Planar Graphs

Planar Graphs

Definition:

A graph that can be drawn in the plane without any of its edges intersecting is called a planar graph. A graph that is so drawn in the plane is also said to be embedded (or imbedded) in the plane.

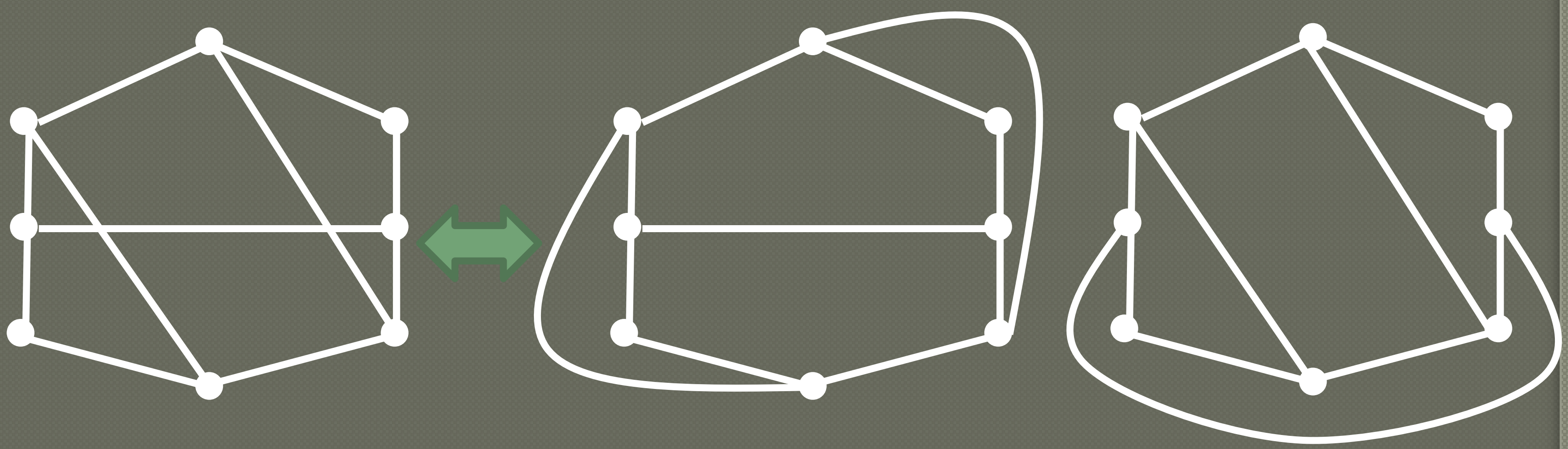
Applications:

(1) circuit layout problems

(2) Three house and three utilities problem

Definition: A plane graph can be defined as a planar graph with a mapping from every node to a point on a plane, and from every edge to a plane curve on that plane, such that the extreme points of each curve are the points mapped from its end nodes, and all curves are disjoint except on their extreme points.

