zE=0,\quad \lim_{z\to\infty}\frac{dE}{dz}-ik_zE=0$$

with the solutions

$$\begin{cases} E=c_1e^{-izk_z}, \text{for } -\infty<z<z_0 \\ E=c_2e^{izk_z}+c_3e^{-izk_z}, \text{for } z_0<z<0 \end{cases}$$

Also, from equation 3,

$$\lim_{z\to z_0^+}E=\lim_{z\to z_0^-}E$$

$$\lim_{z\to z_0^+}\frac{\partial E}{\partial z}=\lim_{z\to z_0^-}\frac{\partial E}{\partial z}-h(\omega)$$

and thus,

$$c_2e^{izk_z}+c_3e^{-izk_z}=c_1e^{-izk_z}$$

$$c_2e^{izk_z}-c_3e^{-izk_z}=-c_1e^{-izk_z}+\frac{ih(\omega)}{k_z}$$

solving these equations for c_2 and c_3, one gets

$$c_2=\frac{ih(\omega)}{2k}e^{-iz_0k_z} \text{ and } c_3=c_1-\frac{ih(\omega)}{2k_z}e^{iz_0k_z}$$

so for $z_0<z<0$,

$$E=c_1e^{-izk_z}+\frac{ih(\omega)}{2k_z}e^{i(z-z_0)k_z}-\frac{ih(\omega)}{2k_z}e^{-i(z-z_0)k_z}$$

Finally, for $z=0$ one gets

$$E_z+ik_zE=-h(\omega)e^{-ik_zz_0} \tag{6}$$

For Ricker wavelet h,

$$h = \frac{i\mu_0}{(\pi f_c)^3}\omega^3 e^{-(\frac{\omega^2}{2(\pi f_c)^2} - \frac{i\pi\omega}{f_c})}$$

Because $\sigma = 0$, $\epsilon = 1$, and $\mu = 1$ for $z < 0$, $k_z = -\sqrt{\frac{\omega^2}{c^2} - k_{\mathrm{x}}^2}$, substituting $E_z = IE$ into equation 6 and solving for electric field, one gets

$$E(\omega) = \frac{-he^{-ik_0 z_0}}{I + ik_z} \tag{7}$$

Now that E is found on the top layer, $E_z = IE$ can be found. With this data, one can begin solving the inverse problem.

Inverse Problem

For E and $\frac{\partial E}{\partial z}$ given at $z = 0$, one must find $\epsilon(z)$ and $\sigma(z)$ for $0 < z < L$. The method is based on the analytic continuation of the Fourier transform, $E(\omega)$, of $E(t)$. For the analytic continuation the following formula is used:

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\omega t} E(t)dt = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\omega_1 t} e^{\omega_2 t} E(t)dt$$

From equation 5,

$$k_z = -iI_{j-1}\frac{1 - \alpha e^{2ik_z l}}{1 + \alpha e^{2ik_z l}}$$

If the imaginary component of k_z approaches infinity, then k_z approaches $-iI_{j-1}$. This allows one to evaluate k_z by damping the other terms in the equation. From equation 4, ϵ and σ can be derived. Taking $\omega_2 = 0$, one can evaluate k_z and use it to find E and $\frac{\partial E}{\partial z}$ at the bottom of the layer. From this, I_j can be calculated, and this process can be repeated for each layer.

This theory is implemented in the next section, and a new algorithm is proposed to numerically determine the time between signals.

Method

To create synthetic data, the direct problem must first be presented. One must first define the permittivity and conductivity, along with the thickness of each layer, to describe some scenario. For materials such as air, asphalt, clay, sand, soil, and many more, the relative permittivity ranges from 1 to 40 (Daniels, 2004, p.90). This value can easily be higher, and since water has a permittivity of around 81, dampness can significantly effect it too. For our simulation, we use relative permittivity values ranging between 1 and 10. A typical range for the conductivity of non-conductive materials is from 0 to 0.1. We consider non-conductive materials since they lead to a lower decay, and assume that the conductivity values are left on the order of 10^{-5} for the simulation. This will ensure that every layer will produce a reflection that can be detected. Lastly, one must pick a central frequency such that it will be able to penetrate deep enough, as well as be able to identify each layer. In numerical experiments layer thickness was chosen to be of the order of the wavelength.

