1. The-Real-and-Complex-Number-Systems
   1.1. Definition of Rational Numbers
   1.2. Rationals are Inadequate

# Note Information

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- **References:**
  - Analysis I
  - Rudin W., Principles of Mathematical Analysis

# Main Content

**Main Idea**
Rational numbers are of the form $\frac{m}{n}$, where $m$ and $n$ are integers and $n \neq 0$.

**Explanation**

# Review

1. Define the rational number system.

# Links to Other Notes

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# Main Content

### Main Idea

The rational number system is inadequate for many purposes, both as a field and as an ordered set.

### Explanation

For example, there is no rational $p$ such that $p^2 = 2$.

**Proof 0.1** *Suppose on the contrary there was a $p$ that satisfied $p^2 = 2$. We could write $p = \frac{m}{n}$, where $m$ and $n$ are integers and coprime. The original expression implies*

$$\begin{aligned}(\frac{m}{n})^2 &= 2\\ \frac{m^2}{n^2} &= 2\\ m^2 &= 2n^2\end{aligned}$$

*From this expression, we see that $m^2$ is even, and thus, $m$ is even. Plugging $2k$ in for $m$, it is clear that $m^2$ is divisible by 4. It follows that $2n^2$ is divisible by 4 as well, which implies $n^2$ is even, and thus, $n$ is even. Therefore, our assumption leads to a contradiction that both $m$ and $n$ are even, thus violating the coprime property of $m$ and $n$. Hence, it is impossible for $p$ to be rational.*

**Proof 0.2 (Alternative)** *Let $A$ be the set of all positive rationals $p$ such that $p^2 < 2$ and let $B$ consist of all positive rationals $p$ such that $p^2 > 2$. By showing there is no largest element in $A$ and no smallest element in $B$, we effectively partion the set of rational numbers, thus implying there is no rational $p$ that falls outside these two sets, therefore satisfying $p^2 = 2$.*

*To prove that for every $p$ in $A$ we can find a rational $q$ in $A$ such that $p < q$, we associate with each rational $p > 0$ the number*

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}. \tag{1}$$

*Then*

$$q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}. \tag{2}$$

*If $p$ is in $A$ then $p^2 - 2 < 0$, (1) shows that $q > p$, and (2) shows that $q^2 < 2$. Thus $q$ is in $A$. If $p$ is in $B$ then $p^2 - 2 > 0$, (1) shows that $0 < q < p$, and (2) shows that $q^2 > 2$. Thus $q$ is in $B$.*

## Review

1. Prove that there is no rational $p$ such that $p^2 = 2$ in two different ways.

## Links to Other Notes

- Definition of Rational Numbers

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