STAT 333 Course Note

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1. Fundamental of Probability

1.1 What's Probability

1.1.1 Examples

1. Coin toss
    - "H" - head
    - "T" - tail
2. Roll a dice
    - every number in the set: $\{1, 2, 3, 4, 5, 6\}$
3. Tomorrow weather

- {sunny, rainy, cloudy,...}

4. Randomly pick a number in $[0, 1]$

Although things are random, they are not haphazard/arbitrary. There are "patterns"

Example 1

If we repeat tossing a coin, then the fraction of times that we get a "H" goes to $\frac{1}{2}$ as the number of toss goes to infinity.

$$\frac{\# \ of \ "H"}{total \ \# \ of \ toss} = \frac{1}{2}$$

This number $1/2$ reflects how "likely" a "H" will appear in one toss (if the experiment is not repeated)

1.2 Probability Models

The *Sample space* Ω is the set consisting of al the possible outcomes of a random experiment.

1.2.1 Examples

1. $\{H, T\}$
2. $\{1, 2, 3, 4, 5, 6\}$
3. $\{sunny, rainy, cloudy, ...\}$
4. $[0, 1]$

An *event* $E \in \Omega$ is a subset of Ω

for which we can talk about "likelihood of happening"; for example

- in **2**:
  - {getting an even number} = {2, 4, 6}
- in **4**:
  - $\{the \ point \ is \ between \ 0 \ and \ 1/3\} = [0, \frac{1}{3}]$ is an event
  - $\{the \ point \ is \ rational\} = Q \cap [0, 1]$

We say an event E "happens", if the result of the experiment turns out to belong to E (a subset of Ω)

A probability P is a set function (a mapping from events to real numbers)

$$\begin{aligned} P :& \xi \to R \\ & E \to P(E) \end{aligned}$$

which satisfies the following 3 properties:

1. $\forall E \in \xi, 0 \le P(E) \le 1$
2. $P(\Omega) = 1$
3. For
   - countably many disjoint events $E_1, E_2, ...$, we have $P(U_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$
   - countable: $\exists$ 1-1 mapping to natural numbers $1, 2, 3, ...$

Intuitively, one can think the probability of an event as the "likelihood/chance" for the event happens. If we repeat the experiment for a large number of events, the probability is the fraction of time that the event happens

$$P(E) = \lim_{n \to \infty} \frac{\text{\# of times the E happens in n trials}}{n}$$

1.2.1.1 Example 2

$$\begin{aligned} & P(\{1\}) = P(\{2\}) = \ldots = P(\{6\}) = \frac{1}{6} \\ & E = \{\text{even number}\} = \{2, 4, 6\} \\ \Rightarrow \ & P(E) = P(\{2\} \cup P(\{4\})) \cup P(\{6\}) = \frac{1}{2} \end{aligned}$$

Properties of probability:

1. $P(E) + P(E^c) = 1$

2. $P(\emptyset)=0$
3. $E_1 \subseteq E_2 \Rightarrow P(E_1) \leq P(E_2)$
4. $P(E_1 \cup E_2)=P(E_1)+P(E_2)-P(E_1 \cap E_2)$ -$P(E_1 \cap E_2)$: E_1 and E_2 happen

1.2.2 Remark: why do we need the notion of event?

If the sample space Ω is **discrete**, then everything can has at most countable elements be built from the "atoms"

$$
\begin{aligned}
\Omega=\{w_1, w_2, \ldots\} \\
P(w_1)=P_i \\
P_i \in[0,1], \sum_{i=1}^{\infty} P_i=1
\end{aligned}
$$

Then for any event $E=\{w_1, i \in I\}$, $P(E)=\sum_{i \in I} P_i$

However, if the sample space Ω is continuous; e.g, $[0,1]$ in Example 4, then such a construction can not be done for any $x \in[0,1]$ we get $P(\{x\}=0$ (x: the point is exactly x)

We can not get $P([0, \frac{1}{3}])$ by adding $P(\{x\})$ for $x \leq \frac{1}{3}$.

This is why we need the notion of event; and we define P as a set function from ξ to R rather than a function from Ω to R

To summarize: A **Probability Space** consists of a triplet (Ω, ξ, P):

- Ω: sample space,
- ξ: collection of events
- P: probability

1.3 Conditional Probability

If we know some information, the probability of an event can be updated

Let E, F be two events $P(F)>0$

The conditional probability of E, given F is

$$
P(E \mid F)=\frac{P(E \cap F)}{P(F)}
$$

Again, think probability as the long-run frequency:

$$
\begin{aligned}
P(E \cap F) & =\lim_{n \rightarrow \infty} \frac{\# of \ times \ E \ and \ F \ happen \ in \ n \ trails}{n} \\
P(F) & =\lim_{n \rightarrow \infty} \frac{\# of \ times \ F \ happen \ in \ n \ trails}{n} \\
\Rightarrow \frac{P(E \cap F)}{P(F)} & =\lim_{n \rightarrow \infty} \frac{\# of \ times \ E \ and \ F \ happen}{\# of \ times \ F \ happens}
\end{aligned}
$$

By definition

$$
P(E \cap F)=P(E \mid F) \cdot P(F)
$$

1.4 Independence

Def: Two events E and F are said to be independent, if $P(E \cap F)=P(E) \cdot P(F)$; denoted as $E \perp\!\!\!\perp F$. **This is different rom disjoint.**

Assume $P(F)>0$, then $E \perp\!\!\!\perp F \Leftrightarrow P(E|F)=P(E)$; intuitively, knowing F does not change the probability of E.

Proof:

$$
\begin{aligned}
E \perp\!\!\!\perp F & \Leftrightarrow P(E \cap F)=P(E) \cdot P(F) \\
& \Leftrightarrow \frac{P(E \cap F)}{P(F)}=P(E) \\
& \Leftrightarrow P(E|F)=P(E)
\end{aligned}
$$

More generally, a sequence of events $E_1, E_2, \ldots$ are called independent if for **any** finite index set I,

$$P(\bigcap_{i\in I} E_i) = \prod_{i\in I} P(E_i)$$

1.5 Bayes' rule and law of total probability

Theorem: Let $F_1, F_2, \dots$ be disjoint events, and $\bigcap_{i=1}^{\infty} F_i = \Omega$, we say $\{F_i\}_{i=1}^{\infty}$ forms a "partition" of the sample space Ω

Then $P(E) = \sum_{i=1}^{\infty} P(E|F_i) \cdot P(F_i)$

Proof: Exercise

Intuition: Decompose the total probability into different cases.

$$P(E \cap F_2) = P(E|F_2) \cdot P(F_2)$$

1.5.1 Bayes' rule

$$P(F_i|E) = \frac{P(E|F_i) \cdot P(F_i)}{\sum_{j=1}^{\infty} P(E|F_j) \cdot P(F_j)}$$

Bayes' rule tells us how to find conditional probability by switching the role of the event and the condition.

Proof:

$$\begin{aligned} P(F_i|E) &= \frac{P(F_i \cap E)}{P(E)} && \text{definition of condition probability} \\ &= \frac{P(E|F_i)P(F_i)}{P(E)} \\ &= \frac{P(E|F_i)P(F_i)}{\sum_{j=1}^{\infty} P(E|F_j)P(F_j)} && \text{law of total probability} \end{aligned}$$

2 Random variables and distributions

2.1 Random variables

(Ω, ξ, P): Probability space.

Definition: A random variable X (or r.v.) is a mapping from Ω to $\mathbb{R}$

$$\begin{aligned} X : &\Omega \to \mathbb{R} \\ &\omega \to X(\omega) \end{aligned}$$

A random variable transforms arbitrary "outcomes" into numbers.

X introduces a probability on R. For $A \subseteq R$, define

$$\begin{aligned} P(X \in A) :&= P(\{X(\omega) \in A\}) \\ &= P(\{\omega : X(\omega) \in A\}) \\ &= P(X^{-1}(A)) \end{aligned}$$

From now on, we can often "forget" te original probability space and focus on the random variables and their distributions.

Definition: let X be a random variable. The **CDF**(cumulative distribution function) F of X is defined by

$$F(x) = P(X \le x) = P(X \in (-\infty, x]))$$
$$X : \text{random variable}, x : \text{number}$$

Properties of cdf:

1. F is non-decreasing. $F(x_1) \le F(x_2), x_1 < x_2$
2. limits
   - $\lim_{x\to -\infty} F(x) = 0$
   - $\lim_{x\to \infty} F(x) = 1$
3. $F(x)$ is right continuous

- $lim_{x\downarrow a}F(x) = F(a)$: x decreases to a(approaching from the right)
- Hint: $\{x \leq a\} = \bigcap_{i=1}^{\infty}\{X \leq a_i\}$ for $a_i \downarrow a$

2.2 Discrete random variables and distributions

A random variable X is called **discrete** if it only takes values in an **at most countable** set $\{x_1, x_2, ...\}$ (finite or countable).

The distribution of a discrete random variable is fully characterized by its **probability mass function**(p.m.f)

$$p(x) := P(X = x); x = x_1, x_2, ...$$

Properties of pmf:

1. $p(x) \geq 0, \quad \forall x$
2. $\sum_i p(x_i) = 1$

Q: what does the cdf of a discrete random variable look like?

2.2.1 Examples of discrete distributions

1. Bemoulli distribution

$$\begin{aligned} p(1) &= P(X = 1) = p \\ p(c) &= P(X = c) = 1 - p \\ p(x) &= 0 \quad otherwise \end{aligned}$$

Denote $X \sim Ber(p)$

2.Binomial distribution

$$p(k) = P(X = k) = (^n_k)p^k(1-p)^{n-k}$$

- $X \sim Bin(n, p)$ to choose k successes.
- Binomial distribution is the distribution of number of successes in n independent trials; each having probability p of success.

3.Geometric distribution

$$p(k) = P(X = k) = (1-p)^{k-1}p$$
$$(1-p)^{k-1} \text{ : the first k-1 trials are all failures}, \quad p \text{ : success in } k^{th} \text{ trial}$$

- $X \sim Geo(p)$
- X is the number of trials needed to get the first success in n independent trials with probability p of success each
- X has the memoryless property $P(X > n + m|X > m) = P(x > n) \qquad n, m = 0, 1, ...$

Memoryless property:

$$p(X > n + m|X > m) = P(X > n)$$

Proof:

$$\begin{aligned} P(X > k) &= \sum_{j=k+1}^{\infty} P(X = j) \\ &= \sum_{j=k+1}^{\infty} (1-p)^{j-1}p \\ &= (1-p)^k p \cdot \frac{1}{1-(1-p)} \\ &= (1-p)^k \end{aligned}$$

$$\begin{aligned} P(X > n + m|x > m) &= \frac{P(X > n + m \bigcap X > m)}{P(X > m)} \\ &= \frac{P(X > n + m)}{P(X > m)} = \frac{(1-p)^{n+m}}{(1-p)^m} = (1-p)^n = P(X > n) \end{aligned}$$

Intuition: The failures in the past have no influence on how long we still need to wait to get the first success in the future

4. Poisson distribution

$$p(k) = P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, \ldots, \lambda > 0$$

Other discrete distributions:

- negative binomial
- discrete uniform

2.3 Continuous random variables and distributions

Definition: A random variable X is called **continuous** if there exists a non-negative function f, such that for any interval $[a, b]$, (a, b) or $[a, b)$:

$$P(X \in [a, b]) = \int_a^b f(x)dx$$

The function f is called the *probability density function(pdf)* of X

Remark: probability density function(pdf) is not probability. $P(X = x) = 0$ if X is continuous. The probability density function f only gives probability when it is integrated.

If X is continuous, then we can get cdf by:

$$F(a) = P(X \in (-\infty, a]) = \int_{-\infty}^a f(x)dx$$

hence, $F(x)$ is continuous, and differentiable "almost everywhere".

We can take $f(x) = F'(x)$ when the derivative exists, and $f(x) =$arbitrary number otherwise often to choose a value to make f have some continuity.

Property of pdf:

1. $f(x) \leq 0, x \in R$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. For $A \subseteq R, P(X \in A) = \int_A f(x)dx$

2.3.1 Example of continuous distribution

Exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x \leq 0 \end{cases}$$
$$X \sim Exp(x)$$

Other continuous distributions:

- Normal distribution
- Uniform distribution

Exercises:

1. Find the cdf of $X \sim Exp(x)$

$$\begin{aligned} F(k) = P(X \leq k) &= \int_{-\infty}^k f(x)dx \\ &= \int_0^k \lambda e^{-\lambda x} dx \\ &= -e^{-\lambda x}\Big|_0^k \\ &= -e^{-\lambda k} - (-e^0) \\ &= 1 - e^{-\lambda k} \end{aligned}$$

2. Show that the exponential distribution has the memoryless property:

$$P(X > t + s|X > t) = P(X > s)$$

2.4 Joint distribution of r.v's

Let X and Y be two r.v's. defined on the same probability space (Ω, ξ, P)

For each $\omega \in \Omega$, we have at the same time $X(\omega)$ and $Y(\omega)$. Then we can talk about the joint behavior of X and Y

Two joint distribution of r.v's is characterized by joint cdf, joint pmf(discrete case) or joint pdf(continuous case).

