

1. The-Real-and-Complex-Number-Systems
  - 1.1. Definition of Rational Numbers
  - 1.2. Rationals are Inadequate
  - 1.3. Order
  - 1.4. Ordered Set
  - 1.5. Upper Bounds and Lower Bounds
  - 1.6. Supremum and Infimum
  - 1.7. Least-Upper-Bound Property
  - 1.8. Relation between Supremum and Infimum
  - 1.9. Definition of Field
  - 1.10. Implications of the Addition Axioms

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- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

## Main Content

### Main Idea

Rational numbers are of the form  $\frac{m}{n}$ , where  $m$  and  $n$  are integers and  $n \neq 0$ .

### Explanation

## Review

1. Define the rational number system.

## Links to Other Notes

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  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

## Main Content

### Main Idea

The rational number system is inadequate for many purposes, both as a field and as an ordered set.

### Explanation

For example, there is no rational  $p$  such that  $p^2 = 2$ .

**Proof 0.1** *Suppose on the contrary there was a  $p$  that satisfied  $p^2 = 2$ . We could write  $p = \frac{m}{n}$ , where  $m$  and  $n$  are integers and coprime. The original expression implies*

$$\begin{aligned}\left(\frac{m}{n}\right)^2 &= 2 \\ \frac{m^2}{n^2} &= 2 \\ m^2 &= 2n^2\end{aligned}$$

*From this expression, we see that  $m^2$  is even, and thus,  $m$  is even. Plugging  $2k$  in for  $m$ , it is clear that  $m^2$  is divisible by 4. It follows that  $2n^2$  is divisible by 4 as well, which implies  $n^2$  is even, and thus,  $n$  is even. Therefore, our assumption leads to a contradiction that both  $m$  and  $n$  are even, thus violating the coprime property of  $m$  and  $n$ . Hence, it is impossible for  $p$  to be rational.*

**Proof 0.2 (Alternative)** *Let  $A$  be the set of all positive rationals  $p$  such that  $p^2 < 2$  and let  $B$  consist of all positive rationals  $p$  such that  $p^2 > 2$ . By showing there is no largest element in  $A$  and no smallest element in  $B$ , we effectively partition the set of rational numbers, thus implying there is no rational  $p$  that falls outside these two sets, therefore satisfying  $p^2 = 2$ .*

*To prove that for every  $p$  in  $A$  we can find a rational  $q$  in  $A$  such that  $p < q$ , we associate with each rational  $p > 0$  the number*

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \quad (1)$$

*Then*

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}. \quad (2)$$

*If  $p$  is in  $A$  then  $p^2 - 2 < 0$ , (1) shows that  $q > p$ , and (2) shows that  $q^2 < 2$ . Thus  $q$  is in  $A$ . If  $p$  is in  $B$  then  $p^2 - 2 > 0$ , (1) shows that  $0 < q < p$ , and (2) shows that  $q^2 > 2$ . Thus  $q$  is in  $B$ .*

## Review

1. Prove that there is no rational  $p$  such that  $p^2 = 2$  in two different ways.

## Links to Other Notes

- Definition of Rational Numbers

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  - Rudin W., Principles of Mathematical Analysis

## Main Content

### Main Idea

Let  $S$  be a set. An order on  $S$  is a relation, denoted by  $<$ , with the following two properties:

- If  $x \in S$  and  $y \in S$  then one and only one of the statements

$$x < y, \quad x = y, \quad y < x$$

is true.

- If  $x, y, z \in S$ , if  $x < y$  and  $y < z$ , then  $x < z$ .

### Explanation

## Review

1. Define order.

## Links to Other Notes

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- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

## Main Content

### Main Idea

An ordered set is a set  $S$  in which an order is defined.

### Explanation

For example,  $\mathbb{Q}$  is an ordered set if  $r < s$  is defined to mean that  $s - r$  is a positive rational number.

## Review

1. Define ordered set and give an example.

## Links to Other Notes

- Order

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  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

## Main Content

### Main Idea

Suppose  $S$  is an ordered set, and  $E \subset S$ . If there exists a  $\beta \in S$  such that  $x \leq \beta$  for every  $x \in E$ , we say that  $E$  is bounded above, and call  $\beta$  an upper bound of  $E$ . Lower bounds are defined in the same way.

### Explanation

## Review

1. Define upper bound.
2. Define lower bound.

## Links to Other Notes

- Order
- Ordered Set

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  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

## Main Content

### Main Idea

Suppose  $S$  is an ordered set,  $E \subset S$ , and  $E$  is bounded above. Suppose there exists an  $\alpha \in S$  with the following properties:

- $\alpha$  is an upper bound of  $E$ .
- If  $\gamma < \alpha$  then  $\gamma$  is not an upper bound of  $E$ .

Then  $\alpha$  is called the least upper bound of  $E$  or the supremum of  $E$ , and we write

$$\alpha = \sup E.$$

The greatest lower bound, or infimum, of a set  $E$  which is bounded below is defined in the same manner: If  $\alpha = \inf E$ , then  $\alpha$  is a lower bound of  $E$  and no  $\beta$  with  $\beta > \alpha$  is a lower bound of  $E$ .

### Explanation

For example, consider the sets  $A$  and  $B$  from the alternative proof in Rationals are Inadequate.  $A$  and  $B$  are subsets of the ordered set  $Q$ . The set  $A$  is bounded above by the members of  $B$ . Since  $B$  contains no smallest member,  $A$  has no supremum in  $Q$ .  $B$  is bounded below by the members of  $A$ . Since  $A$  has no largest member,  $B$  has no infimum in  $Q$ .

