(1) $E(x,t) = -(t\log(\hat{t}) + (1-t)\log(1-\hat{t}))$.
Find gradients.

i) $\nabla_\theta E(x,t)$: $\frac{\partial E}{\partial \theta_i} = -\left(\frac{t}{\hat{t}} - \frac{1-t}{1-\hat{t}}\right)\frac{d\hat{t}}{d\theta_i}$

$$\frac{d\hat{t}}{d\theta_i} = \frac{h_i^{(2)} e^{-(b_4+\theta^T h^{(2)})}}{(1+e^{-(b_4+\theta^T h^{(2)})})^2} = -h_i^{(2)}(1-\hat{t})\hat{t}$$

So $\nabla_\theta E(x,t) = -\left(\frac{t}{\hat{t}} - \frac{1-t}{1-\hat{t}}\right)(1-\hat{t})\hat{t}\,h^{(2)}$

$$= -(t(1-\hat{t}) - \hat{t}(1-t))\,h^{(2)}$$

$$= (\hat{t}-t)\,h^{(2)}$$

and $\frac{\partial E}{\partial b_4} = -\left(\frac{t}{\hat{t}} - \frac{1-t}{1-\hat{t}}\right)\frac{\partial \hat{t}}{\partial b_4}$

$$= -\left(\frac{t}{\hat{t}} - \frac{1-t}{1-\hat{t}}\right)\frac{-e^{-(b_4+\theta^T h^{(2)})}}{(1+e^{-(b_4+\theta^T h^{(2)})})^2}$$

$$= -\left(\frac{t}{\hat{t}} - \frac{1-t}{1-\hat{t}}\right)(1-\hat{t})\hat{t}$$

$$= -(t(1-\hat{t}) - \hat{t}(1-t)) = (\hat{t}-t).$$

ii) Find

$\nabla_{w_1} E$, $\nabla_{w_2} E$, $\nabla_{b_1} E$, and $\nabla_{b_2} E$.

$$\frac{dE}{dh_i^{(2)}} = -\left(\frac{t}{\hat{t}} - \frac{1-t}{1-\hat{t}}\right)(1-\hat{t})\hat{t}\,\theta_i = (\hat{t}-t)\theta_i$$

$$\frac{dh_i^{(2)}}{db_i^{(2)}} = 1-\tanh^2(b_2 + W^{(2)}h^{(1)}) = 1-(h_i^{(2)})^2, \text{ so}$$

$$\frac{dE}{db_i^{(1)}} = (\hat{t}-t)(1-(h_i^{(1)})^2)\theta_i$$

and

$$\frac{dE}{dw_i^{(2)}} = (\hat{t}-t)(1-(h_i^{(2)})^2)\theta_i h_i^{(1)} .$$

while $\displaystyle \frac{dE}{dh_i^{(1)}} = (\hat{t}-t)(1-(h_i^{(2)})^2)W_i^{(2)}\theta_i$

and $\displaystyle \frac{dh_i^{(1)}}{db_i^{(1)}} = (1-(h_i^{(1)})^2)$ while $\displaystyle \frac{dh_i}{dW_i^{(1)}} = X_i(1-(h_i^{(1)})^2)$

so $\displaystyle \frac{dE}{db_i^{(1)}} = (\hat{t}-t)(1-(h_i^{(1)})^2)(1-(h_i^{(2)})^2)W_i^{(2)}\theta_i$

