Note 15 - Mar 07

Review

Positive recurrence is related to the existence of the stationary distribution.

generating function:

$$\begin{aligned}\psi(s) &= \mathbb{E}(s^{\xi}) \\ &= \sum_{k=0}^{\infty} \underbrace{P_k}_{\mathbb{P}(\xi=k),k=0,1,\dots} S^k \quad for 0 \le s \le 1\end{aligned}$$

Properties

1. $\psi(0) = p_0, \quad \psi(1) = \sum_{k=0}^{\infty} p_k = 1$
2. Generating function determines the distribution

$$p_k = \frac{1}{k!}\frac{d^k\psi(s)}{ds^k}|_{s=0}$$

generating function determines the ditribution.

$$\frac{d^k\psi(s)}{ds^k} = k!P_k + (\dots)s + (\dots)S^2 + \dots$$

Since $P_k \ge 0$ for all $k = 0, 1, \dots$, $\frac{d^k\psi(s)}{ds^k} \ge 0$ for all $k = 1, 2, \dots, s \in [0, 1]$.

In particular, $\psi(s)$ is increasing and convex between 0 and 1

4. Stochastic Processes (cont'd)

4.6 Generating function and branching processes

Properties of generating function

1. $\psi(0) = p_0, \quad \psi(1) = \sum_{k=0}^{\infty} p_k = 1$
2. Generating function determines the distribution

$$p_k = \frac{1}{k!}\frac{d^k\psi(s)}{ds^k}|_{s=0}$$

   Reason:

$$\psi(s) = p_0 + p_1 s^1 + \cdots + p_{k-1}s^{k-1} + p_k s^k + p_{k+1}s^{k+1} + \cdots$$

$$\frac{d^k\psi(s)}{ds^k} = k!p_k + (\cdots)s + (\cdots)s^2 + \cdots$$

$$\frac{d^k\psi(s)}{ds^k}|_{s=0} = k!p_k$$

   In particular, $p_1 \ge 0 \Rightarrow \psi(s)$ is increasing. $p_2 \ge 0 \Rightarrow \psi(s)$ is climax

3. Let $\xi_1, \dots, \xi_n$ be independent r.b. with generating function $\psi_1, \dots, \psi_n$,

$$X = \xi_1 + \dots + \xi_n \Rightarrow \psi_X(s) = \psi_1(s)\psi_2(s)\dots\psi_n(s)$$

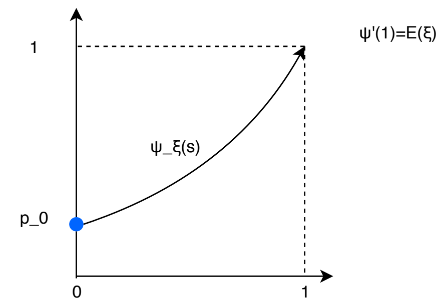
   Proof:

$$\begin{aligned}\psi_X(s) &= s^{\mathbb{X}} \\ (independent) &= \mathbb{E}(s^{\xi_1} s^{\xi_2} ... s^{\xi_n}) \\ &= \mathbb{E}(s^{\xi_1}) ... \mathbb{E}(s^{\xi_n}) \\ &= \psi_1(s) ... \psi_n(s)\end{aligned}$$

4. $$\left.\frac{d^\psi(s)}{ds^k}\right|_{s=1} = \left.\frac{d^k \mathbb{E}(s^\xi)}{ds^k}\right|_{s=1} = \mathbb{E}(\left.\frac{d^k s^\xi}{ds^k}\right|_{s=1} = \mathbb{E}(\xi(\xi-1)(\xi-2)...(\xi-k+1)s^{\xi-k})\Big|_{s=1} = \mathbb{E}(\xi(\xi-1)...(\xi-k+1))$$

In particular, $\mathbb{E}(\xi) = \psi'(1)$ and $Var(\xi) = \mathbb{E}(\xi^2) - (\mathbb{E}(\xi))^2 = \mathbb{E}(\xi^2 - \xi) + \mathbb{E}(\xi) - (\mathbb{E}(\xi))^2 = \psi''(1) + \psi(1) - (\psi'(1))^2$

