STAT 333 Course Note

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1. Fundamental of Probability

1.1 What's Probability

1.1.1 Examples

1. Coin toss
    - "H" - head
    - "T" - tail
2. Roll a dice
    - every number in the set: $\{1,2,3,4,5,6\}$
3. Tomorrow weather
    - {sunny, rainy, cloudy,...}
4. Randomly pick a number in $[0,1]$

Although things are random, they are not haphazard/arbitrary. There are "patterns"

Example 1

If we repeat tossing a coin, then the fraction of times that we get a "H" goes to $\frac{1}{2}$ as the number of toss goes to infinity.

$$\frac{\#\ of\ "H"}{total\ \#\ of\ toss} = \frac{1}{2}$$

This number $1/2$ reflects how "likely" a "H" will appear in one toss (if the experiment is not repeated)

1.2 Probability Models

The *Sample space* Ω is the set consisting of al the possible outcomes of a random experiment.

1.2.1 Examples

1. $\{H,T\}$
2. $\{1,2,3,4,5,6\}$
3. $\{sunny, rainy, cloudy, ...\}$
4. $[0,1]$

An *event* $E \in \Omega$ is a subset of Ω

for which we can talk about "likelihood of happening"; for example

- in **2**:
    - {getting an even number} = {2, 4, 6}
- in **4**:
    - $\{the\ point\ is\ between\ 0\ and\ 1/3\} = [0, \frac{1}{3}]$ is an event
    - $\{the\ point\ is\ rational\} = Q \cap [0,1]$

We say an event E "happens", if the result of the experiment turns out to belong to E (a subset of Ω)

A probability P is a set function (a mapping from events to real numbers)

$$
\begin{aligned}
P :& \xi \to R \\
& E \to P(E)
\end{aligned}
$$

which satisfies the following 3 properties:

1. $\forall E \in \xi, 0 \leq P(E) \leq 1$
2. $P(\Omega) = 1$
3. For
   - countably many disjoint events $E_1, E_2, ...$, we have $P(U_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$
   - countable: $\exists$ 1-1 mapping to natural numbers $1, 2, 3, ...$

Intuitively, one can think the probability of an event as the "likelihood/chance" for the event happens. If we repeat the experiment for a large number of events, the probability is the fraction of time that the event happens

$$
P(E) = \lim_{n \to \infty} \frac{\text{\# of times the E happens in n trials}}{n}
$$

1.2.1.1 Example 2

$$
\begin{aligned}
& P(\{1\}) = P(\{2\}) = \ldots = P(\{6\}) = \frac{1}{6} \\
& E = \{\text{even number}\} = \{2, 4, 6\} \\
\Rightarrow \ & P(E) = P(\{2\} \cup P(\{4\})) \cup P(\{6\}) = \frac{1}{2}
\end{aligned}
$$

Properties of probability:

1. $P(E) + P(E^c) = 1$
2. $P(\emptyset) = 0$
3. $E_1 \subseteq E_2 \Rightarrow P(E_1) \leq P(E_2)$
4. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ -$P(E_1 \cap E_2)$: E_1 and E_2 happen

1.2.2 Remark: why do we need the notion of event?

If the sample space Ω is **discrete**, then everything can has at most countable elements be built from the "atoms"

$$
\begin{aligned}
\Omega = \{w_1, w_2, \ldots\} \\
P(w_1) = P_i \\
P_i \in [0, 1], \sum_{i=1}^{\infty} P_i = 1
\end{aligned}
$$

Then for any event $E = \{w_1, i \in I\}$, $P(E) = \sum_{i \in I} P_i$

However, if the sample space Ω is continuous; e.g, $[0, 1]$ in Example 4, then such a construction can not be done for any $x \in [0, 1]$ we get $P(\{x\} = 0$ (x: the point is exactly x)

We can not get $P([0, \frac{1}{3}])$ by adding $P(\{x\})$ for $x \leq \frac{1}{3}$.

This is why we need the notion of event; and we define P as a set function from ξ to R rather than a function from Ω to R

To summarize: A **Probability Space** consists of a triplet (Ω, ξ, P):

- Ω: sample space,
- ξ: collection of events
- P: probability

1.3 Conditional Probability

If we know some information, the probability of an event can be updated

Let E, F be two events $P(F) > 0$

The conditional probability of E, given F is

$$
P(E \mid F) = \frac{P(E \cap F)}{P(F)}
$$

Again, think probability as the long-run frequency:

$$\begin{aligned} P(E \cap F) &= \lim_{n \to \infty} \frac{\#of\ times\ E\ and\ F\ happen\ in\ n\ trails}{n} \\ P(F) &= \lim_{n \to \infty} \frac{\#of\ times\ F\ happen\ in\ n\ trails}{n} \\ \Rightarrow \frac{P(E \cap F)}{P(F)} &= \lim_{n \to \infty} \frac{\#of\ times\ E\ and\ F\ happen}{\#of\ times\ F\ happens} \end{aligned}$$

By definition

$$P(E \cap F) = P(E \mid F) \cdot P(F)$$

1.4 Independence

Def: Two events E and F are said to be independent, if $P(E \cap F) = P(E) \cdot P(F)$; denoted as $E \perp\!\!\!\perp F$. **This is different rom disjoint.**

Assume $P(F) > 0$, then $E \perp\!\!\!\perp F \Leftrightarrow P(E|F) = P(E)$; intuitively, knowing F does not change the probability of E.

Proof:

$$\begin{aligned} E \perp\!\!\!\perp F &\Leftrightarrow P(E \cap F) = P(E) \cdot P(F) \\ &\Leftrightarrow \frac{P(E \cap F)}{P(F)} = P(E) \\ &\Leftrightarrow P(E|F)) = P(E) \end{aligned}$$

More generally, a sequence of events $E_1, E_2, \ldots$ are called independent if for **any** finite index set I,

$$P(\bigcap_{i \in I} E_i) = \prod_{i \in I} P(E_i)$$

1.5 Bayes' rule and law of total probability

Theorem: Let $F_1, F_2, \ldots$ be disjoint events, and $\bigcap_{i=1}^{\infty} F_i = \Omega$, we say $\{F_i\}_{i=1}^{\infty}$ forms a "partition" of the sample space Ω

Then $P(E) = \sum_{i=1}^{\infty} P(E|F_i) \cdot P(F_i)$

Proof: Exercise

Intuition: Decompose the total probability into different cases.

$$P(E \cap F_2) = P(E|F_2) \cdot P(F_2)$$

1.5.1 Bayes' rule

$$P(F_i|E) = \frac{P(E|F_i) \cdot P(F_i)}{\sum_{j=1}^{\infty} P(E|F_j) \cdot P(F_j)}$$

Bayes' rule tells us how to find conditional probability by switching the role of the event and the condition.

Proof:

$$\begin{aligned} P(F_i|E) &= \frac{P(F_i \cap E)}{P(E)} && \text{definition of condition probability} \\ &= \frac{P(E|F_i)P(F_i)}{P(E)} \\ &= \frac{P(E|F_i)P(F_i)}{\sum_{j=1}^{\infty} P(E|F_j)P(F_j)} && \text{law of total probability} \end{aligned}$$

2 Random variables and distributions

2.1 Random variables

(Ω, ξ, P): Probability space.

Definition: A random variable X (or r.v.) is a mapping from Ω to $\mathbb{R}$

$$
\begin{aligned}
X : &\Omega \to \mathbb{R} \\
&\omega \to X(\omega)
\end{aligned}
$$

A random variable transforms arbitrary "outcomes" into numbers.

X introduces a probability on R. For $A \subseteq R$, define

$$
\begin{aligned}
P(X \in A) : &= P(\{X(\omega) \in A\}) \\
&= P(\{\omega : X(\omega) \in A\}) \\
&= P(X^{-1}(A))
\end{aligned}
$$

From now on, we can often "forget" te original probability space and focus on the random variables and their distributions.

Definition: let X be a random variable. The **CDF**(cumulative distribution function) F of X is defined by

$$
F(x) = P(X \leq x) = P(X \in (-\infty, x]))
$$
$$
X : \text{random variable}, x : \text{number}
$$

Properties of cdf:

1. F is non-decreasing. $F(x_1) \leq F(x_2), x_1 < x_2$
2. limits
    - $\lim_{x \to -\infty} F(x) = 0$
    - $\lim_{x \to \infty} F(x) = 1$
3. $F(x)$ is right continuous
    - $lim_{x \downarrow a} F(x) = F(a)$: x decreases to a(approaching from the right)
    - Hint: $\{x \leq a\} = \bigcap_{i=1}^{\infty} \{X \leq a_i\}$ for $a_i \downarrow a$

2.2 Discrete random variables and distributions

A random variable X is called **discrete** if it only takes values in an **at most countable** set $\{x_1, x_2, \ldots\}$ (finite or countable).