Starting off with the direct problem, the problem is solved going from the bottom layer to the top layer. Assume that the z-axis is perpendicular to the ground and directed inwards, and for all z, $\mu = 1$ and $k_x = 0$. Let k_z be calculated for each layer using equation 4. We consider real ω when making the calculation because the imaginary part of ω damps the signal.

Next, equation 5 should be reiterated until I is found for the top layer. Since the solution exists for I on [0,L] and $I = \frac{E_z}{E} = ik_z$, we can calculate the electric field on the top layer using equation 7. $E(t)$ and $E_z(t)$ can now be found by taking the inverse fast Fourier transform of $E(\omega)$ and $E_z(\omega)$.

The inverse problem can now be solved. The direct method will be reapplied to find E and E_z again for the top of the next layer, and the process is repeated. Starting off, we return to the complex frequency domain by taking the fast Fourier transform of $E(t)e^{\omega_2 t}$ and $E_z(t)e^{\omega_2 t}$. The complex frequency component is introduced to decrease the impact of the signal for the next layers.

So to reiterate this in more detail, I(ω) is calculated and used to find E and E_z from equation 7. Using the following equations, one can find the permittivity and conductivity of the layer.

$$\begin{aligned}
F_1 &= -\Re(I^2) + k_{\mathrm{x}}^2 \\
F_2 &= -\Im(I^2) \\
\epsilon &= \frac{c^2}{|\omega|^2}(F_1 + \frac{\omega_2}{\omega_1}F_2) \\
\sigma &= \frac{2\omega_2 F_1 + F_2(\frac{\omega_2^2}{\omega 1} - \omega_1)}{\mu_0|\omega|^2}
\end{aligned} \tag{8}$$

Once the permittivity is calculated, the wave speed through the layer can be determined. The electric field signal and its reflections have already been calculated when the inverse fast Fourier transform was taken on $E(\omega)$. The next objective is to calculate the time between the signal and its first reflection. Figure 1 shows the electric field signal, along with the time it takes for each reflected signal to reach the receiver. The speed of an electromagnetic wave through a medium with permeability μ and permittivity ϵ is $v = \frac{c}{\sqrt{\mu\epsilon}}$. Since $\mu = 1$, and the permittivity is known, one can calculate the wave speed. The layer thickness is equal to the wave speed multiplied by the time between the reflected signals. The layer thickness is divided by two since the wave travels through the medium, and is reflected back again through the medium. Finally, two methods will be presented to find the time between the signal and its reflected wave.

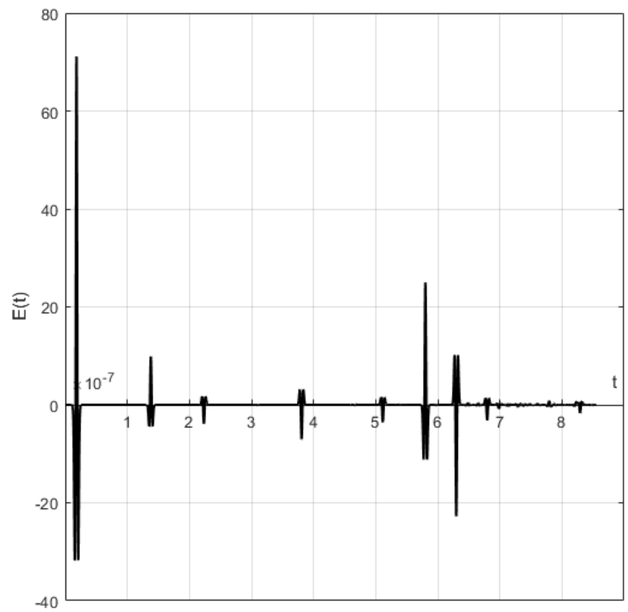

Figure 1. Electromagnetic field signal and its reflections plotted as a function of time.

Method 1: Find time between wavelet maxima

Since there are small oscillations in the electric field, one must first visually determine how many reflected wavelets there are from the data obtained on the top layer. Looking at Figure 1, you can tell that in the presented simulation, there is an initial signal plus seven reflections. This method takes advantage of the findpeaks function in MATLAB. The findpeaks function lists all of the maxima in the entire plot. First, one starts off by finding the time of the absolute maxima. The time is recorded. The absolute value of the plot is taken and multiplied by negative one. Using the findpeaks algorithm, one can then find the time where the Ricker wavelet would intersect the x-axis. The time between the wavelet maxima and where the wavelet equals zero should be found by taking the difference. Since the wavelets are thin and spaced far apart, one can set all values within plus or minus five times the difference between the maxima of the wavelet and the zero of the wavelet to equal zero, without interfering with any of the reflections.

