

# Hidden Markov Model for Cryptocurrency Trading



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This dissertation is submitted for the degree of

April 2022



I would like to dedicate this thesis to my loving parents ...



## **Declaration**

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

Adam Pešek  
April 2022



## **Acknowledgements**

And I would like to acknowledge ...





## **Abstract**

Hidden Markov model (HMM) is a statistical signal prediction model, which has been widely used to predict economic regimes and stock prices. In this paper, we introduce the application of HMM in trading stocks (with S&P 500 index being an example) based on the stock price predictions. The procedure starts by using four criteria, including the Akaike information, the Bayesian information, the Hannan Quinn information, and the Bozdogan Consistent Akaike Information, in order to determine an optimal number of states for the HMM. The selected four-state HMM is then used to predict monthly closing prices of the S&P 500 index. For this work, the out-of-sample  $R^2$ , and some other error estimators are used to test the HMM predictions against the historical average model. Finally, both the HMM and the historical average model are used to trade the S&P 500. The obtained results clearly prove that the HMM outperforms this traditional method in predicting and trading stocks.



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# Chapter 1

## Markov Processes

### 1.1 Markov Processes

In order to fully understand and explore the properties and implications of Hidden Markov Models (hereinafter "HMMs") that will be used for cryptocurrency price prediction it is necessary to apriori state the underlying definition of Markov processes imminently followed by Markov Chains.

#### 1.1.1 Definition and Properties

Definition. Given a measurable space  $(S, B)$  called state space, where  $S$  is a set and  $B$  is a sigma-algebra on  $S$ . A function  $P : S \times B \rightarrow \mathbb{R}$  is called a transition probability function if  $P(x, \cdot)$  is a probability measure on  $(S, B)$  for all  $x \in S$  and if for every  $B \in B$ , the map  $s \rightarrow P(s, B)$  is  $B$ -measurable. Define  $P^1(x, B) = P(x, B)$  and inductively the measures  $P^{n+1}(x, B) = \int_S P^n(y, B)P(x, dy)$ , where we write  $\mathbb{R}$

### 1.2 Discrete-time Markov Chains

Let  $\Omega \neq \emptyset$  and  $\mathcal{A} \subseteq 2^\Omega$  be a  $\sigma$ -algebra on  $\Omega$ , and  $P$  a measure on  $\mathcal{A}$  with  $P(\Omega) = 1$ , i.e.  $P$  is a *probability measure*. Then the triplet  $(\Omega, \mathcal{A}, P)$  is called a *probability space*. Where  $\Omega$  denotes a sure event and it holds that  $\forall \omega \in \Omega$  is called an elementary event. Furthermore,  $\forall A \in \mathcal{A}$  is a random event so that  $P(A)$  is a probability of such a random event.

Let  $I$  be a countable set and  $\Theta$  a  $\sigma$ -algebra on  $I$ . Each  $i \in I$  is called a *state* and  $(I, \Theta)$  a *state-space*. We say that  $\lambda = (\lambda_i : i \in I)$  is a measure on  $I$  if  $0 \leq \lambda_i \leq \infty$ . If in addition the *total mass*  $\sum_{i \in I} \lambda_i = 1$  then  $\lambda$  is a *distribution* (or probability measure).

Suppose now that we have two measurable spaces  $(\Omega, \mathcal{A})$  and  $(I, \Theta)$  and a random variable  $X : \Omega \rightarrow I$  assuming that  $X$  is measurable. Thus we call  $(I, \Theta)$  a state space and  $(\Omega, \mathcal{A})$  an underlying space. Therefore we may set:

$$\lambda_X(i) = P(X = i) = P(\{\omega : X(\omega) = i\}) \quad (1.1)$$

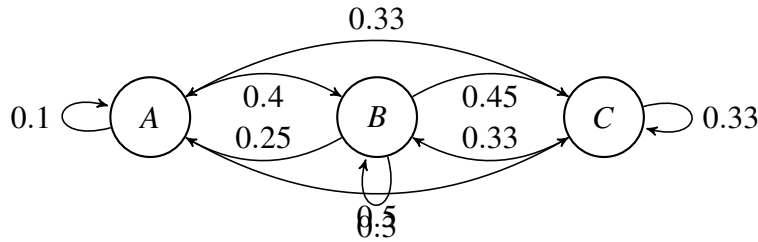
Since we are allowing only for the discrete realisations of the random variable  $X$ , given previous assumptions,  $\lambda_X(i)$  is a *probability mass function*.

