CS4102 Algorithms

Spring 2021 – Floryan and Horton

Module 4, Day 3: Live Session, April 26

Roadmap: Where We've Been and Why

- ***Reductions*** between problems
  - Why? Can be a practical way of solving a new problem
  - **Coming soon:** A proof about one problem's complexity can be applied to another
  - Formal definition of a reduction
- Examples
  - Bipartite matching and max network flow
- **Today:** example problems: vertex cover and independent set
- **Next Lecture:** classes of problems: P, NP, NP-Hard, NP-complete

- Today: Two graph problems that are similar:
  minimum vertex cover and ***maximum independent set***

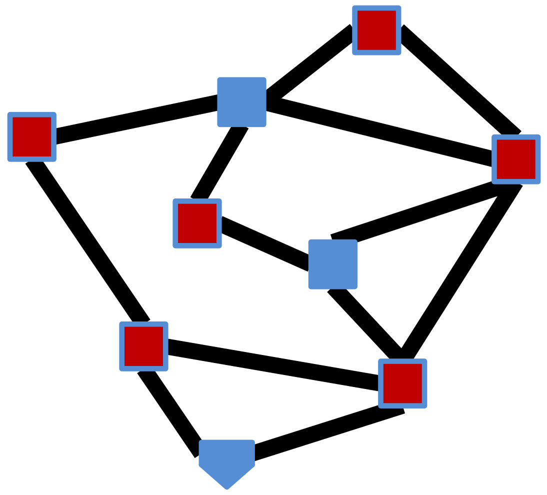
  – We'll do a formal proof that these are ***polynomial-time equivalent***
    - Each reduces to the other

  – BTW, vertex cover is needed for Basic Written HW, Question 2

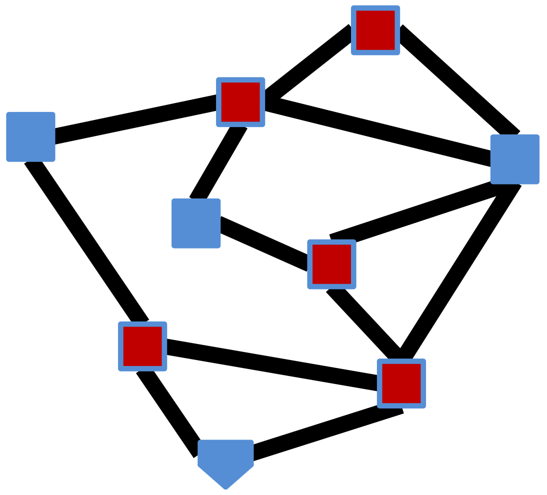
Minimum Vertex Cover

- Vertex Cover: $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- Minimum Vertex Cover Problem: Given a graph $G = (V, E)$ find the minimum vertex cover C

Vertex cover of size 6



Vertex cover of size 5



Is 5 the smallest?