Graph of a g.f.:



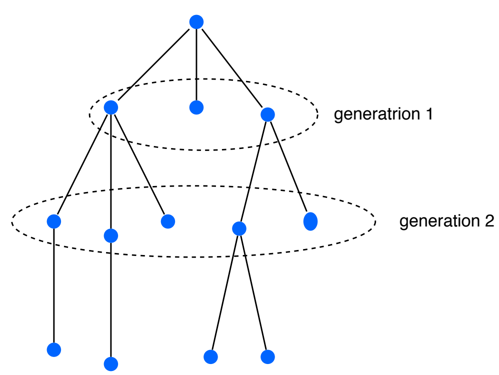
4.6.1 Branching Process

Each organism, at the end of its life, produces a random number Y of offsprings.

$$\mathbb{P}(Y=k) = P_k, \quad k = 0, 1, 2, ..., \quad P_k \geq 0, \quad \sum_{k=0}^{\infty} P_k = 1$$

The number of offsprings of different individuals are independent.

Start from one ancestor $X_0 = 1$, X_n : # of individuals(population in n-th generation)



Then $X_n + 1 = Y_1^{(n)} + Y_2^{(n)} + ... + Y_{X_n}^{(n)}$, where $Y_1^{(n)}, ..., Y_{X_n}^{(n)}$ are independent copies of Y, $Y_i^{(n)}$ is the number of offsprings of the i-th individual in the n-th generation

4.6.1.2 Mean and Variance

Mean: $\mathbb{E}(X_n)$ and Variance: $Var(X_n)$

Assume, $\mathbb{E}(Y) = \mu, Var(Y) = \sigma^2$.

$$\begin{aligned}\mathbb{E}(X_{n+1}) &= \mathbb{E}(Y_1^{(n)} + ... + Y_{X_n}^{(n)}) \\ &= \mathbb{E}(\mathbb{E}(Y_1^{(n)} + ... + Y_{X_n}^{(n)} | X_n)) \\ &= \mathbb{E}(X_n \mu) \\ \text{Wald's identity(tutorial 3)} \quad &= \mu \mathbb{E}(X_n)\end{aligned}$$

$$\begin{aligned}\Rightarrow \mathbb{X}_n &= \mu\mathbb{E}(X_{n-1})\\ &= \mu^2\mathbb{X}_{n-2}\\ &\vdots\\ &= \mu^n\mathbb{E}(X_0) = \mu^n, \quad n = 0, 1, ...\end{aligned}$$

$$Var(X_{n+1}) = \mathbb{E}(Var(X_{n+1}|X_n) + Var(\mathbb{E})X_{n+1}|X_n)$$

$$\mathbb{E}(Var(X_{n+1}|X_n)) = \mathbb{E}(Var(Y_1^{(n)+...+Y_{X_N}^{(N)}}|X_N)) \\ = \mathbb{E}(X_n \cdot \sigma^2) \\ = \sigma^2\mu^n$$

$$\begin{aligned}Var(\mathbb{E}(X_{n+1}|X_n)) &= Var(\mu X_n)\\ &= \mu^2 Var(X_u)\\ &\Rightarrow Var(X_{n+1}) = \sigma^2\mu^n + \mu^2 Var(X_n))\\ &\quad Var(X_1) = \sigma^2\\ &\quad Var(X_2) = \sigma^2\mu + \mu^2\sigma^2 = \sigma^2(\mu^1 + \mu^2)\\ &\quad Var(X_3) = \sigma^2\mu^2 + \mu^2(\sigma^2(\mu^1 + \mu^2)) = \sigma^2(\mu^2 + \mu^3 + \mu^4)\\ &\quad\quad \vdots\\ &\quad \text{In general, (can be proved by induction)}\\ &\quad Var(X_n) = \sigma(\mu^{n-1} + ... + \mu^{2n-2})\\ &\quad\quad = \begin{cases}\sigma^2\mu^{n-1}\dfrac{1-\mu^n}{1-\mu} & \cancel{\mu} = 1\\ \sigma^2 n & \mu = 1\end{cases}\end{aligned}$$