- Joint cdf:
  - $F(x, y) = P(X < x, Y < y)$
- Joint pmf:
  - $f(x, y) = P(X = x, Y = y)$
- joint pdf $f(x, y)$ such that for $a < b, c < d$
  - $P(X, Y) \in (a, b] \times (c, d] = P(X \in (a, b], Y \in (c, d]) = \int_a^b \int_c^d f(x, y)dydx$
  - Equivalently:
    1. $F(x, y) = \int_{-\infty}^x \int_{-\infty}^y f(s, t)dtds$
       and
       $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$
    2. $P((X, Y) \in A) = \int \int_A f(x, y)dxdy$ for $A \subseteq R^2$

Definition: Two r.v's X and Y are called independent, if for all sets $A, B \subseteq R$,

$$P(X < A, Y < B) = P(X \in A) \cdot P(Y \in B)$$

($\{X \in A\}$ and $\{Y \in B\}$ are independent events)

Theorem: Two r.v's X and Y are

1. independent, if and only if
2. $F(x, y) = F_x(x)F_y(y); x, y \in R$; where F_x: cdf of x; F_y: cdf of y
3. $f(x, y) = f_x(x)f_y(y); x, y \in R$; where f is the joint pmf/pdf of X and Y; f_x, f_y are marginal pmf/pdf of X and Y, respectively

Proof:

1.$\Rightarrow$ 2.

If $X \perp\!\!\!\perp Y$, then by definition,

$$F(x, y) = P(X \in (-\infty, x], Y \in (-\infty, y])) = P(X \in (-\infty, x])) \cdot P(Y \in (-\infty, y])) = F_x(x)F_y(y)$$

2.$\Rightarrow$ 3.

Assume $F(x, y) = F_x(x) \cdot F_y(y)$,

$$\begin{aligned} f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) &= \frac{\partial^2}{\partial x \partial y} F_x(x)F_y(y) \\ &= (\frac{\partial}{\partial x} F_x(x))(\frac{\partial}{\partial y} F_y(y)) \\ &= f_x(x)f_y(y) \end{aligned}$$

3.$\Rightarrow$ 1.

Assume $f(x, y) = f_x(x)f_y(y)$; For $A, B \subseteq R$,

$$\begin{aligned} P(X \in A, Y \in B) &= \int_{y \in B} \int_{x \in A} f(x, y)dxdy \\ &= \int_{y \in B} \int_{x \in A} f_x(x)f_y(y)dxdy \\ &= (\int_{x \in A} f_x(x)dx)(\int_{y \in B} f_y(y)dy) \\ &= P(X \in A)P(Y \in B) \end{aligned}$$

2.5 Expectation

Definition: For a r.v X, the expectation of $g(x)$ is defined as

$$\mathbb{E}(g(X)) = \begin{cases} \sum_{i=1}^{\infty} g(x_i)P(X = x_i) & \text{for discrete } X \\ \int_{-\infty}^{\infty} g(x)f(x)dx & \text{for continuous} X \end{cases}$$

Let X,Y be two r.v's; then the expectation of $g(X, Y)$ is defined in a similar way.

$$\mathbb{E}(g(X,Y)) = \begin{cases} \sum_i \sum_j g(x_i, y_j)P(X = x_i, Y = y_j) \\ \int\int g(x_i, y_j)f(x,y)dxdy \end{cases}$$

2.5.1 Properties of expectation

1. Linearity:expectation of X: $\mathbb{E}(X) = \begin{cases} \sum x_i P(X = x_i) \\ \int_{-\infty}^{\infty} xf(x)dx \end{cases}$, $g(X) = x$
   - $\mathbb{E}(ax + b) = a\mathbb{E}(x) + b$
   - $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$
2. If $X \perp\!\!\!\perp Y$, then $\mathbb{E}(g(X)h(Y)) = \mathbb{E}(g(X)) \cdot \mathbb{E}(h(Y))$
   - **proof**: (continuous case)
   - $$\begin{aligned} \mathbb{E}(g(X)h(Y)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty} g(x)f_X(x) \cdot \int_{-\infty}^{\infty} h(y)f_Y(y)dy \end{aligned}$$
   - In particular, $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ if $X \perp\!\!\!\perp Y$

2.5.2 Definitions

Definition: The expectation $\mathbb{E}(X^n)$ is called the n-th moment of X:

- 1st moment: $\mathbb{E}(X)$
- 2st moment: $\mathbb{E}(X^2)$

Definition: The variance of a r.v X is defined as:

$$Var(x) = \mathbb{E}((X - \mathbb{E}(X))^2) \text{ also denoted as } \sigma^2, \sigma_x^2$$

Definition: the covariance of the r.v's X and Y is defined as:

$$Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]$$

Thus $Var(X) = Cov(X, X)$

Definition: the correlation between X and Y is defined as:

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Fact: $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

Proof:

$$\begin{aligned}
Var(X) &= \mathbb{E}((X-\mathbb{E}(X))^2) \\
&= \mathbb{E}(X^2 - 2X\mathbb{E}(X) + (\mathbb{E}(X))^2) \\
&= \mathbb{E}(X^2) - 2\mathbb{E}(X\mathbb{E}(X)) + (\mathbb{E}(X))^2 \\
&= \mathbb{E}(X^2) - 2(\mathbb{E}(X))^2 + (\mathbb{E}(X))^2 \\
&= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \qquad \blacksquare
\end{aligned}$$

Fact: $Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

Proof:

$$\begin{aligned}
Cov(X,Y) &= \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\
&= \mathbb{E}[XY - X\mathbb{E}[Y] - Y\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y]] \\
&= \mathbb{E}[XY] - \mathbb{E}[X\mathbb{E}[Y]] - \mathbb{E}[Y\mathbb{E}[X]] + \mathbb{E}[E]X]\mathbb{E}[Y]) \\
&= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \qquad \blacksquare
\end{aligned}$$

Variance and covariance are **translation invariant**. Variance is quadratic, covariance is bilinear.

$$Var(aX+b) = a^2 \cdot Var(X)$$

$$Cov(aX+b, cY+d) = ac \cdot Cov(X,Y)$$

Proof: $Var(aX+b) = a^2 \cdot Var(X)$

$$\begin{aligned}
Var(aX+b) &= \mathbb{E}((aX+b)^2) - (\mathbb{E}(aX+b))^2 \\
&= \mathbb{E}(a^2X^2 + 2abX + b^2) - (a\mathbb{E}(X)+b)^2 \\
&= a^2\mathbb{E}(X^2) + 2ab\mathbb{E}(X) + b^2 - a^2\mathbb{E}^2(X) - ab\mathbb{E}(X) - b^2 \\
&= a^2\mathbb{E}(X^2) - a^2\mathbb{E}^2(X) \\
&= a^2 Var(X) \qquad \blacksquare
\end{aligned}$$

Proof: $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

$$\begin{aligned}
Var(X+Y) &= \mathbb{E}[(X+Y)^2] - E^2[X+Y] \\
&= \mathbb{E}[X^2 + XY + Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 \\
&= \mathbb{E}[X^2] + \mathbb{E}[XY] + \mathbb{E}[Y^2] - E^2[X] - 2\mathbb{E}[X]\mathbb{E}[Y] - E^2[Y] \\
&= (\mathbb{E}[X^2] - E^2[X]) + (\mathbb{E}[Y^2] - E^2[Y]) + (\mathbb{E}[XY] - 2\mathbb{E}[X]\mathbb{E}[Y]) \\
&= Var(X) + Var(Y) + 2Cov(X,Y) \qquad \blacksquare
\end{aligned}$$

If $X \perp\!\!\!\perp Y$, then $Cov(X,Y) = 0$ and $Var(X+Y) = Var(X) + Var(Y)$

Proof:

$$\begin{aligned}
&Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \\
&\text{we know:} \\
&X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \\
&\text{Thus, } Cov(X,Y) = 0 \Rightarrow Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) \\
&\text{So we see independence} \Rightarrow \text{Covariance is 0: "uncorrelated"} \\
&\text{the converse is not true.} \\
&Cov(X,Y) = 0 \not\Rightarrow \text{independence}
\end{aligned}$$

Remarks

We have $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.

If $X \perp\!\!\!\perp Y$, we also have:

- $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$, and
- $Var(X+Y) = Var(X) + Var(Y)$

It's important to remember that the first result and the other two results are of very different nature. While $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$ is a property of expectation and holds unconditionally;

the other two, $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ and $Var(X+Y) = Var(X) + Var(Y)$, only hold if $X \perp\!\!\!\perp Y$.

It is more appropriate to consider them as **properties of independence** rather than properties of expectation and variance

2.6 Indicator

A random variable I is called an indicator, if

$$I(w) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \not\in A \end{cases}$$

$$E(I_A) = P(A)$$

for some event A

For A given, I is also elevated as I_A

The most important property of indicator is its expectation gives the probability of the event $\mathbb{E}(I_A) = \mathbb{P}(A)$

Proof:

$$\begin{aligned} \mathbb{P}(I_A = 1) &= \mathbb{P}(\omega : I_A(\omega = 1)) \\ &= \mathbb{P}(\omega : \omega \in A) \\ &= \mathbb{P}(A) \end{aligned}$$

$$\mathbb{P}(I_A = 0) = 1 - \mathbb{P}(A) \Rightarrow \mathbb{E}(I_A) = 1 \cdot \mathbb{P}(A) + c \cdot (1 - \mathbb{P}(A)) = \mathbb{P}(A)$$

2.6.1 Example

we see $I_A \sim Ber(\mathbb{P}(A))$

Let $X \sim Bin(n, p)$, X is number of successes in n Bernoulli trials, each with probability p of success

$$\Rightarrow X = I_1 + \cdots + I_n$$

where $I_1, \cdots, I_n$ are indicators for independent events. $I_i = 1$ if th i the trial is a success. $I_i = 0$ if the i th trial is a failure.

Hence I_i are **idd**(independent and identically distributed) r.v's

$$\begin{aligned} \Rightarrow \mathbb{E}(X) &= \mathbb{E}(I_1 + \cdots + I_N) \\ &= \mathbb{E}(I_1) + \cdots + \mathbb{E}(I_n) \\ &= p + \cdots + p = n \cdot p \end{aligned}$$

$$\begin{aligned} Var(X) &= Var(I_1 + \cdots + I_n) \\ &= Var(I_1) + \cdots + Var(I_n) \\ &= n \cdot Var(I_i) \\ &= n \cdot p(1-p) \end{aligned}$$

$$Var(I_1) = \mathbb{E}(I_1^2) - (\mathbb{E}(I_1))^2 = \mathbb{E}(I_1) - (\mathbb{E}(I_1))^2 = p - p^2 = p(1-p)$$

2.6.1 Example 3

Let X be a r.v. taking values in non-negative integers, then

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} P(X > n)$$

Proof:

Note that $X = \sum_{n=0}^{\infty} I_n$ where $I_n = I_{x>n}$. ($x > n$ is an event)

$$
\begin{aligned}
\mathbb{E}(X) &= \mathbb{E}(\sum_{n=0}^{\infty} I_n) \\
&= \sum_{n=0}^{\infty} \mathbb{E}(I_n) \\
&= \sum_{n=0}^{\infty} P(X > n)
\end{aligned}
$$

In particular, let $X \sim Geo(p)$. As we have seen, $P(X > n) = (1-p)^n \Rightarrow$

$$
\begin{aligned}
\mathbb{E}(X) &= \sum_{n=0}^{\infty} P(X > n) \\
&= \sum_{n=0}^{\infty} (1-p)^n \\
&= \frac{1}{1-(1-p)} = \frac{1}{p}
\end{aligned}
$$

2.7 Moment generating function

Definition: Let X be a r.v. Then the function $M(t) = \mathbb{E}(e^{tX})$ is called the *moment generating function(mgf)* of X, if the expectation exists for all $t \in (-h, h)$ for some $h > 0$.

Remark: The mgf is not always well-defined. It is important to check the existence of the expectation.

