1. The-Real-and-Complex-Number-Systems
    1.1. Definition of Rational Numbers
    1.2. Rationals are Inadequate
    1.3. Order
    1.4. Ordered Set
    1.5. Upper Bounds and Lower Bounds
    1.6. Supremum and Infimum

Note Information

- **ID:** `202501131947`
- **Timestamp:** `Tuesday 14th January, 2025 16:22`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea
Rational numbers are of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$.

Explanation

Review

1. Define the rational number system.

Links to Other Notes

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Note Information

- **ID:** `202501132004`
- **Timestamp:** `Tuesday 14th January, 2025 16:22`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-System`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea

The rational number system is inadequate for many purposes, both as a field and as an ordered set.

Explanation

For example, there is no rational p such that $p^2 = 2$.

Proof 0.1 *Suppose on the contrary there was a p that satisfied $p^2 = 2$. We could write $p = \frac{m}{n}$, where m and n are integers and coprime. The original expression implies*

$$(\frac{m}{n})^2 = 2$$
$$\frac{m^2}{n^2} = 2$$
$$m^2 = 2n^2$$

From this expression, we see that m^2 is even, and thus, m is even. Plugging $2k$ in for m, it is clear that m^2 is divisible by 4. It follows that $2n^2$ is divisible by 4 as well, which implies n^2 is even, and thus, n is even. Therefore, our assumption leads to a contradiction that both m and n are even, thus violating the coprime property of m and n. Hence, it is impossible for p to be rational.

Proof 0.2 (Alternative) *Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$. By showing there is no largest element in A and no smallest element in B, we effectively partion the set of rational numbers, thus implying there is no rational p that falls outside these two sets, therefore satisfying $p^2 = 2$.*

To prove that for every p in A we can find a rational q in A such that $p < q$, we associate with each rational $p > 0$ the number

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{1}$$

Then

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}. \tag{2}$$

If p is in A then $p^2 - 2 < 0$, (1) shows that $q > p$, and (2) shows that $q^2 < 2$. Thus q is in A. If p is in B then $p^2 - 2 > 0$, (1) shows that $0 < q < p$, and (2) shows that $q^2 > 2$. Thus q is in B.

Review

1. Prove that there is no rational p such that $p^2 = 2$ in two different ways.

Links to Other Notes

- Definition of Rational Numbers

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Note Information

- **ID:** `202501141228`
- **Timestamp:** `Tuesday 14th January, 2025 16:22`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea

Let S be a set. An order on S is a relation, denoted by $<$, with the following two properties:

- If $x \in S$ and $y \in S$ then one and only one of the statements

$$x < y, \qquad x = y, \qquad y < x$$

  is true.
- If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Explanation

Review

1. Define order.

Links to Other Notes

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Note Information

- **ID:** `202501141241`
- **Timestamp:** `Tuesday 14th January, 2025 16:22`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea
An ordered set is a set S in which an order is defined.

Explanation
For example, Q is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

Review

1. Define ordered set and give an example.

Links to Other Notes

- Order

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Note Information

- **ID:** `202501141250`
- **Timestamp:** `Tuesday 14th January, 2025 16:22`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea

Suppose S is an orderd set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is bounded above, and call β an upper bound of E. Lower bounds are defined in the same way.

Explanation

Review

1. Define upper bound.
2. Define lower bound.

Links to Other Notes

- Order
- Ordered Set

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Note Information

- **ID:** `202501141546`
- **Timestamp:** `Tuesday 14th January, 2025 16:22`
- **Tags:** `Mathematics, Analysis-I, The-Real-and-Complex-Number-Systems`
- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

Main Content

Main Idea

Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

- α is an upper bound of E.
- If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the least upper bound of E or the supremum of E, and we write

$$\alpha = \sup E.$$

The greatest lower bound, or infimum, of a set E which is bounded below is defined in the same manner: If $\alpha = \inf E$, then α is a lower bound of E and no β with $\beta > \alpha$ is a lower bound of E.

Explanation

For example, consider the sets A and B from the alternative proof in Rationals are Inadequate. A and B are subsets of the ordered set Q. The set A is bounded above by the members of B. Since B contains no smallest member, A has no supremum in Q. B is bounded below by the members of A. Since A has no largest member, B has no infimum in Q.

Review

1. Define supremum.
2. Define infimum.

3. If $\alpha = \sup E$ exists, then must α be a member of E? Give an example to justify your answer.

4. Let E consist of all numbers $1/n$, where $n = 1, 2, 3, ...$ What is $\sup E$ and $\inf E$? Are $\sup E$ and $\inf E$ members of E?

Links to Other Notes

- Definition of Rational Numbers
- Rationals are Inadequate
- Order
- Ordered Set
- Upper Bounds and Lower Bounds

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