2 Random variables and distributions (cont'd)

2.2 Discrete random variables and distributions(cont'd)

3.Geometric distribution

$$p(k) = P(X = k) = (1-p)^{k-1}p$$

Memoryless property:

$$p(X > n + m | X > m) = P(X > n)a$$

Proof:

$$\begin{aligned}
P(X > k) &= \sum_{j=k+1}^{\infty} P(X = j) \\
&= \sum_{j=k+1}^{\infty} (i-p)^{j-1}p \\
&= (1-p)^k p \cdot \frac{1}{1-(1-p)} \\
&= (1-p)^k \\
P(X > n+m | x > m) &= \frac{P(X > n+m), X > m}{P(X > m)} \\
&= \frac{P(X > n+m)}{P(X > m)} = \frac{1-p)^{n+m}}{(1-p)^m} = (1-p)^n = P(X > n)
\end{aligned}$$

Intuition: The failures in the past have no influence on how long we still need to wait to get the first success in the future

4. Poisson distribution

$$p(k) = P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, \ldots, \lambda > 0$$

Other discrete distributions:

- negative binomial
- discrete uniform

2.3 Continuous random variables and distributions

Definition: A random variable X is called **continuous** if there exists a non-negative function f, such that for any interval $[a, b]$, (a, b) or $[a, b)$:

$$P(X \in [a,b]) = \int_a^b f(x)dx$$

The function f is called the *probability density function(pdf)* of X

Remark: probability density function(pdf) is not probability. $P(X = x) = 0$ if X is continuous. The probability density function f only gives probability when it is integrated.

If X is continuous, then we can get cdf by:

$$F(a) = P(X \in (-\infty, a]) = \int_\infty^a f(x)dx$$

hence, $F(x)$ is continuous, and differentiable "almost everywhere".

We can take $f(x) = F'(x)$ when the derivative exists, and $f(x) =$arbitrary number otherwise often to choose a value to make f have some continuity.

Property of pdf:

1. $f(x) \leq 0$, $x \in R$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. For $A \subseteq R, P(X \in A) = \int_A f(x)dx$

Example of continuous distribution

Exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x \leq 0 \end{cases}$$
$$X \sim Exp(x)$$

Other continuous distributions:

- Normal distribution
- Uniform distribution

Exercises:

1. Find the cdf of $X \sim Exp(x)$
2. Show that the exponential distribution has the memoryless property:

$$P(X > t + s | x > t) = P(X > s)$$

2.4 Joint distribution of r.v's

Let X and Y be two r.v's. defined on the same probability space (Ω, ξ, P)

For each $\omega \in \Omega$, we have at the same time $X(\omega)$ and $Y(\omega)$. Then we can talk about the joint behavior of X and Y

Two joint distribution of r.v's is characterized by joint cdf, joint pmf(discrete case) or joint pdf(continuous case).

- Joint cdf:
  - $F(x, y) = P(X < x, Y < y)$
- Joint pmf:
  - $f(x, y) = P(X = x, Y = y)$
- joint pdf $f(x, y)$ such that for $a < b, c < d$
  - $P(X, Y) \in (a, b] \times (c, d] = P(X \in (a, b], Y \in (c, d]) = \int_a^b \int_c^d f(x, y) dy dx$
  - Equivalently:
    1. $F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s, t) dt ds$
       $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$
    2. $P((X, Y) \in A) = \int \int_A f(x, y) dx dy$ for $A \subseteq R^2$

Definition: Two r.v's X and Y are called independent, if for all sets $A, B \subseteq R$,

$$P(X < A, Y < B) = P(X \in A) P(Y \in B)$$

($\{X \in A\}$ and $\{Y \in B\}$ are independent events)

Theorem: Two r.v's X and Y are

1. independent, if and only if
2. $F(x, y) = F_x(x) F_y(y); x, y \in R$; where F_x: cdf of x; F_y: cdf of y
3. $f(x, y) = f_x(x) f_y(y); x, y \in R$; where f is the joint pmf/pdf of X and Y; f_x, f_y are marginal pmf/pdf of X and Y, respectively

Proof:

1.$\Rightarrow$ 2.

If $X \perp Y$, then by definition,

$F(x, y) = P(X \in (-\infty, x], Y \in (-\infty, y])) = P(X \in (-\infty, x])) \cdot P(Y \in (-\infty, y])) = F_x(x) F_y(y)$

2.$\Rightarrow$ 3.

Assume $F(x, y) = F_x(x) \cdot F_y(y)$,

$$\begin{aligned} f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) &= \frac{\partial^2}{\partial x \partial y} F_x(x) F_y(y) \\ &= (\frac{\partial}{\partial x} F_x(x))(\frac{\partial}{\partial y} F_y(y)) \\ &= f_x(x) f_y(y) \end{aligned}$$

3.⇒ 1.

Assume $f(x,y) = f_x(x)f_y(y)$; For $A, B \subseteq R$,

$$\begin{aligned} P(X \in A, Y \in B) &= \int_{y\in B}\int_{x\in A} f(x,y)dxdy \\ &= \int_{y\in B}\int_{x\in A} f_x(x)f_y(y)dxdy \\ &= (\int_{x\in A} f_x(x)dx)(\int_{y\in B} f_y(y)dy) \\ &= P(X\in A)P(Y\in B) \end{aligned}$$

2.5 Expectation

Definition: For a r.v X, the expectation of $g(x)$ is defined as

$$\exists(g(x)) = \begin{cases} \sum_{i=1}^{\infty} g(x_i)P(X = x_i) & \text{for discrete } X \\ \int_{-\infty}^{\infty} g(x)f(x)dx & \text{for continuous} X \end{cases}$$

Let X,Y be two r.v's; then the expectation of $g(X, Y)$ is defined in a similar way.

$$\exists(g(x,y)) = \begin{cases} \sum\sum g(x_i, y_j)P(X = x_i, Y = y_j) \\ \int\int g(x_i, y_j)f(x,y)dxdy \end{cases}$$