Since each time the largest wavelet is set to equal zero, this can be repeated seven more times in order to find the time at which the maxima for each wavelet occurs. The time interval can be calculated and plugged into the equation for layer thickness by taking the difference between the time of any two successive maxima.

Method 2: Find time between wavelet centroids

This method works by using the max function in MATLAB, to find the absolute maxima throughout all time. Since the reflected waves are less than the first wavelet, initially, this will find the first signal. Taking the absolute value of the electric field and using the findpeaks function, one may find the zeros of the wavelet. Let the left side of the wavelet intersect the x-axis at point A, and the right side of the wavelet intersect it at point B. The griddedInterpolant method is used to create a function to interpolate the vector. The function is then integrated from A to B. The resulting area is divided by two. Choose a small tolerance T and let point C be equal to A. While the integral from A to C is less than half of the integral from A to B, increment C by T. This way, one can determine the time at which the area from A to C equals the area from C to B. Once the time of the centroid is recorded, one can take advantage of the fact that the transmission time of the signal is much less than the time between signals by setting all values near the centroid equal to zero. Since there will be a new largest wavelet, this algorithm can be rerun until every wavelet centroid is found. The time of each wavelet centroid can be sorted in ascending order, and the time between the wavelets can be used to calculate layer thickness.

Since the transmission time of the wave is very small, this method will be more accurate in finding the time where the wave is the strongest. Method 1 heavily depends upon the step size for time. The step size for time can be decreased to improve the accuracy of the synthetic data, however, the computation time significantly increases with a small decrease in step size. The novelty of Method 2 is that the time between signals can accurately be calculated, without significantly increasing the computation time.

Calculating E and E_z for the next layers.

Use equation 1, along with the newly calculated ϵ and σ to calculate the wavenumber. Using the following the electric field and its normal derivative from the previous layer and the following formula, $E(\omega)$ and $E_z(\omega)$ can be calculated at the bottom of the current layer which is also the top of the next layer.

$$\begin{aligned} E(\omega) &= \frac{E_z + ik_zE}{2ik_z}\, e^{ik_zl} - \frac{E_z - ik_zE}{2ik_z}\, e^{-ik_zl} \\ E_z(\omega) &= \frac{E_z + ik_zE}{2}\, e^{ik_zl} + \frac{E_z - ik_zE}{2}\, e^{-ik_zl} \end{aligned} \tag{9}$$

Once $E(\omega)$ and $E_z(\omega)$ are calculated, one can find $E(t)$ and $E_z(t)$. Lastly, $E(t)$ and $E_z(t)$ should be shifted by the time it takes for the electromagnetic wave to travel to the next layer. This shift creates a new time scale, which improves the accuracy of the numerical analytic continuation process. Finally, this allows us to repeat the previous steps to find ϵ and σ on the next layers.

Results

Method 1: Find distance between Ricker wavelet maxima

In our first method, the maximum point of each Ricker wavelet is found. The distance between each wavelet is taken in the time domain. Since this represents the difference in time between reflections, and the wave speed is known, this allows one to estimate the layer thickness. In the following subsections, results showing the accuracy of this method are presented.

Case 1: Electrical conductivity equals zero. The permittivity and layer thickness are shown in the following two tables for $\sigma = 0$.

Table 1
Layer thickness and reconstructed layer thickness using the find maxima algorithm, when $\sigma = 0$

Layer	exact layer thickness (m)	reconstructed layer thickness (m)	percent error in layer thickness
1	6.781	6.782	0.009%
2	6.404	6.427	0.346%
3	10.548	10.596	0.455%
4	7.158	7.371	2.972%
5	3.391	3.498	3.176%
6	5.274	4.980	5.586%
7	16.952	16.741	1.25%

It can be seen in Table 1 that this method is fairly accurate at finding the layer thickness. The maximum error in layer thickness is 5.586%, however if one looks at the error in the permittivity, the error is much larger. The error in permittivity depends on both the error in layer thickness and the difference in permittivity between two successive layers. However, this error can be minimized by using a more efficient algorithm to find the layer thickness.