Then measure  $\lambda$  defines a distribution of  $X$ . Given such a setting random variable  $X$  is assumed to denote random state  $i$  with probability  $\lambda_i$ .

To simply illustrate the idea behind discrete Markov Chains let us assume a situation where the future market movements transition between a countable number of states  $I = \{\text{upward movement, no movement, downward movement}\}$  and there is a transition matrix (also called *stochastic matrix*)  $P = (p_{i,j} : i, j \in I)$  defined as:

$$P = \begin{pmatrix} 0.1 & 0.4 & 0.5 \\ 0.25 & 0.3 & 0.45 \\ 0.33 & 0.33 & 0.33 \end{pmatrix}$$

Each row represents full set of transition probabilities between each state which is visible from  $\sum_{j \in I} p_{i,j} = 1$ , i.e. each row of matrix  $P$  represents a full distribution of transitions over  $I$ . Such a relationship can be represented as a diagram indexing each state by A, B and C respectively as follows:



This may be easily interpreted for each given state. For example if we assume that the market moved upwards on the last trading day there is a 0.1 chance that the market will move in positive direction today, in other words the conditional probability of observing the state A today given the state A yesterday is 0.1. On the hand if we suppose that today the market actually transitioned to the state B with probability 0.4 there is now a probability of 0.45 to transition to state C since the future transition is only conditioned by its previous state.

Till now we have assumed random variable  $X$  that modelled the probability of observing a state  $i \in I$  given the initial distribution  $\lambda$ . We ought to consider thus we have to allow for a

system of discrete random variables that are also identically distributed for each time step  $\{X_t, t \in T\}$ , for  $\{t_0, t_1, \dots, t_n\} \subset T$  where  $n \in \mathbb{N}_0$ .

Suppose now that we have observed a given sequence of states for the last week as  $\{A, B, C, C, A\}$  and we would like to know its probability of observing given the transition matrix  $P$  and distribution  $\lambda$ . Then  $P(X_{t_0}, \dots, X_{t_n} | P, \lambda)$  can be calculated as:

$$\begin{aligned} P(X_{t_0}, \dots, X_{t_n} | P, \lambda) &= P(A, B, C, C, A | P, \lambda) \\ &= P(A) * P(B|A) * P(C|B) * P(C|C) * P(A|C) \\ &= \lambda * p_{1,2} * p_{2,3} * p_{3,3} * p_{3,1} \\ &= \lambda * 0.4 * 0.45 * 0.33 * 0.33 \end{aligned}$$

maybe by the product of  $X$ s

Where the probability of observing state  $A$  is determined by our distribution  $\lambda$  evaluated at respective state  $A$  since we have no prior knowledge about what exactly happened before  $t_0$ .

On the other hand, we might consider a situation in which we have observed such a sequence of events and we need to determine the next state given the sequence. As in the last example, we have our transition matrix  $P$ , distribution  $\lambda$  and a sequence of events observed until now  $\{A, B, C, C, A\}$  for  $t_{k-4}, \dots, t_k$ . Let us also assume that  $t_{k+1}$  is a time of next event for which we are trying to determine its probability.

$$P(X_{t_{k+1}} | X_{t_k}, \dots, X_{t_{k-4}}) = P(X_{t_{k+1}} | X_{t_k}) \quad (1.2)$$

We know that last observed state was  $A$  which directs us straight to the first row of our transition matrix  $P$  since from the properties of Markov Processes (the process is memoryless) we know that the next state will depend solely on the present state so we can abstract from the given sequence of past states and focus only on  $X_{t_k}$ . Finally we may conclude that the most likely future state is  $C$  with probability of 0.5.