2.7.1 Properties of mgf

1. Moment Generating Function generates moments
   - **Theorem**:
     - $M(0) = 1$
     - $M^{(k)}(0) = \mathbb{E}(X^k), k = 1, 2, \dots$ ($M^{(k)} = \frac{d^k}{dt^k} M(t)|_{t=0}$)
       - **Proof**:

$$
M(0) = \mathbb{E}(e^{0 \cdot X}) = \mathbb{E}(1) = 1
$$

$$
\begin{aligned}
M^{(k)}(0) &= \frac{d^k}{dt^k} \mathbb{E}(e^{t \cdot X})|_{t=0} \\
&= \mathbb{E}(\frac{d^k}{dt^k} e^{tX}|_{t=0}) \\
&= \mathbb{E}(X^k)
\end{aligned}
$$

     - As a result, we have: $M(t) = \sum_{k=0}^{\infty} \frac{M^{(k)}(0)}{k!} t^k = \sum_{k=0}^{\infty} \frac{E*X^k}{k!} t^k$ (a method to get moment of a r.v)
2. $X \perp\!\!\!\perp Y$, with mgf's M_x, M_y. Let M_{X+Y} be the mgf of $X+Y$. then

$$
M_{X+Y}(t) = M_X(t) M_Y(t)
$$

   - **Proof**:

$$
\begin{aligned}
M_{X+Y}(t) &= \mathbb{E}(e^{t(X+Y)}) \\
&= \mathbb{E}(e^{tx} e^{ty}) \\
&= \mathbb{E}(e^{tx}) \mathbb{E}(e^{ty}) \\
&= M_X(t) M_Y(t)
\end{aligned}
$$

3. The mgf completely determines the distribution of a r.v.
   - $M_X(t) = M_Y(t)$ for all $t \in (-h, h)$ for some $h > 0$, then $X \stackrel{d}{=} Y$. ($\stackrel{d}{=}$: have the same distribution)
   - Example: Let $X \sim Poi(\lambda_1), Y \sim Poi(\lambda_2)$. $X \perp\!\!\!\perp Y$. Find the distribution of $X+Y$
     - First, derive the mgf of a Poisson distribution.

$$\begin{aligned}
M_X(t) &= \mathbb{E}(e^{tX}) \\
&= \sum_{n=0}^{\infty} e^{tn} \cdot P(X=n) \\
&= \sum_{n=0}^{\infty} e^{tn} \cdot \frac{\lambda_1^n}{n!} e^{-\lambda_1} \\
&= \sum_{n=0}^{\infty} \frac{(e^t \cdot \lambda_1)^n}{n!} \cdot e^{-\lambda_1}
\end{aligned}$$

$$\text{we know that } \sum_{n=0}^{\infty} \frac{(e^t\lambda_1)^n}{n!} = e^{e^t \cdot \lambda_1}. (\text{Since } \frac{(e^t\lambda_1^n)}{n!} e^{-e^t\lambda_1} \text{ is the pmf of } Poi(e^t\lambda_1))$$

$$\Rightarrow M_X(t) = e^{e^t\lambda_1} e^{-\lambda_1} = e^{\lambda_1(e^t-1)}, t \in \mathbb{R}. (e^{\lambda_1(e^t-1)} \text{ is mgf of } Poi(\lambda_1))$$

$$\text{Similarly, } M_Y(t) = e^{\lambda_2(e^t-1)}.$$

We know that

$$\begin{aligned}
M_{X+Y}(t) &= M_X(t) M_Y(t) \\
&= e^{\lambda_1(e^t-1)} e^{\lambda_2(e^-1)} \\
&= e^{(\lambda_1+\lambda_2)(e^t-1)}
\end{aligned}$$

This is the mgf of $Poi(\lambda_1 + \lambda_2)$!

Since the mgf uniquely determines the distribution $X + Y \sim Poi(\lambda_1 + \lambda_2)$

In general, if $X_1, X_2, \ldots, X_n$ independent, $X_i \sim Poi(\lambda_i)$, then $\sum X_i \sim Poi(\sum \lambda_i$

2.7.2 Joint mgf

Definition: Let X, Y be r.v's. Then $M(t_1, t_2) := \mathbb{E}(e^{t_1X+t_2Y})$ is called the joint mgf of X and Y, if the expectation exists for all $t_1 \in (-h_1, h_1)$, $t_2 \in (-h_2, h_2)$ for some $h_1, h_2 > 0$.

More generally, we can define $M(t_1, \ldots, t_n) = \mathbb{E}(exp(\sum_{i=1}^n t_iX_i))$ for r.v's $X_1, \cdots, X_n$, if the expectation exists for $\{(t_1, \cdots, t_n) : t_i \in (-h_i, h_i), i = 1, \cdots, n\}$ for some $\{h_i > 0\}, i = 1, \cdots, n$

2.7.2.1 Properties of the joint mgf

1. $$\begin{aligned}
M_X(t) &= \mathbb{E}(e^{tX}) \\
&= \mathbb{E}(e^{tX+0Y}) \\
&= M(t, 0) \\
M_Y(t) &= M(0, t)
\end{aligned}$$
2. $$\frac{\partial^{m+n}}{\partial t_1^m \partial t_2^n} M(t_1, t_2)|_{(0,0)} = \mathbb{E}(X^mY^n)$$
$$\text{the proof is similar to the single r.v. case}$$
3. If $X \perp\!\!\!\perp Y$, then $M(t_1, t_2) = M_X(t_1)M_Y(t_2)$
   - **Proof**:

$$\begin{aligned}
M(t_1, t_2) &= \mathbb{E}(e^{t_1X+t_2Y}) \\
(X \perp\!\!\!\perp Y) &= \mathbb{E}(e^{t_1X}e^{t_2Y}) \\
&= \mathbb{E}(e^{t_1X}) \cdot \mathbb{E}(e^{t_2Y}) \\
&= M_X(t_1) \cdot M_Y(t_2)
\end{aligned}$$

   - **Remark**: Don't confuse this with the result $X \perp\!\!\!\perp Y \Rightarrow M_{X+Y}(t) = M_X(t)M_Y(t)$.
     - $M_{X+Y}(t) \to$ mgf of $X + Y$; single argument function t
     - $M(t_1, t_2) \to$ joint mgf of (X, Y); two arguments t_1, t_2

3. Conditional distribution and conditional expectation

3.1 Conditional distribution

3.1.1 Discrete case

Definition Let X and Y be discrete r.v's. The conditional distribution of X given Y is given by:

$$P(X=x|Y=y)=\frac{(P(X=x,Y=u))}{P(Y=y)}$$

$$P(X=x|Y=y): f_{X|Y=y}(x). \qquad f_{X|Y}(x|y) \leftarrow \text{conditional probability mass function)}$$

Conditional pmf is a legitimate pmf: given any y, $f_{X|Y=y}(x) \geq 0, \forall x$

$$\sum_x f_{X|Y=y}(x)=1$$

Note that given $Y=y$, as x changes, the value of the function $f_{X|Y=y}(x)$ is proportional to the joint probability.

$$f_{X|Y=y}(x) \propto P(X=x,Y=y)$$

This is useful for solving problems where the denominator $P(Y=y)$ is hard to find.

3.1.1.1 Example

$X_1 \sim Poi(\lambda_1), X_2 \sim Poi(\lambda_2)$. $X_1 \perp\!\!\!\perp X_2$, $Y=X_1+X_2$

Q: $P(X_1=k|Y=n)$?

Note $P(X_1=k|Y=n)=f_{X_1|Y=n}(k)$

A: $P(X_1=k|Y=n)$ can only be non-zero for $k=0,\cdots,n$ in this case,

$$\begin{aligned} P(X_1=k|Y=n) &= \frac{P(X_1=k,Y=n)}{P(Y=n)} \\ &\propto P(X_1=k,Y=n) \\ &= P(X_1=k,X_2=n-k) \\ &= e^{-\lambda_1}\frac{\lambda_1^k}{k!}\cdot e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!} \\ &\propto (\frac{\lambda_1}{\lambda_2})^k/k!(n-k)! \end{aligned}$$

we can get $P(X=k|Y=n)$ by normalizing the above expression.

$$P(X_1=k,Y=n)=\frac{(\frac{\lambda_1}{\lambda_2})^k/k!(n-k)!}{\sum_{k=0}^n(\frac{\lambda_1}{\lambda_2})^k/k!(n-k)!}$$

but then we will need to fine $\sum_{k=0}^n(\frac{\lambda_1}{\lambda_2})^k/k!(n-k)!$

An easier way is to compare $\sum_{k=0}^n(\frac{\lambda_1}{\lambda_2})^k/k!(n-k)!$ with the known results for common distribution. In particular, if $X \sim Bin(n,p)$

$$\begin{aligned} P(X=k) &= (\overset{n}{k})p^k(1-p)^{n-k} \\ &\propto (\frac{p}{1-p})^k/k!(n-k)! \end{aligned}$$

$\Rightarrow P(X_1=k|Y=n)$ follows a binomial distributions with parameters n and p given by $\frac{p}{1-p}=\frac{\lambda_1}{\lambda_2} \Rightarrow p=\frac{\lambda_1}{\lambda_1+\lambda_2}$

Thus, given $Y=X_1+X_2=n$, the conditional distribution of X_1 is binomial with parameter n and $\frac{\lambda_1}{\lambda_1+\lambda_2}$

3.1.2 Continuous case

Definition: Let X and Y be continuous r.v's. The conditional distribution of X given Y is given by

$$f_{X|Y}(x|y)=f_{X|Y=y}(x)=\frac{f(x,y)}{f_Y(y)}$$

A conditional pdf is a legitimate pdf

$$
\begin{aligned}
f_{X|Y}(x|y) &\geq 0 \qquad & x, y \in \mathbb{R} \\
\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx &= 1, & y \in \mathbb{R}
\end{aligned}
$$

3.1.2.1 Example

Suppose $X \sim Exp(\lambda)$, $Y|X = x \sim Exp(x) = f_{Y|X}(y|x) = xe^{-xy}, y = e \leftarrow$ conditional distribution of Y given $X = x$

Q: Find the condition pdf $f_{X|Y}(x|y)$

A:

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\
&\propto f(x,y) \\
&= f_{Y|X}(y|x) \cdot f_X(x) \\
&= xe^{-xy}\lambda e^{-\lambda x} \\
&\propto xe^{-x(y+\lambda)}, \qquad x > 0, y > 0
\end{aligned}
$$

Normalization (make the total probability 1)

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{xe^{-x(y+\lambda)}}{\int_0^\infty xe^{-x(y+\lambda)}dx} \\
\int_0^\infty xe^{-x(y+\lambda)}dx &= (\frac{1}{\lambda + y})^2 \leftarrow \text{integration by parts}
\end{aligned}
$$

Thus, $f_{X|Y}(x|y) = (\lambda + y)^2 xe^{-x(y+\lambda)}$, $x > 0$.

This is a gamma distribution with parameters γ and $\lambda + y$

3.1.2.1. Example 2

Find the distribution of $Z = XY$.

Attention: the following method is wrong:

$$
f_Z(z) = \int_0^\infty f_{Y|X}(\frac{z}{x}|x) \cdot f_X(x)dx
$$

If we want to directly work with pdf's, we will need to use the change of variable formula for multi-variables. The right formula have turns out to be

$$
\begin{aligned}
f_Z(z) &= \int_0^\infty f_{X,Z}(x,z)dx = \int_0^\infty f_{Z|X}(z|x)f_X(x)dx \\
&= \int_0^\infty f(x, \frac{z}{x}) \cdot \frac{1}{x}dx \\
&= f_{Y|X}(\frac{z}{x}|x)f_X(x) \cdot \frac{1}{x}dx
\end{aligned}
$$

As an **easier way** is to use cdf, which gives probability rather than density:

$$
\begin{aligned}
P(Z < z) &= P(XY \leq z) \\
&= \int_0^\infty P(XY \leq z|X = x)f_X(x)dx \qquad \text{(law of total probability)} \\
&= \int_0^\infty P(Y \leq \frac{z}{x}|X = x) \cdot f_X(x)dx \\
Y|X = x \sim Exp(x) \\
&= \int_0^\infty (1 - e^{-x \cdot \frac{z}{x}}) \cdot \lambda e^{-\lambda x}dx \\
&= 1 - e^{-z}\int_0^\infty \lambda e^{-\lambda x}dx \\
&= 1 - e^{-z} \Rightarrow Z \sim Exp(1)
\end{aligned}
$$

Notation $X, Y|\{Z = k\} \overset{iid}{\sim} \cdots$ means that given $Z = k$, X and Y are *conditionally independent*, and they follow certain distribution.

(the conditional joint cdf/pmf/pdf equals the predict of the conditional cdf's/pmf's/pdf's)

3.2 Conditional expectation

We have seen that conditional pmf/pdf are legitimate pmf/pdf. Correspondingly, a conditional distribution is nothing else but a probability distributions. It is simply a (potentially) different distribution, since it takes more information into consideration.

As a result, we can define everything which are previously defined for unconditional distributions also for conditional distributions.

In particular, it is natural to define the conditional expectation.

Definition. The conditional expectation of $g(X)$ given $Y = y$ is defined as

$$\mathbb{E}(g(X)|Y=y) = \begin{cases} \sum_{i_1}^{\infty} g(x_i)P(X = x_u|Y=y) & \text{if } X|Y=y \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x)f_{X|Y}(x|y)dx & \text{if } X|X=y \text{ is continuous} \end{cases}$$

Fix y, the conditional expectation is nothing but the expectation taken under the conditional distribution.

