Note Information

- **ID:** `202501180703`
- **Timestamp:** `Sunday 19th January, 2025 11:39`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea

R is an ordered field and contains Q as a subfield.

Explanation

R is a field, meaning that addition and multiplication of real numbers are commutative, associative, and the distributive property holds. R also has an order, meaning the following two properties hold:

1. If $x \in R$ and $y \in R$, then one and only one of the statements

$$x < y, \qquad x = y, \qquad y < x$$

   is true.

2. If $x, y, z \in R$, if $x < y$ and $y < z$, then $x < z$.

Finally, R is a set containing Q. The operations of addition and multiplication on Q extend to all of R in such a way that every element of R has an additive inverse and every nonzero element of R has a multiplicative inverse.

Review

1. Define the set of real numbers.

Links to Other Notes

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Note Information

- **ID:** `202501180727`
- **Timestamp:** `Sunday 19th January, 2025 11:39`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
Every nonempty set of real numbers that is bounded above has a least upper bound.

Explanation

Review

1. Define the Axiom of Completeness.

Links to Other Notes

- Initial Definition for R

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Note Information

- **ID:** `202501180734`
- **Timestamp:** `Sunday 19th January, 2025 11:39`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
A set $A \subset R$ is bounded above if there exists a number $b \in R$ such that $a \leq b$ for all $a \in A$. The number b is called an upper bound for A. Likewise, the set A is bounded below if there exists a lower bound $l \in R$ such that $l \leq a$ for every $a \in A$.

Explanation

Review

1. Define upper and lower bounds.

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness

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Note Information

- **ID:** `202501180743`
- **Timestamp:** `Sunday 19th January, 2025 11:39`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
A real number s is the least upper bound for a set $A \subset R$ if it meets the following two criteria:

1. s is an upper bound for A;
2. if b is any upper bound for A, then $s \leq b$.

Explanation
The least upper bound is frequently called the supremum of the set A, denoted $s = \sup A$.

Review

1. Define the supremum of a set.
2. Define the infimum, or the greatest lower bound, of a set.
3. Are least upper bounds unique? Explain.
4. Let
$$A = \{\frac{1}{n} : n \in N\} = \{1, \frac{1}{2}, \frac{1}{3}, ...\}.$$
What is $\sup A$ and $\inf A$?

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds

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Note Information

- **ID:** `202501181241`
- **Timestamp:** `Sunday 19th January, 2025 11:39`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
A real number a_0 is a maximum of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$.

Explanation
The supremum can exist and not be a maximum, but when a maximum exists, then it is also the supremum.

Review

1. Define maximum.
2. Define minimum.
3. Consider the open interval
$$(0, 2) = \{x \in R : 0 < x < 2\},$$
and the closed interval
$$[0, 2] = \{x \in R : 0 \leq x \leq 2\}.$$
What are the maximums of the two sets? What are the supremums?

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum

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Note Information

- **ID:** `202501181257`
- **Timestamp:** `Sunday 19th January, 2025 11:39`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
The Axiom of Completeness is not a valid statement about Q.

Explanation
Consider the set

$$S = \{r \in Q : r^2 < 2\}.$$

This set is certainly bounded above, however, when we search for the least upper bound, we can always find a smaller supremum. For example, we might try $b = 2$, $b = 3/2$, $b = 142/100$, $b = 1415/1000$, and so on.

Review

1. Is the Axiom of Completeness a valid statement about Q? Explain.
2. Does the set

   $$S = \{r \in Q : r^2 < 2\}$$

   have a supremum under R?

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum
- Maximum and Minimum

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Note Information

- **ID:** `202501181310`
- **Timestamp:** `Sunday 19th January, 2025 11:39`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
Let $A \subset R$ be nonempty and bounded above, and let $c \in R$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}.$$

Then $\sup(c + A) = c + \sup A$.

Explanation
Let $s = \sup A$. We see that $a \leq s$ for all $a \in A$, which implies $c + a \leq c + s$ for all $a \in A$. Thus $c + s$ is an upper bound for $c + A$ and condition (1) of Supremum and Infimum is verified. For (2), let b be an arbitrary upper bound for $c + A$, thus $c + a \leq b$ for all $a \in A$. This is equivalent to $a \leq b - c$ for all $a \in A$, from which we conclude that $b - c$ is an upper bound for A. Because s is the least upper bound of A, $s \leq b - c$, which can be rewritten as $c + s \leq b$. This verifies part (2) of Supremum and Infimum, and we conclude $\sup(c + A) = c + \sup A$.

Review

1. Prove $\sup(c + A) = c + \sup A$.

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds

- Supremum and Infimum
- Maximum and Minimum
- Q and the Axiom of Completeness

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Note Information

- **ID:** `202501181335`
- **Timestamp:** `Sunday 19th January, 2025 11:39`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
Assume $s \in R$ is an upper bound for a set $A \subset R$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists and element $a \in A$ satisfying $s - \epsilon < a$.

Explanation
For the forward direction, assume $s = \sup A$ and consider $s - \epsilon$, where $\epsilon > 0$ has been arbitrarily chosen. Because $s - \epsilon < s$, part (2) of Supremum and Infimum implies that $s - \epsilon$ is not an upper bound for A. If this is the case, then there must be some element $a \in A$ for which $s - \epsilon < a$.
Conversely, assume s is an upper bound with the property that no matter how $\epsilon > 0$ is chosen, $s - \epsilon$ is no longer an upper bound for A. Notice that what this implies is that if b is any number less than s, then b is not an upper bound. To prove that $s = \sup A$, we must verify part (2) of Supremum and Infimum. Because we have just argued that any number smaller than s cannot be an upper bound, it follows that if b is some other upper bound for A, then $s \leq b$.

