Note 11 - Feb 26

Review

Solving or stationary distribution

$$\begin{cases} \underline{\pi} \cdot P = \underline{\pi} & \text{gives properties between components of } \underline{\pi} \\ \underline{\pi} \cdot \mathbb{1} = 1 & \text{normolize to get exact values} \end{cases}$$

Classification

$$T_i := min\{n > 0 : X_n = i\}$$

$$\begin{array}{rlr} i & recureent & transient \\ & P(T_i < infty | X_0 = i) = 1 & P(T_i < \infty | X_0 = i) < 1 \\ & P(V_i = \infty | X_0 = i) = 1 & P(V_i < \infty | X_0 = i) = 1 \\ & E(V_i | X_0 = i) = \infty & E(V_i | X_0 = i) < \infty \\ \text{easiest to use: } & \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty & \sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty \end{array}$$

Recurrent $\begin{cases} \text{positive recurrent} & \text{if } \mathbb{E}(T_i | X_o = i) < \infty \\ \text{null recurrent} & \text{if } \mathbb{E}(T_i | X_o = i) = \infty \end{cases}$

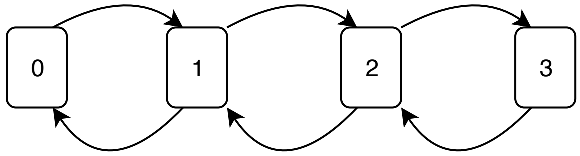
4. Stochastic Processes (cont'd)

4.4. Classification of States (cont'd)

4.4.2 Periodicity

Example:

$$P = \begin{pmatrix} & 1 & & \\ \frac{1}{2} & & \frac{1}{2} & \\ & \frac{1}{2} & & \frac{1}{2} \\ & & 1 & \end{pmatrix}$$



Note that if we starts from 0, we can only get back to 0 in $2, 4, 6, \cdots$, i.t., even number of steps $P_{00}^{(2n+1)} = 0 \forall n$

Definition 4.4.2 : Period

The ***period*** of state i is defined as

$$d_i = \underbrace{gcd}_{\substack{\text{greates} \\ \text{common divisor}}} (\{n : \underbrace{P_{ii}^{(n)} > 0}_{\substack{i \text{ can go back} \\ \text{to } i \text{ in } n \text{ steps}}}\})$$

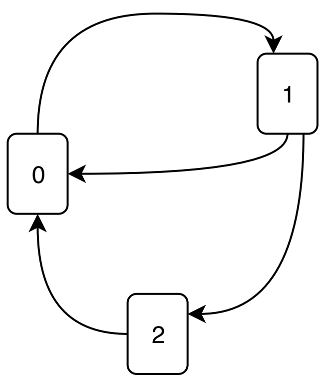
In this example above, $d_0 = gcd(\{\text{even numbers}\}) = 2$

If $d_i = 1$, state i is called ***"aperiodic"***

If $\nexists n > 0$ such that $P_{ii}^{(n)} > 0$, then $d_i = \infty$

Remark 4.4.2

Note that $P_{ii} > 0 \Rightarrow d_i = 1$. The converse is **not true**.



$$P_{00}^{(2)} > 0, P_{00}^{(3)} > 0 \Rightarrow d_0 = 1 \text{ but } P_{00} = 0$$

In general, $d_i = d \not\Rightarrow P_{ii}^{(d)} > 0$

4.4.3 Equivalent classes and irreducibility

Definition 4.4.3.1 : Assessible

Let $\{X_n\}_n = 0, 1, \cdots$ be a DTMC with state space S. State j is said to be $\underline{\text{assessible}}$ from state i, denoted by $i \rightarrow j$, if $P_{ij}^{(n)} > 0$ for some $n \geq 0$.

Intuitively, i can go to state j in finite steps.

Definition 4.4.3.2 : Communicate

If $i \rightarrow j$ and $j \rightarrow i$, we say i and j **communicate**, denoted by $i \leftrightarrow j$.

Fact 4.4.3.1

"Communication" is an equivalence relation.