3.2.1 What is $\mathbb{E}(X|Y)$?

Different ways to understand *conditional expectation*

1. Fix a value y, $\mathbb{E}(g(X)|Y=y)$ is a number
2. As y changes $\mathbb{E}(g(x)|Y=y)$ becomes a function of y (that each y gives a value): $h(y) =: \mathbb{E}(g(x)|Y=y)$
3. since y is actually random, we can define $\mathbb{E}(g(x)|Y) = h(Y)$. This is a random variable

$$\mathbb{E}(g(x)|Y))_{(\omega)} = \mathbb{E}(g(x)|Y = Y(\omega)$$

$\omega \in \Omega$ this random variable takes value $\mathbb{E}(g(x)|Y=y)$ When $Y = y$

$$\Omega \to \mathbb{R}$$
$$h(Y)_{(\omega)} = h(Y(\omega))$$

3.2.2 Properties of conditional expectation

1. Linearity (inherited from expectation)

$$\mathbb{E}(aX + b|Y=y) = a\mathbb{E}(X|Y=y) + b$$

$$\mathbb{E}(X+Z|Y=y) = \mathbb{E}(X|Y=y) + \mathbb{E}(Z|Y=y)$$

2. $\mathbb{E}(g(X,Y)|Y=y) = \mathbb{E}(g(X,y)|Y=y) \not= \mathbb{E}(g(X,y)) \text{ when X and Y are not independent}$

   Proof (Discrete):

$$\mathbb{E}(g(X,Y)|Y=y) = \sum_{x_i}\sum_{y_j} g(x_i,y_j)\cdot P(X=x_i, Y=y_j|Y=y)$$

$$P(X=x_i, Y=y_j|Y=y) = \begin{cases} 0 & \text{if } y_j \not= y \\ \\ P(X=x_1, Y=y_j)/P(Y+y) = P(X=x_i|Y=y) & \text{if } y_j = y \end{cases}$$

$$\begin{aligned} \Rightarrow \mathbb{E}(g(X,Y)|Y=y) &= \sum_{x_i} g(x_i,y)\cdot P(X=x_i|Y=y) \\ &= \mathbb{E}(g(X,y)|Y=y) & g(X,y) \text{ regarded as a function of } x \end{aligned}$$

   In particular,

$$\mathbb{E}(g(X)\cdot h(Y)|Y=y) = h(y)\mathbb{E}(g(X)|Y=y)$$

$$\mathbb{E}(g(X)\cdot h(Y)|Y) = h(Y)\mathbb{E}(g(X)|Y)$$

3. If $X \perp Y$, then $\mathbb{E}(g(X)|Y=y) = \mathbb{E}(g(X))$

   Fact: If $X \perp Y$, then conditional distribution of X given $Y = y$ is the same as the unconditional distribution of X

   Proof(Discrete):

$$\begin{aligned}
&\text{if } X \perp Y \\
&P(X = x_i | Y = y_j) \\
&\quad = P(X = x_i | Y = y_j)/P(Y = y_j) \\
&\quad = P(X = x_i)P(Y = y_j)/P(Y = y_j) \\
&\quad = P(X = x_i)
\end{aligned}$$

4. Law of iterated expectation (or double expectation): Expectation of conditionally expectation is its unconditional expectation

$$\mathbb{E}(\mathbb{E}(X|Y))) = \mathbb{E}(X)$$

$\mathbb{E}(X|Y)$ is a r.v, a function of Y.
Proof(Discrete):

When $Y = y_j$, the r.v. $\mathbb{E}(X|Y) = \mathbb{E}(X|Y = y_j) = \sum_{x_i} x_i P(X = x_i | Y = y_j)$. This happens with probability $P(Y = y_j)$

$$\Rightarrow \quad \begin{aligned}
\mathbb{E}(\mathbb{E}(X|Y)) &= \sum_{y_j} (\sum_{x_i} x_i P(X = x_i | Y = y_j)) P(Y = y_j) \\
&= \sum_{x_i} \sum_{y_j} x_i P(X = x_i | Y = y_j) P(Y = y_j) \\
&= \sum_{x_i} x_i \sum_{y_j} P(X = x_i | Y = y_j) P(Y = y_j) \qquad \text{law of total probability} \\
&= \sum_{x_i} x_i P(X = x_i) = \mathbb{E}(X)
\end{aligned}$$

Alternatively,

$$\begin{aligned}
&\sum_{x_i} \sum_{y_j} x_i P(X = x_i | Y = y_j) P(Y = y_j) \\
&\quad = \sum_{x_i} \sum_{y_j} x_i P(X = x_i, Y = y_j) \qquad g(X, Y) = X \text{ at } (x_i, y_j) \\
&\quad = \mathbb{E}(X)
\end{aligned}$$

Example:

Y: # of claims received by insurance company
X: some random parameter

$$Y|X \sim Poi(X), X \sim Exp(\lambda)$$

a) $\mathbb{E}(Y)$?
b) $P(Y = n)$?

a)

$$Y|X \sim Poi(X) \Rightarrow \mathbb{E}(Y|X = x) = x \Rightarrow \mathbb{E}(Y|X) = X$$

$$\therefore \begin{aligned}
\mathbb{E}(Y) &= \mathbb{E}(\mathbb{E}(Y|X)) \\
&= \mathbb{E}(X) = \frac{1}{\lambda}
\end{aligned}$$

b)

$$
\begin{aligned}
P(Y=n) &= \int_0^\infty P(Y=n|X=x)f_X(x)dx \\
&= \int_o^\infty \frac{e^{-x}x^n}{n!}\cdot \lambda e^{-\lambda x}dx \\
&= \frac{\lambda}{n!}\int_0^\infty x^n e^{-(\lambda+1)x}dx \\
&= \frac{\lambda}{(\lambda+1)^{n+1}n!}\int_0^\infty ((\lambda+1)x)^n e^{-(\lambda+1)x}d(\lambda+1)x \\
&= \frac{\lambda}{(\lambda+1)^{n+1}n!}\Gamma(n+1) && \Gamma(n+1)=n!\text{ ; formula for gamma function or integration by parts} \\
&= \frac{\lambda}{(\lambda+1)^{n+1}} = (\frac{1}{\lambda+1})^n\cdot\frac{1}{\lambda+1} \\
&\Rightarrow Y+1\sim Geo(\lambda/(\lambda+1))
\end{aligned}
$$

We verify that $\mathbb{E}(Y)=\frac{\lambda+1}{\lambda}-1=\frac{1}{\lambda}$

3.3 Decomposition of variance (EVVE's low)

Definition: The conditional variance of Y given $X=x$ is defined as

$$Var(Y|X=x)=\mathbb{E}((Y-\mathbb{E}(Y|X=x))^2|X=x)$$

$$Var(Y|X)_{(\omega)}=Var(Y|X=X_{(\omega)})\qquad Var(Y|X)_{(\omega)}\text{: a r.v, a function of }X$$

The conditional variance is simply the variance taken under the conditional distribution

$$\Rightarrow V(Y|X=x)=\mathbb{E}(Y^2|X=x)-(\mathbb{E}(Y|X=x))^2$$

Thus

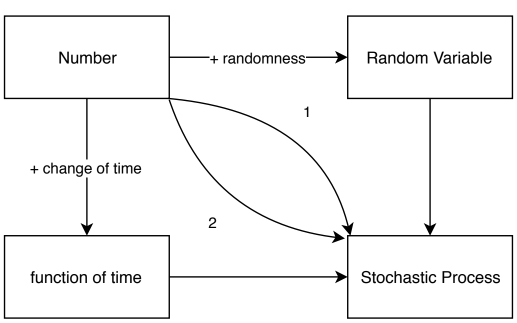
$$Var(Y)=\mathbb{E}(Var(Y|X))+Var(\mathbb{E}(Y|X))$$

$$\mathbb{E}(Var(Y|X))\text{ : "intra-group variance"}\qquad Var(\mathbb{E}(Y|X))\text{ : "inter-group variance"}$$

Proof:

$$
\begin{aligned}
RHS &= E(E(Y^2|X)-(E(Y|X))^2)+E((E(Y|X))^2)-(E(E(Y|X)))^2 \\
&= E(E(Y^2|X))-\cancel{E((E(Y|X))^2)}+\cancel{E((E(Y|X))^2)}-(E(E(Y|X)))^2 \\
&= E(Y^2)-(E(Y))^2 \\
&= Var(Y)
\end{aligned}
$$

4. Stochastic Processes



1. sequence / family of random variables
2. a random function (hard to formulate)

Definition: A **stochastic process** $\{X_t\}_{t\in T}$ is a collection of random variables, defined on a common probability space.

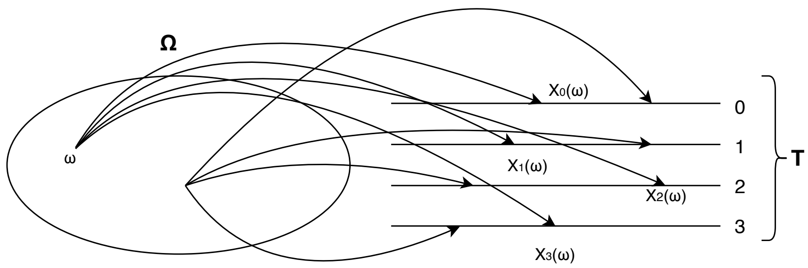
T: index set. In most cases, T corresponds to time, and is either discrete $\{0,1,2,\cdots\}$ or continuous $[0,\infty)$

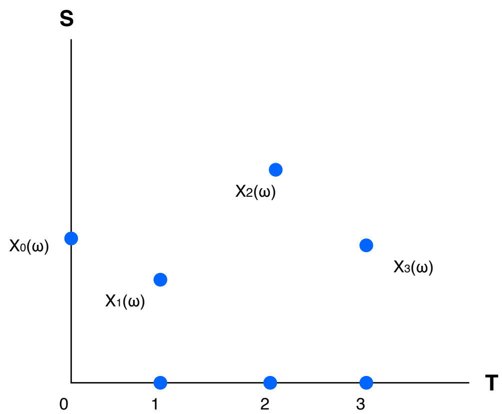
In discrete case, we writes $\{X_n\}_{n=0,1,2,\ldots}$

This **state space** S os a stochastic process is the set of all possible value of $X_t, t\in T$

S can also be either discrete or continuous. In this course, we typically deal with **discrete** state space. Then we relabel the states so that $S = \{0, 1, 2, \cdots\}$ (countable state space) or $S = \{0, 1, 2, \cdots, M\}$ (finite state space)

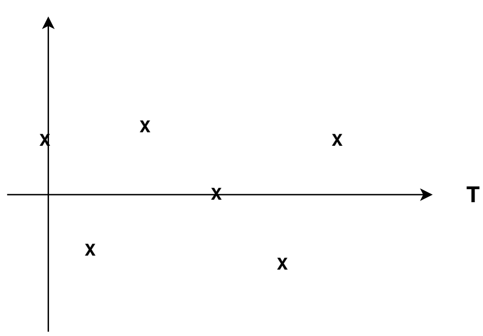
Remark: As in the case of the joint distribution, we need the r.v's in a stochastic process to be defined on a common probability space, because we want to discuss their joint behaviours, i.t, how things change over time.





Thus, we can identify each point ω in the sample space Ω with a function defined on T and taking value in S. Each function is called a **path** of this stochastic process

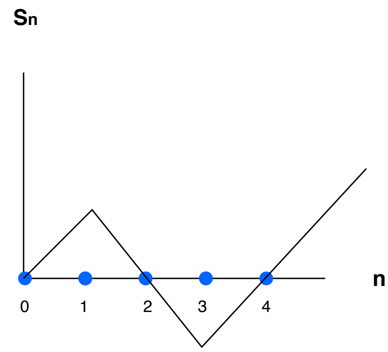
Example Let $X_0, X_1, \cdots$ be independent and identical r.v's following some distribution. Then $\{X_n\}_{n=0,1,2,\ldots}$ is a stochastic process



Example Let $X_1, X_2, \ldots$ be independent and identical r.v.'s. $P(X_1 = 1) = p$, and $P(X_1 = -1) = 1 - p$. Define $S_0 = 0, S_n = \sum_{i=1}^{n} X_i, n \leq 1$, e.g.

- $S_0 = 0$
- $S_1 = X_1$
- $S_2 = X_1 + X_2$
- $\cdots\cdots$

Then $\{S_n\}_{n=0,1,\ldots}$ is a stochastic process, with state space $S = \mathbb{Z}$ (integer)



4.1 Markov Chain

4.1.1 Simple Random Walk

$\{S_n\}_{n=0,1,...}$ is called a **"simple random walk"**. ($S_n = S_{n-1} + X_n$)

$$S_n = \begin{cases} S_{n-1} + 1 \\ S_{n-1} - 1 \end{cases}$$

Remark: Why we need the concept of "stochastic process"? Why don't we just look at the joint distribution of $(X_0, X_1, ..., X_n)$?

Answer: The joint distribution of a large number of r.v's is very complicated, because it does not take advantage of the special structure of T(time).

For example, simple random walk. The full distribution of $(S_0, S_1, ..., S_n)$ is complicated for n large. However, the structure is actually simple if we focus on the adjacent times:

$$S_{n+1} = S_n + X_{n+1}$$

$$S_n : \text{ last value.} \quad X_{n+1} : \text{related to } Ber(p). \text{ They are independent}$$

By introducing time into the framework, we can greatly simplify many things.

More precisely, we find that for simple random walk, $\{S_n\}_{n=0,1,...}$, if we know S_n the distribution of $S_n + 1$ will not depend on the history $(S_0, ..., S_{n-1})$. This is a very useful property

In general for a stochastic process $\{X_n\}_{n=0,1,...}$, at time n, we already know $X_0, X_1, ..., X_n$, S_0; our best estimate of the distribution of X_{n+1} should be the conditional distribution:

$$X_{n+1}|X_n, ..., X_n$$

given by:

$$P(X_{n+1} = x_{n+1}|X_n = x_n, ..., X_0 = x_0)$$

As time passes, the expression becomes more and more complicated $\to$ impossible to handle.

However, if we know that this conditional distribution is actually the same as the conditional distribution only given X_n, then the structure will remain simple for any time. This motivates the notion of *Markov chain*.