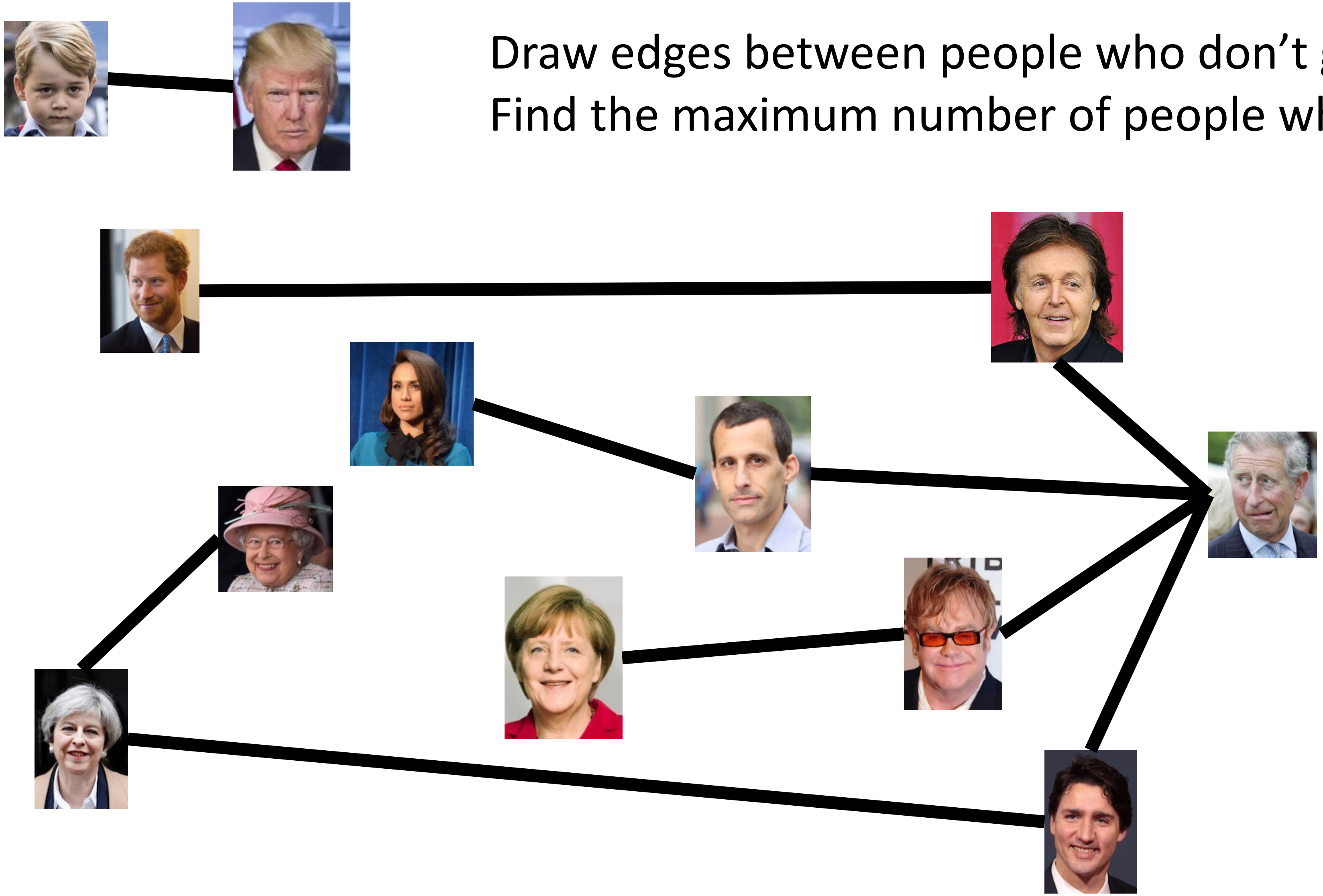
Is there one with 4?

Maximum Independent Set

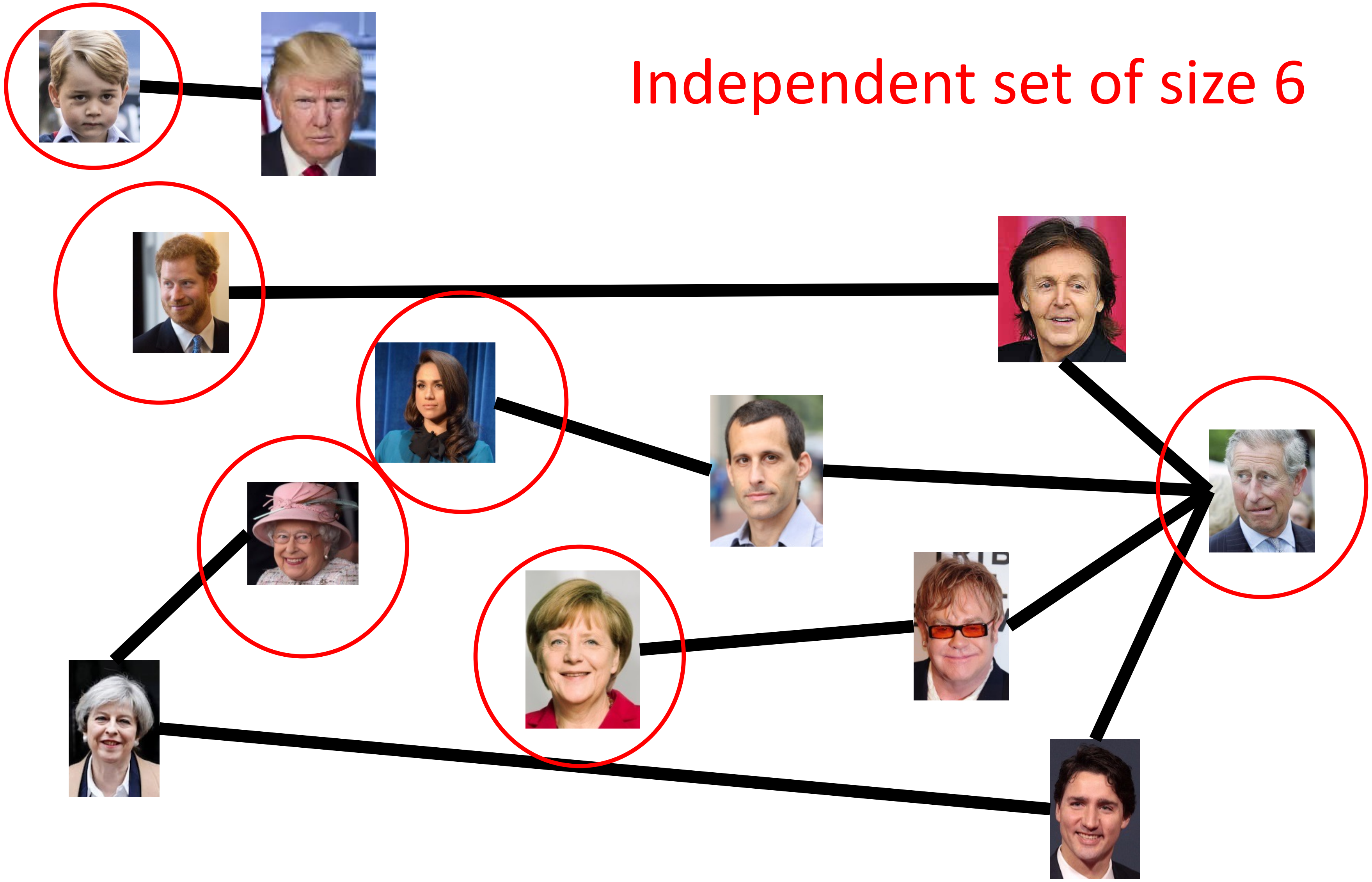
- Independent set: $S \subseteq V$ is an independent set if no two nodes in S share an edge
- Maximum Independent Set Problem: Given a graph $G = (V, E)$ find the maximum independent set S