1. $i \leftrightarrow j$ then $P_{ii}^{(0)} = 1 {=} P(X_0 = i | X_0 = i)$ (Identity)
2. $i \leftrightarrow j$ then $j \leftrightarrow i$ (symmetry)
3. $i \leftrightarrow j, j \leftrightarrow k$, then $i \leftrightarrow k$ (transitivity)

Definition 4.4.3.3 : Class

As a result, we can use "$\leftrightarrow$" to divide the state space into different ***classes***, each containing only the states which communicate with each other.

$$\begin{cases} S = \bigcup_k C_k & (\{C_n\} \text{ is a partition of } S) \\ C_k \bigcap C'_k = \emptyset, k \not= k' \end{cases}$$

- For state i and j in the same class C_k, $i \leftrightarrow j$.
- For i j in different classes, $i \not\leftrightarrow j$ ($i \not\rightarrow j$ or $j \not\rightarrow i$)

Definition 4.4.3.4 : Irreducible

A MC is called **irreducible**, if it has only one class. In other words, $i \leftrightarrow j$ for any $i, j \in S$

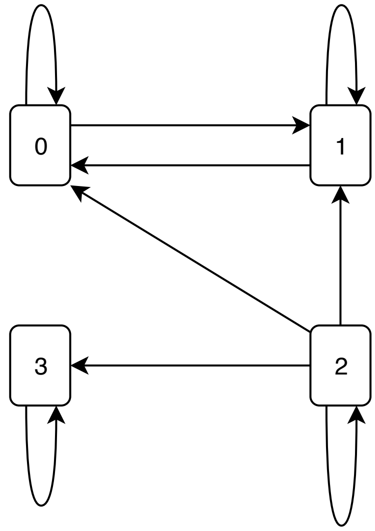
-Q: How to find equivalent classes?

-A: "Draw a graph and find the loops"

Example 4.4.3.1 : Find the classes

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \\ & & & 1 \end{pmatrix}$$

Draw an arrow from i to j if $P_{ij} > 0$

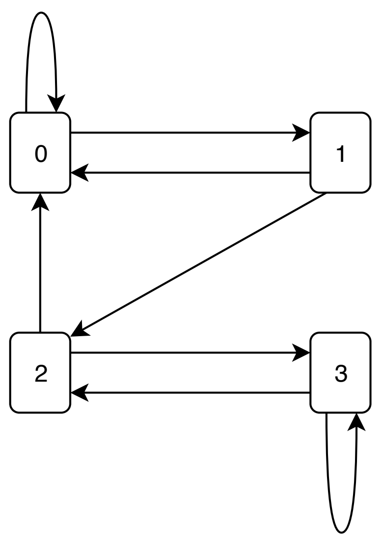


- $P_{01} > 0, P10 > 0 \Rightarrow 0 \leftrightarrow 1$
- State 2 does not communicate with any other state, since $P_{i2} = 0, i \neq 2$
- State 3 does not communicate with any other state, since $P_{i3} = 0, i \neq 3$

$\Rightarrow$ 3 classes: $\{0, 1\}, \{2\}, \{3\}$

Example 4.4.3.2 : Find the classes

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & & \frac{1}{2} & \\ \frac{1}{2} & & & \frac{1}{2} \\ & & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



- $P_{01}, P_{12}, P_{20} > 0 \Rightarrow 0, 1, 2$ are in the same class
- $P_{23}, P_{32} > 0 \Rightarrow 2, 3$ are in the same class
- Transitivity $\Rightarrow$ 0,1,2,3 are all in the same class.

$\Rightarrow$ This MC is irreducible

Fact 4.4.3.2

Preposition Transience/Recurrence are class properties. That is, if $i \leftrightarrow j$, then j is transient/recurrent if and only if i is transient/recurrent

Proof:

Suppose i is recurrent, then $\sum_{k=1}^{\infty} P_{ii}^{(k)} = \infty$

Since $i \leftrightarrow j$, $\exists m, n$ such that $P_{ij}^{(m)} > 0, P_{ij}^{(n)} > 0$

Note that

$$\underbrace{P_{jj}^{(m+n+k)}}_{P(X_{m+n+k}=j|X_0=j)} \geq \underbrace{P_{ji}^{(n)} P_{ii}^{(k)} P_{ij}^{(m)}}_{P(X_{m+n+k}=j, X_{n+k}=i, X_n=i|X_0=j)} \Rightarrow \sum_{l=1}^{\infty} P_{jj}^{(l)} \geq \sum_{l=m+n+1}^{\infty} P_{jj}^{(l)}$$

$$= \sum_{k=1}^{\infty} P_{jj}^{(m+n+k)}$$

$$\geq \sum_{k=1}^{\infty} P_{ji}^{(n)} P_{ii}^{(k)} P_{ji}^{(m)}$$

$$= \underbrace{P_{jj}^{(n)}}_{0} \underbrace{P_{ij}^{(m)}}_{0} \underbrace{\sum_{k=1}^{\infty} P_{ii}^{(k)}}_{\infty} = \infty$$

Thus, j is recurrent. Symmetrically, j is recurrent $\Rightarrow$ i is recurrent

Thus,

- i recurrent $\Leftrightarrow$ j recurrent
- i transient $\Leftrightarrow$ j transient

For irreducible MC, since recurrence and transience are class properties, we also say the Markov Chain is recurrent/transient