Fig 1

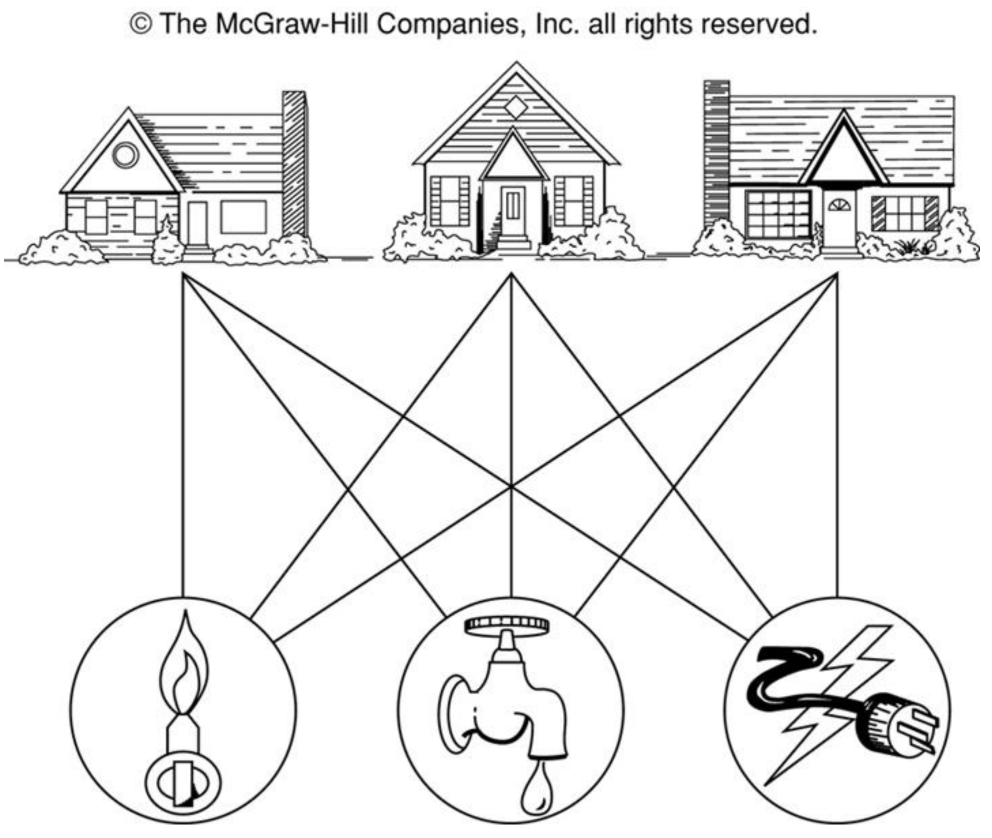


(a) planar, not a plane graph

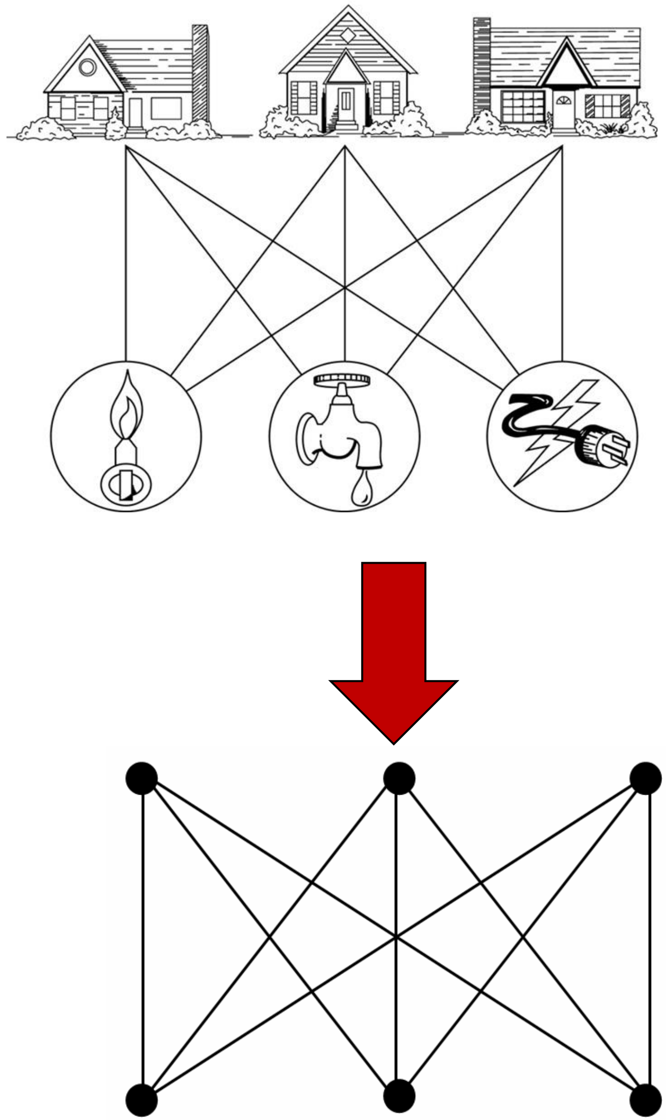
(b) a plane graph

(c) another plane graph

The House-and-Utilities Problem



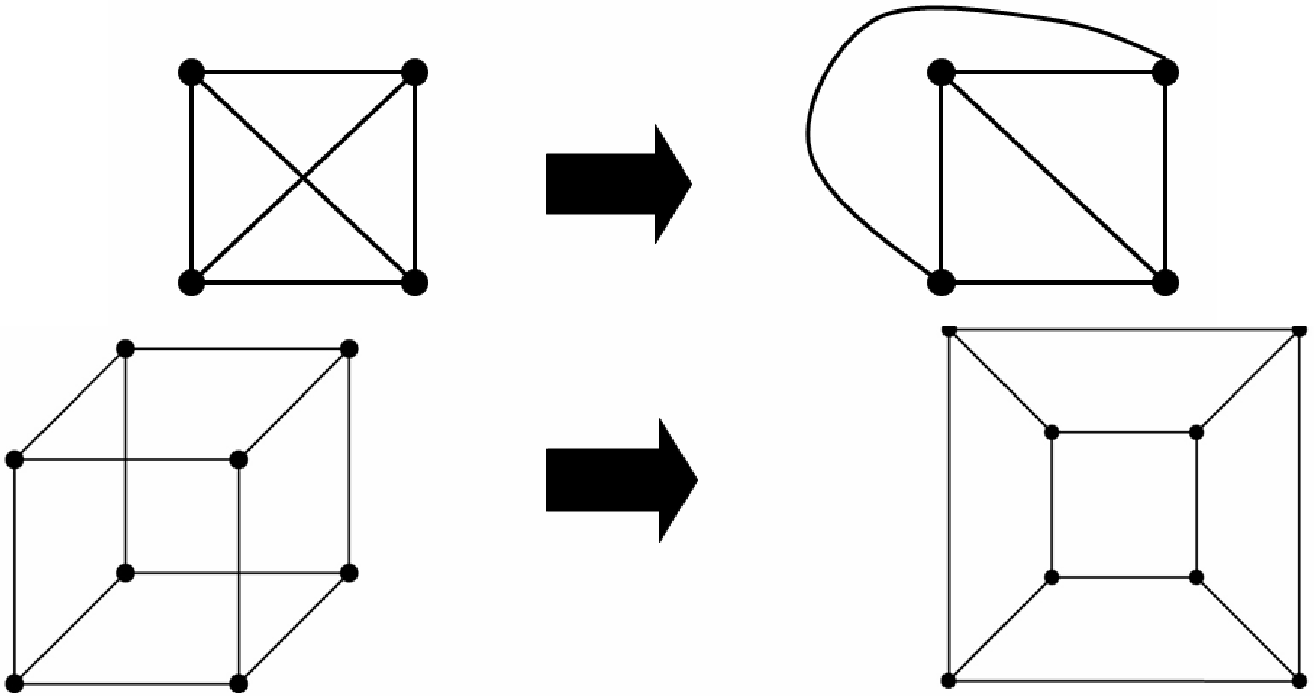
Is it possible to join the three houses to the three utilities in such a way that none of the connections cross?



$K_{3,3}$

If complete bipartite graph $K_{3,3}$ is planar?

Planar Graph example



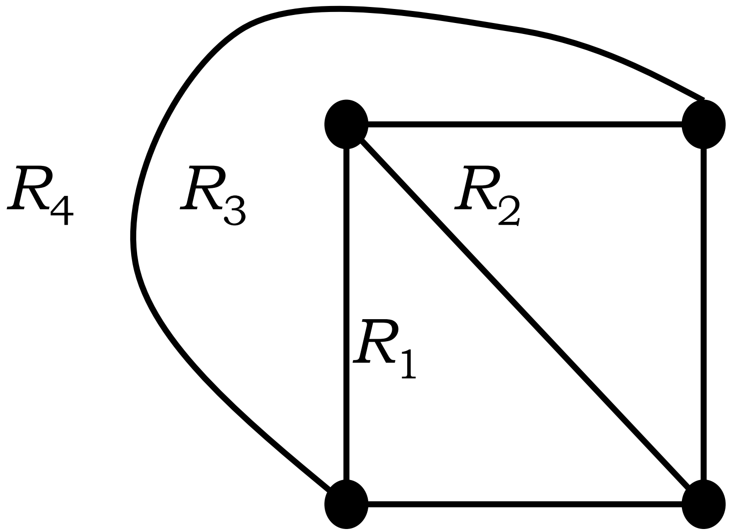
- not all graphs are planar
- It may be difficult to show that a graph is non-planar

→show that there is *no way* to draw the graph without any edges crossing

How?

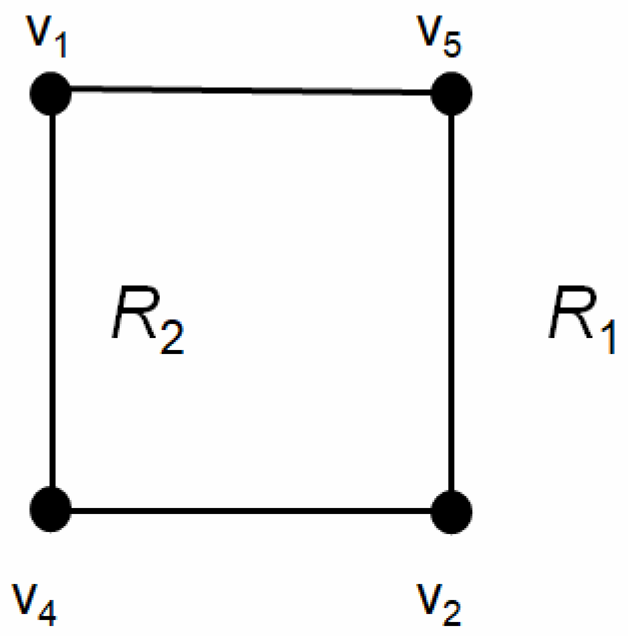
Regions

- Euler showed that <u>all planar representations of a graph</u> split the plane into the <u>same number of *regions*</u>, including an unbounded region

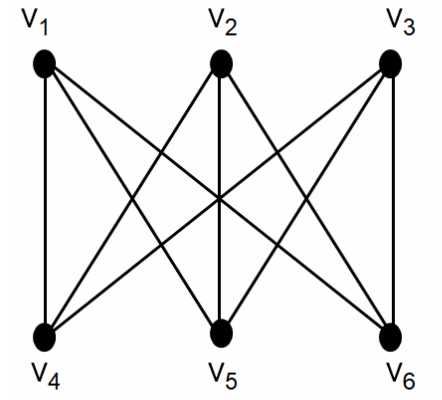


Regions

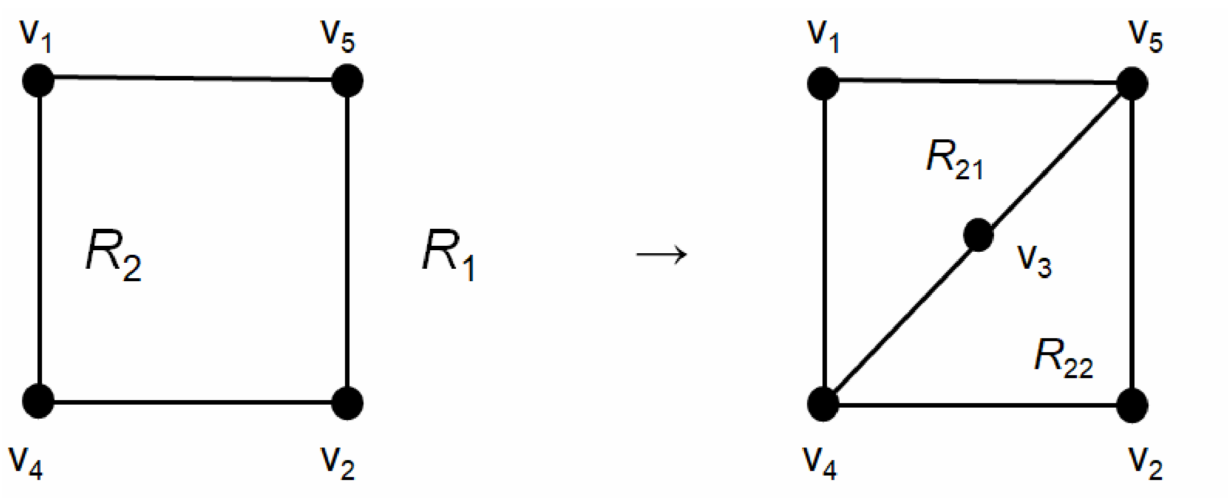
- The four edges $\{v_1, v_4\}$, $\{v_4, v_2\}$, $\{v_2, v_5\}$, $\{v_5, v_1\}$ form a closed curve that splits the plane into two regions, R_1 and R_2