4.1.2 Markov Chain

4.1.2.1 Discrete-time Markov Chain

Definition and Examples

Definition: A discrete-time Stochastic process $\{X_n\}_{n=0,1,...}$ is called a **discrete-time Markov Chain (DTMC)**, if its state space S is discrete, and it has the Markov property:

$$\begin{aligned} &P(X_{n+1} = x_{n+1}|X_n = x_n, ..., X_o = x_o) \\ &= P(X_{n+1} = x_{n+1}|X_n = x_n) \end{aligned}$$

for all $n, x_0, ..., x_n, x_{n+1} \in S$

If $X_{n+1}|\{x_n = i\}$ does not change over time, $P(X_{n+1} = j|N_n = i) = P(X_1 = j|X_0 = i)$, then we call this Markov chain **time-homogeneous** (default setting for this course).

$$\begin{aligned} &P(X_{n+1} = x_{n+1}|X_n = x_n, ..., X_0 = x_0) \qquad && X_{n+1} = x_{n+1}\text{: future; } X_n = x_n\text{: present(state)} \\ &= P(X_{n+1} = x_{x+1}|X_n = x_n) && X_{n-1} = x_{n-1}, ..., X_0 = x_0\text{: past(history)} \end{aligned}$$

Intuition: Given the present state, the past and the future are independent. In other words, the future depends on the previous results only through the current state.

Example: simple random walk

The simple random walk $\{S_n\}_{n=0,1,...}$ is a Markov chain

Proof:

Recall that $S_{n+1} = S_n + X_{n+1}$

$$\begin{aligned} &P(S_{n+1} = s_{n+1}|S_n = s_n, ..., S_0 = s_0) \\ &= 0 \\ &= P(S_{n+1} = s_{n+1}|S_n = s_n) \end{aligned}$$

if $s_{n+1} \neq s_n \pm$

$$\begin{aligned} &P(S_{n+1} = s_n + 1|S_n = s_n, ..., s_0 = 0) \\ &= P(X_{n+1}|S_n = s_n, ..., S_0 = 0) \\ &= P(X_{n+1} = 1) \qquad X_{n+1} \perp (X_1, ..., X_n) \text{ hence also } (S_0, ..., S_n) \end{aligned}$$

Similarly,

$$\begin{aligned} &P(S_{n+1} = s_n + 1|S_n = s_n) \\ &= P(X_{n+1} = 1|S_n = s_n) \\ &= P(X_{n+1} = 1) \\ &\Rightarrow P(S_{n+1}|S_n = s_n, ..., S_0 = s_0) \end{aligned}$$

Similarly,

$$\begin{aligned} &P(S_{n+1} = s_n - 1|S_n = s_n, ..., S_0 = 0) \\ &= P(S_{n+1} = s_n - 1|S_n = s_n) \\ &= P(X_{n+1} = -1) \\ &\Rightarrow \{S_n\}_{n=0,1,...} \text{ is a DTMC} \qquad \blacksquare \end{aligned}$$

4.1.3 One-step transition probability matrix

For a time-homogeneous DTMC, define

$$\begin{aligned} P_{ij} &= P(X_1 = j|X_0 = i) \\ &= P(X_{n+1} = j|X_n = i) \qquad n = 0, 1, ... \end{aligned}$$

P_{ij}: one step transition probability

The collection of $P_{ij}, i, j \in S$ governs all the one-step transitions of the DTMC. Since it has two indices i and j; it naturally forms a matrix $P = \{P_{ij}\}_{i,j \in S}$, called the **(one-setp) transition (probability) matrix** or **transition matrix**

Property of a transition matrix $P = \{P_{ij}\}_{i,j \in S}$:

$$\begin{aligned} &P_{ij} \geq 0 \quad \forall i, j \in S \\ &\sum_{j \in S} P_{ij} = 1 \quad \forall i \in S \quad \rightarrow \text{ the row some of } P \text{ are all } 1 \end{aligned}$$

Reason:

$$
\begin{aligned}
\sum_{j \in S} P_{ij} &= \sum_{j \in S} P(X_1 = j | X_0 = i) \\
&= P(X_1 \in S | X_o = i) \\
&= 1
\end{aligned}
$$

Example 1 : simple random walk

There will be 3 cases:

$$
\begin{aligned}
&P_{i,i+1} = P(S_1 = i+1 | S_0 = i) = P(X_1 = 1) = p \\
&P_{i,i-1} = P(S_1 = i-1 | S_0 = i) = P(X_1 = -1) = 1 - p =: q \\
&P_{i,j} = 0 \qquad\qquad \text{for } j \not= i \pm 1
\end{aligned}
$$

$$
\Rightarrow \text{(infinite dimension)} p = \left\{ \begin{matrix}
... & ... & ... & ... & ... & ... & ... \\
... & 0 & p & 0 & ... & ... & ... \\
... & q & 0 & p & ... & ... & ... \\
... & ... & q & 0 & p & ... & ... \\
... & ... & ... & q & 0 & p & ... \\
... & ... & ... & ... & ... & ... & ...
\end{matrix} \right\}
$$

Example 2: Ehrenfest's urn

Two urns A, B, total M balls. Each time, pick one ball randomly(uniformly), and move it to the opposite urn.

$$X_n : \# \text{ of balls in } A \text{ after step } n$$

$$S = \{0, 1, ..., M\}$$

$$
\begin{aligned}
P_{ij} &= P(X_1 = j | X_0 = j) \qquad (i \text{ balls in } A,\, M - i \text{ balls in } B) \\
&= \begin{cases}
i/M & j = i - 1 \\
(M - i)/M & j = i + 1 \\
0 & j \not= i \pm 1
\end{cases}
\end{aligned}
$$

$$
p = \left\{ \begin{matrix}
0 & 1 & & & & & \\
1/M & 0 & (M-1)/M & & & & \\
 & 1/M & 0 & (M-1)/M & & & \\
 & & 2/M & 0 & (M-2)/M & & \\
... & ... & ... & ... & ... & ... & ... \\
 & & & & (M-1)/M & 0 & 1/M \\
 & & & & & 1 & 0
\end{matrix} \right\}
$$

Example 3: Gambler's ruin

A gambler, each time wins 1 with probability p, losses 1 with probability $1 - p = q$. Initial wealth $S_0 = a$; wealth at time n: S_n. The gambler leaves if $S_n = 0$ (loses all money) or $S_n = M > a$ (wins certain amount of money and gets satisfied)

This is a variant of the simple random walk, where we have absorbing barriers($P_{ii} = 1$) at 0 and M

$$S = \{0, ..., M\}$$

$$
P_{ij} = \begin{cases}
p & j = i + 1, i = 1, ..., M - 1 \\
q & j = i - 1, i = 1, ..., M - 1 \\
1 & i = j = 0 \text{ or } i = j = M \\
0 & \text{otherwise}
\end{cases}
$$

$$
p = \left\{ \begin{matrix}
1 & 0 & ... & & & & \\
q & 0 & p & ... & & & \\
... & q & 0 & p & ... & & \\
 & ... & q & 0 & p & ... & \\
... & ... & ... & ... & ... & ... & ... \\
 & & & ... & q & 0 & p \\
 & & & & ... & 0 & 1
\end{matrix} \right\}
$$

Example 4: Bonus-Malus system

Insurance company has 4 premium levels: 1, 2, 3, 4

Let $X_n \in \{1, 2, 3, 4\}$ be the premium level for a customer at year n

$$Y_n \overset{iid}{\sim} Poi(\lambda): \text{ \# of claims at year } n$$

- If $Y_n = 0$ (no claims)
  - $X_{n+1} = max(X_{n-1}, 1)$
- If $Y_n > 0$
  - $X_{n+1} = min(X_n + Y_n, 4)$

Denote $a_k = P(Y_n = k), k = 0, 1, ...$

$$p = \begin{Bmatrix} a_0 & a_1 & a_2 & (1 - a_0 - a_1 - a_2) \\ a_0 & 0 & a_1 & (1 - a_0 - a_1) \\ 0 & a_0 & 0 & (1 - a_0) \\ 0 & 0 & a_0 & (1 - a_0) \end{Bmatrix}$$

4.2 Chapman-Kolmogorov equations

Q: Given the (one-step) transition matrix, $P = \{P_{ij}\}_{i,j \in S}$, how can we decide the n-step transition probability

$$\begin{aligned} P_{ij}^{(n)} &:= P(X_n = j | X_0 = i) \\ &= P(X_{n+m} = j | X_m = i), \quad m = 0, 1, ... \end{aligned}$$

As a special case, let us start with $P_{ij}^{(2)}$ and their collection $p^{(2)} = \{P_{ij}^{(2)}\}_{i,j \in S}$ (also a square matrix, same dimension as P)

Condition on what happens at time 1:

$$\begin{aligned} P_{ij}^{(2)} &= P(X_2 = j | X_0 = i) \\ &= \sum_{j \in S} P(X_2 = j | X_0 = i, X_1 = k) \cdot P(X_1 = k | X_0 = i) \quad \text{conditional law of total probability} \end{aligned}$$

4.2.1 Conditional Law of total probability

$$\begin{aligned} & P(X_2 = j | X_0 = i) \\ &= \sum_{k \in S} P(X_2 = j, X_1 = k | X_0 = i) \\ &= \sum_{k \in S} \frac{P(X_2 = j, X_1 = k, X_0 = i)}{P(X_0 = i)} \\ &= \sum_{k \in S} \frac{P(X_2 = j, X_1 = k, X_0 = i)}{P(X_1 = k, X_0 = i)} \cdot \frac{P(X_1 = k, X_0 = i)}{P(X_0 = i)} \\ &= \sum_{k \in S} P(X_2 = j | X_0 = i, X_1 = k) \cdot P(X_1 = k | X_0 = i) \end{aligned}$$

continue on $P_{ij}^{(2)}$

$$\begin{aligned} P_{ij}^{(2)} &= P(X_2 = j | X_0 = i) \\ &= \sum_{j \in S} P(X_2 = j | X_0 = i, X_1 = k) \cdot P(X_1 = k | X_0 = i) \quad \text{conditional law of total probability} \\ &= \sum_{k \in S} P(X_2 = j | X_1 = k) \cdot P(X_1 = k | X_0 = i) \\ &= \sum_{k \in S} P(X_1 = j | X_0 = k) \cdot P(X_1 = k | X_0 = i) \\ &= \sum_{k \in S} P_{ik} \cdot P_{kj} \\ &= (P \cdot P)_{ij} \end{aligned}$$

Thus, $p^{(2)} = P \cdot P = p^2$

Using the smae idea, for $n, m = 0, 1, 2, 3...$:

$$\begin{aligned}
P_{ij}^{(n+m)} &= P(X_{n+m} = j | X_0 = i) \\
&= \sum_{k \in S} P(X_{n+m} = j | X_0 = i, X_m = k) \cdot P(X_m = k | X_0 = i) \\
&= \sum_{k \in S} P(X_{n+m} = j | X_m = k) \cdot P(X_m = k | X_0 = i) \quad \text{Markov property} \\
&= \sum_{k \in S} P(X_n = j | X_0 = k) \cdot P(X_m = k | X_0 = i) \\
&= \sum_{k \in S} p_{ik}^{(m)} \cdot P_{kj}^{(n)} \\
&= (p^{(m)} \cdot p^{(n)})_{ij} \\
&\Rightarrow p^{(n+m)} = p^{(m)} \cdot p^{(n)} \qquad (*)
\end{aligned}$$

By definition, $p^{(1)} = p$

- $\Rightarrow p^{(2)} = p^{(1)} \cdot p^{(1)} = p^2$
- $\Rightarrow p^{(3)} = p^{(2)} \cdot p^{(1)} = p^3$
- $\cdots\cdots\cdots$
- $\Rightarrow p^{(n)} = p^n$

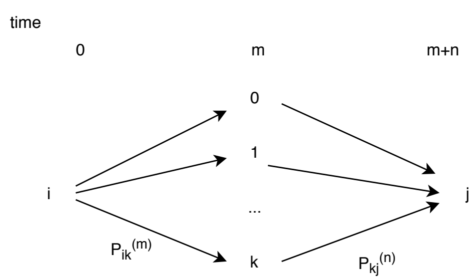
Note:

- n from $p^{(n)}$: n-step transition probability matrix
    - $p^{(n)} = \{p_{ij}^{(n)}\}_{i,j \in S}$
      $p_{ij}^{(n)} = P(X_n = j | X_0 = i)$
- n from p^n: n-th power of the (one-step) transition matrix
    - $p^n = p \cdot ... \cdot p$
      $p = \{P_{ij}\}_{i,j \in S}$
      $p_{ij} = P(X_1 = j | X_0 = i)$

$(*)$ is called the **Chapman-Kolmogorov equations** (c-k equation). Entry-wise:

$$P_{ij}^{n+m} = \sum_{k \in S} P_{ik}^{(m)} P_{kj}^{(n)}$$

Intuition:



"Condition at time m (on X_m) and sum p all the possibilities"

4.2.2 Distribution of X_n

So far, we have seen transition probability $P_{ij}^{(m)} = P(X_n = j | X_0 = i)$. This is not the probability $P(X_n = j)$. In order to get this distribution, we need the information about which state the Markov chain starts with.

