STAT 333 Course Note

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1. Fundamental of Probability

1.1 What's Probability

1.1.1 Examples

1. Coin toss
    - "H" - head
    - "T" - tail
2. Roll a dice
    - every number in the set: $\{1, 2, 3, 4, 5, 6\}$
3. Tomorrow weather
    - {sunny, rainy, cloudy,...}
4. Randomly pick a number in $[0, 1]$

Although things are random, they are not haphazard/arbitrary. There are "patterns"

Example 1

If we repeat tossing a coin, then the fraction of times that we get a "H" goes to $\frac{1}{2}$ as the number of toss goes to infinity.

$$\frac{\# \ of \ "H"}{total \ \# \ of \ toss} = \frac{1}{2}$$

This number $1/2$ reflects how "likely" a "H" will appear in one toss (if the experiment is not repeated)

1.2 Probability Models

The *Sample space* Ω is the set consisting of al the possible outcomes of a random experiment.

1.2.1 Examples

1. $\{H, T\}$
2. $\{1, 2, 3, 4, 5, 6\}$
3. $\{sunny, rainy, cloudy, ...\}$
4. $[0, 1]$

An *event* $E \in \Omega$ is a subset of Ω

for which we can talk about "likelihood of happening"; for example

- in **2**:
    - {getting an even number} = {2, 4, 6}
- in **4**:
    - $\{the \ point \ is \ between \ 0 \ and \ 1/3\} = [0, \frac{1}{3}]$ is an event
    - $\{the \ point \ is \ rational\} = Q \cap [0, 1]$

We say an event E "happens", if the result of the experiment turns out to belong to E (a subset of Ω)

A probability P is a set function (a mapping from events to real numbers)

$$
\begin{aligned}
P :& \xi \to R \\
& E \to P(E)
\end{aligned}
$$

which satisfies the following 3 properties:

1. $\forall E \in \xi, 0 \leq P(E) \leq 1$
2. $P(\Omega) = 1$
3. For
   - countably many disjoint events $E_1, E_2, ...$, we have $P(U_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$
   - countable: $\exists$ 1-1 mapping to natural numbers $1, 2, 3, ...$

Intuitively, one can think the probability of an event as the "likelihood/chance" for the event happens. If we repeat the experiment for a large number of events, the probability is the fraction of time that the event happens

$$
P(E) = \lim_{n \to \infty} \frac{\text{\# of times the E happens in n trials}}{n}
$$

1.2.1.1 Example 2

$$
\begin{aligned}
& P(\{1\}) = P(\{2\}) = \ldots = P(\{6\}) = \frac{1}{6} \\
& E = \{\text{even number}\} = \{2, 4, 6\} \\
\Rightarrow \ & P(E) = P(\{2\} \cup P(\{4\})) \cup P(\{6\}) = \frac{1}{2}
\end{aligned}
$$

Properties of probability:

1. $P(E) + P(E^c) = 1$
2. $P(\emptyset) = 0$
3. $E_1 \subseteq E_2 \Rightarrow P(E_1) \leq P(E_2)$
4. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ -$P(E_1 \cap E_2)$: E_1 and E_2 happen

1.2.2 Remark: why do we need the notion of event?

If the sample space Ω is **discrete**, then everything can has at most countable elements be built from the "atoms"

$$
\begin{aligned}
\Omega = \{w_1, w_2, \ldots\} \\
P(w_1) = P_i \\
P_i \in [0, 1], \sum_{i=1}^{\infty} P_i = 1
\end{aligned}
$$

Then for any event $E = \{w_1, i \in I\}$, $P(E) = \sum_{i \in I} P_i$

However, if the sample space Ω is continuous; e.g, $[0, 1]$ in Example 4, then such a construction can not be done for any $x \in [0, 1]$ we get $P(\{x\} = 0$ (x: the point is exactly x)

We can not get $P([0, \frac{1}{3}])$ by adding $P(\{x\})$ for $x \leq \frac{1}{3}$.