Party Problem

Draw edges between people who don't get along.
Find the maximum number of people who DO get along.

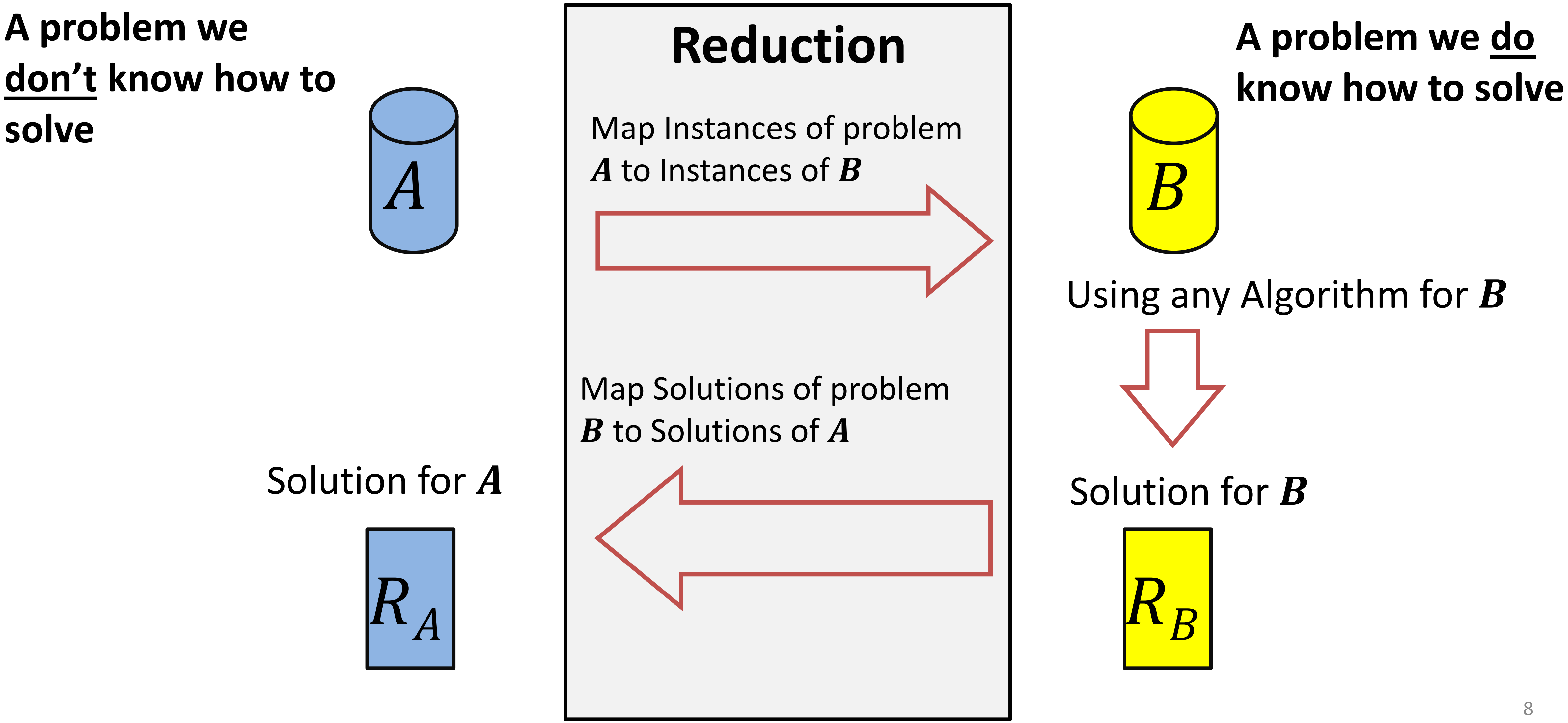


Example



Independent set of size 6

In General: A Reduction



In General: Reduction

Problem we <u>don't</u> know how to solve

A

Solution for $\boldsymbol{A}$

R_A

Reduction

Map Instances of problem $\boldsymbol{A}$ to Instances of $\boldsymbol{B}$

Map Solutions of problem $\boldsymbol{B}$ to Solutions of $\boldsymbol{A}$

Problem we <u>do</u> know how to solve

B

Using any Algorithm for $\boldsymbol{B}$

Solutio

R_B

For now: we are NOT focusing on an algorithm to solve one of these, <u>just on the reduction</u>!

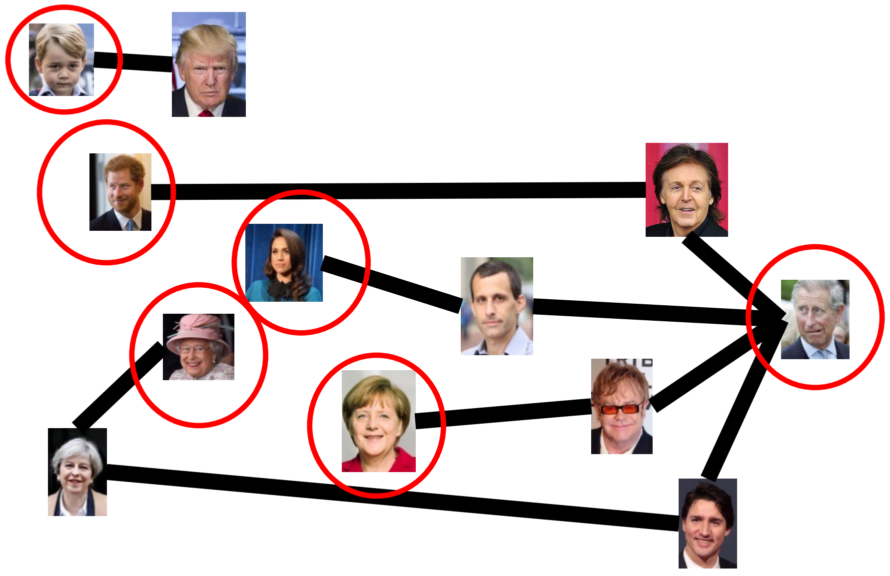
Polynomial-Time Equivalence

- We'll show a reduction in both directions:
  - MinVertCov $\leq_p$ MaxIndSet
  - MaxIndSet $\leq_p$ MinVertCov
- Because they reduce both ways, they are ***polynomial-time equivalent***
  - *(We'll justify the following claims later.)*
  - If we find a polynomial solution for one, the other is also polynomial
  - What if we prove an exponential lower-bound for one?
    Is it possible that the other one could have a polynomial solution?

