

1. The-Axiom-of-Completeness
  - 1.1. Initial Definition for  $\mathbb{R}$
  - 1.2. Axiom of Completeness
  - 1.3. Upper and Lower Bounds
  - 1.4. Supremum and Infimum
  - 1.5. Maximum and Minimum
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  - 1.9. Additional Review

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- **References:**
  - Abbott, S., Understanding Analysis

## Main Content

### Main Idea

$\mathbb{R}$  is an ordered field and contains  $\mathbb{Q}$  as a subfield.

### Explanation

$\mathbb{R}$  is a field, meaning that addition and multiplication of real numbers are commutative, associative, and the distributive property holds.  $\mathbb{R}$  also has an order, meaning the following two properties hold:

1. If  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$ , then one and only one of the statements

$$x < y, \quad x = y, \quad y < x$$

is true.

2. If  $x, y, z \in \mathbb{R}$ , if  $x < y$  and  $y < z$ , then  $x < z$ .

Finally,  $\mathbb{R}$  is a set containing  $\mathbb{Q}$ . The operations of addition and multiplication on  $\mathbb{Q}$  extend to all of  $\mathbb{R}$  in such a way that every element of  $\mathbb{R}$  has an additive inverse and every nonzero element of  $\mathbb{R}$  has a multiplicative inverse.

## Review

1. Define the set of real numbers.

## Links to Other Notes

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- **References:**
  - Abbott, S., Understanding Analysis

## Main Content

### Main Idea

Every nonempty set of real numbers that is bounded above has a least upper bound.

### Explanation

## Review

1. Define the Axiom of Completeness.

## Links to Other Notes

- Initial Definition for  $\mathbb{R}$

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- **References:**
  - Abbott, S., Understanding Analysis

## Main Content

### Main Idea

A set  $A \subset \mathbb{R}$  is bounded above if there exists a number  $b \in \mathbb{R}$  such that  $a \leq b$  for all  $a \in A$ . The number  $b$  is called an upper bound for  $A$ . Likewise, the set  $A$  is bounded below if there exists a lower bound  $l \in \mathbb{R}$  such that  $l \leq a$  for every  $a \in A$ .

### Explanation

## Review

1. Define upper and lower bounds.

## Links to Other Notes

- Initial Definition for  $\mathbb{R}$
- Axiom of Completeness

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  - Abbott, S., Understanding Analysis

## Main Content

### Main Idea

A real number  $s$  is the least upper bound for a set  $A \subset \mathbb{R}$  if it meets the following two criteria:

1.  $s$  is an upper bound for  $A$ ;
2. if  $b$  is any upper bound for  $A$ , then  $s \leq b$ .

### Explanation

The least upper bound is frequently called the supremum of the set  $A$ , denoted  $s = \sup A$ .

## Review

1. Define the supremum of a set.
2. Define the infimum, or the greatest lower bound, of a set.
3. Are least upper bounds unique? Explain.
4. Let

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}.$$

What is  $\sup A$  and  $\inf A$ ?

## Links to Other Notes

- Initial Definition for  $\mathbb{R}$
- Axiom of Completeness
- Upper and Lower Bounds

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  - Abbott, S., Understanding Analysis

## Main Content

### Main Idea

A real number  $a_0$  is a maximum of the set  $A$  if  $a_0$  is an element of  $A$  and  $a_0 \geq a$  for all  $a \in A$ .

### Explanation

The supremum can exist and not be a maximum, but when a maximum exists, then it is also the supremum.

## Review

1. Define maximum.
2. Define minimum.
3. Consider the open interval

$$(0, 2) = \{x \in \mathbb{R} : 0 < x < 2\},$$

and the closed interval

$$[0, 2] = \{x \in \mathbb{R} : 0 \leq x \leq 2\}.$$

What are the maximums of the two sets? What are the supremums?

## Links to Other Notes

- Initial Definition for  $\mathbb{R}$
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum



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## Main Content

### Main Idea

The Axiom of Completeness is not a valid statement about  $\mathbb{Q}$ .

### Explanation

Consider the set

$$S = \{r \in \mathbb{Q} : r^2 < 2\}.$$

This set is certainly bounded above, however, when we search for the least upper bound, we can always find a smaller supremum. For example, we might try  $b = 2$ ,  $b = 3/2$ ,  $b = 142/100$ ,  $b = 1415/1000$ , and so on.

## Review

1. Is the Axiom of Completeness a valid statement about  $\mathbb{Q}$ ? Explain.
2. Does the set

$$S = \{r \in \mathbb{Q} : r^2 < 2\}$$

have a supremum under  $\mathbb{R}$ ?

## Links to Other Notes

- Initial Definition for  $\mathbb{R}$
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum
- Maximum and Minimum

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## Main Content

### Main Idea

Let  $A \subset \mathbb{R}$  be nonempty and bounded above, and let  $c \in \mathbb{R}$ . Define the set  $c + A$  by

$$c + A = \{c + a : a \in A\}.$$

Then  $\sup(c + A) = c + \sup A$ .