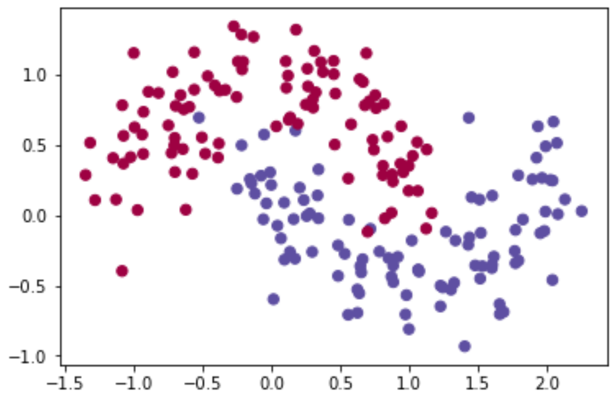
and $\displaystyle \frac{dE}{dw_i^{(1)}} = (\hat{t}-t)(1-(h_i^{(1)})^2)(1-(h_i^{(2)})^2)W_i^{(2)}\theta_i X_i$

Implement a 2-layer Neural Network using Backpropagation

In [8]:
```
# Import packages
import numpy as np
import sklearn
from sklearn import datasets
import matplotlib.pyplot as plt
%matplotlib inline
```

In [9]:
```
# Generate a toy dataset and plot it
np.random.seed(0)
X, y = sklearn.datasets.make_moons(200, noise=0.20)
plt.scatter(X[:,0], X[:,1], s=40, c=y, cmap=plt.cm.Spectral)
```

Out[9]: <matplotlib.collections.PathCollection at 0x7fc2cc028438>



In [10]:
```
# Define helper functions

def predict(W, X):

    # Forward pass
    z_h1 = np.dot(X, W['W_input_h1']) + W['b_input_h1']
    a_h1 = np.tanh(z_h1)

    z_h2 = np.dot(a_h1, W['W_h1_h2']) + W['b_h1_h2']
    a_h2 = np.tanh(z_h2)
    output = np.dot(a_h2, W['W_h2_output']) + W['b_h2_output']

    probs = sigmoid(output)
    probs = probs.squeeze()

    predictions = np.zeros(probs.shape[0])
    predictions[probs>0.5] = 1

    return predictions

def cross_entropy_loss(t_hat, t):
    return np.sum(-(t*np.log(t_hat) + (1-t)*np.log(1-t_hat)))

def sigmoid(x):
    return 1/(1 + np.exp(-x))

def sigmoid_grad(x):
    return sigmoid(x)*(1-sigmoid(x))

def tanh_grad(x):
    return 1 - np.tanh(x)**2

def plot_decision_boundary(W):
    # Set min and max values and give it some padding
    x_min, x_max = X[:, 0].min() - .5, X[:, 0].max() + .5
    y_min, y_max = X[:, 1].min() - .5, X[:, 1].max() + .5
    h = 0.01
    # Generate a grid of points with distance h between them
    xx, yy = np.meshgrid(np.arange(x_min, x_max, h), np.arange(y_min, y_max
, h))
    # Predict the function value for the whole gid
    Z = predict(W, np.c_[xx.ravel(), yy.ravel()])
    Z = Z.reshape(xx.shape)
    # Plot the contour and training examples
    plt.contourf(xx, yy, Z, cmap=plt.cm.Spectral)
    plt.scatter(X[:, 0], X[:, 1], c=y, cmap=plt.cm.Spectral)
```

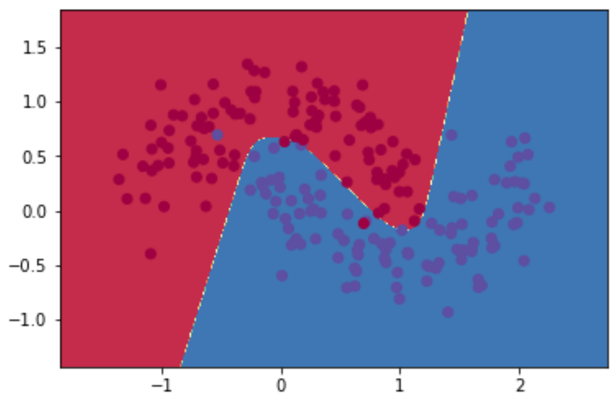
In [11]:
```
# Create the model architecture
input_dim = 2
nodes_hidden_1 = 3
nodes_hidden_2 = 3
output_dim = 1

W = {
    'W_input_h1' : np.random.rand(input_dim, nodes_hidden_1),
    'b_input_h1' : np.random.rand(nodes_hidden_1),

    'W_h1_h2' : np.random.rand(nodes_hidden_1, nodes_hidden_2),
    'b_h1_h2' : np.random.rand(nodes_hidden_2),

    'W_h2_output' : np.random.rand(nodes_hidden_2, output_dim),
    'b_h2_output' : np.random.rand(output_dim)
}
```