Table 2
Relative permittivity and reconstructed relative permittivity using the find maxima algorithm, when $\sigma = 0$

Layer	exact relative permittivity	reconstructed relative permittivity	percent error in permittivity
1	7	7	4.28×10^{-10} %
2	4	3.971	0.735%
3	5	4.950	1.003%
4	7.5	7.080	5.596%
5	9.2	8.625	6.247%
6	2	2.249	12.429%
7	10	10.247	2.472%

Case 2: Electrical conductivity is greater than zero. This section shows how the error behaves when one assigns a conductivity to each layer in the direct problem, and try to reconstruct it. The program accurately reconstructs the conductivity of the first layer, however, for subsequent layers a significantly larger error is produced. The program tries to fix this by assigning the conductivity equal to zero, if the reconstructed value is negative, or if the reconstructed value is greater than 1×10^{-4}. Regardless of the program only working to sufficiently find the conductivity on the first layer, the numerical accuracy of the algorithm to find layer thickness is presented.

Table 3
Layer thickness and reconstructed layer thickness using the find maxima algorithm, when $\sigma \neq 0$

Layer	exact layer thickness (m)	reconstructed layer thickness (m)	percent error in layer thickness
1	6.781	6.782	0.009%
2	6.404	6.427	0.346%
3	10.548	10.638	0.851%
4	7.158	7.536	5.285%
5	3.391	3.635	7.206%
6	5.274	4.589	12.984%
7	16.952	18.845	11.162%

Table 4
Relative permittivity and reconstructed relative permittivity using the find maxima algorithm, when $\sigma \neq 0$

Layer	exact relative permittivity	reconstructed relative permittivity	percent error in permittivity
1	7	7	2.35×10^{-10} %
2	4	3.971	0.736%
3	5	4.911	1.779%
4	7.5	6.773	9.698%
5	9.2	7.989	13.163%
6	2	2.647	32.358%
7	10	8.087	19.134%

Table 5
Conductivity and reconstructed conductivity using the find maxima algorithm

Layer	exact conductivity	reconstructed conductivity	percent error in conductivity
1	9×10^{-5}	9×10^{-5}	1.733×10^{-6} %
2	3×10^{-5}	0	100%
3	4×10^{-5}	0	100%
4	7×10^{-5}	0	100%
5	1×10^{-5}	0	100%
6	5×10^{-5}	0	100%
7	8×10^{-5}	0	100%

It can be seen in both Table 3 and 4 that the error in conductivity leads to a much higher error in layer thickness and permittivity. Currently, the programs ability to find conductivity accurately is not good. Table 5 shows that most of the error was so large that it had to be assumed to be zero in the inverse problem, even though it was not zero in the direct problem. The program needs to be improved upon in its ability to find the conductivity, however, the data was included in this paper to demonstrate the significance of accurately finding the layer thickness. It can be seen that with incorrect guesses for conductivity values, the maximum error in the relative permittivity is 32.359% and the maximum error in layer thickness is 12.984%.

In the next subsection, a new method for finding layer thickness will significantly reduce the error in both permittivity and layer thickness.

Method 2: Find distance between Ricker wavelet centroids

In this method the centroid of each Ricker wavelet was used to calculate the distance between each wavelet. Similarly to the first method, an estimation for layer thickness is made. This method gave more accurate results than the first method, regardless of the conductivity, as shown below.

Case 1: Electrical conductivity equals zero. When the electrical conductivity equals zero, the error for the conductivity is zero. This is the best case scenario, since the error will be small. Comparing it to the previous method for calculating layer thickness, this method performs much better. The largest error in layer thickness is 0.743%, which is much less than the 5.586% error that was introduced from the other method. The largest error in the permittivity also decreases from 12.429% down to 1.524%. This decrease in error in calculating epsilon comes directly from having a more accurate layer thickness because the permittivity values did not change.

