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Refined inertias of strongly connected orientations of the Petersen graph

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ABSTRACT

The 18 non-isomorphic strongly connected orientations of the Petersen graph give rise to matrix patterns in which nonzero entries can be taken to be strictly positive, of arbitrary sign, or of fixed sign. The allowed refined inertias, in which the number of zero eigenvalues are split from others on the imaginary axis, are considered for each such pattern. Each nonnegative pattern is shown to have unique refined inertia determined by the number of required zero eigenvalues. For zero–nonzero patterns, a complete list of allowed refined inertias is given for each orientation. One particular sign pattern is presented that allows only two distinct refined inertias out of a possible 161 for a sign pattern of order 10.

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1. Introduction

We are interested in studying the refined inertias of patterns corresponding to strongly connected digraphs, and in particular we focus on digraphs arising from the well-known Petersen graph; see, for example, [1]. We begin with some pertinent definitions. A *zero–nonzero pattern* $\mathcal{A} = [\alpha_{ij}]$ is a matrix with entries from $\{0, *\}$, where $*$ represents any nonzero real number. A *matrix realization* $A = [a_{ij}]$ of a zero–nonzero pattern $\mathcal{A}$ has $a_{ij} = 0$ if $\alpha_{ij} = 0$, and a_{ij} nonzero (either a constant or a variable) if $\alpha_{ij} = *$. A *sign pattern* $\mathcal{A} = [\alpha_{ij}]$ is a matrix with entries from $\{0, +, -\}$, and a *matrix realization* $A = [a_{ij}]$ of $\mathcal{A}$ has $\mathrm{sgn}(a_{ij}) = \mathrm{sgn}(\alpha_{ij})$. Each nonzero entry of such a realization can be either a constant or a variable having a value of fixed sign. A *pattern* refers to either a sign pattern or a zero–nonzero pattern. A pattern $\mathcal{A}$ *allows* property Q if there is a realization A of $\mathcal{A}$ with property Q. A pattern $\mathcal{A}$ *requires* property Q if every realization A of $\mathcal{A}$ has property Q. A pattern $\mathcal{A}$ is *sign nonsingular* if $\det(A) \neq 0$ for all realizations A of $\mathcal{A}$.

As introduced by Kim et al. [2], the *refined inertia* of an n-by-n real matrix A is an ordered 4-tuple denoted by $\mathrm{ri}(A) = (n_+, n_-, n_z, 2n_p)$, where n_+ is the number of eigenvalues with positive real part, n_- is the number of eigenvalues with negative real part, n_z is the number of zero eigenvalues, and $2n_p$ is the number of nonzero pure imaginary eigenvalues with $n = n_+ + n_- + n_z + 2n_p$. The refined inertia of a pattern $\mathcal{A}$ is the set of refined inertias for all realizations A of $\mathcal{A}$ and is denoted by $\mathrm{ri}(\mathcal{A})$. Refined inertias of sign patterns have been considered, for example, in [3–5], of zero–nonzero patterns in [6,7], and of more general patterns in [8].

The *weighted digraph* of an n-by-n matrix $A = [a_{ij}]$, denoted $\mathcal{D}(A)$, is a digraph on n vertices labelled $1, ..., n$ that has an arc $i \to j$ if and only if $a_{ij} \neq 0$, where this arc is labelled with weight a_{ij}. If A is a matrix realization of a zero–nonzero pattern, then each weight of $\mathcal{D}(A)$ has arbitrary sign. If A is a matrix realization of a sign pattern, then each weight of $\mathcal{D}(A)$ has a fixed sign. Since it is labelled, any weighted digraph with n vertices uniquely determines an n-by-n matrix with nonzero entries corresponding to the weights on the arcs of the digraph. If A is a matrix realization of a pattern $\mathcal{A}$, then $\mathcal{D}(\mathcal{A})$ is equal to $\mathcal{D}(A)$ with each weight replaced by $*$ (for a zero–nonzero pattern), or a $+$ or $-$ sign (for a sign pattern). The underlying digraph of $\mathcal{D}(A)$ or $\mathcal{D}(\mathcal{A})$ refers to the digraph with the weights on the arcs removed.

We now focus on the Petersen graph, which is 3-regular, has 10 vertices, and 15 edges. Using SAGE it can be shown that there are 18 non-isomorphic strongly connected orientations of the Petersen graph, denoted $P_1, ..., P_{18}$. For $i = 1, ..., 18$, we label the vertices of P_i with $1, ..., 10$ and label the 15 arcs with real-valued variables to give weighted digraphs $\mathcal{D}(A_i)$ having underlying digraph P_i. Each digraph $\mathcal{D}(A_i)$ determines a unique matrix A_i (with exactly 15 nonzero entries). In matrix A_i the entries corresponding to the arcs of a directed spanning tree may be set to one (see, for example, [9, Theorem 2.3]); i.e., there is a diagonal similarity that reduces the number of variables in A_i to six while keeping the same refined inertia. Hence, each matrix A_i with weighted digraph $\mathcal{D}(A_i)$, having underlying digraph P_i, can be specified in terms of nonzero parameters a, b, c, d,

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e, f. See Appendix A for these matrices A_i and their weighted digraphs $\mathcal{D}(A_i)$ with their

directed spanning trees. If each parameter can be either positive or negative, then each

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A_i corresponds to one of $2^6 = 64$ realizations. Thus, there exist a total of $64 * 18 = 1152$ different sign patterns, each one isomorphic to a strongly connected orientation of the Petersen graph. If each parameter is a nonzero real number, then A_i corresponds to a realization of a zero–nonzero pattern that has a digraph isomorphic to a strongly connected orientation of the Petersen graph (18 different zero–nonzero patterns). In Section 2 we take $a, ..., f$ positive and in Section 3 we consider zero–nonzero patterns taking $a, ..., f$ nonzero. In Section 4 we illustrate a sign pattern with a particular refined inertia set, taking a, b, c, d, f positive and e negative.

For use in our analysis, we introduce the elementary symmetric functions (see [10, Definition 1.2.9])

$$S_\ell(a_1, ..., a_m) = \sum_{1 \le i_1 < i_2 < ... < i_\ell \le m} a_{i_1} a_{i_2} a_{i_3} ... a_{i_\ell},$$

e.g., $S_2(a_1, ..., a_4) = a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4$. Let $S_i(A)$ be the coefficient of the x^{n-i} term of the characteristic polynomial of an n-by-n matrix A; then

$$S_i(A) = (-1)^i S_i(\lambda_1, ..., \lambda_n),$$

where $\lambda_1, ..., \lambda_n$ are the eigenvalues A (see [10, Theorem 1.2.12]). Note that if the real parts of $a_1, ..., a_m > 0$, then

$$S_{k-1}(a_1, ..., a_m) S_1(a_1, ..., a_m) > S_k(a_1, ..., a_m), \tag{1}$$

where $k \ge 2$.

Several of the coefficients in all of the characteristic polynomials C_i of the matrices A_i are zero; see Appendix B. The following three theorems eliminate the possibility of certain refined inertias for arbitrary real matrices having some zero coefficients in their characteristic polynomials, i.e., certain elementary symmetric functions are equal to zero.