Review

1. What is an alternative phrasing for part (2) in Supremum and Infimum? Explain.

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum

- Maximum and Minimum
- Q and the Axiom of Completeness
- sup(c + A) = c + sup A

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Note Information

- **ID:** `202501181521`
- **Timestamp:** `Sunday 19th January, 2025 11:39`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Questions

1. (a) Write a formal definition in the style of Supremum and Infimum for the infimum or greatest lower bound of a set.

   (b) Now, state and prove a version of Alternative Phrasing for Supremum for greatest lower bounds.

2. Give an example of each of the following, or state that the request is impossible.

   (a) A set B with $\inf B \geq \sup B$.

   (b) A finite set that contains its infimum but not its supremum.

   (c) A bounded subset Q that contains its supremum but not its infimum.

3. (a) Let A be nonempty and bounded below, and define $B = \{b \in R : b \text{ is a lower bound for } A\}$. Show that $\sup B = \inf A$.

   (b) Use (a) to explain why there is no need to assert that greatest lower bounds exist as part of the Axiom of Completeness.

4. As in sup(c + A) = c + sup A, let $A \subset R$ be nonempty and bounded above, and let $c \in R$. This time define the set $cA = \{ca : a \in A\}$.

   (a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

   (b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solutions

1. (a) A real number n is the greatest lower bound for a set $A \subset R$ if it meets the following two criteria:

1. n is a lower bound for A;
2. if b is any lower bound for A, then $b \leq n$.

(b) Assume $n \in R$ is a lower bound for a set $A \subset R$. Then, $n = \inf A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $a < n + \epsilon$.

Proof. Assume $n = \inf A$ and consider $n + \epsilon$, where $\epsilon > 0$ has been chosen arbitrarily. Because $n < n + \epsilon$, the definition for infimum implies that $n + \epsilon$ is not a lower bound for A. Thus, there must be some element $a \in A$ such that $a < n + \epsilon$.
Conversely, assume there exists an element $a \in A$ that satisfies $a < n+\epsilon$. In other words, for any number b that is greater than n, b is not a lower bound. Thus, according to the definition, n is the greatest lower bound for A. □

2. (a) Consider $B = \{0\}$; $\sup B = \inf B = 0$, thus, $\inf B \geq \sup B$.

(b) Impossible, finite sets must have both a maximum and minimum, and thus, must contain their infimum and supremum.

(c) Consider $B = \{b \in Q : 0 < b \leq 1\}$; $\sup B = 1 \in B$ and $\inf B = 0 \notin B$.

3. (a) Since every $b \in B$ is a lower bound for A, we have $b \leq a$ for all $a \in A$. In particular, $\inf A$, being the greatest lower bound of A, satisfies $b \leq \inf A$ for all $b \in B$. Thus, $\sup B \leq \inf A$, since $\sup B$ is the least upper bound of B.
Conversely, by definition of $\inf A$, $\inf A$ is a lower bound for A, so $\inf A \in B$. Since $\sup B$ is the least upper bound for B, it must satisfy $\sup B \geq \inf A$.
Therefore, $\sup B = \inf A$.

(b) The existence of the infimum for a bounded below set A can always be derived from the Axiom of Completeness as follows:

- Define B to be the set of all lower bounds of A.
- The Axiom of Completeness guarantees that B has a supremum $\sup B$.
- By definition and part (a), $\sup B = \inf A$.

Thus, the existence of greatest lower bounds (infima) is already implicit in the Axiom of Completeness, as every bounded below set can be ”reduced” to a problem of finding the supremum of its set of lower bounds.

4. (a) Let $s = \sup A$. We see that $a \leq s$ for all $a \in A$, which implies $ca \leq cs$ for all $a \in A$. Thus, cs is an upper bound for cA and condition (1) of Supremum and Infimum is verified. For (2), let b be an arbitrary upper bound for cA, thus $ca \leq b$ for all $a \in A$. This is equivalent to $a \leq b/c$ for all $a \in A$, from which we conclude that b/c is an upper bound for A. Because s is the least upper bound of A, $s \leq b/c$, which can be rewritten as $cs \leq b$. This verifies part (2) of Supremum and Infimum, and we conclude $\sup (cA) = c \sup A$.

(b) If $c < 0$, $\sup(cA) = c \inf A$.

Review

1.

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum
- Maximum and Minimum
- Q and the Axiom of Completeness
- sup(c + A) = c + sup A
- Alternative Phrasing for Supremum

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Note Information

- **ID:** `202501191115`
- **Timestamp:** `Sunday 19`$^{\text{th}}$ `January, 2025 11:39`
- **Tags:** `Mathematics, Analysis-I-Abbott, Consequences-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
For each $n \in N$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in R : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals

$$I_1 \supset I_2 \supset I_3 \supset I_4 \supset ...$$

has a nonempty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Explanation
Consider the set

$$A = \{a_n : n \in N\}$$

of left-hand endpoints of the intervals. Because the intervals are nested, we see that every b_n serves as an upper bound for A. Thus, we are justified in setting

$$x = \sup A.$$

Now, consider a particular $I_n = [a_n, b_n]$. Because x is an upper bound for A, we have $a_n \leq x$. The fact that each b_n is an upper bound for A and that x is the least upper bound implies $x \leq b_n$.
Therefore, we have $a_n \leq x \leq b_n$, which means $x \in I_n$ for every choice of $n \in N$. Hence, $x \in \bigcap_{n=1}^{\infty} I_n$, and the intersection is not empty.

Review

1. State and prove the Nested Interval Property.

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum
- Maximum and Minimum
- Q and the Axiom of Completeness
- sup(c + A) = c + sup A
- Alternative Phrasing for Supremum

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