Note Information

- **ID:** `202501131947`
- **Timestamp:** `Thursday 16th January, 2025 06:37`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea
Rational numbers are of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$.

Explanation

Review

1. Define the rational number system.

Links to Other Notes

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Note Information

- **ID:** `202501132004`
- **Timestamp:** `Thursday 16th January, 2025 06:37`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-System`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea

The rational number system is inadequate for many purposes, both as a field and as an ordered set.

Explanation

For example, there is no rational p such that $p^2 = 2$.

Proof 0.1 *Suppose on the contrary there was a p that satisfied $p^2 = 2$. We could write $p = \frac{m}{n}$, where m and n are integers and coprime. The original expression implies*

$$(\frac{m}{n})^2 = 2$$
$$\frac{m^2}{n^2} = 2$$
$$m^2 = 2n^2$$

From this expression, we see that m^2 is even, and thus, m is even. Plugging $2k$ in for m, it is clear that m^2 is divisible by 4. It follows that $2n^2$ is divisible by 4 as well, which implies n^2 is even, and thus, n is even. Therefore, our assumption leads to a contradiction that both m and n are even, thus violating the coprime property of m and n. Hence, it is impossible for p to be rational.

Proof 0.2 (Alternative) *Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$. By showing there is no largest element in A and no smallest element in B, we effectively partion the set of rational numbers, thus implying there is no rational p that falls outside these two sets, therefore satisfying $p^2 = 2$.*

To prove that for every p in A we can find a rational q in A such that $p < q$, we associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{1}$$

Then

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}. \tag{2}$$

If p is in A then $p^2 - 2 < 0$, (1) shows that $q > p$, and (2) shows that $q^2 < 2$. Thus q is in A. If p is in B then $p^2 - 2 > 0$, (1) shows that $0 < q < p$, and (2) shows that $q^2 > 2$. Thus q is in B.

Review

1. Prove that there is no rational p such that $p^2 = 2$ in two different ways.

Links to Other Notes

- Definition of Rational Numbers

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Note Information

- **ID:** `202501141228`
- **Timestamp:** `Thursday 16th January, 2025 06:37`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea

Let S be a set. An order on S is a relation, denoted by $<$, with the following two properties:

- If $x \in S$ and $y \in S$ then one and only one of the statements

  $$x < y, \qquad x = y, \qquad y < x$$

  is true.
- If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Explanation

Review

1. Define order.

Links to Other Notes

-

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Note Information

- **ID:** `202501141241`
- **Timestamp:** `Thursday 16th January, 2025 06:37`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea
An ordered set is a set S in which an order is defined.

Explanation
For example, Q is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

Review

1. Define ordered set and give an example.

Links to Other Notes

- Order

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Note Information

- **ID:** `202501141250`
- **Timestamp:** `Thursday 16th January, 2025 06:37`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea
Suppose S is an orderd set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is bounded above, and call β an upper bound of E. Lower bounds are defined in the same way.

Explanation

Review

1. Define upper bound.
2. Define lower bound.

Links to Other Notes

- Order
- Ordered Set

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Note Information

- **ID:** `202501141546`
- **Timestamp:** `Thursday 16th January, 2025 06:37`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea

Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

- α is an upper bound of E.
- If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the least upper bound of E or the supremum of E, and we write

$$\alpha = \sup E.$$

The greatest lower bound, or infimum, of a set E which is bounded below is defined in the same manner: If $\alpha = \inf E$, then α is a lower bound of E and no β with $\beta > \alpha$ is a lower bound of E.

Explanation

For example, consider the sets A and B from the alternative proof in Rationals are Inadequate. A and B are subsets of the ordered set Q. The set A is bounded above by the members of B. Since B contains no smallest member, A has no supremum in Q. B is bounded below by the members of A. Since A has no largest member, B has no infimum in Q.

Review

1. Define supremum.
2. Define infimum.

3. If $\alpha = \sup E$ exists, then must α be a member of E? Give an example to justify your answer.

4. Let E consist of all numbers $1/n$, where $n = 1, 2, 3, ...$ What is $\sup E$ and $\inf E$? Are $\sup E$ and $\inf E$ members of E?

Links to Other Notes

- Definition of Rational Numbers
- Rationals are Inadequate
- Order
- Ordered Set
- Upper Bounds and Lower Bounds

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Note Information

- **ID:** `202501141632`
- **Timestamp:** `Thursday 16th January, 2025 06:37`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea

An ordered set S is said to have the least-upper-bound property if the following is true: If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

Explanation

Review

1. Define the least-upper-bound property.
2. Define the greatest-lower bound property.
3. Does Q have the least-upper-bound property? Explain.

Links to Other Notes

- Definition of Rational Numbers
- Rationals are Inadequate
- Order
- Ordered Set
- Upper Bounds and Lower Bounds
- Supremum and Infimum

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Note Information

- **ID:** 202501141654
- **Timestamp:** Thursday 16th January, 2025 06:37
- **Tags:** Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea
If an ordered set has the least-upper-bound property, then it also has the greatest-lower-bound property.

Explanation

Theorem 0.3 *Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B. Then*

$$\alpha = \sup L$$

exists in S, and $\alpha = \inf B$. In particular, $\inf B$ exists in S.