Table 6

Layer thickness and reconstructed layer thickness using the find centroid algorithm, when $\sigma = 0$

Layer	exact layer thickness (m)	reconstructed layer thickness (m)	percent error in layer thickness
1	6.781	6.781	0.001%
2	6.404	6.400	0.063%
3	10.548	10.538	0.102%
4	7.158	7.158	0.007%
5	3.391	3.391	0.008%
6	5.274	5.235	0.743%
7	16.952	16.887	0.389%

Table 7

Relative permittivity and reconstructed relative permittivity using the find centroid algorithm, when $\sigma = 0$

Layer	exact relative permittivity	reconstructed relative permittivity	percent error in permittivity
1	7	7	4.28×10^{-10} %
2	4	4.005	0.137%
3	5	5.010	0.197%
4	7.5	7.499	0.009%
5	9.2	9.197	0.034%
6	2	2.030	1.524%
7	10	10.074	0.735%

Case 2: Electrical conductivity is greater than zero. According to Table 10, the error in conductivity is the same. When compared with the other algorithm for finding layer thickness, the worst error in layer thickness is similar. With the new algorithm, our worst error is 13.143%, which is close to the 12.984% from the previous algorithm. However, it should be noted that there was a significant decrease in error when calculating the permittivity of the layer when using this method. The error introduced by the conductivity is the same in both

methods, while the maximum error in permittivity decreases from 32.358% to 22.306%.

It should be noted that the method for calculating conductivity does need to be improved upon, but even in this worst case scenario it can be seen that this second method for calculating layer thickness provides a significantly better reconstruction of permittivity than the first method.

Table 8
Layer thickness and reconstructed layer thickness using the find centroid algorithm, when $\sigma \neq 0$

Layer	exact layer thickness (m)	reconstructed layer thickness (m)	percent error in layer thickness
1	6.781	6.781	0.002%
2	6.404	6.400	0.064%
3	10.548	10.580	0.297%
4	7.158	7.336	2.495%
5	3.391	3.539	4.387%
6	5.274	4.770	9.568%
7	16.952	19.181	13.143%

Table 9
Relative permittivity and reconstructed relative permittivity using the find centroid algorithm, when $\sigma \neq 0$

Layer	exact relative permittivity	reconstructed relative permittivity	percent error in permittivity
1	7	7	2.35×10^{-10} %
2	4	4.005	0.137%
3	5	5.970	0.599%
4	7.5	7.138	4.823%
5	9.2	8.444	8.213%
6	2	2.446	22.306%
7	10	7.808	21.919%

Table 10

Conductivity and reconstructed conductivity using the find centroid algorithm

Layer	exact conductivity	reconstructed conductivity	percent error in conductivity
1	9×10^{-5}	9×10^{-5}	1.733×10^{-6} %
2	3×10^{-5}	0	100%
3	4×10^{-5}	0	100%
4	7×10^{-5}	0	100%
5	1×10^{-5}	0	100%
6	5×10^{-5}	0	100%
7	8×10^{-5}	0	100%

Figure 2 shows a qualitative comparison between the layer thickness and the reconstruction of that layer thickness using the second method. In the figure there was no error introduced due to conductivity. Assuming that the conductivity is reconstructed with relatively high accuracy, then Method 2 can be used to reconstruct layer thickness for at least seven layers accurately.

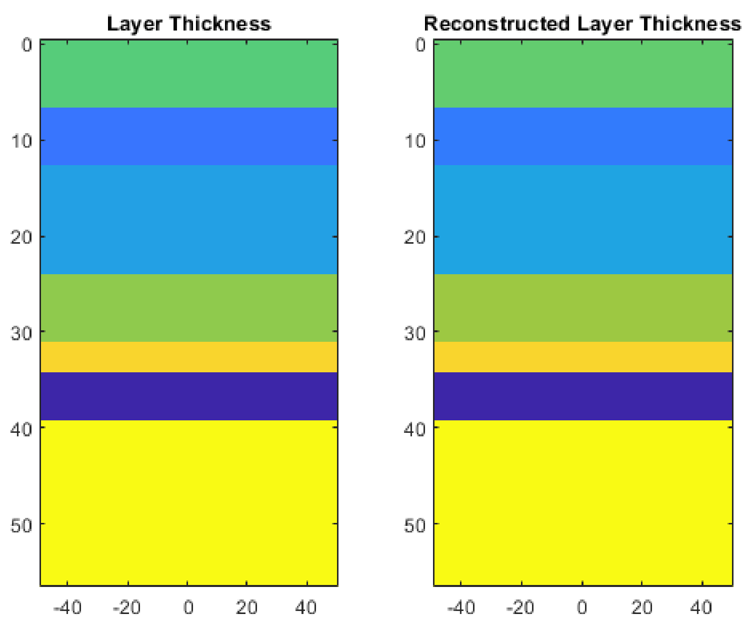

Figure 2. A qualitative comparison between the exact layer thickness and the reconstructed layer thickness using Method 2. In this scenario, the conductivity was assumed to be zero and therefore perfectly reconstructed.