The distribution of a discrete random variable is fully characterized by its **probability mass function**(p.m.f)

$$
p(x) := P(X = x); x = x_1, x_2, \ldots
$$

Properties of pmf:

1. $p(x) \geq 0, \quad \forall x$
2. $\sum_i p(x_i) = 1$

Q: what does the cdf of a discrete random variable look like?

2.2.1 Examples of discrete distributions

1. Bemoulli distribution

$$
\begin{aligned}
&p(1) = P(X = 1) = p \\
&p(c) = P(X = c) = 1 - p \\
&p(x) = 0 \quad otherwise
\end{aligned}
$$

Denote $X \sim Ber(p)$

2.Binomial distribution

$$
p(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}
$$

- $X \sim Bin(n, p)$ to choose k successes.
- Binomial distribution is the distribution of number of successes in n independent trials; each having probability p of success.

3.Geometric distribution

$$
p(k) = P(X = k) = (1-p)^{k-1} p
$$

$$(1-p)^{k-1} : \text{the first k-1 trials are all failures}, \quad p : \text{success in } k^{th} \text{ trial}$$

- $X \sim Geo(p)$
- X is the number of trials needed to get the first success in n independent trials with probability p of success each
- X has the memoryless property $P(X > n+m|X > m) = P(x > n) \qquad n, m = 0, 1, \ldots$

Memoryless property:

$$p(X > n+m|X > m) = P(X > n)$$

Proof:

$$\begin{aligned}
P(X > k) &= \sum_{j=k+1}^{\infty} P(X = j) \\
&= \sum_{j=k+1}^{\infty} (i-p)^{j-1} p \\
&= (1-p)^k p \cdot \frac{1}{1-(1-p)} \\
&= (1-p)^k \\
P(X > n+m|x > m) &= \frac{P(X > n+m), X > m}{P(X > m)} \\
&= \frac{P(X > n+m)}{P(X > m)} = \frac{1-p)^{n+m}}{(1-p)^m} = (1-p)^n = P(X > n)
\end{aligned}$$

Intuition: The failures in the past have no influence on how long we still need to wait to get the first success in the future

4. Poisson distribution

$$p(k) = P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, \ldots, \lambda > 0$$

Other discrete distributions:

- negative binomial
- discrete uniform

2.3 Continuous random variables and distributions

Definition: A random variable X is called **continuous** if there exists a non-negative function f, such that for any interval $[a, b]$, (a, b) or $[a, b)$:

$$P(X \in [a, b]) = \int_a^b f(x)dx$$

The function f is called the *probability density function(pdf)* of X

Remark: probability density function(pdf) is not probability. $P(X = x) = 0$ if X is continuous. The probability density function f only gives probability when it is integrated.

If X is continuous, then we can get cdf by:

$$F(a) = P(X \in (-\infty, a]) = \int_{\infty}^{a} f(x)dx$$

hence, $F(x)$ is continuous, and differentiable "almost everywhere".

We can take $f(x) = F'(x)$ when the derivative exists, and $f(x) =$arbitrary number otherwise often to choose a value to make f have some continuity.

Property of pdf:

1. $f(x) \le 0, x \in R$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. For $A \subseteq R, P(X \in A) = \int_A f(x)dx$

2.3.1 Example of continuous distribution

Exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ,x \geq 0 \\ 0 & ,x \leq 0 \end{cases}$$
$$X \sim Exp(x)$$

Other continuous distributions:

- Normal distribution
- Uniform distribution

Exercises:

1. Find the cdf of $X \sim Exp(x)$
2. Show that the exponential distribution has the memoryless property:

$$P(X > t + s | x > t) = P(X > s)$$

2.4 Joint distribution of r.v's

Let X and Y be two r.v's. defined on the same probability space (Ω, ξ, P)

For each $\omega \in \Omega$, we have at the same time $X(\omega)$ and $Y(\omega)$. Then we can talk about the joint behavior of X and Y

Two joint distribution of r.v's is characterized by joint cdf, joint pmf(discrete case) or joint pdf(continuous case).

- Joint cdf:
  - $F(x, y) = P(X < x, Y < y)$
- Joint pmf:
  - $f(x, y) = P(X = x, Y = y)$
- joint pdf $f(x, y)$ such that for $a < b, c < d$
  - $P(X, Y) \in (a, b] \times (c, d] = P(X \in (a, b], Y \in (c, d]) = \int_a^b \int_c^d f(x, y) dy dx$
  - Equivalently:
    1. $F(x, y) = \int_{-\infty}^x \int_{-\infty}^y f(s, t) dt ds$
       $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$
    2. $P((X, Y) \in A) = \int \int_A f(x, y) dx dy$ for $A \subseteq R^2$

Definition: Two r.v's X and Y are called independent, if for all sets $A, B \subseteq R$,

$$P(X < A, Y < B) = P(X \in A) P(Y \in B)$$

($\{X \in A\}$ and $\{Y \in B\}$ are independent events)

Theorem: Two r.v's X and Y are

1. independent, if and only if
2. $F(x, y) = F_x(x) F_y(y); x, y \in R$; where F_x: cdf of x; F_y: cdf of y
3. $f(x, y) = f_x(x) f_y(y); x, y \in R$; where f is the joint pmf/pdf of X and Y; f_x, f_y are marginal pmf/pdf of X and Y, respectively

Proof:

1.$\Rightarrow$ 2.

If $X \perp Y$, then by definition,

$$F(x, y) = P(X \in (-\infty, x], Y \in (-\infty, y])) = P(X \in (-\infty, x])) \cdot P(Y \in (-\infty, y])) = F_x(x) F_y(y)$$

2.$\Rightarrow$ 3.

Assume $F(x, y) = F_x(x) \cdot F_y(y)$,

$$\begin{aligned} f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) &= \frac{\partial^2}{\partial x \partial y} F_x(x) F_y(y) \\ &= (\frac{\partial}{\partial x} F_x(x))(\frac{\partial}{\partial y} F_y(y)) \\ &= f_x(x) f_y(y) \end{aligned}$$

3.⇒ 1.

Assume $f(x,y) = f_x(x)f_y(y)$; For $A, B \subseteq R$,

$$\begin{aligned} P(X \in A, Y \in B) &= \int_{y\in B}\int_{x\in A} f(x,y)dxdy \\ &= \int_{y\in B}\int_{x\in A} f_x(x)f_y(y)dxdy \\ &= (\int_{x\in A} f_x(x)dx)(\int_{y\in B} f_y(y)dy) \\ &= P(X\in A)P(Y\in B) \end{aligned}$$

2.5 Expectation

Definition: For a r.v X, the expectation of $g(x)$ is defined as

$$\exists(g(x)) = \begin{cases} \sum_{i=1}^{\infty} g(x_i)P(X = x_i) & \text{for discrete } X \\ \int_{-\infty}^{\infty} g(x)f(x)dx & \text{for continuous} X \end{cases}$$

Let X,Y be two r.v's; then the expectation of $g(X,Y)$ is defined in a similar way.

$$\exists(g(x,y)) = \begin{cases} \sum\sum g(x_i,y_j)P(X = x_i, Y = y_j) \\ \int\int g(x_i,y_j)f(x,y)dxdy \end{cases}$$

2.5.1 Properties of expectation

1. Linearity:expectation of X: $\mathbb{E}(X) = \begin{cases} \sum X_i\mathbb{P}(X = x_i) \\ \int_{-\infty}^{x_1} xf(x)dx \end{cases}$, $g(X) = x$
   - $\mathbb{E}(ax+b) = a\mathbb{E}(x) + b$
   - $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$
2. If $X \perp\!\!\!\perp Y$, then $\mathbb{E}(g(X)h(Y)) = \mathbb{E}(g(X)) \cdot \mathbb{E}(h(Y))$
   - **proof**: (continuous case)
   - $$\begin{aligned} \mathbb{E}(g(X)h(Y)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(f)f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty} g(x)f_X(x) \cdot \int_{-\infty}^{\infty} h(y)f_Y(y)dy \end{aligned}$$
   - In particular, $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ if $X \perp\!\!\!\perp Y$

