Circle Inversion for Hyperbolic Geometry

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For the circle Ω with center $z_0 \in \mathbb{C}$ and radius $\rho \in \mathbb{R}$, consider some arbitrary $z_{arb} \in \mathbb{C}$ with $z_{arb} \neq z_0$. Construct the ray from z_0 through z_{arb}. This ray is directed along the unit vector $\hat{\mathbf{z}} = \frac{z_{arb} - z_0}{\|z_{arb} - z_0\|}$.

Let $z_{inv} \in \mathbb{C}$ denote the inverse of z_{arb} with respect to the circle Ω. The definition of Inversion in a Circle implies that the inverse point lies on the ray, i.e.

$$z_{inv} = t * \hat{\mathbf{z}} \tag{1}$$

for some $t \in \mathbb{R}$, and that the relevant norms are related by

$$\|z_{inv}\| * \|z_{arb}\| = \rho * \rho \tag{2}$$

The special case of inversion in the unit circle can be used to simplify derivation of inversion in an arbitrary circle. In particular, the unit circle has $z_0 = 0$ and $\rho = 1$. The unit vector becomes $\hat{\mathbf{z}} = \frac{z_{arb}}{\|z_{arb}\|}$, with corresponding simplification of equations 1 and 2, viz.

$$z_{inv} = t * \frac{z_{arb}}{\|z_{arb}\|} \tag{3}$$

$$\|z_{inv}\| * \|z_{arb}\| = 1 \tag{4}$$

The complex conjugate of the inverse point follows directly from equation 3:

$$\overline{z_{inv}} = t * \frac{\overline{z_{arb}}}{\|z_{arb}\|} \tag{5}$$

Compute the product of z_{inv} (from 3) with its complex conjugate (as in 5):

$$z_{inv} * \overline{z_{inv}} = t * t * \frac{z_{arb} * \overline{z_{arb}}}{\|z_{arb}\| * \|z_{arb}\|} \tag{6}$$

The history of complex analysis is littered with names and notations bestowed upon the quantity $\|w\|^2$, including absolute square, $w * \overline{w}$, squared norm, dot(w, w) (an idiom in computer graphics shader languages) and $\langle w \mid w \rangle$. In terms of this language, the next step is to replace the absolute square in the denominator of equation 6 by its equivalent expression in complex conjugates to get

$$z_{inv} * \overline{z_{inv}} = t * t * \frac{z_{arb} * \overline{z_{arb}}}{z_{arb} * \overline{z_{arb}}} \tag{7}$$

which immediately reduces to

$$\|z_{inv}\|^2 = t^2 \tag{8}$$

Ignoring the negative solution, the scalar t obeys

$$t = \|z_{inv}\| \tag{9}$$

Exploit the reciprocal relationship between $\|z_{inv}\|$ and $\|z_{arb}\|$ expressed in equation (4):

$$t = \frac{1}{\|z_{arb}\|} \tag{10}$$

This explicit solution for parameter t can be used in equation 3 to give

$$z_{inv} = \frac{1}{\|z_{arb}\|} * \frac{z_{arb}}{\|z_{arb}\|} \tag{11}$$

Once again, replace the absolute square by its equivalent expression in complex conjugates:

$$z_{inv} = \frac{z_{arb}}{z_{arb} * \overline{z_{arb}}} \tag{12}$$

whence

$$z_{inv} = \frac{1}{\overline{\boldsymbol{z_{arb}}}} \tag{13}$$

Now the case of inversion in a general circle can be treated. It is one of a large class of problems in the complex plane which can be greatly simplified by the technique of “shifting the problem to the origin”.

Consider three distinct spaces of complex coordinates: the complex-z plane, the complex-u plane and the complex-w plane.

In the complex-z plane, the circle of inversion Ω_z has arbitrary center z_0 and radius ρ.

In the complex-u plane, the circle of inversion Ω_u is centered at the origin of u space u_0 but still has radius ρ. Let T_u denote the linear transformation taking the circle Ω_z to the circle Ω_u . T_u is just a translation for which

$$u = T_u(z) = z + b \tag{14}$$

In order to satisfy the constraint that Ω_u is centered at the origin u_0 it must be true that

$$u_0 = T_u(z_0) = z_0 + b = 0 \tag{15}$$

which gives

$$b = -z_0 \tag{16}$$

In the complex-w plane, the circle of inversion Ω_w is centered at the origin of w space w_0 and has unit radius. Let T_w denote the linear transformation taking the circle Ω_u to the circle Ω_w. T_w is a dilation for which

$$w = T_w(u) = a * u \tag{17}$$

In order to satisfy the constraint that Ω_w has unit radius it must be true that

$$w = T_w(\rho) = a * \rho = 1 \tag{18}$$

which gives

$$a = \frac{1}{\rho} \tag{19}$$

The composition of transforms $T_w * T_u$ is the forward transformation from z-space to w-space:

$$w = T_w(u) = \frac{u}{\rho} = \frac{T_z(z)}{\rho} = \frac{z - z_0}{\rho} \tag{20}$$

Since the circle of inversion in w-space is the unit circle, the equation 13 can be applied to get the inverse point in w-space:

$$w_{inv} = \frac{1}{\overline{w_{arb}}} \tag{21}$$

To get this formula in z terms, apply the inverse of the composition of transforms (equation 20):

$$\frac{z_{inv} - z_0}{\rho} = \frac{1}{\overline{\frac{z_{arb} - z_0}{\rho}}} \tag{22}$$

$$\frac{z_{inv} - z_0}{\rho} = \frac{\rho}{\overline{z_{arb} - z_0}} \tag{23}$$

$$z_{inv} = z_0 + \frac{\rho * \rho}{\overline{\boldsymbol{(z_{arb} - z_0)}}} \tag{24}$$

References

[1] Michael P. Hitchman, *Geometry with an Introduction to Cosmic Topology*, CreateSpace Publishing Platform, 2018.