Let $\alpha_{0,i} = P(X_0 = i)$. The row vector $\alpha_0 = (\alpha_0, 0, \alpha_0, 1, ...)$ is called the **initial distribution** of the Markov chain. This is the distribution of the initial state X_0

Similarly, we define distribution of X_n: $\alpha_n = (\alpha_n, 0, \alpha_n, 1, ...)$ where $\alpha_{n,i} = P(X_n = i)$

Fact: $\alpha_n = \alpha_0 \cdot p^n$

Proof:

$$\forall j \in S$$

$$
\begin{aligned}
\alpha_{n,j} &= P(X_n = j) \\
&= \sum_{i \in S} P(X_n = j | X_0 = i) \cdot P(X_0 = i) \\
&= \sum_{i \in S} \alpha_{0,i} \cdot P_{ij}^{(n)} \\
&= (\alpha_0 \cdot P^{(n)})_j = (\alpha_0 \cdot p^n)_j
\end{aligned}
$$

$$\Rightarrow \alpha_n = \alpha 0 \cdot p^n$$

- α_n: distribution of X_n
- α_0: initial distribution
- p^n: transition matrix

Remark: The distribution of a DTMC is completely determined by two things:

- the initial distribution α_0 (row vector), and
- the transition matrix p (square matrix)

4.3 Stationary distribution (invariant distribution)

Definition: A probability distribution $\pi = (\pi_0, \pi_1, ...)$ os ca;;ed a **stationary distribution**(invariant distribution) of the DTMC $\{X_n\}_{n=0,1,...}$ with transition matrix P, if :

1. $\underline{\pi} = \pi \cdot P$
2. $\sum_{i \in S} \pi_i = 1 (\Leftrightarrow \underleftarrow{\pi} \cdot \mathbb{1})$. ($\mathbb{1}$: a column of all 1's)

Why such $\underline{\pi}$ is called stationary/invariant distribution?

$$
\begin{gathered}
\sum_{i \in S} \pi_i = 1, \pi_i \geq 0, i = 0, 1, ... \Rightarrow \text{distribution} \\
\underline{\pi} = \pi \cdot P \Rightarrow \text{invariant/stationary.}
\end{gathered}
$$

Assume the MC starts from the initial distribution $\alpha_0 = \underline{\pi}$. hen the distribution of X_1 is

$$\alpha_1 = \alpha_0 \cot P = \underline{\pi} \cdot P = \underline{\pi} = \alpha_0$$

The distribution of X_2:

$$\alpha_2 = \alpha_0 \cdot P^2 = \underline{\pi} \cdot P \cdot P = \underline{\pi} \cdot P = \underline{(}\pi) = \alpha_0$$

$$\cdots\cdots$$

$$\alpha_n = \alpha_0$$

Thus, if the MC starts from a stationary distribution, then its distribution will not change over time.

Example 4.3.1

An electron has two states: $ground(0), excited(1)$. Let $X_n \in \{0, 1\}$ be the state at time n.

At each step, changes state with probability:

- α if it is in state 0.
- β if it is in state 1.

THen $\{X_n\}$ is a DTMC. ITs transitional matrix is:

$$P = \begin{Bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{Bmatrix}$$

Now let us solve for the stationary distribution $\underline{\pi} = \underline{\pi} \cdot P$.

$$(\pi_0, \pi_1) \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} = (\pi_0, \pi_1)$$

$$\Rightarrow \begin{cases} \pi_0(1-\alpha) + \pi_1\beta = \pi_0 & (1) \\ \pi_0\alpha + \pi_1(1-\beta) = \pi_1 & (2) \end{cases}$$

We have two equations and two unknowns. However, note that they are not linearly independent:

sum of LHS $= \pi_0 + \pi_1 =$ sum of RHS. Hence (2) can be derived from (1). By (1), we have:

$$\alpha\pi_0 = \beta\pi_1 \quad \text{or} \quad \frac{\pi_0}{\pi_1} = \frac{\beta}{\alpha}$$

This where we need $\underline{\pi} \cdot \mathbb{1}$:

$$\pi_0 + \pi_1 = 1 \Rightarrow \pi_0 = \frac{\beta}{\alpha+\beta}, \quad \pi_1 = \frac{\alpha}{\alpha+\beta}$$

Thus, we conclude that there exists a unique stationary distribution $\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right) = \underline{\pi}$

The above procedure for solving for stationary distribution is typical:

1. Use $\underline{\pi} = \underline{\pi}P$ to get the properties between different components of $\underline{\pi}$
2. Use $\underline{\pi} \cdot \mathbb{1} = 1$ to normalize (get exact values)

4.4. Classification of States

4.4.1. Transience and recurrence

Let T: be the waiting for a MC to visit/revisit state i for the first time

$$T_i := min\{n > 0 : X_n = i\} \qquad T_i \text{ is a r.v.}$$

$T_i = \infty$ if the MC never (re)visits state i.

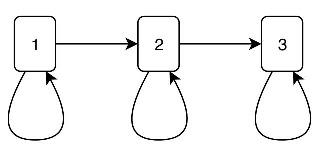
Definition 4.4.1

A state i is called:

- transient, if $P(T_1 < \infty | X_0 = i) < 1$ (never goes back to i positive probability)
- recurrence, if $P(T_i < \infty | X_0 = i) = 1$ (always goes back to state i)
  - positive recurrent, if $E(T_i | X_0 = i) < \infty$
  - null recurrent, if $E(T_i | X_0 = i) = \infty$
  - (note: a r.v. is finite with probability $\not\Rightarrow$ its expectation is finite)
    - Example: $T = 2, 4, ..., 2^n, p = \frac{1}{2}, \frac{1}{4}, ..., 2^{-n}$
      $E(T_= 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + ... + 2^n \cdot 2^{-n} = \infty$

Example 4.4.1

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \\ & \frac{1}{2} & \frac{1}{2} \\ & & 1 \end{pmatrix}$$



Given $X_0 = 0$,

$$P(\underbrace{X_1 = 0}_{T_0=1} | X_0 = 0) = P(\underbrace{X_1 = 1}_{\substack{T_0=\infty \text{ since state 1} \\ \text{and 2 do not go to 0}}} | X_0 = 0) = \frac{1}{2} \quad \Rightarrow \quad P(T_0 < \infty | X_0 = 0) = \frac{1}{2} < 1$$

Thus, state 0 is transient

Similarly, state 1 is transient.

Given $X_0 = 2$,

$$P(X_1 = 2 | X_0 = 2) \Rightarrow P(T_2 < \infty | X_0 = 2) = 1$$

As $E(T_2 | X_0 = 2) = 1$Thus, state 2 is a positive recurrence.

In general, the distribution of T_i is very hard to determine $\Rightarrow$ need better criteria for recurrence/transience.

Criteria (1): Define $f_{ii} = P(T_i < \infty | X_0 = i)$, and

$$V_i = \# \text{ of times that the MC (revisits) state i} = \sum_{n=1}^{\infty} 1\!\!1_{\{X_n = i\}}$$

If state i is transient

$$P(V_i = k | X_0 = j) = \underbrace{f_{ii}^k}_{\substack{\text{goes back to} \\ i \text{ for } k \text{ times}}} \underbrace{(1 - f_{ii})}_{\substack{\text{never visits} \\ i \text{ again}}}$$
$$\Rightarrow V_i + 1 \sim Geo(1 - f_{ii})$$

In particular, $P(V_i < \infty | X_0 = i) = 1 \Rightarrow$ If state i is transient, it is visited away finitely many times with probability 1. The MC will leave state i forever sooner or later.

On the other hand, if state i is recurrent, then $f_{ii} = 1$

$$P(V_i = k) = 0 \quad k = 0, 1, ... \Rightarrow P(V_1 = \infty) = 1$$

If the MC starts at a recurrent state i, it will visit that state infinitely many times.

Criteria (2): In terms of $E(V_i | X_0 = i)$:

$$\begin{aligned} E(V_i | X_0 = i) &= \frac{1}{1 - f_{ii}} - 1 = \frac{f_{ii}}{1 - f_{ii}} < \infty \quad & \text{if } f_{ii} < 1, (i \text{ transient}) \\ E(V_i | X_0 = i) &= \infty, & \text{if } f_{ii} = 1, (i \text{ recurrent}) \end{aligned}$$

Criteria (3): Note that

$$\begin{aligned} E(V_i | X_0 = i) &= E(\sum_{n=1}^{\infty} 1\!\!1_{\{X_n = i\}} | X_0 = i) \\ &= \sum_{n=1}^{\infty} E(1\!\!1_{\{X_n = i\}} | X_0 = i) \\ &= \sum_{n=1}^{\infty} P(X_n = i | X_0 = i) \\ &= \sum_{n=1}^{\infty} P_{ii}^{(n)} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty \quad \text{if } i \text{ transient} \\ &\Rightarrow \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \quad \text{if } i \text{ recurrent} \end{aligned}$$

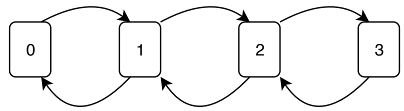
To conclude,

$$\begin{array}{rlr} & i \quad recureent & transient \\ & P(T_i < infty | X_0 = i) = 1 & P(T_i < \infty | X_0 = i) < 1 \\ define: & P(V_i = \infty | X_0 = i) = 1 & P(V_i < \infty | X_0 = i) = 1 \\ & E(V_i | X_0 = i) = \infty & E(V_i | X_0 = i) < \infty \\ \text{easiest to use: } & \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty & \sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty \end{array}$$

4.4.2 Periodicity

Example:

$$P = \begin{pmatrix} & 1 & & \\ \frac{1}{2} & & \frac{1}{2} & \\ & \frac{1}{2} & & \frac{1}{2} \\ & & 1 & \end{pmatrix}$$



Note that if we starts from 0, we can only get back to 0 in $2, 4, 6, \cdots$, i.t., even number of steps $P_{00}^{(2n+1)} = 0 \forall n$

Definition 4.4.2 : Period

The ***period*** of state i is defined as

$$d_i = \underbrace{gcd}_{\substack{\text{greates} \\ \text{common divisor}}} (\{n : \underbrace{P_{ii}^{(n)} > 0}_{\substack{i \text{ can go back} \\ \text{to } i \text{ in } n \text{ steps}}}\})$$

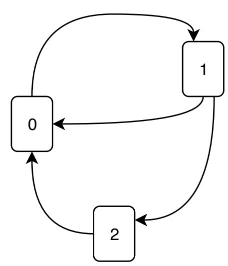
In this example above, $d_0 = gcd(\{\text{even numbers}\}) = 2$

If $d_i = 1$, state i is called ***"aperiodic"***

I~~f~~ $\exists n > 0$ such that $P_{ii}^{(n)} > 0$, then $d_i = \infty$

Remark 4.4.2

Note that $P_{ii} > 0 \Rightarrow d_i = 1$. The converse is **not true**.



$$P_{00}^{(2)} > 0, P_{00}^{(3)} > 0 \Rightarrow d_0 = 1 \text{ but } P_{00} = 0$$

In general, $d_i = \cancel{d} \Rightarrow P_{ii}^{(d)} > 0$

4.4.3 Equivalent classes and irreducibility

Definition 4.4.3.1 : Assessable

Let $\{X_n\}_n = 0, 1, \cdots$ be a DTMC with state space S. State j is said to be $\underline{\text{assessable}}$ from state i, denoted by $i \rightarrow j$, if $P_{ij}^{(n)} > 0$ for some $n \geq 0$.

Intuitively, i can go to state j in finite steps.

Definition 4.4.3.2 : Communicate

If $i \rightarrow j$ and $j \rightarrow i$, we say i and j **communicate**, denoted by $i \leftrightarrow j$.

Fact 4.4.3.1

"Communication" is an equivalence relation.

1. $i \leftrightarrow j$ then $P_{ii}^{(0)} = 1 = P(X_0 = i | X_0 = i)$ (Identity)
2. $i \leftrightarrow j$ then $j \leftrightarrow i$ (symmetry)
3. $i \leftrightarrow j, j \leftrightarrow k$, then $i \leftrightarrow k$ (transitivity)

Definition 4.4.3.3 : Class

As a result, we can use "$\leftrightarrow$" to divide the state space into different ***classes***, each containing only the states which communicate with each other.

$$\begin{cases} S = \bigcup_k C_k & (\{C_n\} \text{ is a partition of } S) \\ C_k \bigcap C_k' = \emptyset, \not k = k' \end{cases}$$

- For state i and j in the same class C_k, $i \leftrightarrow j$.
- For i j in different classes, $\not i \leftrightarrow j$ ($\not i \rightarrow j$ or $\not j \rightarrow i$)

Definition 4.4.3.4 : Irreducible

A MC is called **irreducible**, if it has only one class. In other words, $i \leftrightarrow j$ for any $i, j \in S$

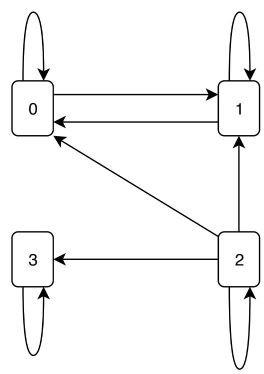
-Q: How to find equivalent classes?