Theorem 1. *If A is a real n-by-n matrix with $S_1(A) = \mathrm{Tr}(A) = 0$, then $\mathrm{ri}(A)$ cannot equal $(0, n_-, n_z, 2n_p)$ or $(n_+, 0, n_z, 2n_p)$, where n_+ and n_- are positive; i.e., $n_+ = 0$ if and only if $n_- = 0$.*

Proof. Let A be a matrix with $S_1(A) = \mathrm{Tr}(A) = 0$, hence the sum of the eigenvalues of A is equal to zero. Since pure imaginary complex conjugate eigenvalues have zero sum, it follows that if A has eigenvalues with only positive or negative real parts, then $\mathrm{Tr}(A) \ne 0$. Therefore, $\mathrm{ri}(A)$ cannot equal $(0, n_-, n_z, 2n_p)$ or $(n_+, 0, n_z, 2n_p)$ with n_-, n_+ positive. □

Theorem 2. *Let A be a real n-by-n matrix with $n \ge 3$. If A has elementary symmetric functions $S_1(A) = S_3(A) = 0$, then $\mathrm{ri}(A)$ cannot equal $(n_+, 1, n_z, 2n_p)$ with $n_+ \ge 2$, or $(1, n_-, n_z, 2n_p)$ with $n_- \ge 2$; i.e., $n_+ = 1$ if and only if $n_- = 1$.*

Proof. Let A be an n-by-n matrix with exactly one negative eigenvalue, $k \geq 2$ eigenvalues with positive real parts, $2m \geq 0$ nonzero pure imaginary complex conjugate eigenvalues, and $n-2m-1-k$ zero eigenvalues. The characteristic polynomial of A factors into the form

$$x^{n-2m-1-k}(x+a)\prod_{i=1}^{k}(x-b_i)\prod_{j=1}^{m}(x^2+\phi_j)$$

for $a, \phi_1, ..., \phi_m > 0$ and the real parts of $b_1, ..., b_k > 0$. Since the elementary symmetric function $S_1(A) = 0$, it follows that

$$-S_1(A) = a - S_1(b_1, ..., b_k) = 0.$$

First assume $n \geq 4$ and $k \geq 3$. Then the function $S_3(A)$ becomes

$$\begin{aligned}-S_3(A) &= S_1(\phi_1, ..., \phi_m)\overbrace{[a - S_1(b_1, ..., b_k)]}^{S_1(A)=0} + aS_2(b_1, ..., b_k) - S_3(b_1, ..., b_k)\\ &= aS_2(b_1, ..., b_k) - S_3(b_1, ..., b_k) = S_1(b_1, ..., b_k)S_2(b_1, ..., b_k) - S_3(b_1, ..., b_k),\end{aligned}$$

which is positive by (1), a contradiction.

If $n \geq 3$ and $k = 2$, then

$$-S_3(A) = S_1(\phi_1, ..., \phi_m)\overbrace{[a - S_1(b_1, b_2)]}^{S_1(A)=0} + aS_2(b_1, b_2) = ab_1b_2 > 0,$$

again a contradiction.

Therefore, if $S_1(A) = S_3(A) = 0$, then $\mathrm{ri}(A)$ cannot equal $(k, 1, n-2m-1-k, 2m)$ for $k \geq 2$. A similar argument holds for A having exactly one positive eigenvalue, and it follows that $\mathrm{ri}(A)$ cannot equal $(1, k, n-2m-1-k, 2m)$ for $k \geq 2$. $\square$

Theorem 3. *Let A be a real n-by-n matrix with $n \geq 4$. If A has elementary symmetric functions $S_2(A) = S_4(A) = 0$, then* $\mathrm{ri}(A)$ *cannot equal* $(1, 1, n_z, 2n_p)$ *with* $n_p \geq 1$.

Proof. Let A be an n-by-n matrix with $n \geq 4$. Let $p(x)$ be the characteristic polynomial of A with elementary symmetric function $S_2(A) = 0$. Assume that A has exactly one negative eigenvalue, exactly one positive eigenvalue, and $m \geq 1$ pairs of nonzero pure imaginary complex conjugate eigenvalues. Then $p(x)$ can be factored into the form

$$x^{n-2m-2}(x+a)(x-b)\prod_{j=1}^{m}(x^2+\phi_j)$$

for $a, b, \phi_1, ..., \phi_m > 0$, and thus

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$$S_2(A) = S_1(\phi_1, ..., \phi_m) - ab,$$
$$S_4(A) = S_2(\phi_1, ..., \phi_m) - abS_1(\phi_1, ..., \phi_m).$$

If $S_2(A) = 0$, then $S_1(\phi_1, ..., \phi_m) = ab$. Since $S_1(\phi_1, ...\phi_m)^2 > S_2(\phi_1, ...\phi_m)$ by (1), it follows that $S_4(A) < 0$, a contradiction. Hence, if $S_2(A) = S_4(A) = 0$, then $\mathrm{ri}(A) \neq (1, 1, n - 2m - 2, 2m)$ for $m \geq 1$. □

2. Nonnegative patterns

For $i = 1, ..., 18$, let $\mathcal{A}_i$ denote the nonnegative pattern having a matrix realization A_i in Appendix A with all parameters $a, ..., f$ positive. Thus $\mathcal{D}(\mathcal{A}_i)$ is a strongly connected orientation of the Petersen graph having each arc weighted with a + sign. We show that these nonnegative patterns have fixed refined inertias.

Theorem 4. *Let $\mathcal{A}$ be a nonnegative pattern with $\mathcal{D}(\mathcal{A})$ having underlying digraph isomorphic to a strongly connected orientation of the Petersen graph. Then $\mathcal{A}$ has unique refined inertia. Specifically, the refined inertias are as follows:*

i. $(3, 3, 4, 0)$ *if $\mathcal{A}$ requires exactly four zero eigenvalues,*
ii. $(5, 3, 2, 0)$ *if $\mathcal{A}$ requires exactly two zero eigenvalues,*
iii. $(5, 4, 1, 0)$ *if $\mathcal{A}$ requires a single zero eigenvalue,*
iv. $(6, 4, 0, 0)$ *if $\mathcal{A}$ is sign nonsingular.*

Proof. Let $\mathcal{A}$ be a sign pattern with entries from the set $\{0, +\}$, where $\mathcal{D}(\mathcal{A})$ has underlying digraph isomorphic to a strongly connected orientation of the Petersen graph. If A is a realization of $\mathcal{A}$, then the characteristic polynomial of A corresponds to one of the C_i given in Appendix B, with all of the parameters in C_i positive. We first show that the C_i do not allow nonzero pure imaginary zeros. If $x = ik$ for $k > 0$ is a zero of any one of the polynomials C_1 to C_4, then the imaginary part gives $-\alpha ik^5 = 0$ with $\alpha > 0$, a contradiction. Similarly, for the remaining polynomials C_5 to C_{18}, if $x = ik$ with $k > 0$, then either

$$-\alpha ik^5 - Dik = 0 \qquad \text{or} \qquad -\alpha ik^5 - \zeta ik = 0$$

with $\alpha, D, \zeta > 0$, giving a contradiction.

Since these polynomials do not allow nonzero pure imaginary zeros and the polynomials have 0 as a zero of fixed multiplicity, it is impossible to transition from one refined inertia to another by continuity (i.e., nonzero eigenvalues of A are fixed in their respective regions, the left or right half of the complex plane). With all parameter values $a, ..., f$ set to one, the refined inertias for each matrix realization A_i of $\mathcal{A}_i$ are given in Table 1. □

Table 1
Characteristic polynomials of C_i from Appendix A with all parameters a, b, c, d, e, f set to one, and their corresponding refined inertias.

i	C_i	$\mathrm{ri}(A_i)$
1	$x^{10} - 4x^5 - 2x^4$	$(3, 3, 4, 0)$
2	$x^{10} - 4x^5 - 2x^4$	$(3, 3, 4, 0)$
3	$x^{10} - 3x^5 - 2x^4 - x^2$	$(5, 3, 2, 0)$
4	$x^{10} - 3x^5 - 2x^4 - x^2$	$(5, 3, 2, 0)$
5	$x^{10} - 2x^5 - 2x^4 - x^2 - x$	$(5, 4, 1, 0)$
6	$x^{10} - 3x^5 - 2x^4 - x^2 - x$	$(5, 4, 1, 0)$
7	$x^{10} - 2x^5 - 2x^4 - x^2 - x$	$(5, 4, 1, 0)$
8	$x^{10} - 3x^5 - 2x^4 - x^2 - x$	$(5, 4, 1, 0)$
9	$x^{10} - 3x^5 - x^4 - x^2 - x$	$(5, 4, 1, 0)$
10	$x^{10} - 3x^5 - x^4 - x^2 - x$	$(5, 4, 1, 0)$
11	$x^{10} - 2x^5 - 3x^4 - 2x^2 - 2x$	$(5, 4, 1, 0)$
12	$x^{10} - 2x^5 - 3x^4 - 2x^2 - 2x$	$(5, 4, 1, 0)$
13	$x^{10} - 3x^5 - x^4 - x^2 - x + 1$	$(6, 4, 0, 0)$
14	$x^{10} - 3x^5 - x^4 - x^2 - x + 1$	$(6, 4, 0, 0)$
15	$x^{10} - 4x^5 - x^4 - x^2 - 2x + 1$	$(6, 4, 0, 0)$
16	$x^{10} - 4x^5 - x^4 - x^2 - 2x + 1$	$(6, 4, 0, 0)$
17	$x^{10} - 3x^5 - 2x^4 - 2x^2 - x + 1$	$(6, 4, 0, 0)$
18	$x^{10} - 3x^5 - 2x^4 - 2x^2 - x + 1$	$(6, 4, 0, 0)$

3. Zero–nonzero patterns

For $i = 1, ..., 18$, we now consider zero–nonzero patterns $\mathcal{A}_i$ with matrix realizations A_i having the weighted digraphs $\mathcal{D}(A_i)$ in Appendix A. Thus each of the parameters $a, ..., f$ in A_i can have any nonzero value. The refined inertia $(n_-, n_+, n_z, 2n_p)$ is called the *reversal* of the refined inertia $(n_+, n_-, n_z, 2n_p)$, and the following result is well known; see, for example, [6].