Proof 0.4 *Since B is bounded below, L is not empty. Since L consists of exactly those $y \in S$ which satisfy the inequality $y \leq x$ for every $x \in B$, we see that every $x \in B$ is an upper bound of L. Thus L is bounded above. Our hypothesis about S implies therefore that L has a supremum in S, namely α. If $\gamma < \alpha$ then γ is not an upper bound of L, hence $\gamma \notin B$. It follows that $\alpha \leq x$ for every $x \in B$. Thus $\alpha \in L$. If $\alpha < \beta$ then $\beta \notin L$, since α is an upper bound of L. We have shown that $\alpha \in L$ but $\beta \notin L$ if $\beta > \alpha$. In other words α is a lower bound of B, but β is not if $\beta > \alpha$. This means that $\alpha = \inf B$.*

Review

1. Prove that if an ordered set has the least-upper-bound property, then it also has the greatest-lower-bound property.

Links to Other Notes

- Definition of Rational Numbers
- Rationals are Inadequate
- Order
- Ordered Set
- Upper Bounds and Lower Bounds
- Supremum and Infimum
- Least-Upper-Bound Property

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Note Information

- **ID:** `202501150657`
- **Timestamp:** `Thursday 16th January, 2025 06:37`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea
A field is a set F with two operations, called addition and multiplication, which satisfy the following field axioms (A), (M), and (D):

(A) Axioms for addition

- (A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F.
- (A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.
- (A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
- (A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
- (A5) To every $x \in F$ corresponds an element $-x \in F$ such that $x + (-x) = 0$.

(M) Axioms for multiplication

- (M1) If $x \in F$ and $y \in F$, then their product xy is in F.
- (M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.
- (M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
- (M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
- (M5) If $x \in F$ and $x \neq 0$ then there exists an element $1/x \in F$ such that $x \cdot (1/x) = 1$.

(D) The distributive law

$$x(y + z) = xy + xz \text{ holds for all } x, y, z \in F$$

Explanation

Review

1. Define field.
2. Is Q a field?

Links to Other Notes

- Definition of Rational Numbers
- Rationals are Inadequate

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Note Information

- **ID:** `202501150717`
- **Timestamp:** `Thursday 16th January, 2025 06:37`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea
The axioms for addition imply the following statements.

(a) If $x + y = x + z$ then $y = z$.

(b) If $x + y = x$ then $y = 0$.

(c) If $x + y = 0$ then $y = -x$.

(d) $-(-x) = x$.

Explanation

Proof 0.5 *If $x + y = x + z$, the axioms (A) give*

$$\begin{aligned} y = 0 + y = (-x + x) + y = -x + (x + y) \\ = -x + (x + z) = (-x + x) + z = 0 + z = z. \end{aligned}$$

This proves (a). Take $z = 0$ in (a) to obtain (b). Take $z = -x$ in (a) to obtain (c). Since $-x + x = 0$, (c) with $-x$ in place of x gives (d).

Review

1. What do the axioms for addition imply? Explain.

Links to Other Notes

- Definition of Field

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Note Information

- **ID:** 202501150809
- **Timestamp:** Thursday 16th January, 2025 06:37
- **Tags:** Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea
The axioms for multiplication imply the following statements.

(a) If $x \neq 0$ and $xy = xz$ then $y = z$.

(b) If $x \neq 0$ and $xy = x$ then $y = 1$.

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.

(d) If $x \neq 0$ then $1/(1/x) = x$.

Explanation

Proof 0.6 *Suppose $x \neq 0$. Let $xy = xz$. The axioms (M) give*

$$\begin{aligned} y = 1 \cdot y = (x \cdot (1/x)) \cdot y &= (1/x) \cdot (x \cdot y) \\ &= (1/x) \cdot (x \cdot z) = ((1/x) \cdot x) \cdot z = 1 \cdot z = z \end{aligned}$$

This proves (a). Take $z = 1$ in (a) to obtain (b). Take $z = (1/x)$ in (a) to obtain (c). Since $x(1/x) = 1$, (c) with $1/x$ in place of y gives (d).

Review

1. What do the axioms for multiplication imply? Explain.

Links to Other Notes

- Definition of Field
- Implications of Addition Axioms

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Note Information

- **ID:** `202501152115`
- **Timestamp:** `Thursday 16th January, 2025 06:37`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea

The field axioms imply the following statements, for any $x, y, z \in F$.

(a) $0x = 0$.

(b) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

(c) $(-x)y = -(xy) = x(-y)$.

(d) $(-x)(-y) = xy$.

Explanation

Proof 0.7 $0x + 0x = (0 + 0)x = 0x$. *Hence statement (b) from Implications of the Addition Axioms implies that* $0x = 0$, *and (a) holds. Next, assume* $x \neq 0$, $y \neq 0$, *but* $xy = 0$. *Then (a) gives*

$$1 = (1/y)(1/x)xy = (1/y)(1/x)0 = 0,$$

a contradiction. Thus (b) holds. The first equality in (c) comes from

$$(-x)y + xy = (-x + x)y = 0y = 0,$$

combined with statement (c) from Implications of the Addition Axioms; the other half of (c) is proved in the same way:

$$xy + x(-y) == x(y - y) = 0x = 0$$

Finally,

$$(-x)(-y) = -[x(-y)] = -[-(xy)] = xy$$

by (c) and statement (d) from Implications of the Addition Axioms.

Review

1. What other statements do the field axioms imply? Explain.

Links to Other Notes

- Definition of Field
- Implications of Addition Axioms
- Implications of Multiplication Axioms

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Note Information

- **ID:** `202501160624`
- **Timestamp:** `Thursday 16th January, 2025 06:37`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea
An ordered field is a field F which is also an ordered set, such that

- $x + y < x + z$ if $x, y, z \in F$ and $y < z$,
- $xy > 0$ if $x \in F$, $y \in F$, $x > 0$, and $y > 0$.

Explanation

Review

1. Define ordered field.
2. Is Q an ordered field?

Links to Other Notes

- Definition of Rational Numbers
- Rationals are Inadequate
- Order
- Ordered Set
- Definition of Field
- Implications of Addition Axioms
- Implications of Multiplication Axioms
- Further Implications of Field Axioms

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