2.5.2 Definitions

Definition: The expectation $\mathbb{E}(X^n)$ is called the n-th moment of X:

- 1st moment: $\mathbb{E}(X)$
- 2st moment: $\mathbb{E}(X^2)$

Definition: The variance of a r.v X is defined as:

$$Var(x) = \mathbb{E}((X - \mathbb{E}(X))^2) \text{ also denoted as } \sigma^2, \sigma_x^2$$

Definition: the covariance of the r.v's X and Y is defined as:

$$Cov(X,Y) = \mathbb{E}((X - \mathbb{E}(X)))\mathbb{E}((Y - \mathbb{E}(Y)))$$

Thus $Var(X) = Cov(X,X)$

Definition: the correlation between X and Y is defined as:

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Fact: $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

Proof:

$$\begin{aligned} Var(X) &= \mathbb{E}((X - \mathbb{E}(X))^2) \\ &= \mathbb{E}(X^2 - 2X\mathbb{E}(X) + (\mathbb{E}(X)^2)) \\ &= \mathbb{E}(X^2) - 2\mathbb{E}(X\mathbb{E}(X)) + (\mathbb{E}(X))^2 \\ &= \mathbb{E}(X^2) - 2(\mathbb{E}(X))^2 + (\mathbb{E}(X))^2 \\ &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \end{aligned}$$

Fact: $Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

Proof: similar to previous

Variance and covariance are **translation invariant**. Variance is quadratic, covariance is bilinear.

$$Var(aX + b) = a \cdot Var(X)$$

$$Cov(aX + b, cY + d) = ac \cdot Cov(X,Y)$$

Proof:

$$\begin{aligned} Var(aX + b) &= \mathbb{E}((aX + b0\mathbb{E}(aX + b)^2)) \\ &= \mathbb{E}([a(X - \mathbb{E}(X))]^2) \\ &= a^2\mathbb{E}((X - \mathbb{E}(X)^2)) \\ &= a^2\mathbb{E}(X) \end{aligned}$$

Proof: $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

Exercise

If $X \perp\!\!\!\perp Y$, then $Cov(X,Y) = 0$ and $Var(X + Y) = Var(X) + Var(Y)$

Proof:

$$\begin{aligned} &Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \\ &\text{we know:} \\ &X\ Y \Rightarrow \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \\ &\text{Thus, } Cov(X,Y) = 0 \Rightarrow Var(X + Y) = Var(X) + Var(Y) + 2Cov(X,Y) \\ &\text{So we see independence} \Rightarrow \text{Covariance is 0: "uncorrelated"} \\ &\text{the converse is not true.} \\ &Cov(X,Y) = 0 \Rightarrow \not\text{independence} \end{aligned}$$

Remarks

We have $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.

If $X \perp\!\!\!\perp Y$, we also have:

- $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$, and
- $Var(X + Y) = Var(X) + Var(Y)$

It's important to remember that the first result and the other two results are of very different nature. While $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$ is a property of expectation and holds unconditionally;

the other two, $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ and $Var(X + Y) = Var(X) + Var(Y)$, only hold if $X \perp\!\!\!\perp Y$.

It is more appropriate to consider them as **properties of independence** rather than properties of expectation and variance

2.6 Indicator

A random variable I is called an indicator, if

$$I(w) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \not\in A \end{cases}$$

$$P(I_A) = P(A)$$

for some event A

For A given, I is also elevated as I_A

The most important property of indicator is its expectation gives the probability of the event $\mathbb{E}(I_A) = \mathbb{P}(A)$

Proof:

$$\begin{aligned} \mathbb{P}(I_A = 1) &= \mathbb{P}(\omega : I_A(\omega = 1)) \\ &= \mathbb{P}(\omega : \omega \in A) \\ &= \mathbb{P}(A) \end{aligned}$$

$$\mathbb{P}(I_A = 0) = 1 - \mathbb{P}(A) \Rightarrow \mathbb{E}(I_A) = 1 \cdot \mathbb{P}(A) + c \cdot (1 - \mathbb{P}(A)) = \mathbb{P}(A)$$

2.6.1 Example

we see $I_A \sim Ber(\mathbb{P}(A))$

Let $X \sim Bin(n,p)$, X is number of successes in n Bernoulli trials, each with probability p of success

$$\Rightarrow X = I_1 + \cdots + I_n$$

where $I_1, \cdots, I_n$ are indicators for independent events. $I_i = 1$ if th i the trial is a success. $I_i = 0$ if the i th trial is a failure.

Hence I_i are **i.d.**(independent and identically distributed) r.v's

$$\begin{aligned} \Rightarrow \mathbb{E}(X) &= \mathbb{E}(I_1 + \cdot, I_N) \\ &= \mathbb{E}(I_1) + \cdots | \mathbb{E}(I_n) \\ &= p + \cdots + p = n \cdot p \end{aligned}$$

$$\begin{aligned} Var(X) &= Var(I_1 + \cdots + I_n) \\ &= Var(I_1) + \cdots + Var(I_n) \\ &= n \cdot Var(I_i) \\ &= n \cdot p(1-p) \end{aligned}$$

$$Var(I_1) = \mathbb{E}(I_1^2) - (\mathbb{E}(I_1))^2 = \mathbb{E}(I_1) - (\mathbb{E}(I_1))^2 = p - p^2 = p(1-p)$$

2.6.1 Example 3

Let X be a r.v. taking values in non-negative integers, then

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} P(X > n)$$

Proof:

Note that $X = \sum_{n=0}^{\infty} I_n$ where $I_n = I_{x>n}$. ($x > n$ is an event)

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}(\sum_{n=0}^{\infty} I_n) \\ &= \sum_{n=0}^{\infty} \mathbb{E}(I_n) \\ &= \sum_{n=0}^{\infty} P(X > n) \end{aligned}$$

In particular, let $X \sim Geo(p)$. As we have seen, $P(X > n) = (1-p)^n \Rightarrow$

$$
\begin{aligned}
\mathbb{E}(X) &= \sum_{n=0}^{\infty} P(X > n) \\
&= \sum_{n=0}^{\infty} (1-p)^n \\
&= \frac{1}{1-(1-p)} = \frac{1}{p}
\end{aligned}
$$

2.7 Moment generating function

Definition: Let X be a r.v. Then the function $M(t) = \mathbb{E}(e^{tx})$ is called the *moment generating function(mgf)* of X, if the expectation exists for all $t \in (-h, h)$ for some $h > 0$.

Remark: The mgf is not always well-defined. It is important to check the existence of the expectation.

2.7.1 Properties of mgf

1. Moment Generating Function generates moments
   - **Theorem**:
     - $M(0) = 1$
     - $M^{(k)}(0) = \mathbb{E}(X^k), k = 1, 2, \dots$ ($M^{(k)} = \frac{d^k}{dt^k} M(t)|_{t=0}$)
       - **Proof**:

$$
\begin{aligned}
M(0) = \mathbb{E}(e^{0 \cdot X}) = \mathbb{E}(1) = 1 \\
M^{(k)}(0) = \frac{d^k}{dt^k} \mathbb{E}(e^{t \cdot X}))|_{t=0} \\
= \mathbb{E}(\frac{d^k}{dt^k} e^{tX}|_{t=0}) \\
= \mathbb{E}(X^k)
\end{aligned}
$$

     - As a result, we have: $M(t) = \sum_{k=0}^{\infty} \frac{M^{(k)}(0)}{k!} t^k = \sum_{k=0}^{\infty} \frac{E*X^k}{k!} t^k$ (a method to get moment of a r.v)
2. $X \perp\!\!\!\perp Y$, with mgf's M_x, M_y. Let M_{X+Y} be the mgf of $X + Y$. then

$$
M_{X+Y}(t) = M_X(t) M_Y(y)
$$

   - **Proof**:

$$
\begin{aligned}
M_{X+Y}(t) &= \mathbb{E}(e^{t(X+Y)}) \\
&= \mathbb{E}(e^{tx} e^{ty}) \\
&= \mathbb{E}(e^{tx}) \mathbb{E}(e^{ty}) \\
&= M_X(y) M_Y(t)
\end{aligned}
$$

3. The mgf completely determines the distribution of a r.v.
   - $M_X(t) = M_Y(t)$ for all $t \in (-h, h)$ for some $h > 0$, then $X \stackrel{d}{=} Y$. ($\stackrel{d}{=}$: have the smae distribution)
   - Example: Let $X \sim Poi(\lambda_1), Y \sim Poi(\lambda_2)$. $X \perp\!\!\!\perp Y$. Find the distribution of $X + Y$
     - First, derive the mgf of a Poisson distribution.

$$
\begin{aligned}
M_X(t) &= \mathbb{E}(e^{tX}) \\
&= \sum_{n=0}^{\infty} e^{tn} \cdot P(X = n) \\
&= \sum_{n=0}^{\infty} e^{tn} \cdot \frac{\lambda_1^n}{n!} e^{-\lambda_1} \\
&= \sum_{n=0}^{\infty} \frac{(e^t \cdot \lambda_1)^n}{n!} \cdot e^{-\lambda_1}
\end{aligned}
$$

$$
\begin{aligned}
\text{we know that } \sum_{n=0}^{\infty} \frac{(e^t \lambda_1)^n}{n!} = e^{e^t \cdot \lambda_1}.(\text{Since } \frac{(e^t \lambda_1^n)}{n!} e^{-e^t \lambda_1} \text{ is the pmf of } Poi(e^t \lambda_1)) \\
\Rightarrow M_X(t) = e^{e^t \lambda_1} e^{-\lambda_1} = e^{\lambda_1 (e^t - 1)}, t \in \mathbb{R}.(e^{\lambda_1 (e^t - 1)} \text{ is mgf of } Poi(\lambda_1)) \\
\text{Similarly, } M_Y(t) = e^{\lambda_2 (e^t - 1)}.
\end{aligned}
$$

We know that

$$\begin{aligned} M_{X+Y}(t) &= M_X(t)M_Y(t) \\ &= e^{\lambda_1(e^t-1)}e^{\lambda_2(e^-1)} \\ &= e^{(\lambda_1+\lambda_2)(e^t-1)} \end{aligned}$$

This is the mgf of $Poi(\lambda_1 + \lambda_2)$!

Since the mgf uniquely determines the distribution $X + Y \sim Poi(\lambda_1 + \lambda_2)$

In general, if $X_1, X_2, \ldots, X_n$ independent, $X_i \sim Poi(\lambda_i)$, then $\sum X_i \sim Poi(\sum \lambda_i$

2.7.2 Joint mgf

Definition: Let X, Y be r.v's. Then $M(t_1, t_2) := \mathbb{E}(e^{t_1X+t_2Y}$ is called the joint mgf of X and Y, if the expectation exists for all $t_1 \in (-h_1, h_1)$, $t_2 \in (-h_2, h_2)$ for some $h_1, h_2 > 0$.

More generally, we can define $M(t_1, \ldots, t_n) = \mathbb{E}(exp(\sum_{i=1}^n t_iX_i))$ for r.v's $X_1, \cdots, X_n$, if the expectation exists for $\{(t_1, \cdots, t_n) : t_i \in (-h_i, h_i), i = 1, \cdots, n\}$ for some $\{h_i > 0\}, i = 1, \cdots, n$

2.7.2.1 Properties of the joint mgf

1. $$\begin{aligned} M_X(t) &= \mathbb{E}(e^{tX}) \\ &= \mathbb{E}(e^{tX+oY}) \\ &= M(t, o) \\ M_Y(t) &= M(o, t) \end{aligned}$$

2. $$\frac{\partial^{m+n}}{\partial t_1^m \partial t_2^n} M(t_1, t_2)|_{(0,0)} = \mathbb{E}(X^mY^n$$
$$\text{the proof is similar to the single r.v. case}$$

3. If $X \perp\!\!\!\perp Y$, then $M(t_1, t_2) = M_X(t_1)M_Y(t_2)$
   - **Proof**:

$$\begin{aligned} M(t_1, t_2) &= \mathbb{E}(e^{t_1X+t_2Y}) \\ (X \perp\!\!\!\perp Y) &= \mathbb{E}(e^{t_1X}e^{t_2Y}) \\ &= \mathbb{E}(e^{t_1X}) \cdot \mathbb{E}(e^{t_2Y}) \\ &= M_X(t_1) \cdot M_Y(t_2) \end{aligned}$$

   - **Remark**: Don't confuse this with the result $X \perp\!\!\!\perp Y \Rightarrow M_{X+Y}(t) = M_X(t)M_Y(t)$.
     - $M_{X+Y}(t) \rightarrow$ mgf of $X + Y$; single argument function t
     - $M(t_1, t_2) \rightarrow$ joint mgf of (X, Y); two arguments t_1, t_2

3. Conditional distribution and conditional expectation

3.1 Conditional distribution

3.1.1 Discrete case

Definition Let X and Y be discrete r.v's. The conditional distribution of X given Y is given by:

$$P(X = x|Y = y) = \frac{(P(X = x, Y = u))}{P(Y = y)}$$

$$P(X = x|Y = y) : f_{X|Y} = y(x), f_{X|Y}(x|y) \leftarrow \text{conditional probability mass function})$$

Conditional pmf is a legitimate pmf: given any y, $f_{X|Y=y}(x) \geq 0, \forall x$

$$\sum_x f_{X|Y=y}(x) = 1$$

Note that given $Y = y$, as x changes, the value of the function $f_{X|Y=y}(x)$ is proportional to the joint probability.

$$f_{X|Y=y}(x) \propto P(X = x, Y = y)$$

This is useful for solving problems where the denominator $P(Y=y)$ is hard to find.

3.1.1.1 Example

$X_1 \sim Poi(\lambda_1), X_2 \sim Poi(\lambda_2)$. $X_1 \perp\!\!\!\perp X_2, Y = X_1 + X_2$

Q: $P(X_1 = k|Y = n)$?

Note $P(X_1 = k|Y = u) = f_{X_1|Y=n}(k)$

A: $P(X_1 = k|Y = n)$ can only be non-zero for $k = 0, \cdots, n$ in this case,

$$\begin{aligned}
P(X_1 = k|Y = n) &= \frac{P(X_1 = k, Y = n)}{P(Y = n)} \\
&\propto P(X_1 = k, Y = n) \\
&= P(X_1 = k, X_2 = n-k) \\
&= e^{-\lambda_1}\frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!} \\
&\propto (\frac{\lambda_1}{\lambda_2})^k / k!(n-k)!
\end{aligned}$$

we can get $P(X = k|Y = n)$ by normalizing the above expression.

$$P(X_1 = k, Y = n) = \frac{(\frac{\lambda_1}{\lambda_2})^k / k!(n-k)!}{\sum_{k=0}^n (\frac{\lambda_1}{\lambda_2})^k / k!(n-k)!}$$

but then we will need to fine $\sum_{k=0}^n (\frac{\lambda_1}{\lambda_2})^k / k!(n-k)!$

An easier way is to compare $\sum_{k=0}^n (\frac{\lambda_1}{\lambda_2})^k / k!(n-k)!$ with the known results for common distribution. In particular, if $X \sim Bin(n, p)$

$$\begin{aligned}
P(X = k) &= (\overset{n}{k}) p^k (1-p)^{n-k} \\
&\propto (\frac{p}{1-p})^k / k!(n-k)!
\end{aligned}$$

$\Rightarrow P(X_1 = k|Y = n)$ follows a binomial distributions with parameters n and p given by $\frac{p}{1-p} = \frac{\lambda_1}{\lambda_2} \Rightarrow p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

Thus, given $Y = X_1 + X_2 = n$, the conditional distribution of X_1 is binomial with parameter n and $\frac{\lambda_1}{\lambda_1 + \lambda_2}$

3.1.2 Continuous case

Definition: Let X and Y be continuous r.v's. The conditional distribution of X given Y is given by

$$f_{X|Y}(x|y) = f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

A conditional pdf is a legitimate pdf

$$\begin{aligned}
&f_{X|Y}(x|y) \geq 0 &\quad x, y \in \mathbb{R} \\
&\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = 1, &\quad y \in \mathbb{R}
\end{aligned}$$

3.1.2.1 Example

Suppose $X \sim Exp(\lambda), Y|X = x \sim Exp(x) = f_{Y|X}(y|x) = xe^{-xy}, y = e \leftarrow$ conditional distribution of Y given $X = x$

Q: Find the condition pdf $f_{X|Y}(x|y)$

A:

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\
&\propto f(x,y) \\
&= f_{Y|X}(y|x) \cdot f_X(x) \\
&= xe^{xy}\lambda e^{-\lambda xa} \\
&\propto xe^{-x(y+\lambda)}, \qquad x > 0, y > 0
\end{aligned}$$

Normalization (make the total probability 1)

$$f_{X|Y}(x|y) = \frac{xe^{-x(y+\lambda)}}{\int_0^\infty xe^{-x(y+\lambda)}dx}$$
$$\int_0^\infty xe^{-x(y+\lambda)}dx = \frac{1}{\lambda+t}^2 \leftarrow \text{integration by parts}$$

Thus, $f_{X|Y}(x|y) = (\lambda+y)^2 xe^{-x(y+\lambda)}$, $x > 0$.