Theorem 5. *A zero–nonzero pattern allows refined inertia $(n_+, n_-, n_z, 2n_p)$ if and only if it allows its reversal $(n_-, n_+, n_z, 2n_p)$.*

Theorem 6. *If $\mathcal{A}$ is a zero–nonzero pattern with weighted digraph $\mathcal{D}(\mathcal{A})$ having its underlying digraph isomorphic to P_1 or P_2, then*

$$\mathrm{ri}(\mathcal{A}) = \{(0, 0, 10, 0), (2, 3, 5, 0), (3, 2, 5, 0), (4, 2, 4, 0), (2, 4, 4, 0), (3, 3, 4, 0), (2, 2, 4, 2)\}.$$

Proof. Let A be a matrix realization of either $\mathcal{A}_1$ or $\mathcal{A}_2$. For nonzero values of a, b, c, d, e, f in A_1 or A_2 (see Appendix A), the characteristic polynomial is of the form

$$p(x) = x^4(x^6 - \alpha x - \beta), \tag{2}$$

with $\alpha, \beta \in \mathbb{R}$. If $\alpha = \beta = 0$, then $p(x) = x^{10}$ and $\mathrm{ri}(A) = (0, 0, 10, 0)$. If $\beta = 0$, then $p(x) = x^5(x^5 - \alpha)$. If in addition $\alpha > 0$, then the solutions to $x^5 = \alpha$ are evenly distributed around a scaled unit circle (with one positive root); therefore, $\mathrm{ri}(A) = (3, 2, 5, 0)$. By

Table 2
Examples of parameter values for all possible refined inertias of zero–nonzero patterns $\mathcal{A}_1$ and $\mathcal{A}_2$.

A_1 $(a = c = d = 1)$			A_2 $(a = c = e = 1)$			α	β	$\mathrm{ri}(A_1), \mathrm{ri}(A_2)$
b	e	f	b	d	f			
1	-1	1	1	-1	-1	0	0	$(0,0,10,0)$
$\frac{1}{2}$	-2	2	1	-1	-2	-1	0	$(2,3,5,0)$
$\frac{1}{2}$	-2	4	2	-1	-1	1	0	$(3,2,5,0)$
2	-1	1	$\frac{1}{2}$	-2	$\frac{1}{2}$	0	-1	$(2,2,4,2)$
-1	-1	1	-1	1	-1	0	2	$(3,3,4,0)$
2	-1	2	1	-2	1	1	-1	$(4,2,4,0)$
1	-2	2	1	-2	-1	-1	-1	$(2,4,4,0)$

Theorem 5, $\mathcal{A}$ also allows $(2,3,5,0)$, which is attained if $\beta = 0$ and $\alpha < 0$. If $\alpha = 0$, then $p(x) = x^4(x^6 - \beta)$. If in addition $\beta > 0$, then $\mathrm{ri}(A) = (3,3,4,0)$, whereas if $\beta < 0$, then $\mathrm{ri}(A) = (2,2,4,2)$.

Suppose $\alpha \neq 0$ and $\beta \neq 0$. Since A has four zero eigenvalues and $S_i(A) = 0$ for $i = 1, ..., 4$, by Theorems 1, 2, and 3 the only other possible refined inertias of A are $(4,2,4,0)$, $(2,4,4,0)$, and $(0,0,4,6)$. If the refined inertia $(0,0,4,6)$ can occur, then the factored polynomial $x^4(x^2+\phi_1)(x^2+\phi_2)(x^2+\phi_3)$ has elementary symmetric function $S_2(A) = \phi_1 + \phi_2 + \phi_3 > 0$, a contradiction. Therefore, refined inertia $(0,0,4,6)$ cannot occur. The remaining refined inertias, $(4,2,4,0)$ and $(2,4,4,0)$, are found numerically, with parameter values demonstrating the allowed refined inertias of $\mathcal{A}_1$ and $\mathcal{A}_2$ given in Table 2. □

Theorem 7. *If $\mathcal{A}$ is a zero–nonzero pattern with weighted digraph $\mathcal{D}(\mathcal{A})$ having its underlying digraph isomorphic to P_3 or P_4, then*

$$\mathrm{ri}(\mathcal{A}) = \{(3,5,2,0), (5,3,2,0), (4,4,2,0), (3,3,2,2), (2,2,2,4)\}.$$

Proof. Let A be a matrix realization of either $\mathcal{A}_3$ or $\mathcal{A}_4$. For nonzero values of a, b, c, d, e, f in A_3 or A_4, the characteristic polynomials C_3 and C_4 are of the form

$$x^{10} - \alpha x^5 - \beta x^4 - Cx^2, \tag{3}$$

where $\alpha, \beta \in \mathbb{R}$ and $C \neq 0$. Since (3) has a factor x^2, by Theorems 1, 2, and 3 the only possible refined inertias of $\mathcal{A}$ are $(6,2,2,0)$, $(5,3,2,0)$, $(4,2,2,2)$, $(4,4,2,0)$, $(3,3,2,2)$, $(2,2,2,4)$, $(0,0,2,8)$, and their reversals.

If A has a nonzero pure imaginary eigenvalue, then $\alpha = 0$, and (3) contains of only even powers of x. It follows that if λ is an eigenvalue, then $-\lambda$ is also an eigenvalue; therefore, $(4,2,2,2)$ and its reversal cannot occur as $n_+ \neq n_-$.

If $\mathrm{ri}(A) = (6,2,2,0)$, then the characteristic polynomial of A can be factored into the form

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Table 3
Examples of parameter values for all possible refined inertias of zero–nonzero patterns $\mathcal{A}_3$ and $\mathcal{A}_4$.

A_3 $(e=2, b=\frac{1}{2})$				A_4 $(c=1, e=3)$				α	β	C	$\mathrm{ri}(A_3), \mathrm{ri}(A_4)$
a	c	d	f	a	b	d	f				
$-\frac{2}{3}$	$\frac{1}{3}$	-3	$-\frac{3}{2}$	1	-2	-2	-2	-1	1	1	$(3,5,2,0)$
-2	1	-1	$-\frac{1}{2}$	1	-1	-2	-1	1	1	1	$(5,3,2,0)$
$\frac{2}{3}$	1	3	$-\frac{3}{2}$	-1	-2	2	-2	-1	-1	-1	$(4,4,2,0)$
-1	1	-3	-1	1	-2	-2	-1	0	1	1	$(3,3,2,2)$
1	2	2	-1	-1	-2	1	-1	0	-2	-1	$(2,2,2,4)$

$$x^2(x+a_1)(x+a_2)\prod_{i=1}^{6}(x-b_i),$$

where the real parts of $a_1, a_2, b_1, ..., b_6 > 0$. Then

$$-S_1(A) = S_1(a_1,a_2) - S_1(b_1,...,b_6) = 0,$$

$$S_2(A) = S_2(b_1,...,b_6) - S_1(a_1,a_2)S_1(b_1,...,b_6) + S_2(a_1,a_2) = 0,$$

$$-S_3(A) = -S_1(b_1,...,b_6)S_2(a_1,a_2) + S_2(b_1,...,b_6)S_1(a_1,a_2) - S_3(b_1,...,b_6) = 0.$$