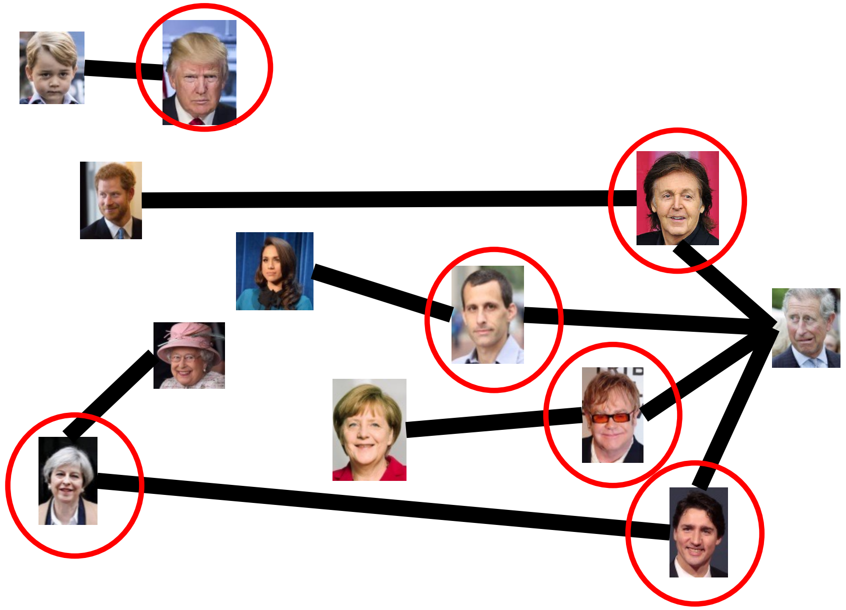
Reduction Idea

S is an independent set of G iff $V - S$ is a vertex cover of G

Independent Set



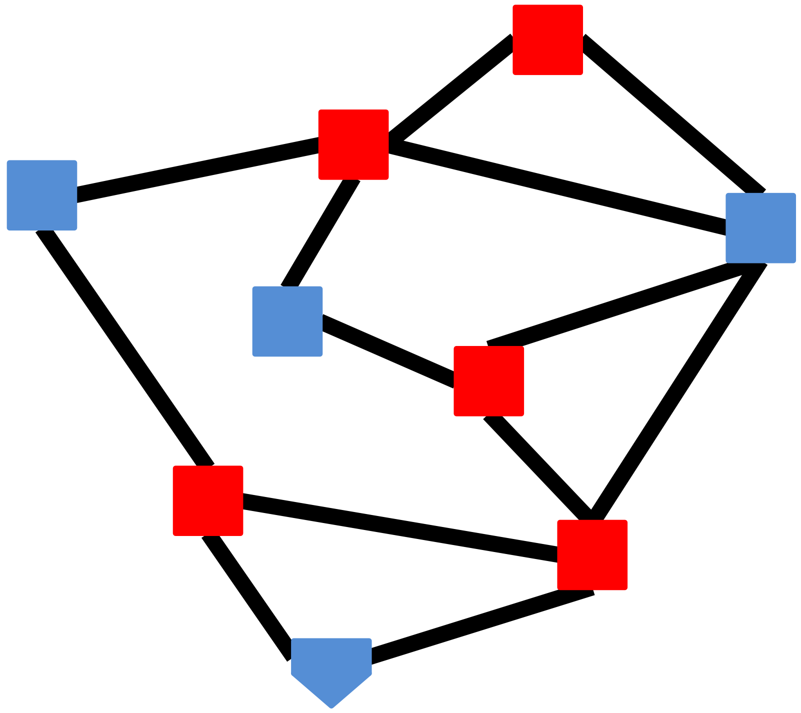
Vertex Cover



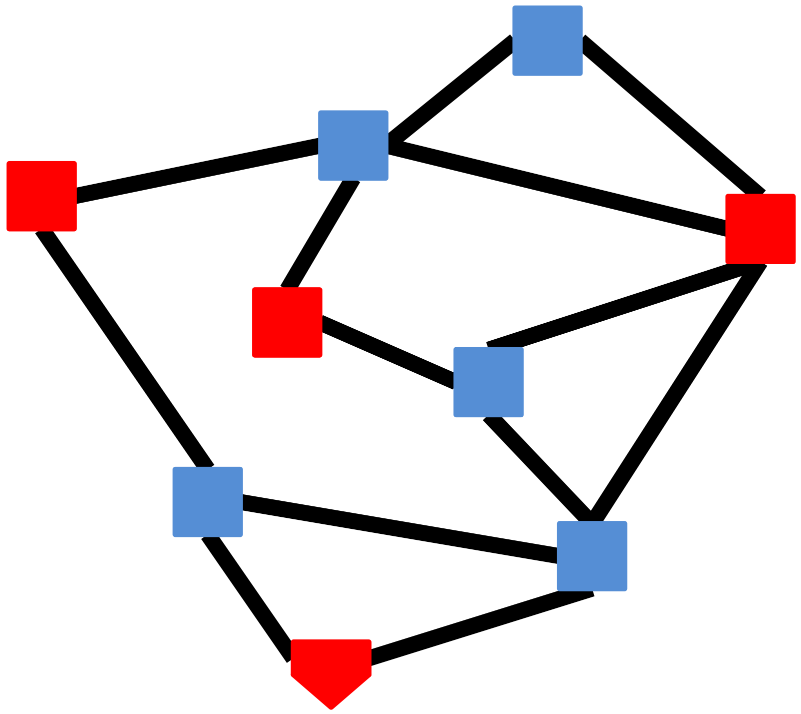
Reduction Idea

S is an independent set of G iff $V - S$ is a vertex cover of G

Vertex Cover