Obviously, the probabilities in previous transition matrix  $P$  were imaginary and served only as a mere example of the main properties of Discrete-time Markov Chains. From now on we consider a dataset of BTC-USD daily close prices calculated by CoinMarketCap website from 5th March 2017 to 5th March 2022. First of all we ought to make several assumptions about the data in order to apply the logic and properties of Markov Chains.

Assumption 1: The close daily prices of BTC-USD trading pair are discrete, meaning that we only record prices at exactly 0:00 of each day since there are no closing hours of the major cryptocurrency exchanges. This assumption will be elevated in the next section with Continuous-time Markov Chains.

Assumption 2: We will assume that future prices of Bitcoin depend only on their pre Theorems:

## **1.3 Continuous-time Markov Chains**

Viterbi algorithm should, in its most general form, provide a solution to the maximum a posteriori probability (MAP) estimation of the state sequence of a finite-state discrete-time Markov process introduced in the last chapter.

# Chapter 2

## Parameter Estimation for Hidden Markov Models

### 2.1 Hidden Markov Model

A Hidden Markov Model

### 2.2 Expectation–maximization algorithm

Also abbreviated as EM algorithm is an iterative approach in computing maximum likelihood estimates. It is used in situations where incomplete data are present therefore the part of a complete data set is hidden and we may not be able to apply straightforward analytical procedure for computing maximum likelihood estimates. This is of great importance and efficiency for hidden Markov models where we have a sample space of observed variables, let us denote it as  $X$  and a hidden sample space  $Z$ . We assume that there is a mapping from  $X$  into  $Z$ , where  $\forall x \in X$  is a realization from sample space  $X$  and the mapping from  $X$  into  $Z$  is many-one. We also assume that given our model the realizations  $z \in Z$  are not observable directly but only as a projection from  $X$  into  $Z$ .

The EM algorithm involves two main stages:

- 1) Expectation step (E-step) -
- 2) Maximization step (M-step) -

When it comes to the definition of the EM algorithm we start by making several assumptions that will be removed subsequently to introduce more general properties and definitions. Suppose we are given the probability density function of exponential family form as follows:

$$f(z|\theta) = b(z)\exp(\theta t(z)^T)/a(\theta) \quad (2.1)$$

where  $\theta \in \Theta$  is a vector of parameters,  $t(z)$  denotes vector of complete-data sufficient statistics

### 2.2.1 Baum-Welsch algorithm

## 2.3 Observable states

There is a huge number of observable variables that one could abstract from cryptocurrency market. A possibility of discrete states within the given state space is plausible and feasible, it would, given our model constraints, provide poor inference since additional information would remain hidden. Imagine a situation where our observable states are defined as a relative change in price or a sudden drop/uprise in the traded volume on the exchange. In order to discretize our states and construct transition and emission probabilities we are forced to construct intervals that would well represent the boundaries upon which the model defines structure and predictions.

Assuming that the price increase in the idea predefined Hidden Markov Model will assume

However it is much more efficient to assume continuity in our predefined observable states. It is nowadays empirically proved, as in (citace) that using technical indicators as a predictors for the future spot price yields more accurate machine learning models. As will be demonstrated each of the technical indicators can be classified into several families of indicators, such as momentum, volume, volatility and cycle indicators. For our purposes we will consider mainly momentum indicators that are calculated using Open, High, Low, Close prices (hereinafter "OHLC") and Volume indicators. There is a huge variety of technical indicators to choose from therefore the selection was made according the most used and well known indicators or their transformed versions.

In our case we will consider following observable states that will be defined and elaborated on in the upcoming sections:

- 1) Moving Average Convergence/Divergence (MACD)
- 2) Stochastic Oscillator
- 3) Chaikin Oscillator
- 4) Relative Strength Index (RSI)

## 5) Aroon Oscillator

## 2.4 Moving Average Convergence Divergence

Also known as MACD is a trend-following momentum indicator that represents the differences between two exponential moving averages (hereinafter "EMA"). The most common and traditional moving averages are 26-period EMA and 12-period EMA.

The indicator is often used with so called "signal line" that is constructed as a 9-period EMA and is used as a trigger for a buy and sell signal. In practical application a trader decides to buy a stock if the signal line crosses MACD line from above and sell if it crosses from below, assuming simplistic trading strategy using only MACD. EMA also called exponentially weighted moving average is a type of moving average that differs from weighted moving average WMA by the distribution of weights to past observations. While WMA considers the linearly decreasing distribution of weights, the EMA assumes exponential decrease in weights. Furthermore it is necessary to elaborate over the values of weights because it might not always be unambiguous. WMA distributes weights chronologically and linearly, e.g. 10-period WMA gives weight 1 to the earliest observation and 10 to the most recent observation, the case within EMA is often not that simple. The weights given to each observation are computed as  $(1 - \lambda)^i$  where  $i \in \mathbb{N}_0$  and is bounded from above by the assumed period of interest, e.g. 3, 10, 26-period denoted as  $T$  for the sake of . As  $i$  increases identically with the time lag the value of weights decreases. The important role that ought to be questioned is the parameter  $\lambda$  that is defined as  $\frac{k}{T+1}$  where  $k$  represents the so called "smoothing" parameter. Traders and analysts use value 2 for the smoothing parameter but the number may be defined on the interval  $(0, T)$ . Higher values of  $k$  mean bigger weights given to most recent observations.

The Figure 2.1 illustrates MACD line, signal line as well as "MACD histogram" which is displayed as a bar chart indicating the difference of the former ones. Traders use such a distance to identify whether the bullish or bearish momentum is high, i.e. bigger the distances of these two lines higher the price momentum.

MACD has its unfortunate limitations that mainly arise from the non-trending moments. When the price enters sideways movement the MACD histogram signals decreases distances between MACD and signal line, the trend reversal is possible but the price moves sideways which eventually results in false positive signal. Moreover when the price moves sideways for longer periods MACD may signal too many false trend reversals. The most common practice for traders is to combine MACD signals with other indicators such as Relative Strength index (RSI) that measures overbought or oversold market. The RSI uses average price gains and

losses usually over 14 periods and yields values between 0 and 100, indicating overbought market for values 70 (80) to 100 and 30 (20) to 0 for oversold market. The idea is that when the distances between MACD line and signal line increase and RSI signals overbought market the trader might consider this as a strong trend reversal signal. The idea is that signals from MACD strategy often produce false signals when price suddenly moves sideways and RSI helps to indicate the false positive signal.



Fig. 2.1 Candlestick graph of daily spot price of BTC/USD from March 2021 to March 2022 with subplot containing two lines for MACD and Signal line and a bar chart as their difference

## 2.5 Stochastic Oscillator

A Stochastic Oscillator is a momentum indicator that compares the most recent closing price of a security with its predeceasing ones. Naturally, the range of preceding closing prices or the range of closing prices is 14 periods but it is regular that such an assumption is often edited to best fit the current needs of a trader. Also slight variation in taking the (weighted) moving average of the oscillator values is often introduced. The indicator is used to generate trading signals that refer to the current overbought state of the market, which means that the indicator values range from 0 to 100 where the values closer to the number 0 indicate oversold market and inversely values closer to 100 overbought market.



Stochastic Oscillator is computed as follows:

$$SO_t = \frac{C_{t-1} - L_{14}}{H_{14} - L_{14}} \quad (2.2)$$

where  $C_{t-1}$  denotes the most recent closing price of a security,  $H_{14}$  and  $L_{14}$  are the highest and lowest price traded during 14-period interval respectively.  $SO_t$  is sometimes referred to as a "fast" stochastic indicator. As said before this interval may be changed arbitrarily. Traders also developed so called "slow" Stochastic Oscillator which is defined as a 3-period moving average of  $SO_t$ . Thus when Stochastic Oscillator crosses the smooth "Slow" Stochastic Oscillator a trading signal is generated.

Considering values above 80, the indicator signals overbought market and oversold market when the value drops below 20. Although it remains to hold true that the indicator often produces false indications that may be caused by periods of time where the price remains overbought/oversold for some time and trading with respect to such oscillator may result in losses. It is rather recommended to observe the values of stochastic oscillator and use it for trend reversal indication.



Fig. 2.2 Candlestick graph of daily spot price of BTC/USD from March 2021 to March 2022 with subplot with line indicating Stochastic Oscillator

## 2.6 Chaikin Oscillator

Chaikin Oscillator is a momentum based indicator of the Accumulation/Distribution Line (hereinafter "A/D line"), which is a cumulative indicator that aims to identify potential divergences between stock price and trading volume. The oscillator is calculated as a difference between 3- day and 10-day Exponential Moving Average of A/D line.

The calculation of the Chaikin Oscillator may be broken down into several steps:

- (i) First of all we ought to calculate the Money Flow Multiplier for each time step denoted by "N".

$$N_t = \frac{(Close_t - Low_t) - (High_t - Close_t)}{High_t - Low_t} \quad (2.3)$$

- (ii) Now we may multiply  $N_t$  by the trading volume in given period of time to get the Money Flow Volume denoted as  $M_t$ . With that we recursively construct the A/D line as:

$$ADL_t = M_{t-1} + M_t \quad (2.4)$$

- (iii) Given the constructed A/D line, we compute the Chaikin Oscillator values as a difference of 3-day and 10-day exponential moving averages.

$$CO_t = \frac{\sum_{i=0}^3 (1 - \alpha)^i * Close_{t-i}}{\sum_{i=0}^3 (1 - \alpha)^i} - \frac{\sum_{i=0}^{10} (1 - \beta)^i * Close_{t-i}}{\sum_{i=0}^{10} (1 - \beta)^i} \quad (2.5)$$

where we assume that the weights denoted as  $\alpha$  and  $\beta$  are computed as  $2/(days + 1)$ . Numerator as a smoothing factor is often declared as 2. However the indicator may be set to absolutely different number between 0 and 1 according to the needs and assumptions made by the trader/analyst, therefore setting the parameter close to 1 is putting more weight to the most recent price.

One way to interpret the indicator is to trade with respect to the time when the Chaikin Oscillator crosses zero from below and above which signals buy and sell signals respectively.



Fig. 2.3 Candlestick graph of daily spot price of BTC/USD from March 2021 to March 2022 with subplot indicating Chaikin Oscillator

## 2.7 Relative Strength Index

## 2.8 Aroon Indicator

Aroon indicator is used for trend reversal identification and a measure of its strength. Indicator is composed out of two lines aroon up and aroon down that measure the time between new highs or lows respectively. Alternatively, they measure the strength of a bullish or bearish trend. Obviously the main idea of the indicator is based upon the fact that bullish trends are naturally formed by subsequently creating new highs while bearish trends form new lows. Aroon Up and Aroon Down are computed as follows:

$$AroonUp = \frac{25 - h}{25} * 100 \quad (2.6)$$

$$AroonDown = \frac{25 - l}{25} * 100 \quad (2.7)$$

Where  $h$  represents the number of periods from the last 25-period High and  $l$  the number of periods from last 25-period Low.

The interpretation of the indicator is very intuitive since the situation in which the Aroon Up line is above Aroon Down line signals bullish trend and when these two lines cross the signal of the trend reversal is generated. That also implies that for higher values of Aroon Up the bigger the strength and for lower values the uptrend is weaker and vice versa. In practice the crossover of these two lines is what generates the buy or sell signals, i.e. if Aroon Up crosses Aroon Down line from below a buy signal is generated and vice versa.

Although, Figure 2.4 graphically illustrates the Aroon Up and Down lines well, it is simpler to transform these two lines into one oscillator that would produce buy or sell signal in the case of zero crossover from above and from below. That is achieved by subtracting Aroon Up and Aroon Down line creating Aroon Oscillator.



Fig. 2.4 Candlestick graph of daily spot price of BTC/USD from March 2021 to March 2022 with subplots containing two lines for Aroon Up/Down Indicator and Aroon Oscillator

## **2.9 Trading using trading**



# Chapter 3

## My third chapter

### 3.1 First section of the third chapter

And now I begin my third chapter here ...

And now to cite some more people Read [2], Ancey et al. [1]

#### 3.1.1 First subsection in the first section

...and some more

#### 3.1.2 Second subsection in the first section

...and some more ...

##### First subsub section in the second subsection

...and some more in the first subsub section otherwise it all looks the same doesn't it? well we can add some text to it ...

#### 3.1.3 Third subsection in the first section

...and some more ...

##### First subsub section in the third subsection

...and some more in the first subsub section otherwise it all looks the same doesn't it? well we can add some text to it and some more and some more and some more and some more and some more and some more and some more ...

### Second subsub section in the third subsection

... and some more in the first subsub section otherwise it all looks the same doesn't it? well we can add some text to it ...

## 3.2 Second section of the third chapter

and here I write more ...

## 3.3 The layout of formal tables

This section has been modified from “Publication quality tables in L<sup>A</sup>T<sub>E</sub>X<sup>\*</sup>” by Simon Fear.

The layout of a table has been established over centuries of experience and should only be altered in extraordinary circumstances.

When formatting a table, remember two simple guidelines at all times:

1. Never, ever use vertical rules (lines).
2. Never use double rules.

These guidelines may seem extreme but I have never found a good argument in favour of breaking them. For example, if you feel that the information in the left half of a table is so different from that on the right that it needs to be separated by a vertical line, then you should use two tables instead. Not everyone follows the second guideline:

There are three further guidelines worth mentioning here as they are generally not known outside the circle of professional typesetters and subeditors:

3. Put the units in the column heading (not in the body of the table).
4. Always precede a decimal point by a digit; thus 0.1 *not* just .1.
5. Do not use ‘ditto’ signs or any other such convention to repeat a previous value. In many circumstances a blank will serve just as well. If it won't, then repeat the value.

A frequently seen mistake is to use ‘`\begin{center}`’ ... ‘`\end{center}`’ inside a figure or table environment. This center environment can cause additional vertical space. If you want to avoid that just use ‘`\centering`’



Table 3.1 A badly formatted table

	Species I		Species II	
Dental measurement	mean	SD	mean	SD
I1MD	6.23	0.91	5.2	0.7
I1LL	7.48	0.56	8.7	0.71
I2MD	3.99	0.63	4.22	0.54
I2LL	6.81	0.02	6.66	0.01
CMD	13.47	0.09	10.55	0.05
CBL	11.88	0.05	13.11	0.04

Table 3.2 A nice looking table

Dental measurement	Species I		Species II	
	mean	SD	mean	SD
I1MD	6.23	0.91	5.2	0.7
I1LL	7.48	0.56	8.7	0.71
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Table 3.3 Even better looking table using booktabs

Dental measurement	Species I		Species II	
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I1LL	7.48	0.56	8.7	0.71
I2MD	3.99	0.63	4.22	0.54
I2LL	6.81	0.02	6.66	0.01
CMD	13.47	0.09	10.55	0.05
CBL	11.88	0.05	13.11	0.04



# References

- [1] Ancey, C., Coussot, P., and Evesque, P. (1996). Examination of the possibility of a fluid-mechanics treatment of dense granular flows. *Mechanics of Cohesive-frictional Materials*, 1(4):385–403.
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# Appendix A

## How to install L<sup>A</sup>T<sub>E</sub>X

### Windows OS

#### TeXLive package - full version

1. Download the TeXLive ISO (2.2GB) from  
<https://www.tug.org/texlive/>
2. Download WinCDEmu (if you don't have a virtual drive) from  
<http://wincdemu.sysprogs.org/download/>
3. To install Windows CD Emulator follow the instructions at  
<http://wincdemu.sysprogs.org/tutorials/install/>
4. Right click the iso and mount it using the WinCDEmu as shown in  
<http://wincdemu.sysprogs.org/tutorials/mount/>
5. Open your virtual drive and run setup.pl

or

#### Basic MikTeX - T<sub>E</sub>X distribution

1. Download Basic-MiK<sub>T</sub>E<sub>X</sub>(32bit or 64bit) from  
<http://miktex.org/download>
2. Run the installer
3. To add a new package go to Start » All Programs » MikTeX » Maintenance (Admin)  
and choose Package Manager

4. Select or search for packages to install

### **TexStudio - T<sub>E</sub>X editor**

1. Download TexStudio from  
<http://texstudio.sourceforge.net/#downloads>
2. Run the installer

## **Mac OS X**

### **MacTeX - T<sub>E</sub>X distribution**

1. Download the file from  
<https://www.tug.org/mactex/>
2. Extract and double click to run the installer. It does the entire configuration, sit back and relax.

### **TexStudio - T<sub>E</sub>X editor**

1. Download TexStudio from  
<http://texstudio.sourceforge.net/#downloads>
2. Extract and Start

## **Unix/Linux**

### **TeXLive - T<sub>E</sub>X distribution**

#### **Getting the distribution:**

1. TexLive can be downloaded from  
<http://www.tug.org/texlive/acquire-netinstall.html>.
2. TexLive is provided by most operating system you can use (rpm,apt-get or yum) to get TexLive distributions

## Installation

1. Mount the ISO file in the mnt directory

```
mount -t iso9660 -o ro,loop,noauto /your/texlive####.iso /mnt
```

2. Install wget on your OS (use rpm, apt-get or yum install)
3. Run the installer script install-tl.

```
cd /your/download/directory
./install-tl
```

4. Enter command 'i' for installation
5. Post-Installation configuration:  
<http://www.tug.org/texlive/doc/texlive-en/texlive-en.html#x1-320003.4.1>
6. Set the path for the directory of TexLive binaries in your .bashrc file

### For 32bit OS

For Bourne-compatible shells such as bash, and using Intel x86 GNU/Linux and a default directory setup as an example, the file to edit might be

```
edit ~/.bashrc file and add following lines
PATH=/usr/local/texlive/2011/bin/i386-linux:$PATH;
export PATH
MANPATH=/usr/local/texlive/2011/texmf/doc/man:$MANPATH;
export MANPATH
INFOPATH=/usr/local/texlive/2011/texmf/doc/info:$INFOPATH;
export INFOPATH
```

### For 64bit OS

```
edit ~/.bashrc file and add following lines
PATH=/usr/local/texlive/2011/bin/x86_64-linux:$PATH;
export PATH
MANPATH=/usr/local/texlive/2011/texmf/doc/man:$MANPATH;
export MANPATH
```

```
INFOPATH=/usr/local/texlive/2011/texmf/doc/info:$INFOPATH;  
export INFOPATH
```

**Fedora/RedHat/CentOS:**

```
sudo yum install texlive  
sudo yum install psutils
```

**SUSE:**

```
sudo zypper install texlive
```

**Debian/Ubuntu:**

```
sudo apt-get install texlive texlive-latex-extra  
sudo apt-get install psutils
```



# Appendix B

## Installing the CUED class file

$\text{\LaTeX}$ .cls files can be accessed system-wide when they are placed in the  $\langle\text{texmf}\rangle/\text{tex}/\text{latex}$  directory, where  $\langle\text{texmf}\rangle$  is the root directory of the user's  $\text{\TeX}$  installation. On systems that have a local  $\text{texmf}$  tree ( $\langle\text{texmflocal}\rangle$ ), which may be named “ $\text{texmf-local}$ ” or “ $\text{localtexmf}$ ”, it may be advisable to install packages in  $\langle\text{texmflocal}\rangle$ , rather than  $\langle\text{texmf}\rangle$  as the contents of the former, unlike that of the latter, are preserved after the  $\text{\LaTeX}$  system is reinstalled and/or upgraded.

It is recommended that the user create a subdirectory  $\langle\text{texmf}\rangle/\text{tex}/\text{latex}/\text{CUED}$  for all CUED related  $\text{\LaTeX}$  class and package files. On some  $\text{\LaTeX}$  systems, the directory look-up tables will need to be refreshed after making additions or deletions to the system files. For  $\text{\TeX}$ Live systems this is accomplished via executing “ $\text{texhash}$ ” as root.  $\text{MikTeX}$  users can run “ $\text{initexmf -u}$ ” to accomplish the same thing.

Users not willing or able to install the files system-wide can install them in their personal directories, but will then have to provide the path (full or relative) in addition to the filename when referring to them in  $\text{\LaTeX}$ .