4.6.1.2 Extinction Probability

Q: What is the probability that the population size is eventually reduced to 0

Note that for a branching process, $X_n = 0 \Rightarrow X_k = 0$ for all $k \geq n$. Thus, state 0 is absorbing. $(P_{00} = 1)$. Let N be the time that extinction happens.

$$N = min\{n : X_n = 0\}$$

Define

$$U_n = \mathbb{P}(\underbrace{N \leq n}_{\substack{\text{extinction happens}\\ \text{before or at n}}}) = \mathbb{P}(X_n = 0)$$

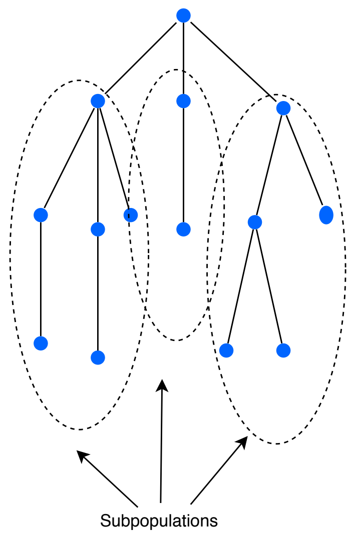
Then U_n is increasing in n, and

$$\begin{aligned}u = \lim_{n\to\infty} U_n &= \mathbb{P}(N < \infty)\\ &= P(\text{the extinction eventually happens})\\ &= \text{extinction probability}\end{aligned}$$

Out goal : find u

We have the following relation between U_n and U_{n-1}:

$$U_n = \sum_{k=0}^{\infty} P_k(U_{n-1})^k = \underbrace{\psi}_{\text{gf of Y}}(U_{n-1})$$



Each subpopulation has the same distribution as the whole population.

Total population dies out in n steps if and only if each subpopulation dies out int $n-1$ steps

$$
\begin{aligned}
U_n &= \mathbb{P}(N \leq n) \\
&= \sum_k \mathbb{P}(N \leq n | X_1 = k) \underbrace{\mathbb{P}(X_1 = k)}_{=P_k} \\
&= \sum_k \mathbb{P}(\underbrace{N_1 \leq n-1}_{\#\text{ of steps for subpopulation 1 to die out}}, \cdots, N_k \leq n-1 | X_1 = k) \cdot P_k \\
&= \sum_k P_k \cdot U_{n-1}^k \\
&= \mathbb{E}(U_{n-1}^Y) \\
&= \psi(U_{n-1})
\end{aligned}
$$

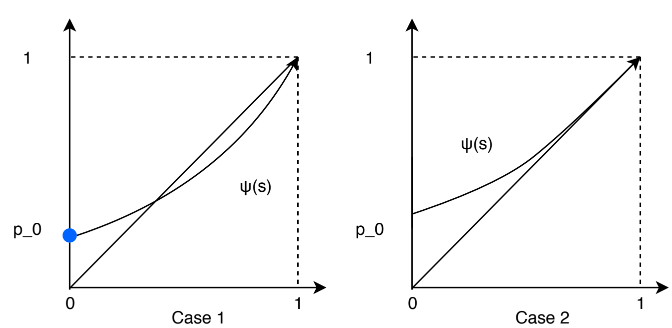
Thus, the question is :

> With initial value $U_0 = 0$ (since $X_0 = 1$), relation $U_n = \psi(U_{n-1})$. What is $\lim_{n\to\infty} U_N = u$?

Recall that we have

1. $\psi(0) = P_0 \geq 0$
2. $\psi(1) = 1$
3. $\psi(s)$ is increasing
4. $\psi(s)$ is convex

Draw $\psi(s)$ and function $f(s) = s$ between 0 and 1, we have two cases:



The extinction probability u will be the smallest intersection of $\psi(s)$ and $f(s)$. Equivalently, it is the smallest solution of the equation $\psi(s) = s$ between 0 and 1.