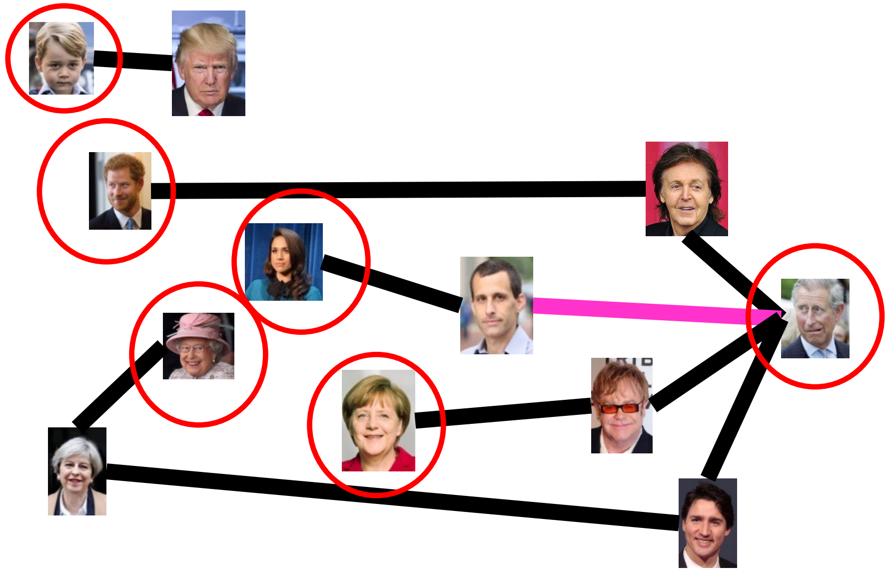
Independent Set



Proof: $\Rightarrow$

S is an independent set of G iff $V - S$ is a vertex cover of G

Let S be an independent set



Consider any edge $(x, y) \in E$

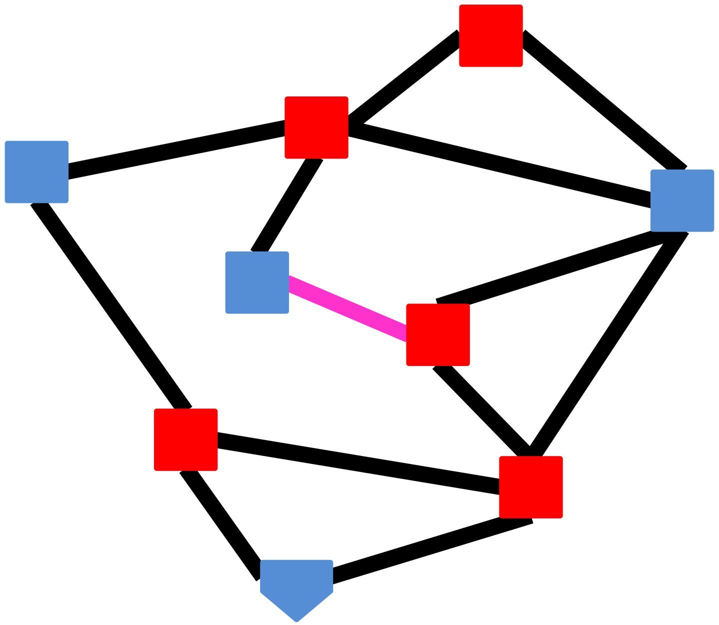
If $x \in S$ then $y \notin S$, because otherwise S would not be an independent set

Therefore $y \in V - S$, so edge (x, y) is covered by $V - S$

Proof: $\Leftarrow$

S is an independent set of G iff $V - S$ is a vertex cover of G

Let $V - S$ be a vertex cover



Consider any edge $(x, y) \in E$

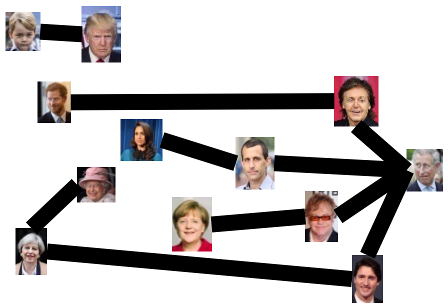
At least one of x and y belong to $V - S$, because $V - S$ is a vertex cover

Therefore x and y are not both in S,

No edge has both end-nodes in S, thus S is an independent set

The Reduction: MaxIndSet $\leq_P$ MinVertCov

MaxIndSet



A

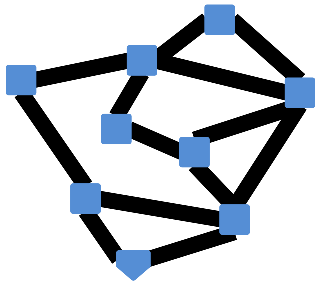
Reduction

Do nothing. $\Theta(1)$

Take complement of solution. $\Theta(V)$

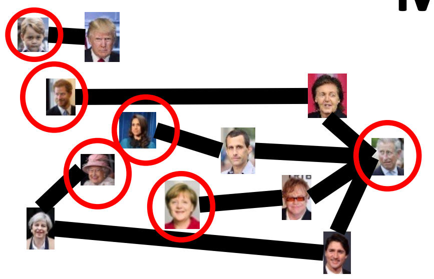
MinVertCov

B



Algorithm for **MinVertCov**

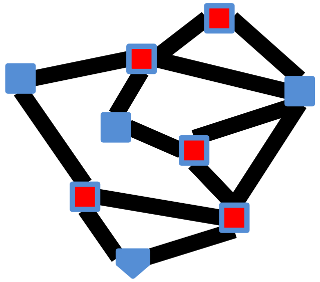
Solution for **MaxIndSet**



R_A

Solution for **MinVertCov**

R_B

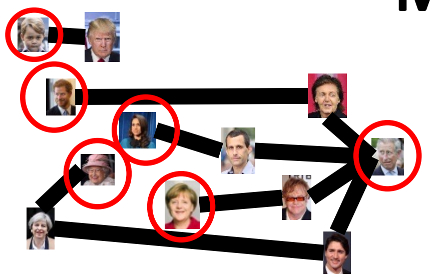


MaxIndSet $\leq_P$ MinVertCov

MaxIndSet

Yes, it's kind of trivial, but we just <u>proved</u> it's a valid reduction. And it's polynomial.

Solution for **MaxIndSet**



R_A

Reduction

Do nothing $\Theta(1)$

Take complement of solution $\Theta(V)$

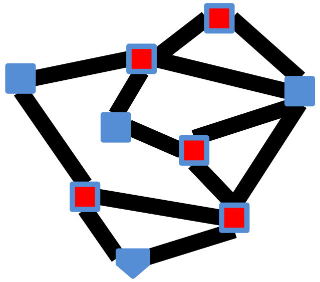
MinVertCov

We won't show it, but showing the other direction MinVertCov $\leq_P$ MaxIndSet is like this slide

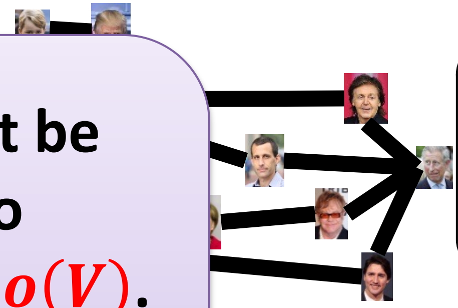
Algorithm for **MinVertCov**

Solution for **MinVertCov**

R_B



MaxIndSet $\leq_V$ MinVertCov



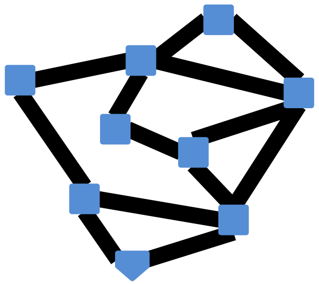
A

Problem **A**

Reduces to, in $\Theta(V)$

B

Problem **B**



…must not be possible to solve B in $o(V)$. Otherwise, reduction gives us a $o(V)$ solution for A.