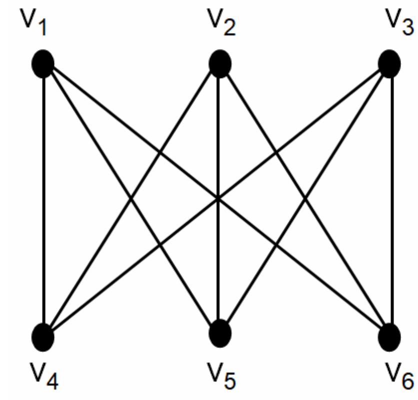
Regions



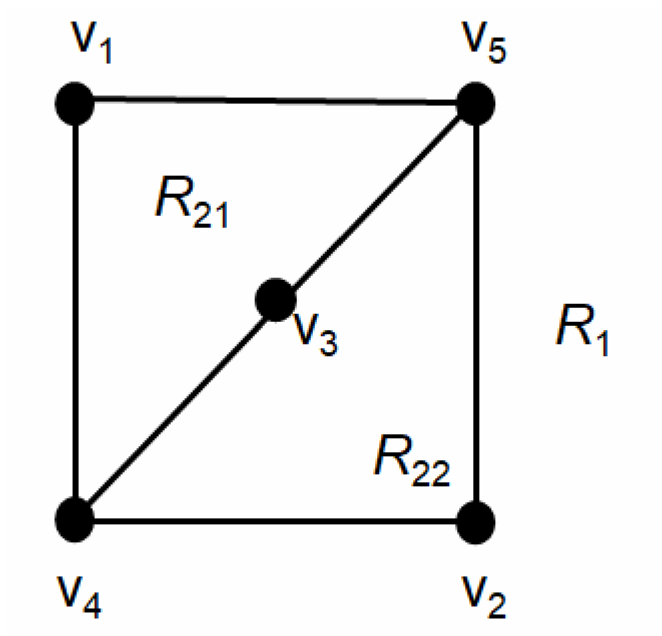
- Note that v_3 must be in either R_1 or R_2
- Assume v_3 is in R_2. Then the edges $\{v_3, v_4\}$ and $\{v_3, v_5\}$ separate R_2 into two subregions → R_{21} and R_{22}



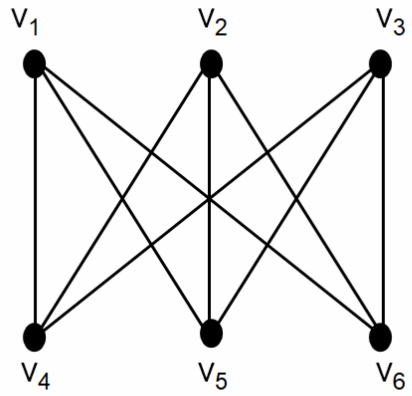
Regions



- Now there is no way to place vertex v_6 without forcing a crossing:
  - If v_6 is in R_1 then $\{v_6, v_3\}$ must cross an edge
  - If v_6 is in R_{21} then $\{v_6, v_2\}$ must cross an edge
  - If v_6 is in R_{22} then $\{v_6, v_1\}$ must cross an edge

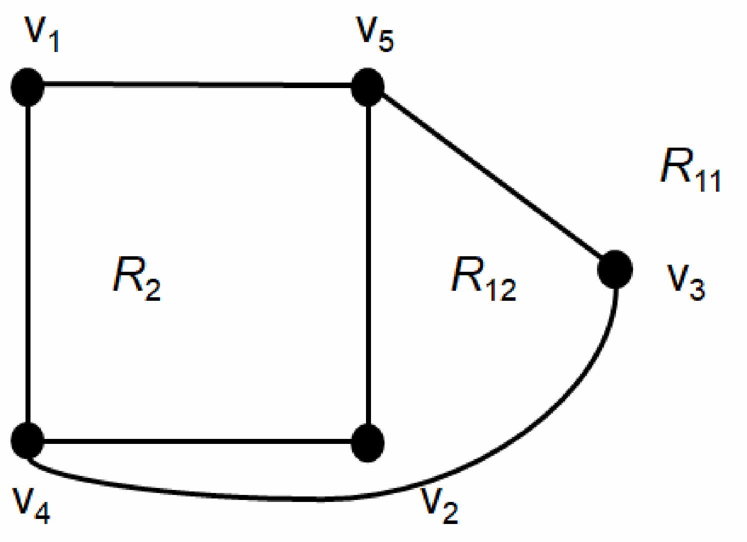


Regions

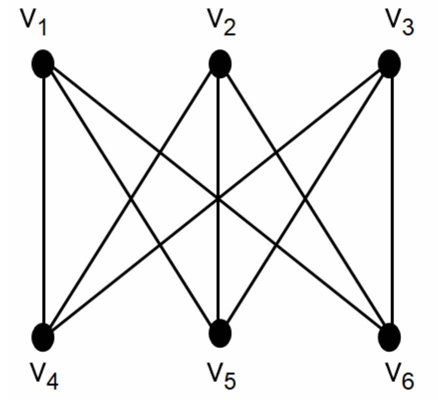


- Alternatively, assume v_3 is in $R_{1;}$

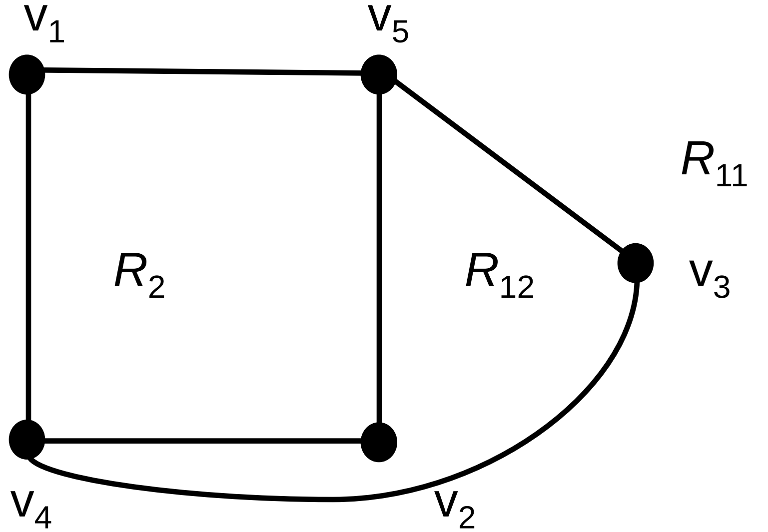
- Then the edges $\{v_3, v_4\}$ and $\{v_3, v_5\}$ separate R_1 into two subregions, R_{11} and R_{12}