This is a gamma distribution with parameters γ and $\lambda + y$

3.1.2.1. Example 2

Find the distribution of $z = XY$.

Attention: the following method is wrong:

$$f_Z(z) = \int_0^\infty f_{Y|X}(\frac{z}{x}|x) \cdot f_X(x)dx$$

If we want to directly work with pdf's, we will need to use the change of variable formula for multi-variables. The right formula have turns out to be

$$\begin{aligned}
f_Z(z) &= \int_0^\infty f_{X,Z}(x,z)dx = \int_0^\infty f_{Z|X}(z|x)f_X(x)dx \\
&= \int_0^\infty f(x,\frac{z}{x}) \cdot \frac{1}{x}dx \\
&= f_{Y|X}(\frac{z}{x}|x)f_X(x) \cdot \frac{1}{x}dx
\end{aligned}$$

As an **easier way** is to use cdf, which gives probability rather than density:

$$\begin{aligned}
P(Z = z) &= P(XY \le z) \\
&= \int_0^\infty P(XY \le z|X = x)f_X(x)dx \qquad \text{(law of total probability)} \\
&= \int_0^\infty P(Y \le \frac{z}{x}|X = x) \cdot f_X(x)dx \\
Y|X = x \sim Exp(x) & \\
&= \int_0^\infty (1 - e^{-x\cdot\frac{z}{x}}) \cdot \lambda e^{-\lambda x}dx \\
&= 1 - e^{-z}\int_0^\infty \lambda e^{-\lambda x}dx \\
\Rightarrow Z \sim Exp(1) &
\end{aligned}$$

Notation $X, Y|\{Z = k\} \overset{iid}{\sim} \cdots$ means that given $Z = k$, X and Y are *conditionally independent*, and they follow certain distribution.

(the conditional joint cdf/pmf/pdf equals the predict of the conditional cdf's/pmf's/pdf's)

3.2 Conditional expectation

We have seen that conditional pmf/pdf are legitimate pmf/pdf. Correspondingly, a conditional distribution is nothing else but a probability distributions. It is simply a (potentially) different distribution, since it takes more information into consideration.

As a result, we can define everything which are previously defined for unconditional distributions also for conditional distributions.

In particular, it is natural to define the conditional expectation.

Definition. The conditional expectation of $g(X)$ given $Y = y$ is defined as

$$\mathbb{E}(g(X)|Y = y) = \begin{cases} \sum_{i_1}^\infty g(x_i)P(X = x_u|Y = y) & \text{if } X|Y = y \text{ is discrete} \\ \int_{-\infty}^\infty g(x)f_{X|Y}(x|y)dx & \text{if } X|X = y \text{ is continuous} \end{cases}$$

Fix y, the conditional expectation is nothing but the expectation taken under the conditional distribution.

3.2.1 What is $\mathbb{E}(X|Y)$?

Different ways to understand *conditional expectation*

1. Fix a value y, $\mathbb{E}(g(X)|Y=y)$ is a number
2. As y changes $\mathbb{E}(g(x)|Y=y)$ becomes a function of y (that each y gives a value): $h(y) =: \mathbb{E}(g(x)|Y=y)$
3. since y is actually random, we can define $\mathbb{E}(g(x)|Y) = h(Y)$. This is a random variable

$$\mathbb{E}(g(x)|Y))_{(\omega)} = \mathbb{E}(g(x)|Y = Y(\omega))$$

$\omega \in \Omega$ this random variable takes value $\mathbb{E}(g(x)|Y=y)$ When $Y=y$

$$\begin{gathered}\Omega \to \mathbb{R} \\ h(Y)_{(\omega)} = h(Y(\omega))\end{gathered}$$

3.2.2 Properties of conditional expectation

1. Linearity (inherited from expectation)

$$\mathbb{E}(aX+b|Y=y) = a\mathbb{E}(X|Y=y)+b$$

$$\mathbb{E}(X+Z|Y=y) = \mathbb{E}(X|Y=y)+\mathbb{E}(Z|Y=y)$$

2. $\mathbb{E}(g(X,Y)|Y=y) = \mathbb{E}(g(X,y)|Y=y) \neq \mathbb{E}(g(X,y)) \text{ when X and Y are not independent}$

Proof (Discrete):

$$\mathbb{E}(g(X,Y)|Y=y) = \sum_{x_i}\sum_{y_j} g(x_i,y_j)\cdot P(X=x_i, Y=y_j|Y=y)$$

$$P(X=x_i,Y=y_j|Y=y) = \begin{cases} 0 & \text{if } y_j \neq y \\ \\ P(X=x_1,Y=y_j)/P(Y+y) = P(X=x_i|Y=y) & \text{if } y_j = y\end{cases}$$

$$\begin{aligned}\Rightarrow \mathbb{E}(g(X,Y)|Y=y) &= \sum_{x_i} g(x_i,y)\cdot P(X=x_i|Y=y) \\ &= \mathbb{E}(g(X,y)|Y=y) \qquad\qquad g(X,y) \text{ regarded as a function of } x\end{aligned}$$

In particular,

$$\mathbb{E}(g(X)\cdot h(Y)|Y=y) = h(y)\mathbb{E}(g(X)|Y=y)$$

$$\mathbb{E}(g(X)\cdot h(Y)|Y) = h(Y)\mathbb{E}(g(X)|Y)$$

3. If $X \perp Y$, then $\mathbb{E}(g(X)|Y=y) = \mathbb{E}(g(X))$

Fact: If $X \perp Y$, then conditional distribution of X given $Y=y$ is the same as the unconditional distribution of X

Proof(Discrete):

$$\begin{aligned}&\text{if } X \perp Y \\ &P(X=x_i|Y=y_j) \\ &\quad = P(X=x_i|Y=y_j)/P(Y=y_j) \\ &\quad = P(X=x_i)P(Y=y_j)/P(Y=y_j) \\ &\quad = P(X=x_i)\end{aligned}$$

4. Law of iterated expectation (or double expectation): Expectation of conditionally expectation is its unconditional expectation

$$\mathbb{E}(\mathbb{E}(X|Y))) = \mathbb{E}(X)$$

$\mathbb{E}(X|Y)$ is a r.v, a function of Y.
Proof(Discrete):

When $Y=y_j$, the r.v. $\mathbb{E}(X|Y) = \mathbb{E}(X|Y=y_j) = \sum_{x_i} x_i P(X=x_i|Y=y_j)$. This happens with probability $P(Y=y_j)$

$$
\begin{aligned}
\mathbb{E}(\mathbb{E}(X|Y)) &= \sum_{y_j}(\sum_{x_i} x_i P(X=x_i|Y=y_j))P(Y=y_j) \\
\Rightarrow \quad &= \sum_{x_i}\sum_{y_j} P(X=x_i|Y=y_j)P(Y=y_j) \\
&= \sum_{x_i} x_i \sum_{y_j} P(X=x_i|Y=y_j)P(Y=y_2) \qquad \text{law of total probability} \\
&= \sum_{x_i} x_i P(X=x_i) = \mathbb{E}(X)
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
&\sum_{x_i}\sum_{y_j} x_i P(X=x_i|Y=y_j)P(Y=y_j) \\
&= \sum_{x_i}\sum_{y_j} x_i P(X=x_i, Y=y_j) \qquad\qquad g(X,Y)=X \text{ at } (x_i,y_j) \\
&= \mathbb{E}(X)
\end{aligned}
$$

Example:

Y: # of claims receive by insurance company
X: some random parameter

$$Y|X \sim Poi(X), X \sim Exp(\lambda)$$

a) $\mathbb{E}(Y)$?
b) $P(Y=n)$?