In [12]:
```
# Train using Gradient Descent

learning_rate = 0.01
no_epochs = 2000

for epoch in range(no_epochs):

    # Forward pass
    z_h1 = np.dot(X, W['W_input_h1']) + W['b_input_h1']
    #print(z_h1[1,1])
    a_h1 = np.tanh(z_h1)
    #print( W['W_input_h1'])

    z_h2 = np.dot(a_h1, W['W_h1_h2']) + W['b_h1_h2']
    #print(z_h2.shape)
    a_h2 = np.tanh(z_h2)
    #print(a_h2.shape)
    output = np.dot(a_h2, W['W_h2_output']) + W['b_h2_output']
    #print(W['W_input_h1'].shape)

    probs = sigmoid(output)
    probs = probs.squeeze()

    #~~~~~~~~~~~ YOUR CODE BEGINS HERE ~~~~~~~~~~~~~~~~

    delta3 = (probs - y).reshape(200,1)
    delta2 = delta3 @ W['W_h2_output'].T * (1 - np.power(a_h2, 2))
    delta1 = delta2 @ W['W_h1_h2'].T * (1 - np.power(a_h1, 2))

    dWout = a_h2.T @ delta3
    dbout = np.sum(delta3, axis=0, keepdims=True).reshape(output_dim,)
    dW2 = a_h1.T @ delta2
    db2 = np.sum(delta2, axis=0, keepdims=True).reshape(nodes_hidden_2,)
    dW1 = X.T @ delta1
    db1 = np.sum(delta1, axis=0, keepdims=True).reshape(nodes_hidden_1,)

    # Gradient Descent
    W['W_input_h1'] += -1 * learning_rate * dW1
    W['b_input_h1'] += -1 * learning_rate * db1
    W['W_h1_h2'] += -1 * learning_rate * dW2
    W['b_h1_h2'] += -1 * learning_rate * db2
    W['W_h2_output'] += -1 * learning_rate * dWout
    W['b_h2_output'] += -1 * learning_rate * dbout

    #~~~~~~~~~~~ YOUR CODE ENDS HERE ~~~~~~~~~~~~~~~~~~

    loss = cross_entropy_loss(probs, y)

    if epoch%100 == 0:
        print("Loss after epoch ", epoch, " : ", loss)
```

```
Loss after epoch  0  :  157.625124238
Loss after epoch  100  :  58.0230836875
Loss after epoch  200  :  47.8963790291
Loss after epoch  300  :  14.3334810612
Loss after epoch  400  :  13.2279477282
Loss after epoch  500  :  12.717457716
Loss after epoch  600  :  18.5371437118
Loss after epoch  700  :  12.141548192
Loss after epoch  800  :  11.7234249244
Loss after epoch  900  :  11.5023376869
Loss after epoch  1000  :  11.7606812565
Loss after epoch  1100  :  16.3219752991
Loss after epoch  1200  :  11.2307388833
Loss after epoch  1300  :  10.947078349
Loss after epoch  1400  :  11.0909588534
Loss after epoch  1500  :  14.587772505
Loss after epoch  1600  :  10.7264706409
Loss after epoch  1700  :  10.568560292
Loss after epoch  1800  :  12.7525283002
Loss after epoch  1900  :  10.6848849173
```

In [13]:

```
# Plot the decision boundary
plot_decision_boundary(W)
```



In [ ]: