Note 11 - Feb 14

4. Stochastic Processes (cont'd)

4.3 Stationary distribution (cont'd)

Example 4.3.1

An electron has two states: $ground(0), excited(1)$. Let $X_n \in \{0, 1\}$ be the state at time n.

At each step, changes state with probability:

- α if it is in state 0.
- β if it is in state 1.

THen $\{X_n\}$ is a DTMC. ITs transitional matrix is:

$$P = \begin{Bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{Bmatrix}$$

Now let us solve for the stationary distribution $\underline{\pi} = \underline{\pi} \cdot P$.

$$(\pi_0, \pi_1) \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} = (\pi_0, \pi_1)$$

$$\Rightarrow \begin{cases} \pi_0(1-\alpha) + \pi_1 \beta = \pi_0 & (1) \\ \pi_0 \alpha + \pi_1(1-\beta) = \pi_1 & (2) \end{cases}$$

We have two equations and two unknowns. However, note that they are not linearly independent:

sum of LHS $= \pi_0 + \pi_1 =$ sum of RHS. Hence (2) can be derived from (1). By (1), we have:

$$\alpha \pi_0 = \beta \pi_1 \quad \text{or} \quad \frac{\pi_0}{\pi_1} = \frac{\beta}{\alpha}$$

This where we need $\underline{\pi} \cdot \mathbb{1}$:

$$\pi_0 + \pi_1 = 1 \Rightarrow \pi_0 = \frac{\beta}{\alpha + \beta}, \quad \pi_1 = \frac{\alpha}{\alpha + \beta}$$

Thus, we conclude that there exists a unique stationary distribution $\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right) = \underline{\pi}$

The above procedure for solving for stationary distribution is typical:

1. Use $\underline{\pi} = \underline{\pi} P$ to get the properties between different components of $\underline{\pi}$
2. Use $\underline{\pi} \cdot \mathbb{1} = 1$ to normalize (get exact values)

4.4. Classification of States

4.4.1. Transience and recurrence

Let T: be the waiting for a MC to visit/revisit state i for the first time

$$T_i := min\{n > 0 : X_n = i\} \qquad T_i \text{ is a r.v.}$$

$T_i = \infty$ if the MC never (re)visits state i.

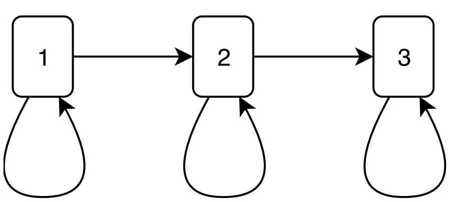
Definition 4.4.1:

A state i is called:

- transient, if $P(T_1 < \infty | X_0 = i) < 1$ (never goes back to i positive probability)
- recurrence, if $P(T_i < \infty | X_0 = i) = 1$ (always goes back to state i)
  - positive recurrent, if $E(T_i | X_0 = i) < \infty$
  - null recurrent, if $E(T_i | X_0 = i) = \infty$
  - (note: a r.v. is finite with probability $\not\Rightarrow$ its expectation is finite)
    - Example: $T = 2, 4, ..., 2^n, p = \frac{1}{2}, \frac{1}{4}, ..., 2^{-n}$
      $E(T_{=} 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + ... + 2^n \cdot 2^{-n} = \infty$

Example 4.4.1

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \\ & \frac{1}{2} & \frac{1}{2} \\ & & 1 \end{pmatrix}$$



Given $X_0 = 0$,

$$P(\underbrace{X_1 = 0}_{T_0 = 1} | X_0 = 0) = P(\underbrace{X_1 = 1}_{\substack{T_0 = \infty \text{ since state 1} \\ \text{and 2 do not go to 0}}} | X_0 = 0) = \frac{1}{2} \quad \Rightarrow \quad P(T_0 < \infty | X_0 = 0) = \frac{1}{2} < 1$$

Thus, state 0 is transient

Similarly, state 1 is transient.

Given $X_0 = 2$,

$$P(X_1 = 2|X_0 = 2) \Rightarrow P(T_2 < \infty|X_0 = 2) = 1$$

As $E(T_2|X_0 = 2) = 1$Thus, state 2 is a positive recurrence.

In general, the distribution of T_i is very hard to determine $\Rightarrow$ need better criteria for recurrence/transience.

Criteria (1): Define $f_{ii} = P(T_i < \infty|X_0 = i)$, and

$$V_i = \# \text{ of times that the MC (revisits) state i} = \sum_{n=1}^{\infty} \mathbb{1}_{\{X_n = i\}}$$

If state i is transient

$$P(V_i = k|X_0 = j) = \underbrace{f_{ii}^k}_{\substack{\text{goes back to} \\ i \text{ for } k \text{ times}}} \underbrace{(1 - f_{ii})}_{\substack{\text{never visits} \\ i \text{ again}}}$$

$$\Rightarrow V_i + 1 \sim Geo(1 - f_{ii})$$

In particular, $P(V_i < \infty|X_0 = i) = 1 \Rightarrow$ If state i is transient, it is visited away finitely many times with probability 1. The MC will leave state i forever sooner or later.

On the other hand, if state i is recurrent, then $f_{ii} = 1$

$$P(V_i = k) = 0 \quad k = 0, 1, ... \Rightarrow P(V_1 = \infty) = 1$$

If the MC starts at a recurrent state i, it will visit that state infinitely many times.

Criteria (2): In terms of $E(V_i|X_0 = i)$:

$$\begin{aligned} E(V_i|X_0 = i) &= \frac{1}{1 - f_{ii}} - 1 = \frac{f_{ii}}{1 - f_{ii}} < \infty \quad && \text{if } f_{ii} < 1, (i \text{ transient}) \\ E(V_i|X_0 = i) &= \infty, && \text{if } f_{ii} = 1, (i \text{ recurrent}) \end{aligned}$$

Criteria (3): Note that

$$
\begin{aligned}
E(V_i|X_0=i) &= E(\sum_{n=1}^{\infty} \mathbb{1}_{\{X_n=i\}}\,|X_0=i) \\
&= \sum_{n=1}^{\infty} E(\mathbb{1}_{\{X_n=i\}}\,|X_0=i) \\
&= \sum_{n=1}^{\infty} P(X_n=i|X_0=i) \\
&= \sum_{n=1}^{\infty} P_{ii}^{(n)}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty \quad \text{if } i \text{ transient} \\
&\Rightarrow \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \quad \text{if } i \text{ recurrent}
\end{aligned}
$$

To conclude,

$$
\begin{array}{rll}
i & recureent & transient \\
 & P(T_i < infty|X_0=i) = 1 & P(T_i < \infty|X_0=i) < 1 \\
define: & P(V_i=\infty|X_0=i) = 1 & P(V_i < \infty|X_0=i) = 1 \\
 & E(V_i|X_0=i) = \infty & E(V_i|X_0=i) < \infty \\
\text{easiest to use:} & \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty & \sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty
\end{array}
$$