Note 21 - Mar 28

Review

Generator

$$R = \lim_{n \to 0^+} \frac{p(n) - I}{n}$$

$$R_{ii} = -\lambda_i, \quad R_{ij} = \lambda_i q_{ij}$$
$$\text{Row sums of } R \text{ are } 0$$

$$\lambda_i = -R_{ii}, \quad q_{ij} = -\frac{R_{ij}}{R_{ii}}, \quad i \not= j$$

6. Continuous-Time Markov Chain (cont'd)

6.3. Classification of States

The matrix Q is a transition matrix of a DTMC ($q_{ij} \geq 0, \quad \sum_{j \in S} q_{ij} = 1$), It contains all the information about the state changes, but "forget" the time.

Since ***accessibility, communication, irreducibility, recurrence/transience*** are only related to the change of states, not the time, these properties will be the same for the CTMC and its discrete skeleton.

For a CTMC $\{X(t)\}_{t \geq 0}$, we call "state j is accessible from state i", "i and j communicate", "the process is irreducible", "i is recurrent/transient" if and only if this is the case for its discrete skeleton.

6.3.1. Positive / Null Recurrence

Note that since **positive/null recurrence** do involve the (expected) amount of time, we can indeed have different results for a CTMC and its discrete skeleton.

Let R_i be the amount of (continuous, random) time the MC (re)visits state i.

A state i is called positive recurrent, if it is recurrent, and $\mathbb{E}(R_i | X(0) = i) < \infty$. It is called null recurrent, if it is recurrent and $\mathbb{E}(R_i | X(0) = i) = \infty$

As in the discrete-time case, the positive recurrence, null recurrence and transience are class property

6.4. Stationary Distribution

Definition 6.4.1. Stationary Distribution

A distribution $\underline{\pi} = (\pi_0, \pi_1, \cdots)$ is called a stationary distribution of a CTMC $\{X(t)\}_{t\geq 0}$ with generator R is it satisfies:

1. $\underline{\pi} \cdot R = 0 \rightarrow (0, 0, \cdots)$
2. $\sum_{i\in S} \pi_i = 1 \quad (\underline{\pi} \cdot \mathbb{1} = 1)$

Q: Why such a $\underline{\pi}$ is called stationary?

A: Assume the process starts from the initial distribution $\underline{\alpha^{(0)}} = \underline{\pi}$:

$$\mathbb{P}(X(0) = i) = \pi_i$$

Then the distribution at time t is given by

$$\underline{\alpha}^{(t)} = \underline{\alpha}^{(0)} \cdot P(t) = \underline{\pi} \cdot P(t)$$

Reason:

$$\begin{aligned}
\alpha_j^{(t)} &= \mathbb{P}(X(t) = j) \\
&= \sum_{i\in S} \mathbb{P}(X(t) = j | X(0) = i)\mathbb{P}(X(0) = i) \\
&= \sum_{i\in S} P_{ij}(t) \cdot \alpha_i^{(0)} \\
&= (\underline{\alpha}^{(i)} \cdot P(t))_j
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{d}{dt}((\underline{\alpha})^{(t)}) &= \frac{d}{dt}(\underline{\pi} \cdot P(t)) \\
&= \underline{\pi}(\frac{d}{dt}P(t)) \\
\text{c-k equation} &= \underline{\pi} \lim_{n\to 0^+} \frac{P(t+n) - p(t)}{t} \\
&= \underline{\pi} \lim_{n\to 0^+} \frac{P(n)P(t) - P(t)}{n} \\
&= \underline{\pi}(\lim_{n\to 0^+} \frac{P(n) - I}{n})P(t) \\
\underline{0} &= \underline{\pi} \cdot R \cdot P(t) \\
&= \underline{(0)}
\end{aligned}$$

$\Rightarrow \underline{\alpha}(t)$ is a constant (vector)

In other words, the distribution of $X(t)$ will not change over time, if the MC start from the stationary distribution.

Fact 6.4.1. Stationary Distribution

If a CTMC starts from a stationary distribution, then its distribution will never change.

Remark 6.4.1. Kolmogorov's Backward Equation

In the above derivation, we also see that

$$\frac{d}{dt}(P(t)) = P'(t) = R \cdot P(t)$$

This is called the **Kolmogorov's Backward Equation**

6.4.1. Basic Limit Theorem for CTMC

Let $\{X(t)\}_{t \geq 0}$ be an irreducible, recurrent CTMC. Then

$$\lim_{t \to \infty} P_{ij}(t) =: \pi'_j = \frac{\mathbb{E}(T_j)}{\mathbb{E}(R_j | X(0) = j)} = \frac{1/\lambda_i}{\mathbb{E}(R_j | X(0) = j)}$$

In addition, the MC is positive recurrent if and only if an unique stationary distribution exists. In this case, the stationary distribution is $\underline{\pi}' = (\pi'_0, \pi'_1, \cdots)$

Remark 6.4.1.1

$$\frac{\mathbb{E}(T_j)}{\mathbb{E}(R_j | X(0) = j)} = \text{long-run fraction of time spent in state } j$$

Thus, π'_j is also the long-run fraction of time that the process spends in stat j.

6.5. Birth and Death Processes

Definition 6.5.1. Birth and Death Process

A **Birth and Death Process** is a CTMC such that $S = \{0, 1, \cdots, M\}$, or $S = \{0, 1, \cdots\}$, and $\underline{q_{ij} = 0}$ if $\underline{|j - i| > 1}$.

The process can only change to neighbouring states:

$$q_{i,i-1} + q_{i,i+1} = 1, \quad i \geq 1$$

$$q_{01} = 1$$