Solving these for $S_1(a_1,a_2), S_2(a_1,a_2)$, and $S_3(b_1,...,b_6)$, and substituting into the last term of

$$S_4(A) = S_2(b_1,...,b_6)S_2(a_1,a_2) + S_4(b_1,...,b_6) - S_3(b_1,...,b_6)S_1(a_1,a_2),$$

gives

$$\begin{aligned}
&S_4(A) = S_2(b_1,...,b_6)S_2(a_1,a_2) + S_4(b_1,...,b_6)\\
&-S_1(b_1,...,b_6)\left(S_2(b_1,...,b_6)S_1(b_1,...,b_6) - S_1(b_1,...,b_6)\left[S_1(b_1,...,b_6)^2 - S_2(b_1,...,b_6)\right]\right),\\
&= S_2(b_1,...,b_6)S_2(a_1,a_2) + S_1(b_1,...,b_6)^2\left[S_1(b_1,...,b_6)^2 - 2S_2(b_1,...,b_6)\right] + S_4(b_1,...,b_6),\\
&= S_2(b_1,...,b_6)S_2(a_1,a_2) + S_1(b_1,...,b_6)^2(b_1^2 + ... + b_6^2) + S_4(b_1,...,b_6) > 0. \qquad (4)
\end{aligned}$$

This contradicts the fact that $S_4(A) = 0$ in (3); hence, $\mathcal{A}$ does not allow refined inertia $(6,2,2,0)$ or its reversal.

If the refined inertia $(0,0,2,8)$ occurs, then the characteristic polynomial $x^2\prod_{i=1}^{4}(x^2+\phi_i)$ with all $\phi_i > 0$ has elementary symmetric function $S_2(A) = S_1(\phi_1,...,\phi_4) > 0$, a contradiction since $S_2(A) = 0$. Hence, refined inertia $(0,0,2,8)$ cannot occur. Parameter values that demonstrate the allowed refined inertias of $\mathcal{A}_3$ and $\mathcal{A}_4$ are given in Table 3. □

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Theorem 8. *If $\mathcal{A}$ is a zero–nonzero pattern with weighted digraph $\mathcal{D}(\mathcal{A})$ having its underlying digraph isomorphic to one of the digraphs $P_5, ..., P_{10}$, then*

$$\mathrm{ri}(\mathcal{A}) = \{(6,3,1,0),(3,6,1,0),(4,3,1,2),(3,4,1,2),(5,4,1,0),(4,5,1,0)\}.$$

Proof. Let A be a matrix realization of $\mathcal{A}_i$ for $i = 5, ..., 8$. For nonzero values of a, b, c, d, e, f in $A_5, ..., A_8$, the characteristic polynomials are of the form

$$p(x) = x^{10} - \alpha x^5 - \beta x^4 - Cx^2 - Dx, \tag{5}$$

with $\alpha, \beta \in \mathbb{R}$ and $C, D \neq 0$. Since (5) has a factor of x, by Theorems 1, 2, and 3 the only possible refined inertias of A are $(7,2,1,0)$, $(6,3,1,0)$, $(5,4,1,0)$, $(5,2,1,2)$, $(4,3,1,2)$, $(3,2,1,4)$, and their reversals.

If (5) has a complex zero $x = i\sqrt{\phi}$ with $\sqrt{\phi} > 0$, then $p(i\sqrt{\phi}) = 0$ implies that $\alpha = -D/\phi^2$ and $C = \phi^4 + \beta\phi$. Substituting these conditions into (5) gives

$$p(x) = x\left(x^9 + \frac{D}{\phi^2}x^4 - \beta x^3 - (\phi^3 + \beta)\phi x - D\right),$$

which factors into the form

$$p(x) = x(x^2 + \phi)\underbrace{\left(x^7 - \phi x^5 + \phi^2 x^3 + \frac{D}{\phi^2}x^2 - (\phi^3 + \beta)x - \frac{D}{\phi}\right)}_{q(x)}.$$

Since $D \neq 0$ and $\phi > 0$, if $q(x)$ has a complex zero $x = it$ with $t > 0$, then

$$-\frac{D}{\phi^2}t^2 - \frac{D}{\phi} = 0,$$

which is a contradiction. Thus, since $q(x)$ has no eigenvalues on the imaginary axis for all $\phi > 0, D \neq 0$, and $\beta \in \mathbb{R}$, the refined inertia corresponding to the zeros of $q(x)$ is of the form $(\hat{n}_+, \hat{n}_-, 0, 0)$ for some fixed values of $\hat{n}_+, \hat{n}_-$ with $\hat{n}_+ + \hat{n}_- = 7$. Taking any value of $D > 0$ gives $(\hat{n}_+, \hat{n}_-) = (3,4)$, and taking any value of $D < 0$ gives $(\hat{n}_+, \hat{n}_-) = (4,3)$. Consequently, if A has fixed eigenvalues 0 and $\pm i\sqrt{\phi}$ for any $\phi > 0$, then since $D \neq 0$, the other seven eigenvalues are such that $\mathrm{ri}(A) = (4,3,1,2)$ or $(3,4,1,2)$ as β and D vary. In particular, the refined inertias $(5,2,1,2)$, $(3,2,1,4)$ cannot occur.

If $\mathrm{ri}(A) = (7,2,1,0)$, then the characteristic polynomial of A can be factored into the form

$$x(x + a_1)(x + a_2)\prod_{i=1}^{7}(x - b_i)$$

Table 4
Examples of values for the coefficients of the characteristic polynomials C_5 to C_{10} that can be obtained by choosing nonzero values of a, b, c, d, e, and f to give all possible refined inertias of $\mathcal{A}_5$ to $\mathcal{A}_{10}$. For brevity we do not list the parameter values $a, \ldots, f$.

α	β,B	C	D	$\mathrm{ri}(A_5)$ to $\mathrm{ri}(A_{10})$
1	1	3	−1	(6, 3, 1, 0)
−1	1	3	1	(3, 6, 1, 0)
−1	1	1	1	(5, 4, 1, 0)
1	1	1	−1	(4, 5, 1, 0)
1	1	2	−1	(4, 3, 1, 2)
−1	1	2	1	(3, 4, 1, 2)

where the real parts of $a_1, a_2, b_1, \ldots, b_7 > 0$. The same argument as for refined inertia $(6, 2, 2, 0)$ in Theorem 7 on replacing $b_1, \ldots, b_6$ by $b_1, \ldots, b_7$ gives a contradiction, namely $S_4(A) > 0$ as in (4). Thus refined inertia $(7, 2, 1, 0)$ and its reversal are impossible.

If A is a realization of $\mathcal{A}_9$ or $\mathcal{A}_{10}$, then the characteristic polynomials of A are of the form

$$x^{10} - \alpha x^5 - Bx^4 - Cx^2 - Dx,$$

with $\alpha \in \mathbb{R}$ and $B, C, D \neq 0$. The above analysis holds for this polynomial as the nonzero restriction on the coefficient $-\beta = S_6(A)$ of x^4 does not affect the analysis. Parameter values are given in Table 4 showing the allowed refined inertias of $\mathcal{A}_5$ to $\mathcal{A}_{10}$. □

Theorem 9. *If $\mathcal{A}$ is a zero–nonzero pattern with weighted digraph $\mathcal{D}(\mathcal{A})$ having its underlying digraph isomorphic to digraph P_{11} or P_{12}, then*

$$\mathrm{ri}(\mathcal{A}) = \mathrm{ri}(\mathcal{A}_1) \cup \mathrm{ri}(\mathcal{A}_3) \cup \mathrm{ri}(\mathcal{A}_5) - \{(0, 0, 10, 0), (2, 2, 4, 2)\}.$$