Algorithm for **B**

can be used to make

With $\Theta(V)$ overhead

Algorithm for **A**

If impossible to solve A in $o(V)$…

Given that A $\leq_V$ B:

If we prove A requires time $\Omega(V)$ time,

then B must also require $\Omega(V)$ time.

MaxIndSet $\leq_P$ MinVertCov

For two problems A and B, if we show A $\leq_p$ B then:

1. If **we prove A is exponential**, then B must be exponential.
   – Otherwise, the polynomial reduction from A to B gives us a polynomial solution to A.
2. If **we find a polynomial algorithm to B**, then A is polynomial.
   – Use the reduction: Convert input for A to input for B, solve B in polynomial time, and you have solution for A (in polynomial time)

Therefore:

MaxIndSet and MinVertCov are either both polynomial, or both exponential

MaxIndSet $\leq_P$ MinVertCov

MaxIndSet and MinVertCov are either both polynomial, or both exponential

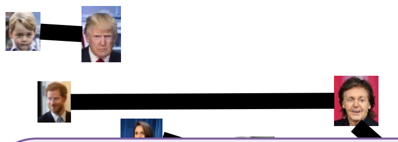
Which is true?

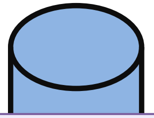
– Spoiler alert: We don't know which is true! (But we think they're both exponential.)

– Both problems are NP-Complete (We'll explain what that means soon!)

MaxIndSet $\leq_P$ MinVertCov

MaxIndSet





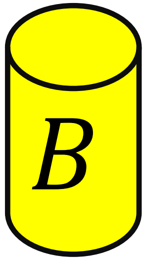
How about we just find a polynomial solution for MinVertCov? ☺

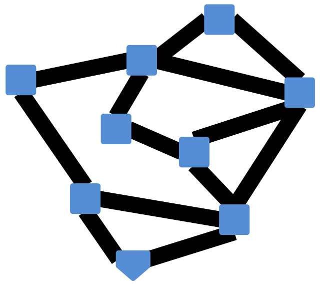
Reduction

Do nothing $\Theta(1)$

Take complement of solution $\Theta(V)$

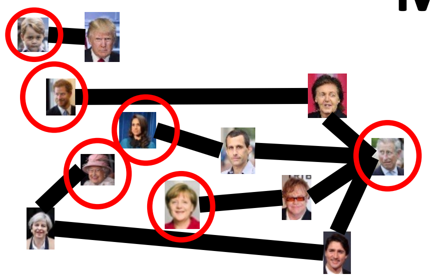
MinVertCov





Algorithm for **MinVertCov**

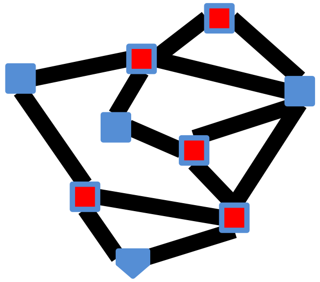
Solution for **MaxIndSet**



R_A

Solution for **MinVertCov**

R_B

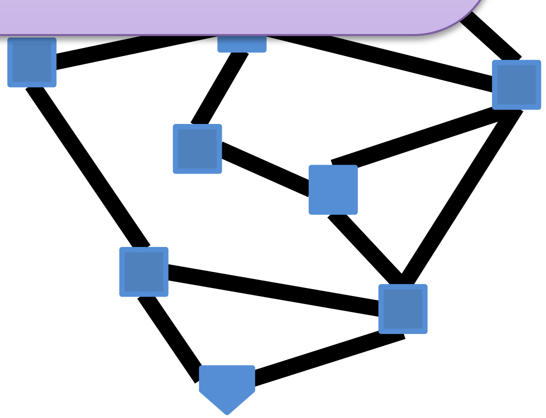


Minimum Vertex Cover

- Vertex Cover: $C \subseteq V$ is a vertex cover if every edge in E has one of its endpoints in C
- Cover Problem: Given a graph $G = (V, E)$ find the over C

Guess what?

We can find counterexamples for each of these to show they don't always work.



Strategy? How about greedy?

Greedy choice?

(1) Pick remaining vertex of max degree.
Remove its incident edges.
(2) CLRS 35.1. Pick any edge (u,v).
Add both to result.
Remove all incident edges for u and v.

Wrapping Up

- No one has found a polynomial algorithm for MinVertCov

- Summary of what we've done
  - Two new problems: MinVertCov and MaxIndSet
  - Proved a reduction between them
  - Examined the consequences of polynomial-time equivalence for two problems with regard to whether they're polynomial or exponential