Lastly, a 10 percent error in conductivity was introduced at each layer. This corresponded to a maximum of 2.23 percent error in layer thickness, occurring at the last layer. Figure 3 shows a qualitative comparison between the exact layer thickness and the reconstructed layer thickness. This demonstrates that, even with a sizable error in conductivity, the layer thickness can be reconstructed relatively well with this method. If a method to reconstruct the conductivity within 10 percent of its exact value is implemented, then Method 2 will more accurately reconstruct the layer thickness with this error when compared to Method 1 and no error in conductivity.

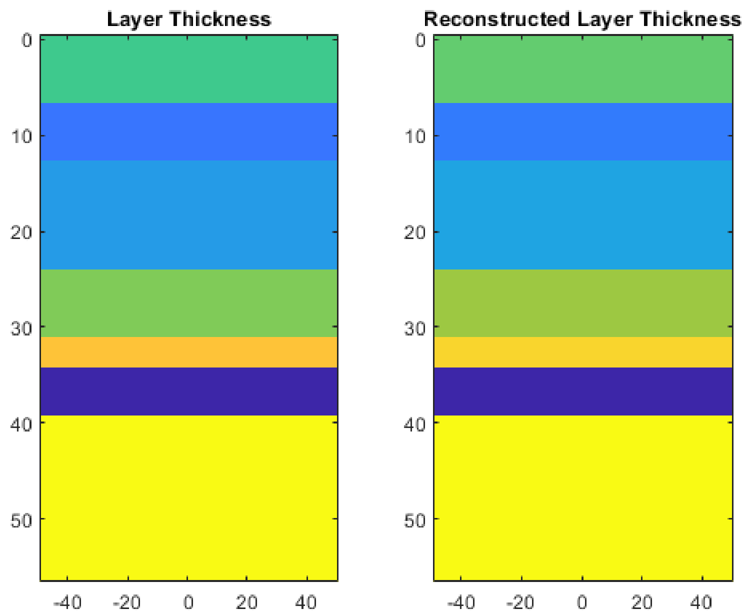

Figure 3. A qualitative comparison between the exact layer thickness and the reconstructed layer thickness using Method 2. The conductivity was assumed to have a 10 percent error at each layer, in this scenario.

Discussion

For GPR data processing, the numerical method for finding the interval between centroids of the wavelets proves to be more reliable. Furthermore, when the conductivity is equal to zero the error in layer thickness with the new method did not pass one percent. When finding the interval between maxima and using it to determine layer thickness in the same scenario, the error in layer thickness is above 5 percent. From a practical standpoint, using the less accurate method gives an error on the order of decimeters for more than one layer, while using the more accurate method gave an error on the order of centimeters. When determining the layout of the ground, the errors can either add up or cancel depending on whether the reconstructed layer thickness was an overestimate or an underestimate. Even if the errors all are summed up in the more accurate method, the resulting error is less than that introduced in just one layer of the less accurate method.

In order for the program to model the real world more accurately, the method for the reconstruction of conductivity will have to be reworked to find the layers past the first one more accurately. It is worth noting that an accurate reconstruction of epsilon provides sufficient information to determine what material the layer is, so the inability to reconstruct the conductivity is not detrimental to the program.

One hindrance in the layer stripping method is it does not show promising results when two successive layers have a large difference in permittivity. For example, if there were an underground cavity that contained air ($\epsilon \approx 1$) with water ($\epsilon \approx 80$) beneath it, the program would have difficulty in reconstructing the permittivity and layer thickness of the next layer. This is due to the signal strength decreasing across the boundary of two layers.