Regions

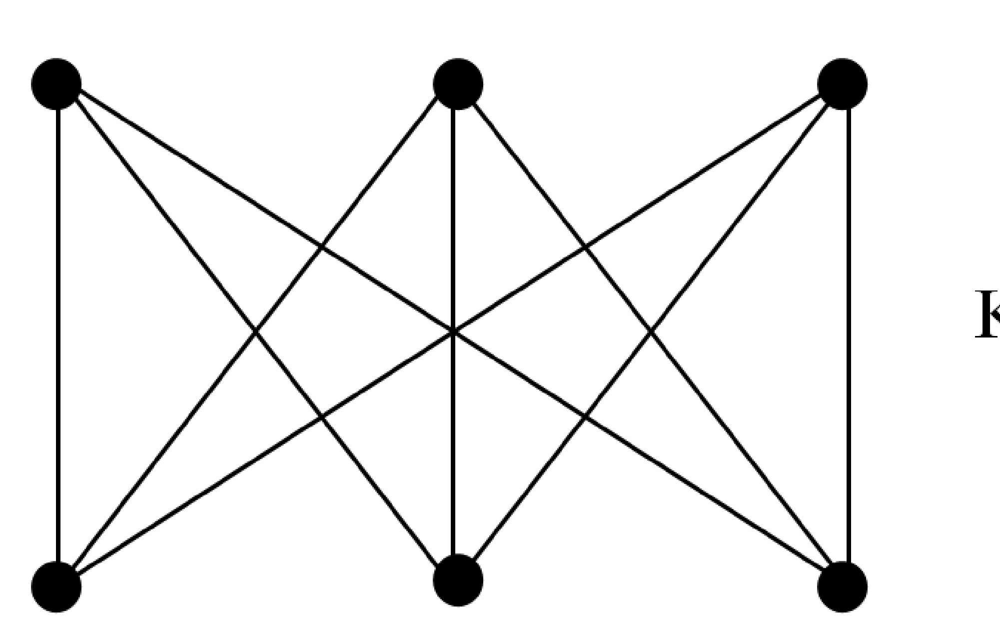


- Now there is no way to place vertex v_6 without forcing a crossing:
  - If v_6 is in R_2 then $\{v_6, v_3\}$ must cross an edge
  - If v_6 is in R_{11} then $\{v_6, v_2\}$ must cross an edge
  - If v_6 is in R_{12} then $\{v_6, v_1\}$ must cross an edge



Planar Graphs

- Consequently, the graph $K_{3,3}$ must be nonplanar



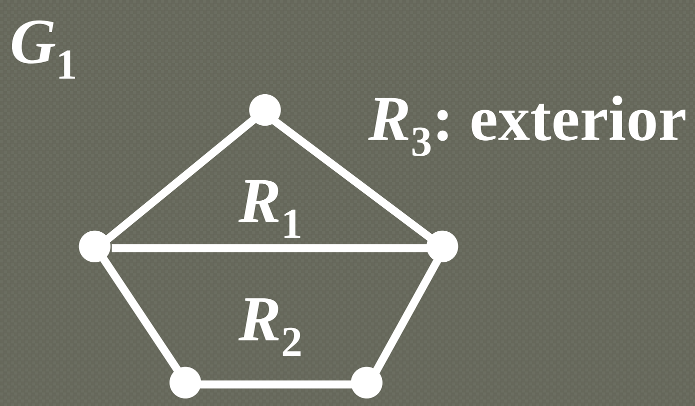
$K_{3,3}$

Note. A given planar graph can give rise to several different plane graphs.

A region is defined to be an area of the plane that is bounded by edges and cannot be further subdivided.

A planar graph divides the plane into one or more regions. One of these regions will be infinite.

Fig 2

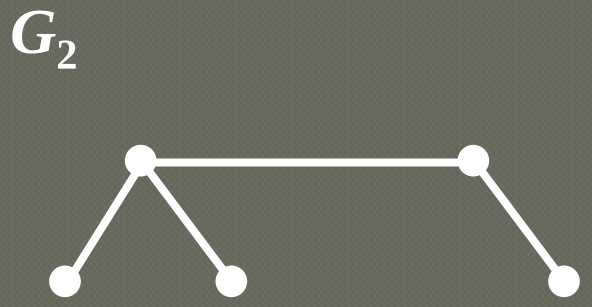


G_1 has 3 regions

Definition:

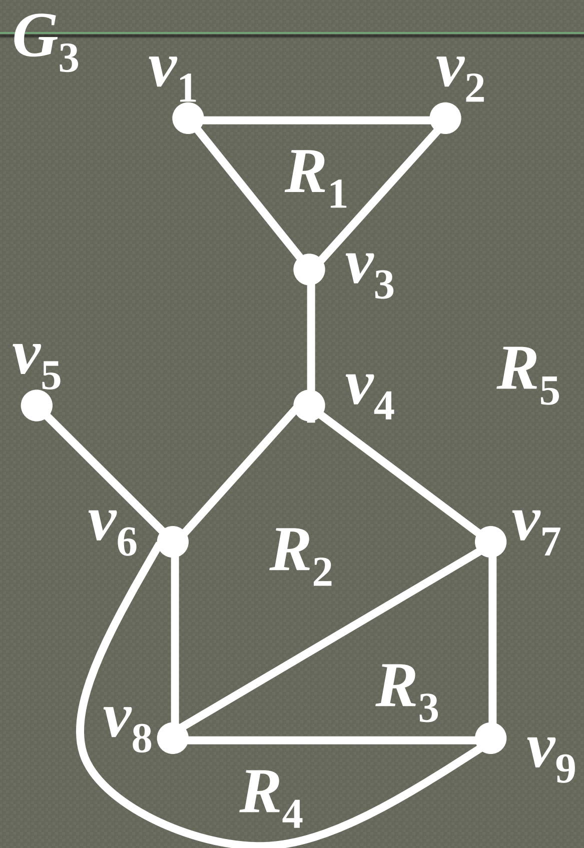
Every plane graph has exactly one unbounded region, called the exterior region. The vertices and edges of G that are incident with a region R form a subgraph of G called the boundary of R.

Fig 2



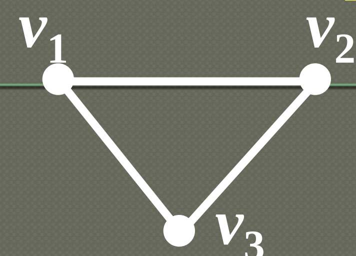
G_2 has only 1 region

Fig 2

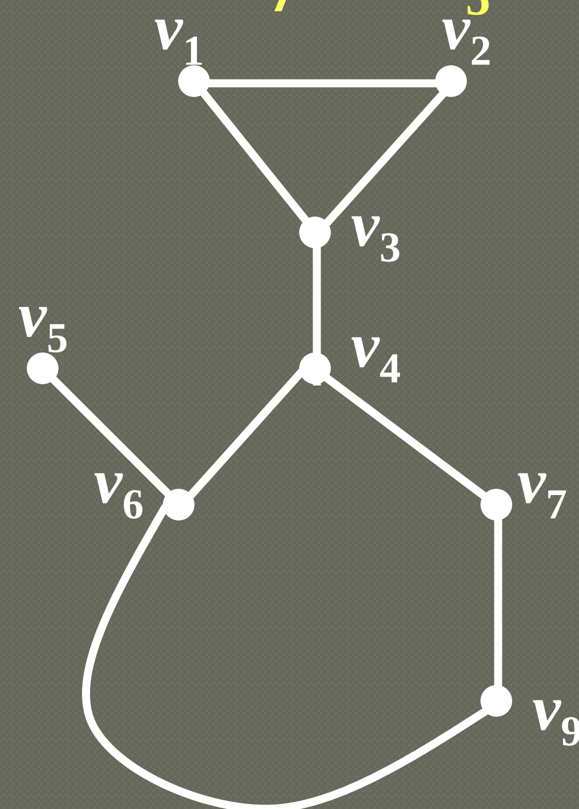


G_3 has 5 regions.