## Review

1. Define supremum.
2. Define infimum.



3. If  $\alpha = \sup E$  exists, then must  $\alpha$  be a member of  $E$ ? Give an example to justify your answer.
4. Let  $E$  consist of all numbers  $1/n$ , where  $n = 1, 2, 3, \dots$ . What is  $\sup E$  and  $\inf E$ ? Are  $\sup E$  and  $\inf E$  members of  $E$ ?

## Links to Other Notes

- Definition of Rational Numbers
- Rationals are Inadequate
- Order
- Ordered Set
- Upper Bounds and Lower Bounds

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## Note Information

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  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

## Main Content

### Main Idea

An ordered set  $S$  is said to have the least-upper-bound property if the following is true: If  $E \subset S$ ,  $E$  is not empty, and  $E$  is bounded above, then  $\sup E$  exists in  $S$ .

### Explanation

## Review

1. Define the least-upper-bound property.
2. Define the greatest-lower bound property.
3. Does  $\mathbb{Q}$  have the least-upper-bound property? Explain.

## Links to Other Notes

- Definition of Rational Numbers
- Rationals are Inadequate
- Order
- Ordered Set
- Upper Bounds and Lower Bounds
- Supremum and Infimum

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  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

## Main Content

### Main Idea

If an ordered set has the least-upper-bound property, then it also has the greatest-lower-bound property.

### Explanation

**Theorem 0.3** *Suppose  $S$  is an ordered set with the least-upper-bound property,  $B \subset S$ ,  $B$  is not empty, and  $B$  is bounded below. Let  $L$  be the set of all lower bounds of  $B$ . Then*

$$\alpha = \sup L$$

*exists in  $S$ , and  $\alpha = \inf B$ . In particular,  $\inf B$  exists in  $S$ .*

**Proof 0.4** *Since  $B$  is bounded below,  $L$  is not empty. Since  $L$  consists of exactly those  $y \in S$  which satisfy the inequality  $y \leq x$  for every  $x \in B$ , we see that every  $x \in B$  is an upper bound of  $L$ . Thus  $L$  is bounded above. Our hypothesis about  $S$  implies therefore that  $L$  has a supremum in  $S$ , namely  $\alpha$ . If  $\gamma < \alpha$  then  $\gamma$  is not an upper bound of  $L$ , hence  $\gamma \notin B$ . It follows that  $\alpha \leq x$  for every  $x \in B$ . Thus  $\alpha \in L$ . If  $\alpha < \beta$  then  $\beta \notin L$ , since  $\alpha$  is an upper bound of  $L$ . We have shown that  $\alpha \in L$  but  $\beta \notin L$  if  $\beta > \alpha$ . In other words  $\alpha$  is a lower bound of  $B$ , but  $\beta$  is not if  $\beta > \alpha$ . This means that  $\alpha = \inf B$ .*

## Review

1. Prove that if an ordered set has the least-upper-bound property, then it also has the greatest-lower-bound property.

## Links to Other Notes

- Definition of Rational Numbers
- Rationals are Inadequate
- Order
- Ordered Set
- Upper Bounds and Lower Bounds
- Supremum and Infimum
- Least-Upper-Bound Property

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  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

## Main Content

### Main Idea

A field is a set  $F$  with two operations, called addition and multiplication, which satisfy the following field axioms (A), (M), and (D):

(A) Axioms for addition

- (A1) If  $x \in F$  and  $y \in F$ , then their sum  $x + y$  is in  $F$ .
- (A2) Addition is commutative:  $x + y = y + x$  for all  $x, y \in F$ .
- (A3) Addition is associative:  $(x + y) + z = x + (y + z)$  for all  $x, y, z \in F$ .
- (A4)  $F$  contains an element  $0$  such that  $0 + x = x$  for every  $x \in F$ .
- (A5) To every  $x \in F$  corresponds an element  $-x \in F$  such that  $x + (-x) = 0$ .

(M) Axioms for multiplication

- (M1) If  $x \in F$  and  $y \in F$ , then their product  $xy$  is in  $F$ .
- (M2) Multiplication is commutative:  $xy = yx$  for all  $x, y \in F$ .
- (M3) Multiplication is associative:  $(xy)z = x(yz)$  for all  $x, y, z \in F$ .
- (M4)  $F$  contains an element  $1 \neq 0$  such that  $1x = x$  for every  $x \in F$ .
- (M5) If  $x \in F$  and  $x \neq 0$  then there exists an element  $1/x \in F$  such that  $x \cdot (1/x) = 1$ .

(D) The distributive law

$$x(y + z) = xy + xz \text{ holds for all } x, y, z \in F$$

### Explanation

## Review

1. Define field.
2. Is  $\mathbb{Q}$  a field?

## Links to Other Notes

- Definition of Rational Numbers
- Rationals are Inadequate

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  - Rudin W., Principles of Mathematical Analysis

## Main Content

### Main Idea

The axioms for addition imply the following statements.

- (a) If  $x + y = x + z$  then  $y = z$ .
- (b) If  $x + y = x$  then  $y = 0$ .
- (c) If  $x + y = 0$  then  $y = -x$ .
- (d)  $-(-x) = x$ .

### Explanation

**Proof 0.5** If  $x + y = x + z$ , the axioms (A) give

$$\begin{aligned} y &= 0 + y = (-x + x) + y = -x + (x + y) \\ &= -x + (x + z) = (-x + x) + z = 0 + z = z. \end{aligned}$$

*This proves (a). Take  $z = 0$  in (a) to obtain (b). Take  $z = -x$  in (a) to obtain (c). Since  $-x + x = 0$ , (c) with  $-x$  in place of  $x$  gives (d).*

## Review

1. What do the axioms for addition imply? Explain.

## Links to Other Notes

- Definition of Field



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