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Appendix

The following program is code for the direct problem. It was saved as GPR_DIRECT.m.

```
clear;
close all;
format long

% Constants
mu0 = 1.25663706e-6;
epsilon0 = 8.85418782e-12;
c=1/sqrt(mu0*epsilon0);
n1 = 20;
n=2^n1;
C=5.e-8;
ht=C*sqrt(2*pi/n);
hom=sqrt(2*pi/n)/C;
t=ht*(0:n-1);
omega1 = hom*(0:n-1);
tmax=t(end);
om1max=omega1(end) %maximal frequency
fc = 2*10^8
kx = 0;
mu = 1;
wavelength=2*pi*c/fc;
```

```
t0 = 1*pi/fc;
x = 0.4*wavelength*[0, 1.8, 3.5, 6.3, 8.2, 9.1, 10.5, 15, inf];% boundary
    of each layer in meters
epsilon = [7, 4, 5, 7.5, 9.2, 2, 10, 18]; % last value is epsilon below
    bottom layer.
sigma = 1.0e-5*[9, 3, 4, 7, 1, 5, 8, 4]; % multiply by zero if zero
    conductivity



layers = length(epsilon) - 1;
nn=n/256;

% Matrix for I, depending on layer and wavelength. (from equation 5)
k = - sqrt( mu*epsilon(layers+1)*omega1.^2/c^2 -...
    1i*mu0*mu*sigma(layers+1)*omega1 - kx^2 );
I_matrix = 1i*k;

for z = layers:-1:1
    k = - sqrt( mu*epsilon(z)*omega1.^2/c^2 -...
    1i*mu0*mu*sigma(z)*omega1 - kx^2 );
    alpha = (I_matrix - 1i*k)./(I_matrix + 1i*k);

    alpha(1)=0;
    I_matrix = 1i.*k.*(1 + alpha.* exp(2*1i*k*...
        (x(z+1) - x(z))) )./(1 - alpha .* exp(2*1i*k*(x(z+1) - x(z))));
    I_matrix(1)=0;
end
```

```
% Finding E and Ez
h = (1i*mu0*omega1.^3/(pi*fc)^3).*exp(-omega1.^2/(2*(pi*fc)^2)...
    - 1i*t0.*omega1);

z0=-0.03*wavelength;

E = - h.* exp((-1i*k*z0)) ./ (I_matrix + 1i*k );
E(1,1)=0;

Ez = I_matrix(1,:).*E;

H=2*sqrt(2*pi)/ht*real(ifft(h));
Et=2*sqrt(2*pi)/ht*real(ifft(E));
Ezt=2*sqrt(2*pi)/ht*real(ifft(Ez));

myprint2(1,nn,t,Et,Ezt)
save gpr
```