a)

$$Y|X \sim Poi(X) \Rightarrow \mathbb{E}(Y|X=x) = x \Rightarrow \mathbb{E}(Y|X) = X$$

$$
\begin{aligned}
\therefore \mathbb{E}(Y) &= \mathbb{E}(\mathbb{E}(Y|X)) \\
&= \mathbb{E}(X) = \frac{1}{\lambda}
\end{aligned}
$$

b)

$$
\begin{aligned}
P(Y=n) &= \int_0^\infty P(Y=n|X=x)f_x(x)dx \\
&= \int_o^\infty \frac{e^{-x}x^n}{n!}\cdot\lambda e^{-\lambda x}dx \\
&= \frac{\lambda}{n!}\int_0^\infty x^n e^{-(\lambda+1)x}dx \\
&= \frac{\lambda}{(\lambda+1)^{n+1}n!}\int_0^\infty ((\lambda+1)x)^n e^{-(\lambda+1)x}d(\lambda+1)x \\
&= \frac{\lambda}{(\lambda+1)^{n+1}n!}\Gamma(n+1) \qquad\qquad \Gamma(n+1)=n!\text{ ; formula for gamma function or integration by parts} \\
&= \frac{\lambda}{(\lambda+1)^{n+1}} = (\frac{1}{\lambda+1})^n\cdot\frac{1}{\lambda+1} \\
&\Rightarrow Y+1 \sim Geo(\lambda/(\lambda+1))
\end{aligned}
$$

We verify that $\mathbb{E}(Y) = \frac{\lambda+1}{\lambda} - 1 = \frac{1}{\lambda}$

3.3 Decomposition of variance (EVVE's low)

Definition: The conditional variance of Y given $X=x$ is defined as

$$Var(Y|X=x) = \mathbb{E}((Y-\mathbb{E}(Y|X=x))^2|X=x)$$

$$Var(Y|X)_{(\omega)} = Var(Y|X=X_{(\omega)}) \qquad Var(Y|X)_{(\omega)}\text{: a r.v, a function of } X$$

The conditional variance is simply the variance taken under the conditional distribution

$$\Rightarrow V(Y|X=x) = \mathbb{E}(Y^2|X=x) - (\mathbb{E}(Y|X=x))^2$$

Thus

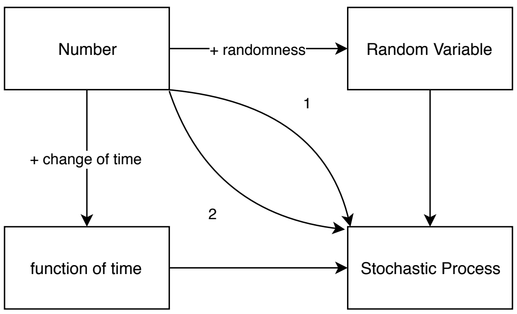
$$Var(Y) = \mathbb{E}(Var(Y|X)) + Var(\mathbb{E}(Y|X))$$

$$\mathbb{E}(Var(Y|X)) : \text{"intra-group variance"} \qquad Var(\mathbb{E}(Y|X)) : \text{"inter-group variance"}$$

Proof:

$$\begin{aligned} RHS &= E(E(Y^2|X) - (E(Y|X))^2) + E((E(Y|X))^2) - (E(E(Y|X)))^2 \\ &= E(E(Y^2|X)) - \cancel{E((E(Y|X))^2)} + \cancel{E((E(Y|X))^2)} - (E(E(Y|X)))^2 \\ &= E(Y^2) - (E(Y))^2 \\ &= Var(Y) \end{aligned}$$

4. Stochastic Processes



1. sequence / family of random variables
2. a random function (hard to formulate)

Definition: A **stochastic process** $\{X_t\}_{t \in T}$ is a collection of random variables, defined on a common probability space.

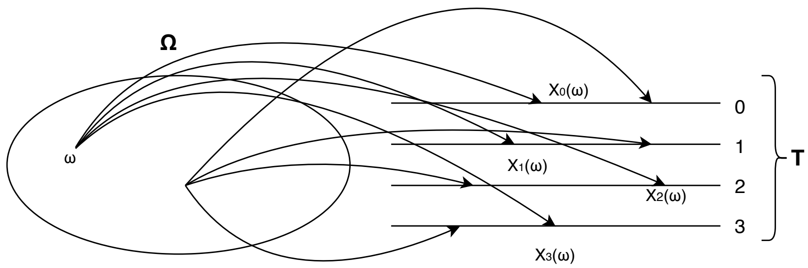
T: index set. In most cases, T corresponds to time, and is either discrete $\{0, 1, 2, \cdots\}$ or continuous $[0, \infty)$

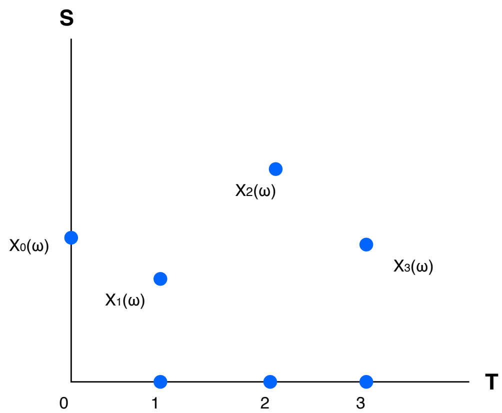
In discrete case, we writes $\{X_n\}_{n=0,1,2,\dots}$

This **state space** S os a stochastic process is the set of all possible value of $X_t, t \in T$

S can also be either discrete or continuous. In this course, we typically deal with **discrete** stat space. Then we relabel the stats so that $S = \{0, 1, 2, \cdots\}$ (countable state space) or $S = \{0, 1, 2, \cdots, M\}$ (finite state space)

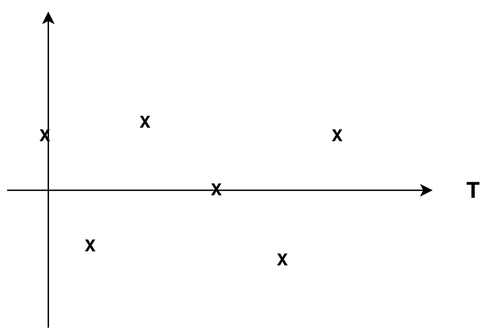
Remark: As in the case of the joint distribution, we need the r.v's in a stochastic process to be defined on a common probability space, because we want to discuss their joint behaviours, i.t, how things change over time.





Thus, we can identify each point ω in the sample space Ω with a function defined on T and taking value in S. Each function is called a **path** of this stochastic process

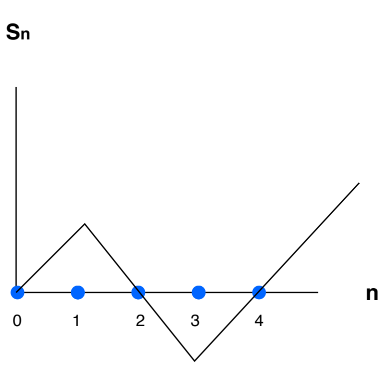
Example Let $X_0, X_1, \cdots$ be independent and identical r.v's following some distribution. Then $\{X_n\}_{n=0,1,2,\ldots}$ is a stochastic process



Example Let $X_1, X_2, \ldots$ be independent and identical r.v.'s. $P(X_1 = 1) = p$, and $P(X_1 = -1) = 1 - p$. Define $S_0 = 0, S_n = \sum_{i=1}^n X_i, n \leq 1$, e.g.

- $S_0 = 0$
- $S_1 = X_1$
- $S_2 = X_1 + X_2$
- $\cdots\cdots$

Then $\{S_n\}_{n=0,1,\ldots}$ is a stochastic process, with state space $S = \mathbb{Z}$ (integer)



4.1 Markov Chain

4.1.1 Simple Random Walk

$\{S_n\}_{n=0,1,...}$ is called a **"simple random walk"**. ($S_n = S_{n-1} + X_n$)

$$S_n = \begin{cases} S_{n-1}+1 \\ S_{n-1}-1 \end{cases}$$

Remark: Why we need the concept of "stochastic process"? Why don't we just look at the joint distribution of $(X_0, X_1, ..., X_n)$?

Answer: The joint distribution of a large number of r.v's is very complicated, because it does not take advantage of the special structure of T(time).

For example, simple random walk. The full distribution of $(S_0, S_1, ..., S_n$ is complicated or n large. However, the structure is actually simple if we focus on the adjacent times:

$$S_{n+1} = S_n + X_{n+1}$$

$$S_n: \text{ last value.} \quad X_{n+1}: \text{related to } Ber(p)\text{. They are independent}$$

By introducing time into the framework, we can greatly simplify many things.

More precisely, we fine that for simple random walk, $\{S_n\}_{n=0,1,...}$, if we know S_n the distribution of S_n+1 will not depend on the history $(S_0, ..., S_n-1$. THis is a very useful property

In general for a stochastic process $\{X_n\}_{n=0,1,...}$, at time n, we already know $X_0, X_1, ..., X_n$, S_0 our best estimate of the distribution of X_{n+1} should be the conditional distribution:

$$X_{n+1}|X_n, ..., X_n$$

given by:

$$P(X_{n+1} = x_{n+1}|X_n = x_n, ..., X_0 = x_0)$$

As time passes, the expression becomes more and more complicated $\rightarrow$ impossible to handle.

However, if we know that this conditional distribution is actually the same as the conditional distribution only given X_n, then the structure will remain simple for any time. This motivates the notion of *Markov chain*.

4.1.2 Markov Chain

4.1.2.1 Discrete-time Markov Chain

Definition and Examples

Definition: A discrete-time Stochastic process $\{X_n\}_{n=0,1,...}$ is called a **discrete-time Markov Chain (DTMC)**, if its state space S is discrete, and it has the Markov property:

$$\begin{aligned} &P(X_{n+1} = x_{n+1}|X_n = x_n, ..., X_o = x_o) \\ =\ &P(X_{n+1} = x_{n+1}|X_n = x_n) \end{aligned}$$

for all $n, x_0, ..., x_n, x_{n+1} \in S$

If $X_{n+1}|\{x_n = i\}$ does not change over time, $P(X_{n+1} = j|N_n = i) = P(X_1 = j|X_0 = i)$, then we call this Markov chain **time-homogeneous** (default setting for this course).

$$\begin{aligned} &P(X_{n+1} = x_{n+1}|X_n = x_n, ..., X_0 = x_0) \\ =\ &P(X_{n+1} = x_{x+1}|X_n = x_n) \end{aligned} \qquad \begin{aligned} X_{n+1} = x_{n+1}\text{: future; } X_n = x_n\text{: present(state)} \\ X_{n-1} = x_{n-1}, ..., X_0 = x_0\text{: past(history)} \end{aligned}$$

Intuition: Given the present state, the past and the future are independent. In other words, the future depends on the previous results only through the current state.

Example: simple random walk

The simple random walk $\{S_n\}_{n=0,1,...}$ is a Markov chain

Proof:

Recall that $S_{n+1} = S_n + X_{n+1}$

$$\begin{aligned}
&P(S_{n+1} = s_{n+1} | S_n = s_n, ..., S_0 = s_0) \\
&= 0 \\
&= P(S_{n+1} = s_{n+1} | S_n = s_n) s
\end{aligned}$$

if $s_{n+1} \neq s_n \pm$

$$\begin{aligned}
&P(S_{n+1} = s_n + 1 | S_n = s_n, ..., s_0 = 0) \\
&= P(X_{n+1} | S_n = s_n, ..., S_0 = 0) \\
&= P(X_{n+1} = 1) \qquad X_{n+1} \perp (X_1, ..., X_n) \text{ hence also } (S_0, ..., S_n)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&P(S_{n+1} = s_n + 1 | S_n = s_n) \\
&= P(X_{n+1} = 1 | S_n = s_n) \\
&= P(X_{n+1} = 1) \\
&\Rightarrow P(S_{n+1} | S_n = s_n, ..., S_0 = s_0)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&P(S_{n+1} = s_n - 1 | S_n = s_n, ..., S_0 = 0) \\
&= P(S_{n+1} = s_n - 1 | S_n = s_n) \\
&= P(X_{n+1} = -1) \\
&\Rightarrow \{S_n\}_{n=0,1,...} \text{ is a DTMC} \qquad \blacksquare
\end{aligned}$$

4.1.3 One-step transition probability matrix

For a time-homogeneous DTMC, define

$$\begin{aligned}
P_{ij} &= P(X_1 = j | X_0 = i) \\
&= P(X_{n+1} = j | X_n = i) \qquad n = 0, 1, ...
\end{aligned}$$

P_{ij}: one step transition probability

The collection of $P_{ij}, i, j \in S$ governs all the one-step transitions of the DTMC. Since it has two indices i and j; it naturally forms a matrix $P = \{P_{ij}\}_{i,k \in S}$, called the **(one-setp) transition (probability) matrix** or **transition matrix**

Property of a transition matrix $P = \{P_{ij}\}_{i,j \in S}$:

$$\begin{aligned}
&P_{ij} \geq 0 \quad \forall i, j \in S \\
&\sum_{j \in S} P_{ij} = 1 \quad \forall i \in S \quad \rightarrow \text{ the row some of } P \text{are all 1}
\end{aligned}$$

Reason:

$$\begin{aligned}
\sum_{j \in S} P_{ij} &= \sum_{j \in S} P(X_1 = j | X_0 = i) \\
&= P(X_1 \in S | X_o = i) \\
&= 1
\end{aligned}$$

Example 1 : simple random walk

There will be 3 cases:

$$\begin{aligned}
&P_{i,i+1} = P(S_1 = i + 1 | S_0 = i) = P(X_1 = 1) = p \\
&P_{i,i-1} = P(S_1 = i - 1 | S_0 = i) = P(X_1 = -1) = 1 - p =: q \\
&P_{i,j} = 0 \qquad \text{for } j \neq i \pm 1
\end{aligned}$$

$$\Rightarrow \text{(infinite dimension)} p = \begin{Bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & p & 0 & \dots & \dots & \dots \\ \dots & q & 0 & p & \dots & \dots & \dots \\ \dots & \dots & q & 0 & p & \dots & \dots \\ \dots & \dots & \dots & q & 0 & p & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{Bmatrix}$$

Example 2: Ehrenfest's urn

Two urns A, B, total M balls. Each time, pick one ball randomly(uniformly), and move it to the opposite urn.

$$X_n : \# \text{ of balls in } A \text{after step } n$$

$$S = \{0, 1, ..., M\}$$

$$\begin{aligned} P_{ij} &= P(X_1 = j | X_0 = j) \qquad (i \text{ balls in } A,\, M - i \text{ balls in } B) \\ &= \begin{cases} i/M & j = i - 1 \\ (M - i)/M & j = i + 1 \\ 0 & j \neq i \pm 1 \end{cases} \end{aligned}$$

$$p = \begin{Bmatrix} 0 & 1 & & & & & \\ 1/M & 0 & (M-1)/M & & & & \\ & 1/M & 0 & (M-1)/M & & & \\ & & 2/M & 0 & (M-2)/M & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & (M-1)/M & 0 & 1/M \\ & & & & & 1 & 0 \end{Bmatrix}$$

Example 3: Gambler's ruin

A gambler, each time wins 1 with probability p, losses 1 with probability $1 - p = q$. Initial wealth $S_0 = a$; wealth at time n: S_n. The gambler leaves if $S_n = 0$ (loses all money) or $S_n = M > a$ (wins certain amount of money and gets satisfied)

This is a variant of the simple random walk, where we have absorbing barriers($P_{ii} = 1$) at 0 and M

$$S = \{0, ..., M\}$$

$$P_{ij} = \begin{cases} p & j = i + 1, i = 1, ..., M - 1 \\ q & j = i - 1, i = 1, ..., M - 1 \\ 1 & i = j = 0 \text{ or } i = j = M \\ 0 & \text{otherwise} \end{cases}$$

$$p = \begin{Bmatrix} 1 & 0 & \dots & & & & \\ q & 0 & p & \dots & & & \\ \dots & q & 0 & p & \dots & & \\ & \dots & q & 0 & p & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & \dots & q & 0 & p \\ & & & & \dots & 0 & 1 \end{Bmatrix}$$

Example 4: Bonus-Malus system

Insurance company has 4 premium levels: 1, 2, 3, 4

Let $X_n \in \{1, 2, 3, 4\}$ be the premium level for a customer at year n

$$Y_n \overset{iid}{\sim} Poi(\lambda) : \ \# \text{ of claims at year } n$$

- If $Y_n = 0$ (no claims)
    - $X_{n+1} = max(X_{n-1}, 1)$
- If $Y_n > 0$
    - $X_{n+1} = min(X_n + Y_n, 4)$

Denote $a_k = P(Y_n = k), k = 0, 1, ...$

$$p = \begin{Bmatrix} a_0 & a_1 & a_2 & (1-a_0-a_1-a_2) \\ a_0 & 0 & a_1 & (1-a_0-a_1) \\ 0 & a_0 & 0 & (1-a_0) \\ 0 & 0 & a_0 & (1-a_0) \end{Bmatrix}$$

4.2 Chapman-Kolmogorov equations

Q: Given the (one-step) transition matrix, $P = \{P_{ij}\}_{i,j\in S}$, how can we decide the n-step transition probability

$$\begin{aligned} P_{ij}^{(n)} &:= P(X_n = j|X_0 = i) \\ &= P(X_{n+m} = j|X_m = i), \quad m = 0, 1, ... \end{aligned}$$

As a special case, let us start with $P_{ij}^{(2)}$ and their collection $p^{(2)} = \{P_{ij}^{(2)}\}_{i,j\in S}$ (also a square matrix, same dimension as P)

Condition on what happens at time 1:

$$\begin{aligned} P_{ij}^{(2)} &= P(X_2 = j|X_0 = i) \\ &= \sum_{j\in S} P(X_2 = j|X_0 = i, X_1 = k)\cdot P(X_1 = k|X_0 = i) \quad \text{conditional law of total probability} \end{aligned}$$

4.2.1 Conditional Law of total probability

$$\begin{aligned} & P(X_2 = j|X_0 = i) \\ &= \sum_{k\in S} P(X_2 = j, X_1 = k|X_0 = i) \\ &= \sum_{k\in S} \frac{P(X_2 = j, X_1 = k, X_0 = i)}{P(X_0 = i)} \\ &= \sum_{k\in S} \frac{P(X_2 = j, X_1 = k, X_0 = i)}{P(X_1 = k, X_0 = i)} \cdot \frac{P(X_1 = k, X_0 = i)}{P(X_0 = i)} \\ &= \sum_{k\in S} P(X_2 = j|X_0 = i, X_1 = k)\cdot P(X_1 = k|X_0 = i) \end{aligned}$$

continue on $P_{ij}^{(2)}$

$$\begin{aligned} P_{ij}^{(2)} &= P(X_2 = j|X_0 = i) \\ &= \sum_{j\in S} P(X_2 = j|X_0 = i, X_1 = k)\cdot P(X_1 = k|X_0 = i) \quad \text{conditional law of total probability} \\ &= \sum_{k\in S} P(X_2 = j|X_1 = k)\cdot P(X_1 = k|X_0 = i) \\ &= \sum_{k\in S} P(X_1 = j|X_0 = k)\cdot P(X_1 = k|X_0 = i) \\ &= \sum_{k\in S} P_{ik}\cdot P_{kj} \\ &= (P\cdot P)_{ij} \end{aligned}$$

Thus, $p^{(2)} = P\cdot P = p^2$

Using the smae idea, for $n, m = 0, 1, 2, 3...$:

$$\begin{aligned} P_{ij}^{(n+m)} &= P(X_{n+m} = j|X_0 = i) \\ &= \sum_{k\in S} P(X_{n+m} = j|X_0 = i, X_m = k)\cdot P(X_m = k|X_0 = i) \\ &= \sum_{k\in S} P(X_{n+m} = j|X_m = k)\cdot P(X_m = k|X_0 = i) \quad \text{Markov property} \\ &= \sum_{k\in S} P(X_n = j|X_0 = k)\cdot P(X_m = k|X_0 = i) \\ &= \sum_{k\in S} p_{ik}^{(m)}\cdot P_{kj}^{(n)} \\ &= (p^{(m)}\cdot p^{(n)})_{ij} \\ &\Rightarrow p^{(n+m)} = p^{(m)}\cdot p^{(n)} \qquad (*) \end{aligned}$$

By definition, $p^{(1)} = p$

- $\Rightarrow p^{(2)} = p^{(1)} \cdot p^{(1)} = p^2$
- $\Rightarrow p^{(3)} = p^{(2)} \cdot p^{(1)} = p^3$
- $\cdots\cdots\cdots$
- $\Rightarrow p^{(n)} = p^n$

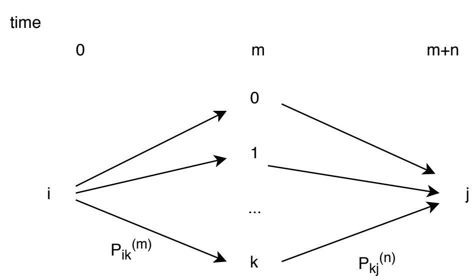
Note:

- n from $p^{(n)}$: n-step transition probability matrix
  - $p^{(n)} = \{p_{ij}^{(n)}\}_{i,j \in S}$
    $p_{ij}^{(n)} = P(X_n = j | X_0 = i)$
- n from p^n: n-th power of the (one-step) transition matrix
  - $p^n = p \cdot ... \cdot p$
    $p = \{P_{ij}\}_{i,j \in S}$
    $p_{ij} = P(X_1 = j | X_0 = i)$

$(*)$ is called the **Chapman-Kolmogorov equations** (c-k equation). Entry-wise:

$$P_{ij}^{n+m} = \sum_{k \in S} P_{ik}^{(m)} P_{kj}^{(n)}$$

Intuition:



"Condition at time m (on X_m) and sum p all the possibilities"

4.2.2 Distribution of X_n

So far, we have seen transition probability $P_{ij}^{(m)} = P(X_n = j | X_0 = i)$. This is not the probability $P(X_n = j)$. In order to get this distribution, we need the information about which state the Markov chain starts with.

Let $\alpha_{0,i} = P(X_0 = i)$. The row vector $\alpha_0 = (\alpha_0, 0, \alpha_0, 1, ...)$ is called the **initial distribution** of the Markov chain. This is the distribution of the initial state X_0

Similarly, we define distribution of X_n: $\alpha_n = (\alpha_n, 0, \alpha_n, 1, ...)$ where $\alpha_{n,i} = P(X_n = i)$

Fact: $\alpha_n = \alpha_0 \cdot p^n$

Proof:

$$\begin{aligned} & \forall j \in S \\ \alpha_{n,j} &= P(X_n = j) \\ &= \sum_{i \in S} P(X_n = j | X_0 = i) \cdot P(X_0 = i) \\ &= \sum_{i \in S} \alpha_{0,i} \cdot P_{ij}^{(n)} \\ &= (\alpha_0 \cdot P^{(n)})_j = (\alpha_0 \cdot p^n)_j \end{aligned}$$

$$\Rightarrow \alpha_n = \alpha 0 \cdot p^n$$

- α_n: distribution of X_n
- α_0: initial distribution
- p^n: transition matrix

Remark: The distribution of a DTMC is completely determined by two things:

- the initial distribution α_0 (row vector), and
- the transition matrix p (square matrix)

4.3 Stationary distribution (invariant distribution)

Definition: A probability distribution $\pi = (\pi_0, \pi_1, ...)$ os ca;;ed a **stationary distribution**(invariant distribution) of the DTMC $\{X_n\}_{n=0,1,...}$ with transition matrix P, if :

1. $\underline{\pi} = \pi \cdot P$
2. $\sum_{i \in S} \pi_i = 1 (\Leftrightarrow \underleftarrow{\pi} \cdot \mathbb{1})$. ($\mathbb{1}$: a column of all 1's)

Why such $\underline{\pi}$ is called stationary/invariant distribution?

$$\sum_{i \in S} \pi_i = 1, \pi_i \geq 0, i = 0, 1, ... \Rightarrow \text{distribution}$$
$$\underline{\pi} = \pi \cdot P \Rightarrow \text{invariant/stationary.}$$

Assume the MC starts from the initial distribution $\alpha_0 = \underline{\pi}$. hen the distribution of X_1 is

$$\alpha_1 = \alpha_0 \cot P = \underline{\pi} \cdot P = \underline{\pi} = \alpha_0$$

The distribution of X_2:

$$\alpha_2 = \alpha_0 \cdot P^2 = \underline{\pi} \cdot P \cdot P = \underline{\pi} \cdot P = (\underline{\pi}) = \alpha_0$$

$$\cdots\cdots$$

$$\alpha_n = \alpha_0$$

Thus, if the MC starts from a stationary distribution, then its distribution will not change over time.