-A: "Draw a graph and find the loops"

Example 4.4.3.1 : Find the classes

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \\ & & & 1 \end{pmatrix}$$

Draw an arrow from i to j if $P_{ij} > 0$

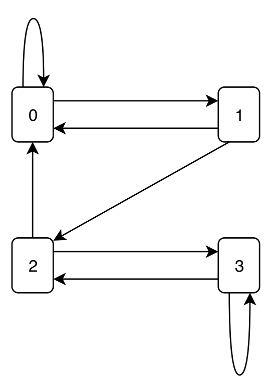


- $P_{01} > 0, P10 > 0 \Rightarrow 0 \leftrightarrow 1$
- State 2 does not communicate with any other state, since $P_{i2} = 0, \not i = 2$
- State 3 does not communicate with any other state, since $P_{i3} = 0, \not i = 3$

$\Rightarrow$ 3 classes: $\{0, 1\}, \{2\}, \{3\}$

Example 4.4.3.2 : Find the classes

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & & \frac{1}{2} & \\ \frac{1}{2} & & & \frac{1}{2} \\ & & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



- $P_{01}, P_{12}, P_{20} > 0 \Rightarrow 0, 1, 2$ are in the same class
- $P_{23}, P_{32} > 0 \Rightarrow 2, 3$ are in the same class
- Transitivity $\Rightarrow$ 0,1,2,3 are all in the same class.

$\Rightarrow$ This MC is irreducible

Fact 4.4.3.2

Preposition Transience/Recurrence are class properties. That is, if $i \leftrightarrow j$, then j is transient/recurrent if and only if i is transient/recurrent

Proof:

Suppose i is recurrent, then $\sum_{k=1}^{\infty} P_{ii}^{(k)} = \infty$

Since $i \leftrightarrow j$, $\exists m, n$ such that $P_{ij}^{(m)} > 0, P_{ij}^{(n)} > 0$

Note that

$$\underbrace{P_{jj}^{(m+n+k)}}_{P(X_{m+n+k}=j|X_0=j)} \geq \underbrace{P_{ji}^{(n)} P_{ii}^{(k)} P_{ij}^{(m)}}_{P(X_{m+n+k}=j, X_{n+k}=i, X_n=i|X_0=j)} \Rightarrow \sum_{l=1}^{\infty} P_{jj}^{(l)} \geq \sum_{l=m+n+1}^{\infty} P_{jj}^{(l)}$$
$$= \sum_{k=1}^{\infty} P_{jj}^{(m+n+k)}$$
$$\geq \sum_{k=1}^{\infty} P_{ji}^{(n)} P_{ii}^{(k)} P_{ji}^{(m)}$$
$$= \underbrace{P_{jj}^{(n)}}_{0} \underbrace{P_{ij}^{(m)}}_{0} \underbrace{\sum_{k=1}^{\infty} P_{ii}^{(k)}}_{\infty} = \infty$$

Thus, j is recurrent. Symmetrically, j is recurrent $\Rightarrow i$ is recurrent

Thus,

- i recurrent $\Leftrightarrow j$ recurrent
- i transient $\Leftrightarrow j$ transient

For irreducible MC, since recurrence and transience are class properties, we also say the Markov Chain is recurrent/transient

Definition 4.4.3.5 : Proposition

If an irreducible MC has a finite state space, then it is recurrent

Idea of proof

If the MC is transient, then with probability 1, each state has a last visit time. Finite states $\Rightarrow \exists$ a last visit time for all the states. As a result, the MC has nowhere to go after that time. $\Rightarrow$ Contradiction.

Remark 4.4.3.1

We can actually prove that the MC must be positive recurrent, if the state space is finite and the MC is irreducible.

Theorem 4.4.3.1

Periodicity is a class property: $i \leftrightarrow j \Rightarrow d_i = d_j$.

For an irreducible MC, its period is defined as the period of any state.

4.5 Limiting Distribution

In this part, we are interested in $lim_{n \to \infty} P_{ij}^{(n)}$ and $lim_{n \to \infty} P(X_u = i)$

To make things simple, we focus on the irreducible case.

Theorem 4.5.1 : Basic Limit Theorem

Let $\{X_n\}_{n=0,1,...}$ be an **irreducible, aperiodic, positive recurrent** DTMC. Then a unique stationary distribution:

$$\underline{\pi} = (\pi_0, \pi_1, \dots) \text{ exits}$$

Moreover:

$$(*) \quad \underbrace{lim_{n-\to\infty} P_{ij}^{(n)}}_{\substack{\text{limiting distribution} \\ \text{(does not depend on the initial state i)}}} = lim_{n\to\infty} \underbrace{\frac{\sum_{k=1}^{n} 1\!\!\perp_{\{X_k=j\}}}{n}}_{\text{long-run fraction of time spent in j}} = \underbrace{\frac{1}{\mathbb{E}(T_j|X_0=j)}}_{\substack{T_j = min\{n>0:X_n=j\} \\ \text{expected revisit time}}} = \pi_j \qquad , i, j \in S$$

Limiting distribution =

- long-run fraction of time
- $1/$ expected revisit time
- stationary distribution

Remark 4.5.1

The result $(*)$ is still true if the MC is null recurrent, where all the terms are **0**, and $\underline{\pi}$ is no longer a distribution. (in other words, there does not exist a stationary distribution)

Remark 4.5.2

If $\{X_n\}_{n=0,1,...}$ has a period $d > 1$:

$$\frac{\lim_{n\to\infty} P_{jj}^{(nd)}}{d} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} 1\!\!\perp_{\{X_k=j\}}}{n} = \frac{1}{\mathbb{E}(T_j|X_0=j)} = \pi_j$$

Back to the aperiodic case. SInce the limit $\lim_{n\to\infty} P_{ij}^{(n)}$ does not depend on i, $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$ is also the limiting(marginal) distribution at state j:

$$\lim_{n\to\infty} \alpha_{n,j} = \lim_{n\to\infty} P(X_n = j) = \pi_j$$

regardless of the initial distribution α_0

Detail:

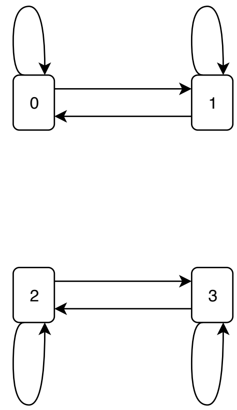
$$\begin{aligned}
\lim_{n\to\infty} \alpha_{n,j} &= \lim_{n\to\infty} (\alpha_0 \cdot p^{(n)})_j \\
&= \lim_{n\to\infty} \sum_{i\in S} \alpha_{0,i} \cdot P_{ij}^{(n)} \\
&= \sum_{i\in S} \lim_{n\to\infty} \alpha_{0,i} \cdot P_{ij}^{(n)} \\
&= \sum_{i\in S} \alpha_{0,i} \lim_{n\to\infty} \cdot P_{ij}^{(n)} \\
&= (\sum_{i\in S} \alpha_{0,i}) \pi_j \\
&= \pi_j
\end{aligned}$$

Why are the conditions in the *Basic Limit Theorem* necessary?

Example 4.5.1

Consider a MC with

$$p = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{2} & \frac{1}{2} \\ & & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



Two classes: $\{0,1\},\{2,3\} \Rightarrow$ it is **not** irreducible. All the states are still aperiodic, positive recurrent

This MC can be decomposed into two MC's:

State $0,1$, with

$$p_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \text{irreducible}$$

State $2,3$, with

$$p_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \text{irreducible}$$

And

$$p = \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}$$

Note that both $(\frac{1}{2},\frac{1}{2},0,0)$ and $(0,0,\frac{1}{2},\frac{1}{2})$ are stationary distributions. Consequently, any convex combination of these two distributions, of the form:

$$a(\frac{1}{2},\frac{1}{2},0,0)+(1-a)(0,0,\frac{1}{2},\frac{1}{2}) \quad ,a \in \{0,1\}$$

is also a stationary distribution

Thus, irreducibility is related to the uniqueness of the stationary distribution.

Correspondingly, the limiting transition probability will depend on i:

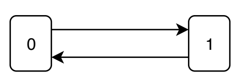
$$\lim_{n\to\infty} P_{00}^{(n)} = (\lim_{n\to\infty} P_1^n)_{00} = \frac{1}{2}$$

but $\lim_{n\to\infty} P_{20}^{(n)} = 0$

Example 4.5.2

Consider a MC with

$$p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



Irreducible, positive recurrent, but not aperiodic: $d=2$

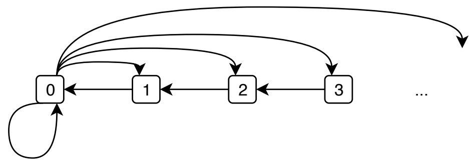
Note that $p^2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = I \Rightarrow p^{2n} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, p^{2n+1} = p = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$p_{00}^{(n)} = 1$ for n even, 0 for n odd $\Rightarrow \lim_{n\to\infty} P_{00}^{(n)}$ does not exist.

Aperiodicity is related to the existence of the limit $\lim_{n\to\infty} P_{ij}^{(n)}$

Example 4.5.3

$$P_{o,j} = p_j, j = 0, 1, \cdots, p_0 > 0$$
$$P_{i,i-1} = 1, i \geq 1$$



Given $X_0 = 0$, $T_0 = n + 1$ if and only if $X_1 = n$. his happens with prob p_n.

$$\begin{aligned} \Rightarrow \mathbb{E}(T_0 | X_0 = 0) &= \sum_{n=0}^{\infty}(n+1)p_n \\ &= 1 + \sum_{n=0}^{\infty} np_n \end{aligned}$$

We can construct p_n such that $\sum_{n=0}^{\infty} np_n = \infty$. (For example, $p_0 = \frac{1}{2}, p_2 = \frac{1}{4}, p_4 = \frac{1}{4}, \cdots$)

In this case, the chain is **null recurrent**. It is irreducible and aperiodic ($P_{00} = p_0 > 0$)

A stationary distribution does not exist. Reason:

$$p = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_i & \cdots \\ 1 & & 0 & & & \\ & 1 & & & & \\ \cdots & & & & & \\ & & & & 1 & \end{pmatrix}$$

$$\underline{\pi} \cdot P = \underline{\pi} \Rightarrow$$

$$\begin{aligned} p_0\pi_0 + \pi_1 &= \pi_0 \\ p_1\pi_0 + \pi_2 &= \pi_1 \\ &\vdots \\ p_{i-1}\pi_0 + \pi_i &= \pi_{i-1} \\ p_i\pi_0 + \pi_{i+1} &= \pi_I \end{aligned}$$

Add the first i equations:

$$(p_0 + \cdots + p_{i-1})\pi_0 + (\cancel{\pi_1} + \cancel{\pi_2} + \cancel{\cdots} + \pi_i) = \pi_0 + \cancel{\cdots} + \cancel{\pi_{i-1}}$$
$$(p_0 + \cdots + p_{i-1})\pi_0 + \pi_i = \pi_0$$
$$\Rightarrow \pi_i = (1 - (p_o + \cdots + p_{i-1}))\pi_0 = \sum_{k=i}^{\infty} p_k \pi_0$$

Try to normalize:

$$\begin{aligned} 1 &= \sum_{i=1}^{\infty} \pi_i \\ &= \sum_{i=0}^{\infty} \sum_{k_i}^{\infty} p_k \pi_0 \\ &= \sum_{k_i}^{\infty} \sum_{i=0}^{\infty} p_k \pi_0 \\ &= \sum_{k_i}^{\infty} p_k \sum_{i=0}^{\infty} \pi_0 \\ &= \underbrace{(\sum_{k_i}^{\infty}(k+1)p_k)}_{\infty}\pi_0 \end{aligned}$$
$$\Rightarrow \pi_0 = 0 \quad , \quad pi_i = 0 \quad \forall i$$

This is not a distribution. Thus, a stationary distribution does not exist.

positive recurrence is related to the existence of the stationary distribution

Example 4.5.4 : Electron

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \quad \alpha, \beta \in (0,1)$$

Irreducible, aperiodic, positive recurrence.

In order to find of P^n; we use the diagonalization technique.

$$P = Q\Lambda Q^{-1} \quad \text{where } \Lambda \text{ is diagonal}$$
$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{pmatrix} \quad Q = \begin{pmatrix} 1 & \alpha \\ 1 & 1-\beta \end{pmatrix} \quad Q^{-1} = \frac{1}{\alpha+\beta} \begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix}$$

Then

$$\begin{aligned}
P^n &= (Q\Lambda \cancel{Q^{-1}})(\cancel{Q}\Lambda \cancel{Q^{-1}}) \cdot (\cancel{Q}\Lambda Q^{-1}) \\
&= Q\Lambda^n Q^{n-1} \\
&= \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix} \begin{pmatrix} 1 & \\ & (1-\alpha-\beta)^n \end{pmatrix} \frac{1}{\alpha+\beta} \begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{\alpha+\beta} \begin{pmatrix} \beta + \alpha(1-\alpha-\beta)^n & \alpha - \alpha(1-\alpha-\beta)^n \\ \beta - \beta(1-\alpha-\beta)^n & \alpha + \beta(1-\alpha-\beta)^n \end{pmatrix} \\
&\Rightarrow \lim_{n\to\infty} P^n = \frac{1}{\alpha+\beta} \begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix}
\end{aligned}$$

Note that $\lim_{n\to\infty} P^n$ has identical rows. This corresponds to the result that $\lim_{n\to\infty} P_{ij}^{(n)}$ does not depend on i

We saw that the stationary distribution $\underline{\pi} = (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$. So we verity that $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$

Also, given $X_0 = 0$, $\mathbb{P}(T_0 = 1 | X_0 = 0) = 1 - \alpha$.

