Note Information

- **ID:** `202501180703`
- **Timestamp:** `Saturday 18th January, 2025 13:54`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
R is an ordered field and contains Q as a subfield.

Explanation
R is a field, meaning that addition and multiplication of real numbers are commutative, associative, and the distributive property holds. R also has an order, meaning the following two properties hold:

1. If $x \in R$ and $y \in R$, then one and only one of the statements

$$x < y, \qquad x = y, \qquad y < x$$

   is true.

2. If $x, y, z \in R$, if $x < y$ and $y < z$, then $x < z$.

Finally, R is a set containing Q. The operations of addition and multiplication on Q extend to all of R in such a way that every element of R has an additive inverse and every nonzero element of R has a multiplicative inverse.

Review

1. Define the set of real numbers.

Links to Other Notes

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Note Information

- **ID:** `202501180727`
- **Timestamp:** `Saturday 18th January, 2025 13:54`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
Every nonempty set of real numbers that is bounded above has a least upper bound.

Explanation

Review

1. Define the Axiom of Completeness.

Links to Other Notes

- Initial Definition for R

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- TOC

Note Information

- **ID:** `202501180734`
- **Timestamp:** `Saturday 18th January, 2025 13:54`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
A set $A \subset R$ is bounded above if there exists a number $b \in R$ such that $a \leq b$ for all $a \in A$. The number b is called an upper bound for A. Likewise, the set A is bounded below if there exists a lower bound $l \in R$ such that $l \leq a$ for every $a \in A$.

Explanation

Review

1. Define upper and lower bounds.

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness

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Note Information

- **ID:** `202501180743`
- **Timestamp:** `Saturday 18th January, 2025 13:54`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea

A real number s is the least upper bound for a set $A \subset R$ if it meets the following two criteria:

1. s is an upper bound for A;
2. if b is any upper bound for A, then $s \leq b$.

Explanation

The least upper bound is frequently called the supremum of the set A, denoted $s = \sup A$.

Review

1. Define the supremum of a set.
2. Define the infimum, or the greatest lower bound, of a set.
3. Are least upper bounds unique? Explain.
4. Let
$$A = \{\frac{1}{n} : n \in N\} = \{1, \frac{1}{2}, \frac{1}{3}, ...\}.$$
What is $\sup A$ and $\inf A$?

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds

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Note Information

- **ID:** `202501181241`
- **Timestamp:** `Saturday 18th January, 2025 13:54`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
A real number a_0 is a maximum of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$.

Explanation
The supremum can exist and not be a maximum, but when a maximum exists, then it is also the supremum.

Review

1. Define maximum.
2. Define minimum.
3. Consider the open interval
$$(0, 2) = \{x \in R : 0 < x < 2\},$$
and the closed interval
$$[0, 2] = \{x \in R : 0 \leq x \leq 2\}.$$
What are the maximums of the two sets? What are the supremums?

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum

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Note Information

- **ID:** `202501181257`
- **Timestamp:** `Saturday 18th January, 2025 13:54`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
The Axiom of Completeness is not a valid statement about Q.

Explanation
Consider the set

$$S = \{r \in Q : r^2 < 2\}.$$

This set is certainly bounded above, however, when we search for the least upper bound, we can always find a smaller supremum. For example, we might try $b = 2$, $b = 3/2$, $b = 142/100$, $b = 1415/1000$, and so on.

Review

1. Is the Axiom of Completeness a valid statement about Q? Explain.
2. Does the set

$$S = \{r \in Q : r^2 < 2\}$$

   have a supremum under R?

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum
- Maximum and Minimum

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Note Information

- **ID:** 202501181310
- **Timestamp:** Saturday 18th January, 2025 13:54
- **Tags:** Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
Let $A \subset R$ be nonempty and bounded above, and let $c \in R$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}.$$

Then $\sup(c + A) = c + \sup A$.

Explanation
Let $s = \sup A$. We see that $a \leq s$ for all $a \in A$, which implies $c + a \leq c + s$ for all $a \in A$. Thus $c + s$ is an upper bound for $c + A$ and condition (1) of Supremum and Infimum is verified. For (2), let b be an arbitrary upper bound for $c + A$, thus $c + a \leq b$ for all $a \in A$. This is equivalent to $a \leq b - c$ for all $a \in A$, from which we conclude that $b - c$ is an upper bound for A. Because s is the least upper bound of A, $s \leq b - c$, which can be rewritten as $c + s \leq b$. This verifies part (2) of Supremum and Infimum, and we conclude $\sup(c + A) = c + \sup A$.

Review

1. Prove $\sup(c + A) = c + \sup A$.

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds

- Supremum and Infimum
- Maximum and Minimum
- Q and the Axiom of Completeness

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Note Information

- **ID:** `202501181335`
- **Timestamp:** `Saturday 18th January, 2025 13:54`
- **Tags:** `Mathematics, Analysis-I-Abbott, The-Axiom-of-Completeness`
- **References:**
  - Abbott, S., Understanding Analysis

Main Content

Main Idea
Assume $s \in R$ is an upper bound for a set $A \subset R$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists and element $a \in A$ satisfying $s - \epsilon < a$.

Explanation
For the forward direction, assume $s = \sup A$ and consider $s - \epsilon$, where $\epsilon > 0$ has been arbitrarily chosen. Because $s - \epsilon < s$, part (2) of Supremum and Infimum implies that $s - \epsilon$ is not an upper bound for A. If this is the case, then there must be some element $a \in A$ for which $s - \epsilon < a$.
Conversely, assume s is an upper bound with the property that no matter how $\epsilon > 0$ is chosen, $s - \epsilon$ is no longer an upper bound for A. Notice that what this implies is that if b is any number less than s, then b is not an upper bound. To prove that $s = \sup A$, we must verify part (2) of Supremum and Infimum. Because we have just argued that any number smaller than s cannot be an upper bound, it follows that if b is some other upper bound for A, then $s \leq b$.

Review

1. What is an alternative phrasing for part (2) in Supremum and Infimum? Explain.

Links to Other Notes

- Initial Definition for R
- Axiom of Completeness
- Upper and Lower Bounds
- Supremum and Infimum

- Maximum and Minimum
- Q and the Axiom of Completeness
- sup(c + A) = c + sup A

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