The following is code for the inverse problem. It was saved as GPR_ INVERSE.m.

```
clear
load gpr
format long

n0=round(fc*2*pi/hom);
omega2= -0.5*omega1(n0);
omega = omega1 + 1i*omega2;

nshift = 0;
m = layers+1; % m should equal the amount of waves that you see.
Epsilon_Reconstructed = zeros(1,m-1);
Sigma_Reconstructed = zeros(1,m-1);
Thickness_Reconstructed = zeros(1,m-1);
Error_epsilon = zeros(1,m-1);
Error_sigma = zeros(1,m-1);
Error_layerThickness = zeros(1,m-1);

% Find Layer Thickness
type = 2;
Eq_TimeOfMax =layThick(Et,m,t,type);

for LAYER = 1: (m-1)

    Eanal=fft(Et.*exp(omega2.*t));
    Ezanal=fft(Ezt.*exp(omega2.*t));
    I1=Ezanal./Eanal;
    F1=-real(I1(n0).^2)+kx^2;
    F2=-imag(I1(n0).^2);
```

```
epsilon_new = c^2*(F1+omega2*F2./omega1(n0))/(omega1(n0)^2+omega2^2);
if epsilon_new <1
    epsilon_new=1;
end
if epsilon_new>40
    epsilon_new=40;
end
sigma_new =
((2*omega2*F1+(omega2^2/omega1(n0)-omega1(n0))*F2)/mu0)/(omega1(n0)^2+omega2^2);

if sigma_new< 0 || sigma_new>1.0e-4
    sigma_new = 0;
end

% uncomment the next line if conductivity is assumed to be zero.
% sigma_new = 0;

Sigma_Reconstructed(LAYER) = real(sigma_new);
Error_sigma(LAYER) = abs(sigma_new - sigma(LAYER))/ sigma(LAYER);

Epsilon_Reconstructed(LAYER) = real(epsilon_new);
Error_epsilon(LAYER) = abs(epsilon_new -
    epsilon(LAYER))/epsilon(LAYER);
```

```
    speed_Layer1 = c/sqrt(epsilon_new);
    dt_Layer1 = Eq_TimeOfMax(LAYER+1) - Eq_TimeOfMax(LAYER);% difference
        in time between first and second maxima
    Layer1_Thickness = speed_Layer1*dt_Layer1/2; % divide by 2 because of
        reflection.
    Thickness_Reconstructed(LAYER) = Layer1_Thickness;
    L_thick = Thickness_Reconstructed(LAYER);
    Error_layerThickness(LAYER) = abs((L_thick - (x(LAYER+1) -
        x(LAYER))))/ (x(LAYER+1) - x(LAYER));

    k = - sqrt( mu*epsilon_new*omega1.^2/c^2 - 1i*mu0*mu*sigma_new*omega1
        - kx^2 );

    E2 = ((Ez + 1i*k.*E)./(2*1i*k)).*(exp(1i*k*L_thick)) - ...
        ((Ez - 1i*k.*E)./(2*1i*k)).*(exp(-1i*k*L_thick));

    Ez2 = ((Ez + 1i*k.*E)/2).*(exp(1i*k*L_thick)) + ...
        ((Ez - 1i*k.*E)/2).*(exp(-1i*k*L_thick));

    E2(1,1)=0;
    Ez2(1,1)=0;
    E=E2;
    Ez=Ez2;
    Et=2*sqrt(2*pi)/ht*real(ifft(E));
    Ezt=2*sqrt(2*pi)/ht*real(ifft(Ez));
    nshift=nshift+max(round(dt_Layer1/ht/2),1);
    Et=[Et(nshift+1:n),zeros(1,nshift)];
    Ezt=[Ezt(nshift+1:n),zeros(1,nshift)];
end
```

```
% Table to format data
Table_col1 =
{'Reconstructed_epsilon','epsilon','percent_error_in_epsilon'};
Table_col2 =
{'Reconstructed_Layer_Thickness','Layer_Thickness','percent_error_in_layer_thickness'};
Table_col3 =
{'Reconstructed_sigma','sigma','percent_error_in_sigma'};

ArrayOfData1 = [Epsilon_Reconstructed', (epsilon(1:m-1))',
   100*real(Error_epsilon)'];
ArrayOfData2 = [real(Thickness_Reconstructed'), (x(2:m) - x(1:m-1))',
   100*Error_layerThickness'];
ArrayOfData3 = [real(Sigma_Reconstructed'), (sigma(1:m-1))',
   100*Error_sigma'];

tab1 = array2table(ArrayOfData1,'VariableNames',Table_col1);
tab2 = array2table(ArrayOfData2,'VariableNames',Table_col2);
tab3 = array2table(ArrayOfData3,'VariableNames',Table_col3);

[tab1]
[tab2]
[tab3]

myprint3(2,layers,round(Thickness_Reconstructed),Epsilon_Reconstructed,...
   'Epsilon Reconstructed',round((x(2:m) - x(1:m-1))),epsilon,'epsilon')
```

This section of code is used to calculate the layer thickness for the inverse problem. It was saved as layThick.m.

```
function Eq_TimeOfMax =layThick(Et_dummy,m,t, type)
    ht=t(2)-t(1);
    TimeOfMax = zeros(1,m);
    for j = 1:m
        [~, nE1] = max(abs(Et_dummy));
        [~, nE2] = findpeaks(-abs(Et_dummy(nE1:end)));

        switch type
            case 1
             Et_dummy(nE1-5*nE2(1):nE1+5*nE2(1)) = 0;
             TimeOfMax(j) = nE1;

            case 2
            [~, nE3] = findpeaks(-abs(Et_dummy(1:nE1)));
            nE4 = nE3(end);
            nE5 = nE1 + nE2(1) - 1;
            % figure(j+1), plot((nE4:nE5), abs(Et_dummy(nE4:nE5)))

            A1 = griddedInterpolant((nE4:nE5), abs(Et_dummy(nE4:nE5)));
            A2 = @(t) A1(t);
            A3 = integral(A2, nE4, nE5);
            A4 = A3/2;
            B1 = 0;
            A5 = A4-1;
                while (A5<A4)
```

```
            A5 = integral(A2, nE4, nE4 + B1);

            B1 = B1 + 0.1;

            end

         Et_dummy(nE1-5*nE2(1):nE1+5*nE2(1)) = 0;

         TimeOfMax(j) = B1 + nE4;

      end

   end

   Eq_TimeOfMax = sort(TimeOfMax*ht, 'ascend');

end
```