Boundary of R_1:



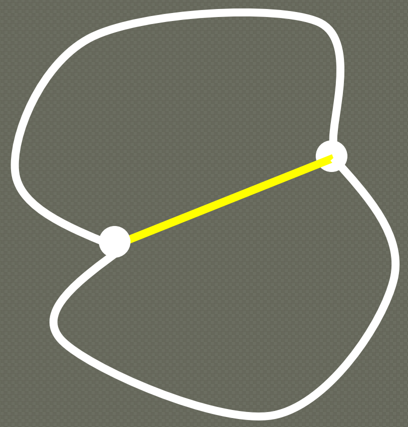
Boundary of R_5:

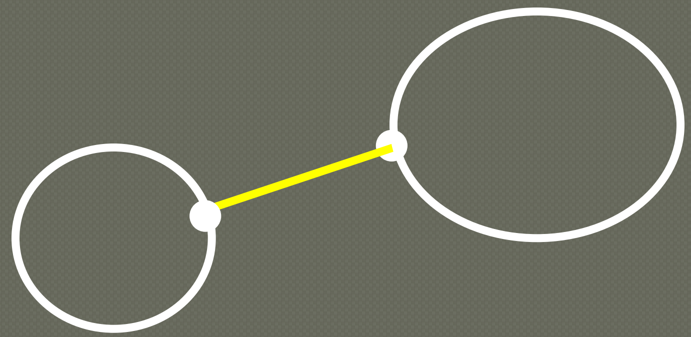


Observe:

(1) Each cycle edge belongs to the boundary of two regions.

(2) Each bridge is on the boundary of only one region. (exterior)





Thm 1: (Euler's Formula)

If G is a connected plane graph with p vertices, q edges, and r regions, then

$$p - q + r = 2$$

pf: (by induction on q)

(basis) If $q = 0$, then $G \cong K_1$; so $p = 1$, $r = 1$, and $p - q + r = 2$ (where, K_1 **is complete graph with vertex 1**)

(inductive) Assume the result is true for any graph with $q = k - 1$ edges, where $k \geq 1$

Let G be a graph with k edges. Suppose G has p vertices and r regions.

If G is a tree, then G has p vertices, $p-1$ edges and 1 region.

$$\Rightarrow p - q + r = p - (p-1) + 1 = 2$$

If G is not a tree, then some edge e of G is on a cycle.

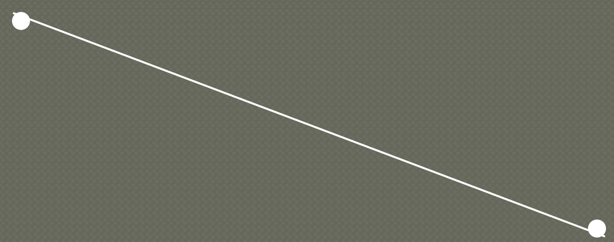
Hence $G-e$ is a connected plane graph having order p and size $k-1$, and $r-1$ regions.

$$\Rightarrow p - (k-1) + (r-1) = 2 \quad \text{(by assumption)}$$

$$\Rightarrow p - k + r = 2$$

Proof of Euler's Formula (cont'd)

- Let's start by drawing G_1



$$v_1 = 2$$
$$e_1 = 1$$
$$r_1 = 1$$

Euler's formula is valid for G_1, since

$r=e-v+2$

1=1-2+2

Can obtain G_2 from G_1 by adding an edge at one of from the vertices of G_1.

Proof of Euler's Formula (cont'd)

In general, we can obtain G_n from G_{n-1} by adding an n^{th} edge to one of the vertices of G_{n-1}

- The new edge might link two vertices already in G_{n-1}

Or, the new edge might add another vertex to G_{n-1}

We will use the method of induction to complete the proof:

We have shown that the theorem is true for G_1

Next, let's assume that it is true for G_{n-1} for any $n > 1$, and prove it is true for G_n

Let (x,y) be the n^{th} edge that is added to G_{n-1} to get G_n

There are two cases to consider….

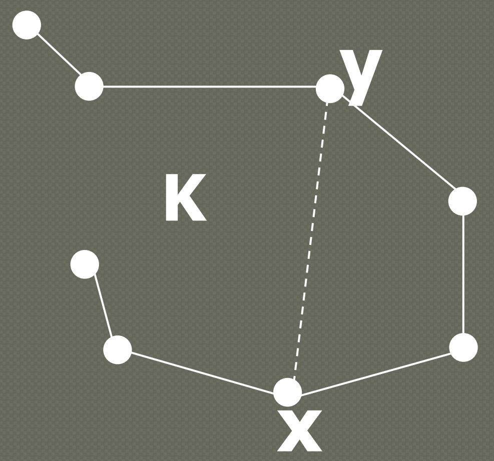
Proof of Euler's Formula (cont'd)

In the first case, x and y are both in G_{n-1}
Then they are on the boundary of a common region K , possibly an unbounded region.

Edge (x, y) splits K into two regions.
Then,

$$r_n = r_{n-1} + 1$$
$$e_n = e_{n-1} + 1$$
$$v_n = v_{n-1}$$

$r = e - v + 2$



Each side of Euler's formula grows by one.
So, if the formula was true for G_{n-1} , it will also be true for G_n

Proof of Euler's Formula (cont'd)

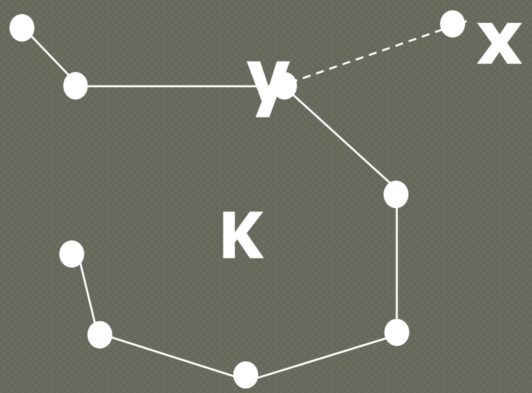
In the second case, one of the vertices *x*,*y* is not in G_{n-1}
Let's say that it is *x*.

Then, adding (*x*,*y*) implies that *x* is also added, but that no new regions are formed (no existing regions are split).

$$r_n = r_{n-1}$$

$$e_n = e_{n-1} + 1$$

$$v_n = v_{n-1} + 1$$



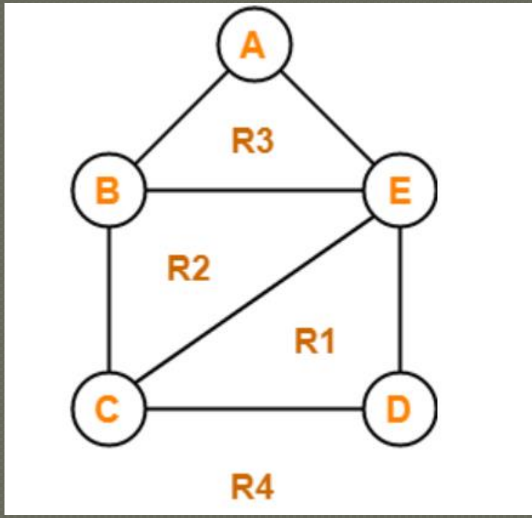
$r = e - v + 2$

So, the value on each side of Euler's equation is unchanged.
The validity of Euler's formula for G_{n-1} implies its validity for G_n

By induction, Euler's formula is true for all G_n's and the full graph *G*.