Proof. Let A be a matrix realization of either $\mathcal{A}_{11}$ or $\mathcal{A}_{12}$. The characteristic polynomial of A is of the form

$$x^{10} - \alpha x^5 - \beta x^4 - \gamma x^2 - \zeta x, \tag{6}$$

where $\alpha, \beta, \gamma, \zeta \in \mathbb{R}$. Ignoring parameter restrictions (i.e., assuming $\alpha, \beta, \gamma, \zeta$ can attain any values in $\mathbb{R}$ independently), if $\gamma = \zeta = 0$, then (6) is the form of (2), and thus $\mathrm{ri}(\mathcal{A}_1) \subseteq \mathrm{ri}(\mathcal{A})$. If $\gamma \neq 0$ and $\zeta = 0$, then (6) is of the form (3), and thus $\mathrm{ri}(\mathcal{A}_3) \subseteq \mathrm{ri}(\mathcal{A})$. If $\gamma \neq 0$ and $\zeta \neq 0$, then (6) is of the form (5), and thus $\mathrm{ri}(\mathcal{A}_5) \subseteq \mathrm{ri}(\mathcal{A})$. This latter inclusion also holds if $\gamma = 0$ and $\zeta \neq 0$ since the proof of Theorem 8 does not require the coefficient $-C$ of x^2 to be nonzero. Since all possible values of γ, ζ have been considered, the characteristic polynomial of A must be equal to that of a realization of either $\mathcal{A}_1, \mathcal{A}_3$, or $\mathcal{A}_5$. Then it follows that

$$\mathrm{ri}(\mathcal{A}) \subseteq \mathrm{ri}(\mathcal{A}_1) \cup \mathrm{ri}(\mathcal{A}_3) \cup \mathrm{ri}(\mathcal{A}_5).$$

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However, A has a characteristic polynomial with restricted coefficients. Specifically, if $\alpha = \beta = \gamma = 0$ in C_{11}, then

$$bf = -adf, \qquad fcd = -e - f, \qquad dbf + afe = 0.$$

Thus since $\zeta = df + fce$, the above three conditions give

$$\zeta ad = ad(df + fce) = (-bf + afce)d = afe + ae(-e - f) = -ae^2 \neq 0,$$

implying that $\zeta \neq 0$. A similar result occurs for C_{12}. Hence, no realization A of either $\mathcal{A}_{11}$ or $\mathcal{A}_{12}$ can have a characteristic polynomial with $\alpha = \beta = \gamma = \zeta = 0$, which eliminates refined inertia $(0, 0, 10, 0)$. If A is a realization of $\mathcal{A}_{11}$ with $\mathrm{ri}(A) = (2, 2, 4, 2)$, then $\alpha = \gamma = \zeta = 0$ and $\beta < 0$, as in the proof of Theorem 6. If $\alpha = 0$, then $bf + adf = 0$, implying $b = -ad$. If $\gamma = 0$, then $bd + ae = 0$ and it follows that $e = d^2$. If $\zeta = 0$, then $df + fce = df + d^2fc = fd(1 + cd) = 0$, implying $c = -1/d$. Thus $\beta = e + f + fcd$, then becomes $d^2 + f + fd(-1/d) = d^2 > 0$, and consequently the refined inertia $(2, 2, 4, 2) \notin \mathrm{ri}(\mathcal{A}_{11})$. A similar analysis and contradiction show that $(2, 2, 4, 2) \notin \mathrm{ri}(\mathcal{A}_{12})$. Parameter values a, b, c, d, e, f have been found for each of the remaining allowed refined inertias in $\mathrm{ri}(\mathcal{A}_1) \cup \mathrm{ri}(\mathcal{A}_3) \cup \mathrm{ri}(\mathcal{A}_5) - \{(0, 0, 10, 0), (2, 2, 4, 2)\}$; see Tables 2, 3 and 4. □

Theorem 10. *If $\mathcal{A}$ is a zero–nonzero pattern with weighted digraph $\mathcal{D}(\mathcal{A})$ having its underlying digraph isomorphic to P_{13}, P_{14}, P_{15}, or P_{16}, then*

$$\mathrm{ri}(\mathcal{A}) = \{(7, 3, 0, 0), (3, 7, 0, 0), (6, 4, 0, 0), (4, 6, 0, 0), (5, 3, 0, 2), (3, 5, 0, 2), (4, 4, 0, 2), (5, 5, 0, 0)\}.$$

Proof. Let A be a matrix realization of $\mathcal{A}_{13}$, $\mathcal{A}_{14}$, $\mathcal{A}_{15}$, or $\mathcal{A}_{16}$. The characteristic polynomial of A is of the form

$$p(x) = x^{10} - \alpha x^5 - Bx^4 - Cx^2 - Dx + E \tag{7}$$

or

$$p(x) = x^{10} - \alpha x^5 - \beta x^4 - \gamma x^2 - Dx + E, \tag{8}$$

where $\alpha, \beta, \gamma, \in \mathbb{R}$ and $B, C, D, E \neq 0$. Since $\det(A) \neq 0$, by Theorems 1, 2, and 3 the only possible refined inertias of A are $(8, 2, 0, 0)$, $(7, 3, 0, 0)$, $(6, 4, 0, 0)$, $(5, 5, 0, 0)$, $(6, 2, 0, 2)$, $(5, 3, 0, 2)$, $(4, 4, 0, 2)$, $(4, 2, 0, 4)$, $(3, 3, 0, 4)$, $(2, 2, 0, 6)$, $(0, 0, 0, 10)$, and their reversals.

If (7) has a complex zero $x = i\sqrt{\phi}$ with $\sqrt{\phi} > 0$, then $p(i\sqrt{\phi}) = 0$ implies that $D = -\alpha\phi^2$ and $B = -\phi^3 + C/\phi + E/\phi^2$. (Note that $D \neq 0$ implies that $\alpha \neq 0$.) Substituting these conditions into $p(x)$, the polynomial becomes

$$x^{10} - \alpha x^5 - (C/\phi + E/\phi^2 - \phi^3)x^4 - Cx^2 + \alpha\phi^2 x + E,$$

which factors into the form

$$(x^2+\phi)\underbrace{\left(x^8-\phi x^6+\phi^2x^4-\alpha x^3-\left(\frac{E}{\phi^2}+\frac{C}{\phi}\right)x^2+\alpha\phi x+\frac{E}{\phi}\right)}_{q(x)}.$$

Since $E, C, \alpha \neq 0$ and $\phi > 0$, if $q(x)$ has a complex zero $x = it$ with $t > 0$, then

$$-\alpha(-it^3)+\alpha\phi(it)=0,$$

which implies that $\alpha = 0$, a contradiction. Thus, since $q(x)$ has no eigenvalues on the imaginary axis for all fixed $\phi > 0$ and $E, C, \alpha \neq 0$, the refined inertia corresponding to the zeros of $q(x)$ is of the form $(\hat{n}_+, \hat{n}_-, 0, 2)$ for some fixed values of $\hat{n}_+, \hat{n}_-$ with $\hat{n}_+ + \hat{n}_- = 8$. Taking any value of $E > 0$ gives $(\hat{n}_+, \hat{n}_-) = (4, 4)$. If E is any negative number, then for $\alpha > 0$ or $\alpha < 0$ the refined inertia corresponding to the zeros of $q(x)$ is $(5, 3, 0, 0)$ or $(3, 5, 0, 0)$, respectively. Consequently, if A has two fixed eigenvalues $\pm i\sqrt{\phi}$ for any $\phi > 0$, then since $E \neq 0$, the other eight eigenvalues are such that $\mathrm{ri}(A) = (4, 4, 0, 2), (5, 3, 0, 2)$, or $(3, 5, 0, 2)$ as α, C and E vary. In particular, the refined inertias $(6, 2, 0, 2)$, $(2, 6, 0, 2)$, $(4, 2, 0, 4)$, $(2, 4, 0, 4)$, $(3, 3, 0, 4)$, $(2, 2, 0, 6)$, and $(0, 0, 0, 10)$ cannot occur.

If $\mathrm{ri}(A) = (8, 2, 0, 0)$, then the characteristic polynomial of A can be factored into the form

$$(x+a_1)(x+a_2)\prod_{i=1}^{8}(x-b_i),$$

where the real parts of $a_1, a_2, b_1, ..., b_8 > 0$. Using the same argument as for the refined inertia $(6, 2, 2, 0)$ in Theorem 7, on replacing $b_1, ..., b_6$ by $b_1, ..., b_8$, a contradiction is found, namely $S_4(A) > 0$ as in (4). Thus $(8, 2, 0, 0)$ and its reversal cannot occur. Parameter values that demonstrate all allowed refined inertias of $\mathcal{A}_{13}$, $\mathcal{A}_{14}$, $\mathcal{A}_{15}$ and $\mathcal{A}_{16}$ are given in Table 5. □

Theorem 11. *If $\mathcal{A}$ is a zero–nonzero pattern with weighted digraph $\mathcal{D}(\mathcal{A})$ having its underlying digraph isomorphic to P_{17} or P_{18}, then*

$$\mathrm{ri}(\mathcal{A}) = \mathrm{ri}(\mathcal{A}_{13}) \cup \{(3, 3, 0, 4)\}.$$

Proof. Let A be a matrix realization of $\mathcal{A}_{17}$ or $\mathcal{A}_{18}$. The characteristic polynomial of A is of the form

$$x^{10}-\alpha x^5-Bx^4-Cx^2-\zeta x+E, \tag{9}$$

where $\alpha, \zeta, \in \mathbb{R}$ and $B, C, E \neq 0$. Many parts in the proof of Theorem 10 are applicable to eliminate certain refined inertias of $\mathcal{A}$. Since $\det(A) \neq 0$, by Theorems 1, 2, and 3

Table 5
Examples of values for the coefficients of the characteristic polynomials C_{13} to C_{18} that can be obtained by choosing nonzero values of a, b, c, d, e, and f to give all possible refined inertias of $\mathcal{A}_{13}$ to $\mathcal{A}_{18}$. For brevity we do not list the parameter values $a, \ldots, f$.

α	β,B	γ,C	D,ζ	E	ri(A_{13}) to ri(A_{18})
2	−1	2	−2	−1	$(7,3,0,0)$
2	−1	1	−2	−1	$(5,3,0,2)$
−1	1	2	1	−1	$(5,5,0,0)$
−1	1	3	1	−1	$(3,5,0,2)$
−1	1	4	1	−1	$(3,7,0,0)$
1	−1	−1	$-\frac{1}{2}$	1	$(6,4,0,0)$
1	−1	−1	−1	1	$(4,4,0,2)$
1	−1	−1	−2	1	$(4,6,0,0)$
α	B	C	ζ	E	In addition ri(A_{17}), ri(A_{18}) can equal
0	1	3	0	−1	$(3,3,0,4)$

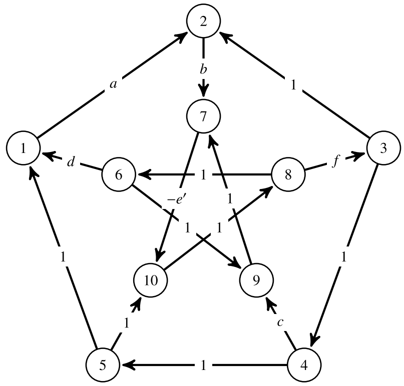
the only possible refined inertias of A are $(8,2,0,0)$, $(7,3,0,0)$, $(6,4,0,0)$, $(5,5,0,0)$, $(6,2,0,2)$, $(5,3,0,2)$, $(4,4,0,2)$, $(4,2,0,4)$, $(3,3,0,4)$, $(2,2,0,6)$, $(0,0,0,10)$, and their reversals. Refined inertias $(8,2,0,0)$, $(2,6,0,2)$, $(4,2,0,4)$, $(2,2,0,6)$, and $(0,0,0,10)$ are eliminated by using the same analysis as in the proof for Theorem 10. However, unlike Theorem 10, pattern $\mathcal{A}$ allows refined inertia $(3,3,0,4)$ since $S_9(A) = \zeta$ can equal zero. If A has parameter values giving $\alpha = \zeta = 0$, $B = 1$, $C = 3$, and $E = -1$, then $\text{ri}(A) = (3,3,0,4)$. Parameter values that demonstrate the other allowed refined inertias for $\mathcal{A}_{17}$ and $\mathcal{A}_{18}$ are given in Table 5. □

4. Refined inertias of a sign pattern

We now consider $\mathcal{A}$ to be a sign pattern, i.e., each nonzero entry of $\mathcal{A}_i$ is signed one of + or −. If sign pattern $\mathcal{A}$ is a signing of one of the zero–nonzero patterns $\mathcal{A}_i$ defined in Section 3, then its weighted digraph $\mathcal{D}(\mathcal{A})$ has an underlying digraph isomorphic to a digraph P_i. It follows from the definitions of zero–nonzero patterns and sign patterns that the refined inertia of a sign pattern is a subset of the refined inertia of its corresponding zero–nonzero pattern. Whereas each nonnegative sign pattern that we considered in Section 2 has a unique refined inertia, we now present a sign pattern example that allows two refined inertias.

Theorem 12. *There exists a sign pattern $\mathcal{A}_c$ that is a signing of $\mathcal{A}_3$ having refined inertia* $\text{ri}(\mathcal{A}_c) = \{(4,4,2,0), (2,2,2,4)\}$.

Proof. Let A_c be the matrix in Fig. 1 with the parameters $a, b, c, d, e', f > 0$, and let $\mathcal{A}_c$ denote the sign pattern of A_c. Then $\mathcal{A}_c$ is a signing of the zero–nonzero pattern $\mathcal{A}_3$ defined in Section 3, and thus $\text{ri}(\mathcal{A}_c) \subseteq \text{ri}(\mathcal{A}_3)$ as given in Theorem 7. The characteristic polynomial of A_c is



$$A_c = \begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & c & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e' \\ 0 & 0 & f & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. 1. $\mathcal{D}(A_c)$, a weighted digraph of P_3.

$$x^{10} + (f - fbe' - e')\, x^5 + (fce' + dbae')\, x^4 + fbae'x^2.$$

Parameter values $a = 1, b = 1/2, c = 1/2, d = 2, e' = 1$, and $f = 2$, which set the coefficient of x^5 equal to zero, give $\mathrm{ri}(A_c) = (2, 2, 2, 4)$. If $f = 3$ with the other parameters fixed as above, then the coefficient of x^5 is nonzero and $\mathrm{ri}(A) = (4, 4, 2, 0)$. Since $fbae' > 0$, it follows that the product of all nonzero eigenvalues of $\mathcal{A}_c$ is positive; hence, there cannot be an odd number of eigenvalues with negative real parts. Thus, the refined inertias $(3, 5, 2, 0)$, $(5, 3, 2, 0)$ and $(3, 3, 2, 2)$ that the nonzero pattern $\mathcal{A}_3$ also allows (see Theorem 7) are not allowed by the sign pattern $\mathcal{A}_c$. $\square$

Note that for any realization of $\mathcal{A}_c$ that has refined inertia $(2, 2, 2, 4)$, any perturbation in the parameters that makes the coefficient of x^5 nonzero results in one pair of pure imaginary eigenvalues moving into the left half plane while the other pair moves into the right half plane.

In conclusion, we remark that each pattern resulting from a weighted strongly connected orientation of the Petersen graph has a very small set of refined inertias, given that the total number of distinct refined inertias for a 10-by-10 pattern is 161, or is 91 for a zero–nonzero pattern if reversals are excluded [6, Theorem 1.1].

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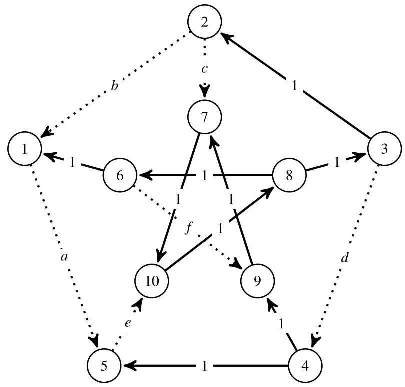
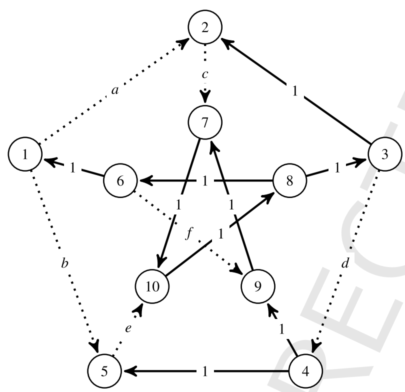
Appendix A. Strongly connected orientations of the Petersen graph

Up to isomorphism, the following are all of the weighted digraphs $\mathcal{D}(A_i)$ with underlying digraphs P_i, together with their matrix representations A_i.