For $k = 2, 3, \cdots$

$$\begin{aligned}
\mathbb{P}(T_0 = k | X_0 = 0) &= \mathbb{P}(X_k = 0, X_{k-1} = 1, \cdots, X_1 = 1 | X_0 = 0) \\
&= \alpha(1-\beta)^{k-2}\beta \\
&\Rightarrow \mathbb{E}(T_0 | X_0 = 0) \\
&= 1 \cdot (1-\alpha) + \sum_{k=2}^{\infty} \alpha(1-\beta)^{k-2}\beta k \\
&= 1 - \alpha + \sum_{k=1}^{\infty} \underbrace{\alpha(1-\beta)^{k-2}\beta(k-1)}_{\mathbb{E}(Geo(\beta))} + \sum_{k=2}^{\infty} \alpha \underbrace{(1-\beta)^{k-2}\beta}_{\text{pmf of Geo}(\beta)} \\
&= 1 - \alpha + \alpha \sum_{k=1}^{\infty} (1-\beta)^{k-2}\beta(k-1) + \sum_{k=2}^{\infty} \alpha(1-\beta)^{k-2}\beta \\
&= 1\alpha + \alpha \cdot \frac{1}{\beta} + \alpha \cdot 1 \\
&= 1 - \cancel{\alpha} + \frac{\alpha}{\beta} + \cancel{\alpha} \\
&= \frac{\alpha+\beta}{\beta}
\end{aligned}$$

Hence we verify that $\mathbb{E}(T_0 | X_0 = 0) = \frac{1}{\pi_0}$

4.6 Generating function and branching processes

Definition 4.6.1

Let $\underline{p} = (p_0, p_1, \cdots)$ be a distribution on $\{0, 1, 2, \cdots\}$. Let ξ be a r.v. following distribution $\underline{p}$. That is $\mathbb{P}(\xi = i) = p_i$. Then the generating function of ξ, or of $\underline{p}$, is defined by

$$\begin{aligned}\psi(s) &= \mathbb{E}(s^\xi) \\ &= \sum_{k=0}^{\infty} p_k s^k \qquad for 0 \leq s \leq 1\end{aligned}$$

Properties of generating function

1. $\psi(0) = p_0, \quad \psi(1) = \sum_{k=0}^{\infty} p_k = 1$
2. Generating function determines the distribution

$$p_k = \frac{1}{k!}\frac{d^k\psi(s)}{ds^k}|_{s=0}$$

Reason:

$$\begin{aligned}\psi(s) = p_0 + p_1 s^1 + \cdots + p_{k-1}s^{k-1} + p_k s^k + p_{k+1}s^{k+1} + \cdots \\ \frac{d^k\psi(s)}{ds^k} = k!p_k + (\cdots)s + (\cdots)s^2 + \cdots \\ \frac{d^k\psi(s)}{ds^k}|_{s=0} = k!p_k\end{aligned}$$

In particular, $p_1 \geq 0 \Rightarrow \psi(s)$ is increasing. $p_2 \geq 0 \Rightarrow \psi(s)$ is climax

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In particular, $p_1 \geq 0 \Rightarrow \psi(s)$ is increasing. $p_2 \geq 0 \Rightarrow \psi(s)$ is climax

3. Let $\xi_1, ..., \xi_n$ be independent r.b. with generating function $\psi_1, ..., \psi_n$,

$$X = \xi_1 + ... + \xi_n \Rightarrow \psi_X(s) = \psi_1(s)\psi_2(s)...\psi_n(s)$$

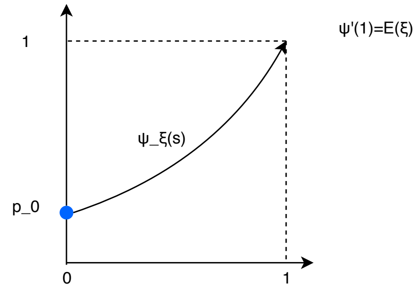
Proof:

$$\begin{aligned}\psi_X(s) &= s^{\mathbb{X}} \\ (independent) &= \mathbb{E}(s^{\xi_1}s^{\xi_2}...s^{\xi_n}) \\ &= \mathbb{E}(s^{\xi_1})...\mathbb{E}(s^{\xi_n}) \\ &= \psi_1(s)...\psi_n(s)\end{aligned}$$

4. $$\frac{d^\psi(s)}{ds^k}\Big|_{s=1} = \frac{d^k\mathbb{E}(s^\xi)}{ds^k}\Big|_{s=1} = \mathbb{E}(\frac{d^k s^\xi}{ds^k}\Big|_{s=1} = \mathbb{E}(\xi(\xi-1)(\xi-2)...(\xi-k+1)s^{\xi-k})\Big|_{s=1} = \mathbb{E}(\xi(\xi-1)...(\xi-k+1))$$

In particular, $\mathbb{E}(\xi) = \psi'(1)$ and $Var(\xi) = \mathbb{E}(\xi^2) - (\mathbb{E}(\xi))^2 = \mathbb{E}(\xi^2 - \xi) + \mathbb{E}(\xi) - (\mathbb{E}(\xi))^2 = \psi''(1) + \psi(1) - (\psi'(1))^2$

Graph of a g.f.:



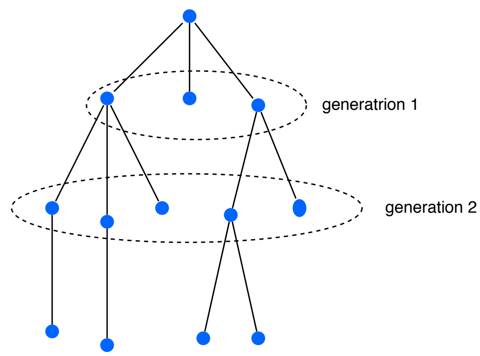
4.6.1 Branching Process

Each organism, at the end of its life, produces a random number Y of offsprings.

$$\mathbb{P}(Y = k) = P_k, \quad k = 0, 1, 2, ..., \quad P_k \geq 0, \quad \sum_{k=0}^{\infty} P_k = 1$$

The number of offsprings of different individuals are independent.

Start from one ancestor $X_0 = 1$, X_n : # of individuals(population in n-th generation)



Then $X_n + 1 = Y_1^{(n)} + Y_2^{(n)} + ... + Y_{X_n}^{(n)}$, where $Y_1^{(n)}, ..., Y_{X_n}^{(n)}$ are independent copies of Y, $Y_i^{(n)}$ is the number of offsprings of the i-th individual in the n-th generation

4.6.1.2 Mean and Variance

Mean: $\mathbb{E}(X_n)$ and Variance: $Var(X_n)$

Assume, $\mathbb{E}(Y) = \mu, Var(Y) = \sigma^2$.

$$\begin{aligned} \mathbb{E}(X_{n+1}) &= \mathbb{E}(Y_1^{(n)} + ... + Y_{X_n}^{(n)}) \\ &= \mathbb{E}(\mathbb{E}(Y_1^{(n)} + ... + Y_{X_n}^{(n)} | X_n)) \\ &= \mathbb{E}(X_n \mu) \\ \text{Wald's identity(tutorial 3)} \quad &= \mu \mathbb{E}(X_n) \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathbb{X}_n &= \mu \mathbb{E}(X_{n-1}) \\ &= \mu^2 \mathbb{X}_{n-2} \\ &\vdots \\ &= \mu^n \mathbb{E}(X_0) = \mu^n, \quad n = 0, 1, ... \end{aligned}$$

$$Var(X_{n+1}) = \mathbb{E}(Var(X_{n+1}|X_n) + Var(\mathbb{E})X_{n+1}|X_n)$$

$$\mathbb{E}(Var(X_{n+1}|X_n)) = \mathbb{E}(Var(Y_1^{(n)+...+Y_{X_N}^{(N)}}|X_N)) \\ = \mathbb{E}(X_n \cdot \sigma^2) \\ = \sigma^2 \mu^n$$

$$\begin{aligned} Var(\mathbb{E}(X_{n+1}|X_n)) &= Var(\mu X_n) \\ &= \mu^2 Var(X_u) \\ &\Rightarrow Var(X_{n+1}) = \sigma^2\mu^n + \mu^2 Var(X_n)) \\ &\quad Var(X_1) = \sigma^2 \\ &\quad Var(X_2) = \sigma^2\mu + \mu^2\sigma^2 = \sigma^2(\mu^1 + \mu^2) \\ &\quad Var(X_3) = \sigma^2\mu^2 + \mu^2(\sigma^2(\mu^1+\mu^2)) = \sigma^2(\mu^2+\mu^3+\mu^4) \\ &\quad \vdots \\ &\quad \text{In general, (can be proved by induction)} \\ &\quad Var(X_n) = \sigma(\mu^{n-1} + ... + \mu^{2n-2}) \\ &\quad = \begin{cases} \sigma^2\mu^{n-1}\dfrac{1-\mu^n}{1-\mu} & \cancel{\mu} = 1 \\ \sigma^2 n & \mu = 1 \end{cases} \end{aligned}$$

4.6.1.2 Extinction Probability

Q: What is the probability that the population size is eventually reduced to 0

Note that for a branching process, $X_n = 0 \Rightarrow X_k = 0$ for all $k \geq n$. Thus, state 0 is absorbing. $(P_{00} = 1)$. Let N be the time that extinction happens.

$$N = min\{n : X_n = 0\}$$

Define

$$U_n = \mathbb{P}(\underbrace{N \leq n}_{\substack{\text{extinction happens} \\ \text{before or at n}}}) = \mathbb{P}(X_n = 0)$$

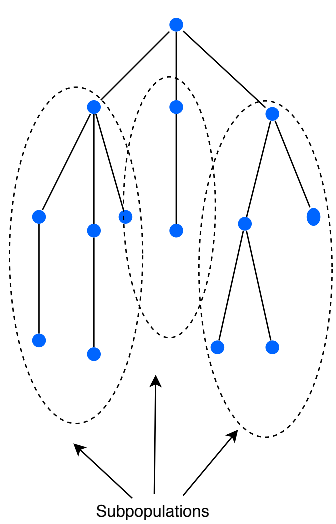
Then U_n is increasing in n, and

$$\begin{aligned} u = \lim_{n\to\infty} U_n &= \mathbb{P}(N < \infty) \\ &= P(\text{the extinction eventually happens}) \\ &= \text{extinction probability} \end{aligned}$$

Out goal : find u

We have the following relation between U_n and U_{n-1}:

$$U_n = \sum_{k=0}^{\infty} P_k(U_{n-1})^k = \underbrace{\psi}_{\text{gf of Y}}(U_{n-1})$$



Each subpopulation has the same distribution as the whole population.

Total population dies out in n steps if and only if each subpopulation dies out int $n-1$ steps

$$\begin{aligned}U_n &= \mathbb{P}(N \le n)\\ &= \sum_k \mathbb{P}(N \le n | X_1 = k)\underbrace{\mathbb{P}(X_1 = k)}_{=P_k}\\ &= \sum_k \mathbb{P}(\underbrace{N_1 \le n-1}_{\#\text{ of steps for subpopulation 1 to die out}}, \cdots, N_k \le n-1 | X_1 = k) \cdot P_k\\ &= \sum_k P_k \cdot U_{n-1}^k\\ &= \mathbb{E}(U_{n-1}^Y)\\ &= \psi(U_{n-1})\end{aligned}$$

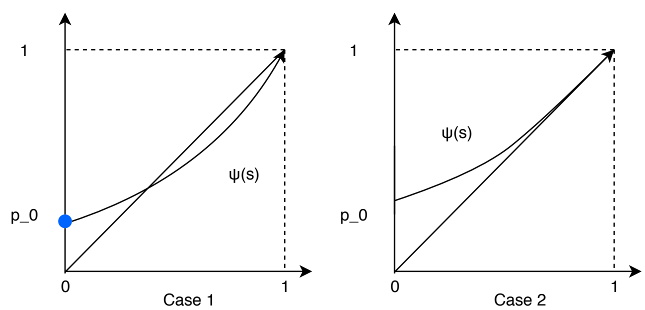
Thus, the question is :

> With initial value $U_0 = 0$ (since $X_0 = 1$), relation $U_n = \psi(U_{n-1})$. What is $\lim_{n\to\infty} U_N = u$?

Recall that we have

1. $\psi(0) = P_0 \ge 0$
2. $\psi(1) = 1$
3. $\psi(s)$ is increasing
4. $\psi(s)$ is convex

Draw $\psi(s)$ and function $f(s) = s$ between 0 and 1, we have two cases:



The extinction probability u will be the smallest intersection of $\psi(s)$ and $f(s)$. Equivalently, it is the smallest solution of the equation $\psi(s) = s$ between 0 and 1.