Degree of a Region

- Each region has some degree associated with it given as-
- Degree of Interior region = Number of edges enclosing that region
- Degree of Exterior region = Number of edges exposed to that region



Degree (R1) = 3
Degree (R2) = 3
Degree (R3) = 3
Degree (R4) = 5

- **Euler's Formula:**
- $\mathbf{r = e - v + 2}$ √

Given-

$v = 25$

$e = 60$

Number of regions (r)

$= 60 - 25 + 2$

$= 37$

→ **For** a planar graph with k components:

- $\mathbf{r = e - v + (k + 1)}$ √

Given-

$v = 10$

$e = 9$

$k = 3$

Number of regions (r)

$= 9 - 10 + (3+1)$

$= 3$

- In any planar graph, Sum of degrees of all the vertices = 2 x Total number of edges in the graph √

$$\sum_{i=1}^{n} \deg(v_i) = 2\,|E|$$

- In any planar graph, Sum of degrees of all the regions = 2 x Total number of edges in the graph √

$$\sum_{i=1}^{n} \deg(r_i) = 2\,|E|$$

Special Cases:

Case 1: In any planar graph, if degree of each region is K, then-

$$K \times |R| = 2 \times |E|$$

Case 2: In any planar graph, if degree of each region is at least K (>=K), then-

$$K \times |R| <= 2 \times |E|$$

Case 3: In any planar graph, if degree of each region is at most K (<=K), then-

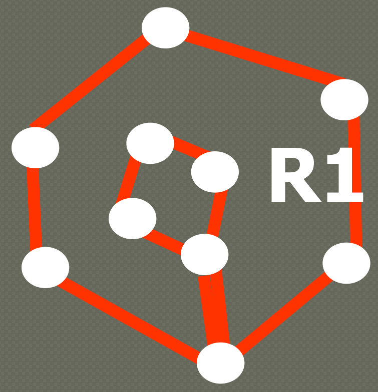
$$K \times |R| >= 2 \times |E|$$

Corollary

- If G is a connected planar graph with $\mathbf{e} > 1$, then $\mathbf{e} \leq 3\mathbf{v} - 6$

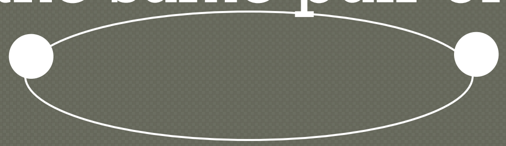
- Proof:
  - Define the degree of a region as the number of edges on its boundary. If an edge occurs twice along the boundary, then count it twice.

  e.g. The region R1 has degree 12

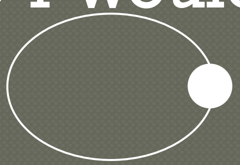


Proof of Corollary continued

- Note that no region can be less than degree 3.
  - A region of degree 2 would be bounded by two edges joining the same pair of vertices (parallel edges)



  - A region of degree 1 would be bounded by a loop edge.



- Neither of these is allowed, and so a region must have at least degree 3

Proof of Corollary continued

Since $2\mathbf{e} = \sum(\text{degrees of r})$, we know that $2\mathbf{e} \geq 3\mathbf{r} \Rightarrow \frac{2}{3}\mathbf{e} \geq \mathbf{r}$

Also, we know $\mathbf{r} = \mathbf{e} - \mathbf{v} + 2$ from Euler's Formula.

Substitute Euler's Formula in to get:

$$\frac{2}{3}\mathbf{e} \geq \mathbf{e} - \mathbf{v} + 2$$

$$\Rightarrow \quad 0 \geq (\frac{1}{3}\mathbf{e} - \mathbf{v} + 2) * 3$$

$$\Rightarrow \quad 0 \geq \mathbf{e} - 3\mathbf{v} + 6$$

$$\Rightarrow \quad \mathbf{e} \leq 3\mathbf{v} - 6$$

Definition:

A plane graph G is called maximal planar if, for every pair u, v of nonadjacent vertices of G, the graph $G+uv$ is nonplanar.

Thus, in any embedding of a maximal planar graph G of order at least 3, the boundary of every region of G is a triangle.

Thm 2: If G is a maximal planar graph with $p \geq 3$ vertices and q edges, then

$$q = 3p - 6$$

pf: Embed the graph G in the plane, resulting in r regions. $\Rightarrow p - q + r = 2$

Since the boundary of every region of G is a triangle, every edge lies on the boundary of two regions.

$$\Rightarrow \sum_{\forall \text{ region } R} |\{\text{the edges of the boundary of } R\}| = 3r = 2q$$

$$\Rightarrow p - q + 2q/3 = 2 \qquad \Rightarrow q = 3p - 6$$

Cor. 2(a): If G is a maximal planar bipartite graph with $p \geq 3$ vertices and q edges, then $q = 2p - 4$

pf: The boundary of every region is a 4-cycle.

$$4r = 2q \;\Rightarrow\; p - q + q/2 = 2 \;\Rightarrow\; q = 2p - 4.$$

Cor. 2(b): If G is a planar graph with $p \geq 3$ vertices and q edges, then

$$q \leq 3p - 6$$

pf: If G is not maximal planar, we can add edges to G to produce a maximal planar graph.

Thm 3: Every planar graph contains a vertex of degree 5 or less.

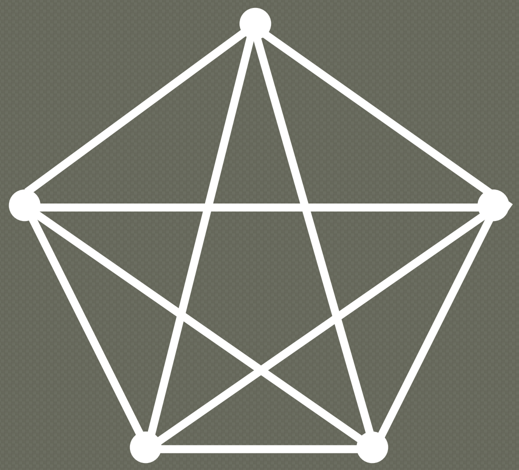
pf: Let $\boldsymbol{G}$ be a planar graph of $\boldsymbol{p}$ vertices and $\boldsymbol{q}$ edges.

If $\deg(\boldsymbol{v}) \geq 6$ for every $\boldsymbol{v} \in \boldsymbol{V}(\boldsymbol{G})$

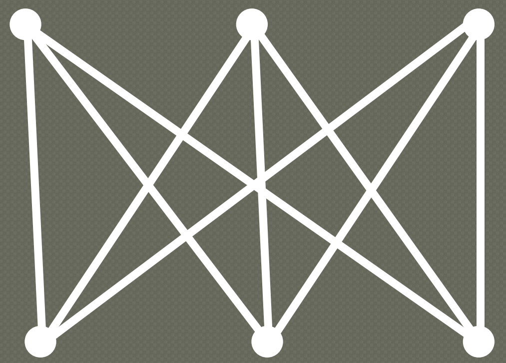
$$\Rightarrow \quad \sum_{v \in V(G)} \deg(v) \geq 6p$$

$$\Rightarrow \quad 2\boldsymbol{q} \geq 6\boldsymbol{p}$$

Fig 5 Two important nonplanar graph



K_5

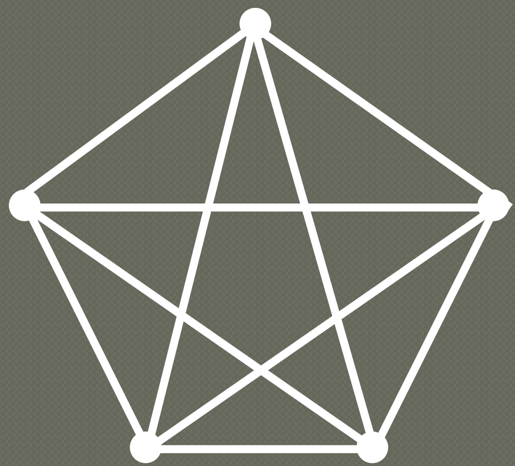


$K_{3,3}$

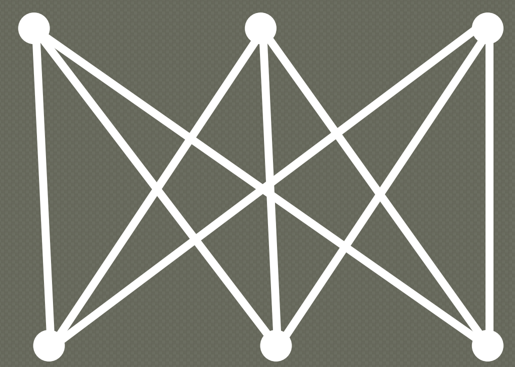
Thm 4: The graphs K_5 and $K_{3,3}$ are nonplanar.

Why?