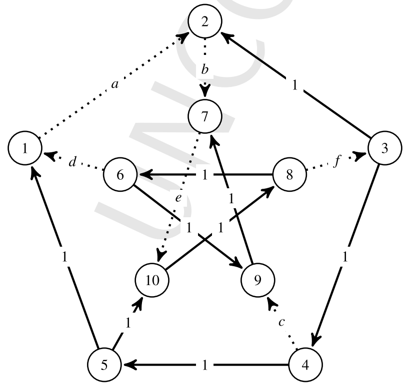
$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 1 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.2. $\mathcal{D}(A_1)$, a weighted digraph of P_1 with directed spanning tree (solid arcs).



$$A_2 = \begin{bmatrix} 0 & a & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 1 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.3. $\mathcal{D}(A_2)$, a weighted digraph of P_2 with directed spanning tree (solid arcs).



$$A_3 = \begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & c & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\ 0 & 0 & f & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.4. $\mathcal{D}(A_3)$, a weighted digraph of P_3 with directed spanning tree (solid arcs).

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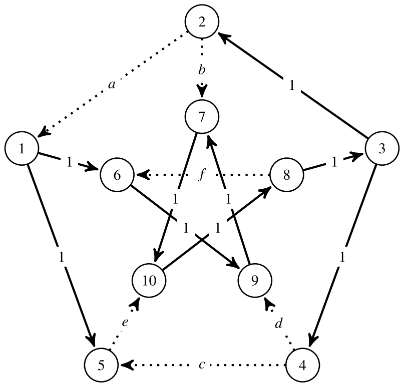
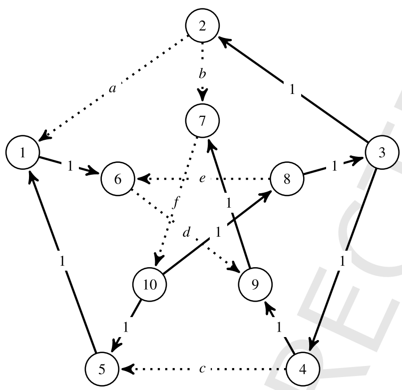


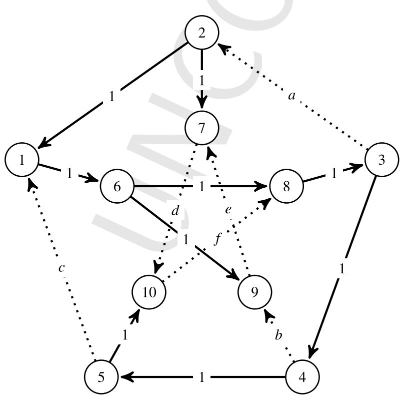
$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.5. $\mathcal{D}(A_4)$, a weighted digraph of P_4 with directed spanning tree (solid arcs).



$$A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f \\ 0 & 0 & 1 & 0 & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.6. $\mathcal{D}(A_5)$, a weighted digraph of P_5 with directed spanning tree (solid arcs).



$$A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 \end{bmatrix}$$

Fig. A.7. $\mathcal{D}(A_6)$, a weighted digraph of P_6 with directed spanning tree (solid arcs).

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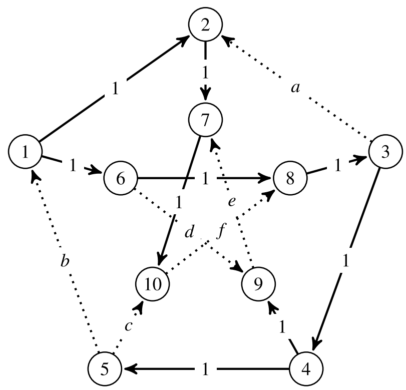
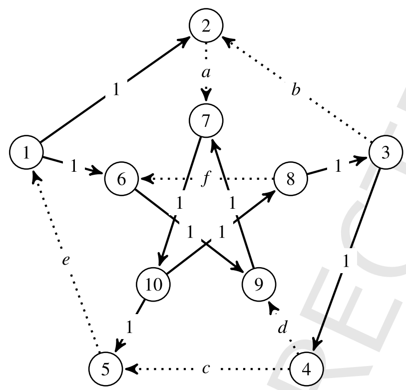


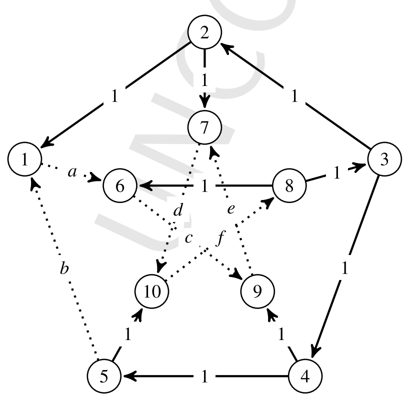
$$A_7 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 \end{bmatrix}$$

Fig. A.8. $\mathcal{D}(A_7)$, a weighted digraph of P_7 with directed spanning tree (solid arcs).



$$A_8 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & b & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & d & 0 \\ e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.9. $\mathcal{D}(A_8)$, a weighted digraph of P_8 with directed spanning tree (solid arcs).



$$A_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 \end{bmatrix}$$

Fig. A.10. $\mathcal{D}(A_9)$, a weighted digraph of P_9 with directed spanning tree (solid arcs).

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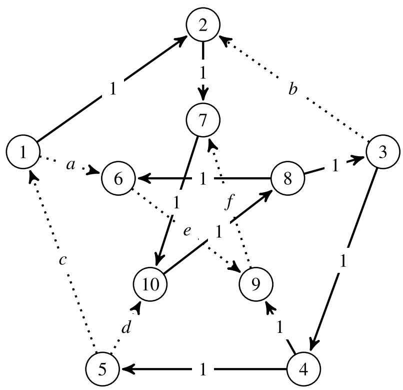
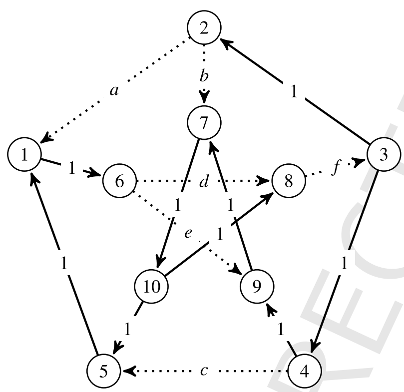


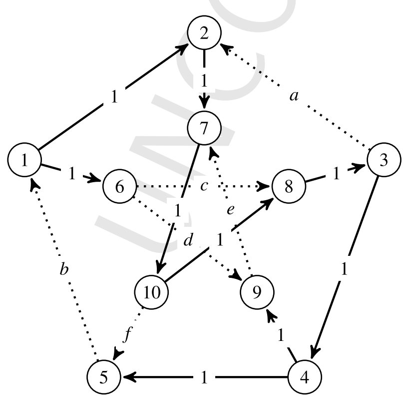
$$A_{10} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & b & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.11. $\mathcal{D}(A_{10})$, a weighted digraph of P_{10} with directed spanning tree (solid arcs).



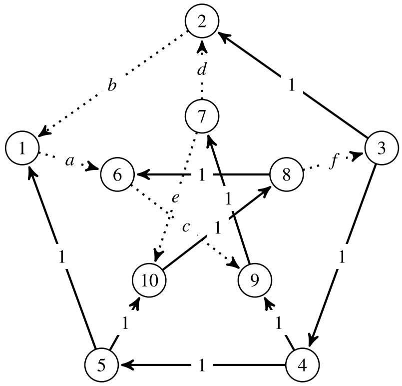
$$A_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & e & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.12. $\mathcal{D}(A_{11})$, a weighted digraph of P_{11} with directed spanning tree (solid arcs).



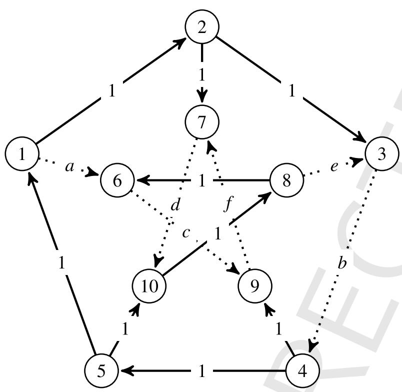
$$A_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.13. $\mathcal{D}(A_{12})$, a weighted digraph of P_{12} with directed spanning tree (solid arcs).



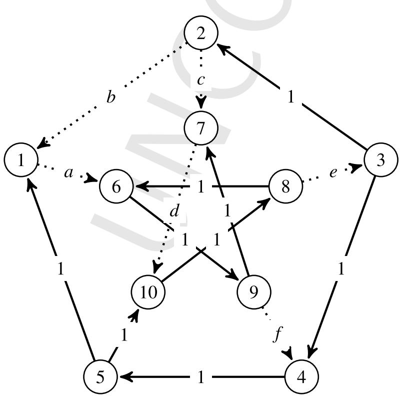
$$A_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\ 0 & 0 & f & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.14. $\mathcal{D}(A_{13})$, a weighted digraph of P_{13} with directed spanning tree (solid arcs).



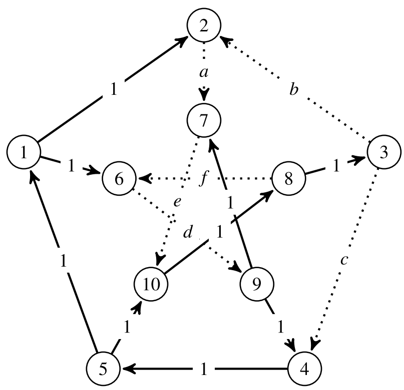
$$A_{14} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & e & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.15. $\mathcal{D}(A_{14})$, a weighted digraph of P_{14} with directed spanning tree (solid arcs).



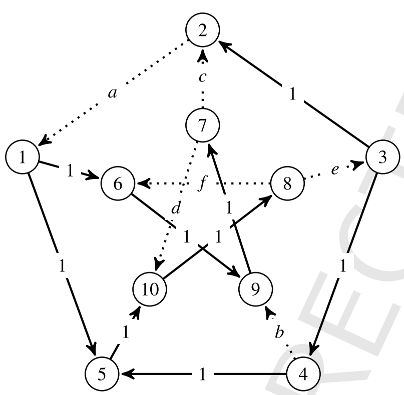
$$A_{15} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & e & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.16. $\mathcal{D}(A_{15})$, a weighted digraph of P_{15} with directed spanning tree (solid arcs).



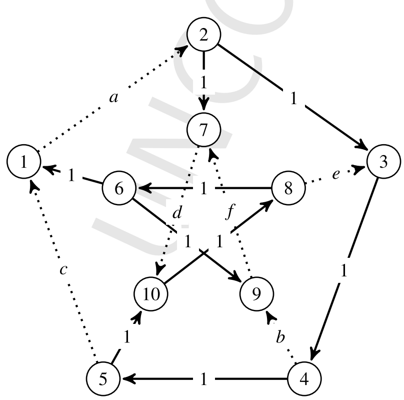
$$A_{16} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & b & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\ 0 & 0 & 1 & 0 & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.17. $\mathcal{D}(A_{16})$, a weighted digraph of P_{16} with directed spanning tree (solid arcs).



$$A_{17} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & e & 0 & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.18. $\mathcal{D}(A_{17})$, a weighted digraph of P_{17} with directed spanning tree (solid arcs).



$$A_{18} = \begin{bmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & e & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fig. A.19. $\mathcal{D}(A_{18})$, a weighted digraph of P_{18} with directed spanning tree (solid arcs).

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Appendix B. Characteristic polynomials

The following are the characteristic polynomials for the matrices A_i in Appendix A. Greek letters denote real-valued parameters (possibly zero) and Roman letters denote nonzero real-valued parameters.

$x^{10}-(c+de+ae+f)\,x^5-(bae+d)\,x^4$	$x^{10}-\alpha x^5-\beta x^4$	(C_1)
$x^{10}-(c+de+be+f)\,x^5-(d+ac)\,x^4$	$x^{10}-\alpha x^5-\beta x^4$	(C_2)
$x^{10}-(f+fbe+e)\,x^5-(fce+dbae)\,x^4-fbaex^2$	$x^{10}-\alpha x^5-\beta x^4-Cx^2$	(C_3)
$x^{10}-(b+ce+f)\,x^5-(ae+d)\,x^4-ax^2$	$x^{10}-\alpha x^5-\beta x^4-Cx^2$	(C_4)
$x^{10}-(bf+edf)\,x^5-(df+f)\,x^4-adfx^2-cdfx$	$x^{10}-\alpha x^5-\beta x^4-Cx^2-Dx$	(C_5)
$x^{10}-(f+adf+a)\,x^5-(fbde+c)\,x^4-fadex^2-fcdex$	$x^{10}-\alpha x^5-\beta x^4-Cx^2-Dx$	(C_6)
$x^{10}-f\,(a+c)\,x^5-(fe+b)\,x^4-bfx^2-fbdex$	$x^{10}-\alpha x^5-\beta x^4-Cx^2-Dx$	(C_7)
$x^{10}-(ae+f+ba)\,x^5-(e+d)\,x^4-caex^2-cex$	$x^{10}-\alpha x^5-\beta x^4-Cx^2-Dx$	(C_8)
$x^{10}-f\,(1+d+cde)\,x^5-edfx^4-facdex^2-fbacdex$	$x^{10}-\alpha x^5-Bx^4-Cx^2-Dx$	(C_9)
$x^{10}-(b+d+fe)\,x^5-fx^4-cx^2-cafex$	$x^{10}-\alpha x^5-Bx^4-Cx^2-Dx$	(C_{10})
$x^{10}-(bf+adf)\,x^5-(e+f+fcd)\,x^4-(dbf+afe)\,x^2-(df+fce)\,x$	$x^{10}-\alpha x^5-\beta x^4-\gamma x^2-\zeta x$	(C_{11})
$x^{10}-(bf+a)\,x^5-(fbde+e+bc)\,x^4-(fbac+b)\,x^2-(fbce+bde)\,x$	$x^{10}-\alpha x^5-\beta x^4-\gamma x^2-\zeta x$	(C_{12})
$x^{10}-(f+ce+dbac)\,x^5-fex^4-fbcaex^2-cafex+dbacf$	$x^{10}-\alpha x^5-Bx^4-Cx^2-Dx+E$	(C_{13})
$x^{10}-(be+fcd+b)\,x^5-fbdex^4-dbex^2-ecdabfx+dbcf$	$x^{10}-\alpha x^5-Bx^4-Cx^2-Dx+E$	(C_{14})
$x^{10}-(e+df+ac)\,x^5-e\,(a+bd)\,x^4-(dae+fac)\,x^2-bcaex+cae$	$x^{10}-\alpha x^5-\beta x^4-\gamma x^2-Dx+E$	(C_{15})
$x^{10}-(e+df+ac)\,x^5-(fbde+ad)\,x^4-(ecad+a)\,x^2-adbfx+fcad$	$x^{10}-\alpha x^5-\beta x^4-\gamma x^2-Dx+E$	(C_{16})
$x^{10}-(e+cde+d+fa)\,x^5-fx^4-dbaex^2-ea\,(bf+d)\,x+facde$	$x^{10}-\alpha x^5-Bx^4-Cx^2-\zeta x+E$	(C_{17})
$x^{10}-(bae+c+edf+d)\,x^5-dfx^4-caex^2-(cde+dafe)\,x+dbae$	$x^{10}-\alpha x^5-Bx^4-Cx^2-\zeta x+E$	(C_{18})

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