### Explanation

Let  $s = \sup A$ . We see that  $a \leq s$  for all  $a \in A$ , which implies  $c + a \leq c + s$  for all  $a \in A$ . Thus  $c + s$  is an upper bound for  $c + A$  and condition (1) of Supremum and Infimum is verified. For (2), let  $b$  be an arbitrary upper bound for  $c + A$ , thus  $c + a \leq b$  for all  $a \in A$ . This is equivalent to  $a \leq b - c$  for all  $a \in A$ , from which we conclude that  $b - c$  is an upper bound for  $A$ . Because  $s$  is the least upper bound of  $A$ ,  $s \leq b - c$ , which can be rewritten as  $c + s \leq b$ . This verifies part (2) of Supremum and Infimum, and we conclude  $\sup(c + A) = c + \sup A$ .

## Review

1. Prove  $\sup(c + A) = c + \sup A$ .

## Links to Other Notes

- Initial Definition for  $\mathbb{R}$
- Axiom of Completeness
- Upper and Lower Bounds

- Supremum and Infimum
- Maximum and Minimum
- $\mathbb{Q}$  and the Axiom of Completeness

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## Main Content

### Main Idea

Assume  $s \in R$  is an upper bound for a set  $A \subset R$ . Then,  $s = \sup A$  if and only if, for every choice of  $\epsilon > 0$ , there exists an element  $a \in A$  satisfying  $s - \epsilon < a$ .

### Explanation

For the forward direction, assume  $s = \sup A$  and consider  $s - \epsilon$ , where  $\epsilon > 0$  has been arbitrarily chosen. Because  $s - \epsilon < s$ , part (2) of Supremum and Infimum implies that  $s - \epsilon$  is not an upper bound for  $A$ . If this is the case, then there must be some element  $a \in A$  for which  $s - \epsilon < a$ .

Conversely, assume  $s$  is an upper bound with the property that no matter how  $\epsilon > 0$  is chosen,  $s - \epsilon$  is no longer an upper bound for  $A$ . Notice that what this implies is that if  $b$  is any number less than  $s$ , then  $b$  is not an upper bound. To prove that  $s = \sup A$ , we must verify part (2) of Supremum and Infimum. Because we have just argued that any number smaller than  $s$  cannot be an upper bound, it follows that if  $b$  is some other upper bound for  $A$ , then  $s \leq b$ .

## Review

1. What is an alternative phrasing for part (2) in Supremum and Infimum? Explain.

## Links to Other Notes

- Initial Definition for  $R$
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum

- Maximum and Minimum
- Q and the Axiom of Completeness
- $\sup(c + A) = c + \sup A$

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- **References:**
  - Abbott, S., Understanding Analysis

## Main Content

### Main Idea

1. (a) Write a formal definition in the style of Supremum and Infimum for the infimum or greatest lower bound of a set.  
(b) Now, state and prove a version of Alternative Phrasing for Supremum for greatest lower bounds.
2. Give an example of each of the following, or state that the request is impossible.
  - (a) A set  $B$  with  $\inf B \geq \sup B$ .
  - (b) A finite set that contains its infimum but not its supremum.
  - (c) A bounded subset  $Q$  that contains its supremum but not its infimum.

### Explanation

1. (a) A real number  $n$  is the greatest lower bound for a set  $A \subset \mathbb{R}$  if it meets the following two criteria:
  1.  $n$  is a lower bound for  $A$ ;
  2. if  $b$  is any lower bound for  $A$ , then  $b \leq n$ .  
(b) Assume  $n \in \mathbb{R}$  is a lower bound for a set  $A \subset \mathbb{R}$ . Then,  $n = \inf A$  if and only if, for every choice of  $\epsilon > 0$ , there exists an element  $a \in A$  satisfying  $a < n + \epsilon$ .

*Proof.* Assume  $n = \inf A$  and consider  $n + \epsilon$ , where  $\epsilon > 0$  has been chosen arbitrarily. Because  $n < n + \epsilon$ , the definition for infimum implies that  $n + \epsilon$  is not a lower bound for  $A$ . Thus, there must be some element  $a \in A$  such that  $a < n + \epsilon$ .

Conversely, assume there exists an element  $a \in A$  that satisfies  $a < n + \epsilon$ . In other words, for any number  $b$  that is greater than  $n$ ,  $b$  is not a lower bound. Thus, according to the definition,  $n$  is the greatest lower bound for  $A$ .  $\square$



## Review

1.

## Links to Other Notes

- Initial Definition for  $\mathbb{R}$
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum
- Maximum and Minimum
- $\mathbb{Q}$  and the Axiom of Completeness
- $\sup(c + A) = c + \sup A$
- Alternative Phrasing for Supremum

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