Fig 5 Two important nonplanar graph



K_5



$K_{3,3}$

i) $\boldsymbol{K_5}$ has $\boldsymbol{v = 5}$ vertices and $\boldsymbol{e}$ = **10** edges

→ $\boldsymbol{e > 3v - 6}$ → is nonplanar

ii) $K_{3,3}$ can't be drawn without line intersection → is nonplanar

Thm 4: The graphs K_5 and $K_{3,3}$ are nonplanar.

pf: (1) K_5 has $p = 5$ vertices and $q = 10$ edges.

$$q > 3p - 6 \quad \Rightarrow K_5 \text{ is nonplanar.}$$

(2) Suppose $K_{3,3}$ is planar, and consider any embedding of $K_{3,3}$ in the plane.
Suppose the embedding has r regions.

$$p - q + r = 2 \quad \Rightarrow \quad r = 5$$

$K_{3,3}$ is bipartite

$\Rightarrow$ The boundary of every region has ≥ 4 edges.

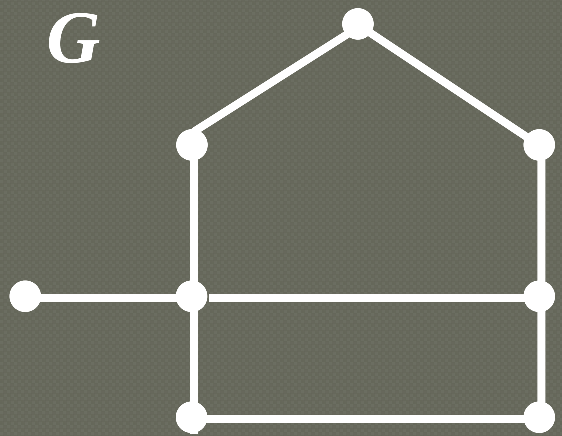
$$\Rightarrow \quad 4r \leq \sum_{\forall \text{ region } R} |\{\text{the edges of the boundary of } R\}| = 2q = 18$$

Definition:

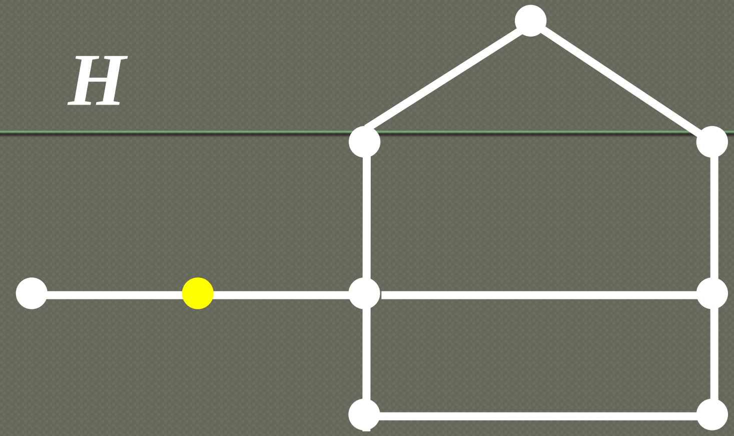
A subdivision of a graph G is a graph obtained by inserting vertices (of degree 2) into the edges of G.

Fig 6 Subdivisions of graphs.

G

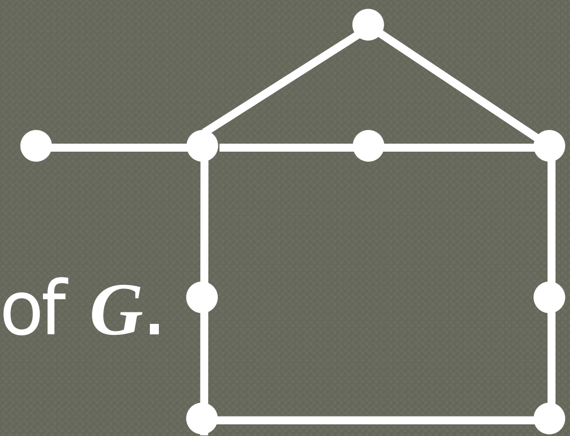


H



H is a subdivision of *G*.

F

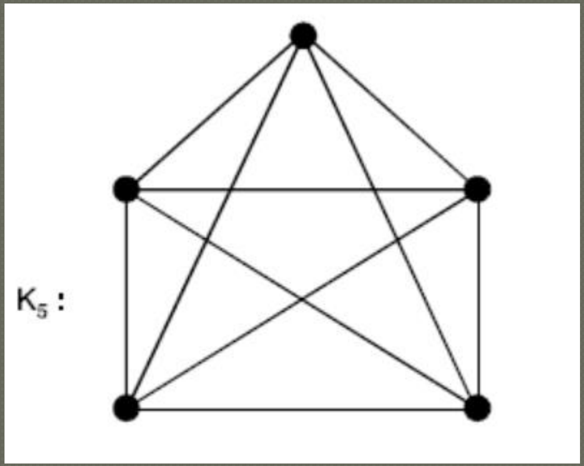


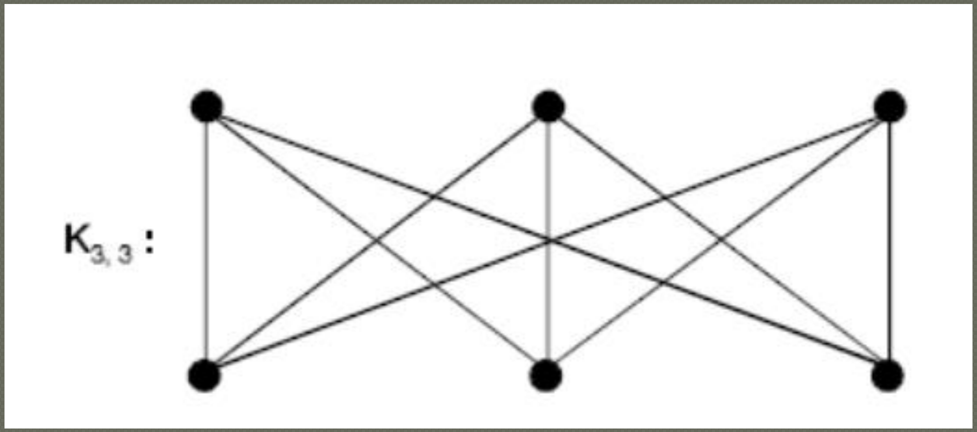
F is not a subdivision of *G*.

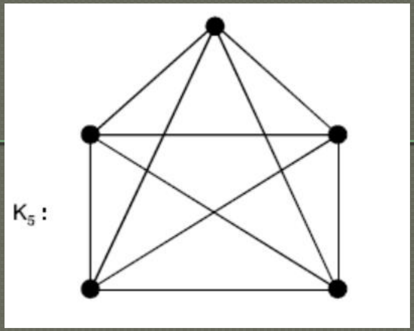
What are Kuratowski's Two Graphs?

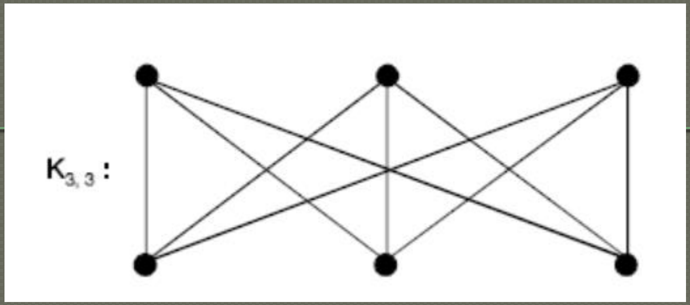
- Study of two specific non-planar graphs

1. The complete graph with 5 vertices
2. A regular, connected graph with 6 vertices and 9 edges







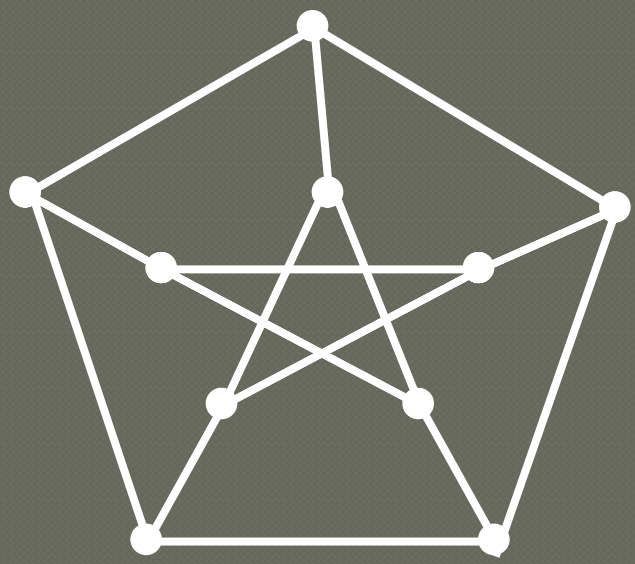


- **Observations**
  (i) Both are regular graphs
  (ii) Both are non-planar graphs
  (iii) Removal of one vertex or one edge makes the graph planar
  (iv) (Kuratowski's) first graph is non-planar graph with smallest number of vertices and (Kuratowski's) second graph is non-planar graph with smallest number of edges → Thus both are simplest non-planar graphs.

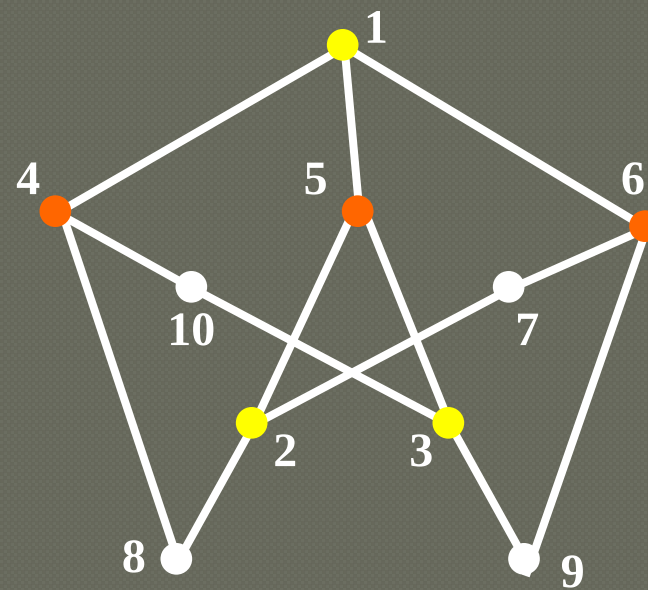
- **Kuratowski's theorem** is a mathematical forbidden graph characterization of planar graphs
- It states that a finite graph is planar iff, it does not contain a subgraph that is a subdivision of K5 or of K3,3

Thm 5: (Kuratowski's Theorem)
A graph is planar if and only if it contains no subgraph that is isomorphic to or is a subdivision of K_5 or $K_{3,3}$

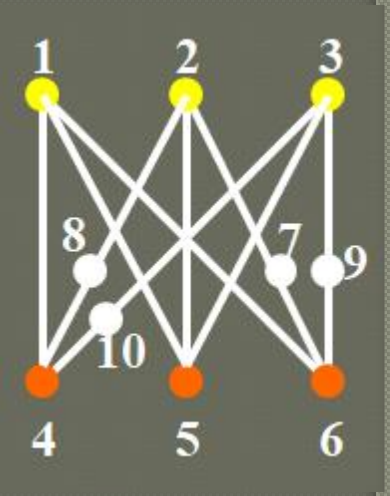
Fig 7 The Petersen graph is nonplanar.

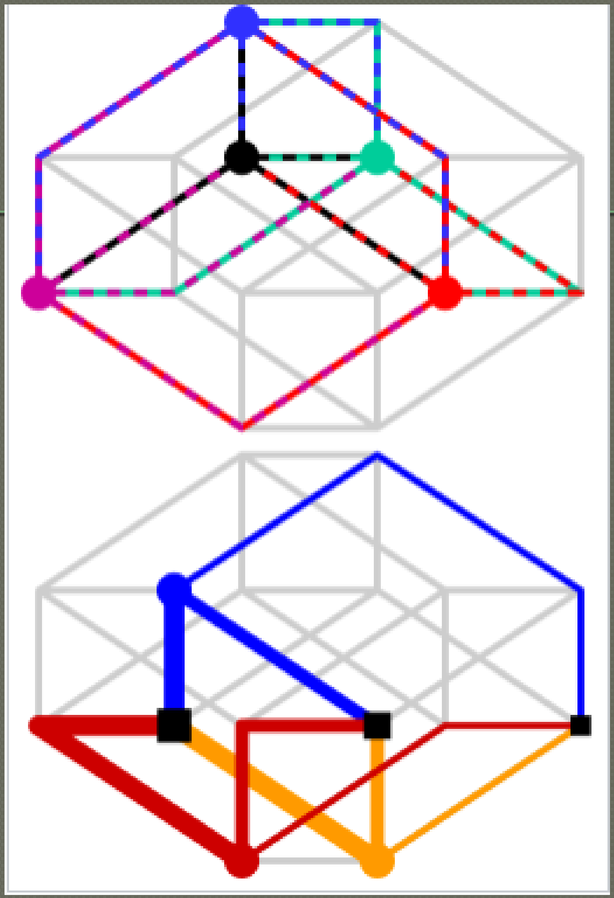


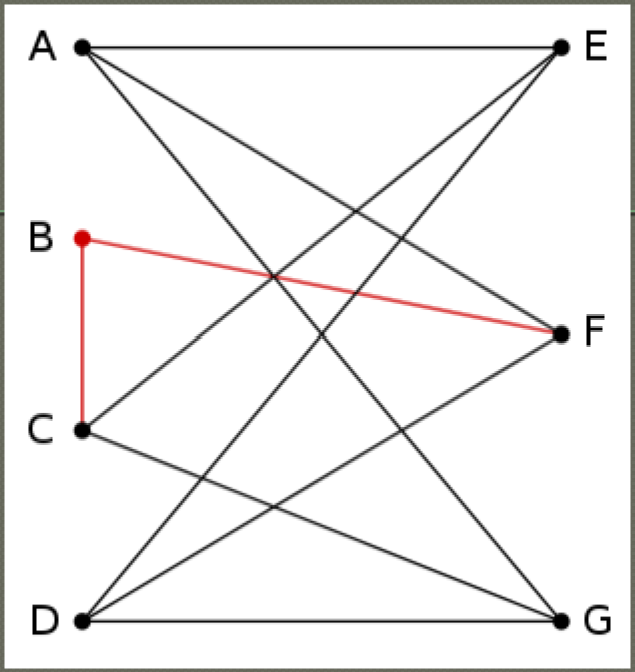
(a) Petersen



(b) Subdivision of $K_{3,3}$







A) is **non-planar** finding either K_5 (top) or $K_{3,3}$ (bottom) subgraphs

B) Is a graph with no K_5 or $K_{3,3}$ subgraph. However, it contains a subdivision of $K_{3,3}$ and hence non-planar

Kuratowski's Theorem _Implications

- A Kuratowski subgraph of a nonplanar graph can be found in linear time as measured by the size of the input graph
- It is possible to extract a large number of Kuratowski subgraphs in time dependent on their total size
- Allows the correctness of a planarity algorithm to be verified for nonplanar inputs
- It is straightforward to test whether a given subgraph Kuratowski subgraph? Usually, non-planar graphs contain a large number of Kuratowski-subgraphs
- The extraction of these subgraphs is needed, e.g., in *branch and cut* algorithms for crossing minimization

Elementary Tests for Planarity

For any G (e, n) ←→ (Edges, Vertices)

1) If $e < 9$ or $n < 5$ then the *graph must be planar*

2) If $e > 3n - 6$ then the *graph must be non-planar*

→ If these tests fail to resolve the question of planarity, then we need to use a more elaborative test….

- Thank You!