This is why we need the notion of event; and we define P as a set function from ξ to R rather than a function from Ω to R

To summarize: A **Probability Space** consists of a triplet (Ω, ξ, P):

- Ω: sample space,
- ξ: collection of events
- P: probability

1.3 Conditional Probability

If we know some information, the probability of an event can be updated

Let E, F be two events $P(F) > 0$

The conditional probability of E, given F is

$$
P(E \mid F) = \frac{P(E \cap F)}{P(F)}
$$

Again, think probability as the long-run frequency:

$$\begin{aligned} P(E\cap F) &= \lim_{n\to\infty}\frac{\#of\ times\ E\ and\ F\ happen\ in\ n\ trails}{n} \\ P(F) &= \lim_{n\to\infty}\frac{\#of\ times\ F\ happen\ in\ n\ trails}{n} \\ \Rightarrow \frac{P(E\cap F)}{P(F)} &= \lim_{n\to\infty}\frac{\#of\ times\ E\ and\ F\ happen}{\#of\ times\ F\ happens} \end{aligned}$$

By definition

$$P(E\cap F) = P(E\mid F)\cdot P(F)$$

1.4 Independence

Def: Two events E and F are said to be independent, if $P(E\cap F) = P(E)\cdot P(F)$; denoted as $E \perp\!\!\!\perp F$. **This is different rom disjoint.**

Assume $P(F) > 0$, then $E \perp\!\!\!\perp F \Leftrightarrow P(E|F) = P(E)$; intuitively, knowing F does not change the probability of E.

Proof:

$$\begin{aligned} E \perp\!\!\!\perp F &\Leftrightarrow P(E\cap F) = P(E)\cdot P(F) \\ &\Leftrightarrow \frac{P(E\cap F)}{P(F)} = P(E) \\ &\Leftrightarrow P(E|F) = P(E) \end{aligned}$$

More generally, a sequence of events $E_1, E_2, \ldots$ are called independent if for **any** finite index set I,

$$P(\bigcap_{i\in I} E_i) = \prod_{i\in I} P(E_i)$$

1.5 Bayes' rule and law of total probability

Theorem: Let $F_1, F_2, \ldots$ be disjoint events, and $\bigcap_{i=1}^{\infty} F_i = \Omega$, we say $\{F_i\}_{i=1}^{\infty}$ forms a "partition" of the sample space Ω

Then $P(E) = \sum_{i=1}^{\infty} P(E|F_i)\cdot P(F_i)$

Proof: Exercise

Intuition: Decompose the total probability into different cases.

$$P(E\cap F_2) = P(E|F_2)\cdot P(F_2)$$

1.5.1 Bayes' rule

$$P(F_i|E) = \frac{P(E|F_i)\cdot P(F_i)}{\sum_{j=1}^{\infty} P(E|F_j)\cdot P(F_j)}$$

Bayes' rule tells us how to find conditional probability by switching the role of the event and the condition.

Proof:

$$\begin{aligned} P(F_i|E) &= \frac{P(F_i\cap E)}{P(E)} && \text{definition of condition probability} \\ &= \frac{P(E|F_i)P(F_i)}{P(E)} \\ &= \frac{P(E|F_i)P(F_i)}{\sum_{j=1}^{\infty} P(E|F_j)P(F_j)} && \text{law of total probability} \end{aligned}$$

2 Random variables and distributions

2.1 Random variables

(Ω, ξ, P): Probability space.

Definition: A random variable X (or r.v.) is a mapping from Ω to $\mathbb{R}$

$$\begin{aligned} X : &\Omega \to \mathbb{R} \\ &\omega \to X(\omega) \end{aligned}$$

A random variable transforms arbitrary "outcomes" into numbers.

X introduces a probability on R. For $A \subseteq R$, define

$$\begin{aligned} P(X \in A) :&= P(\{X(\omega) \in A\}) \\ &= P(\{\omega : X(\omega) \in A\}) \\ &= P(X^{-1}(A)) \end{aligned}$$

From now on, we can often "forget" te original probability space and focus on the random variables and their distributions.

Definition: let X be a random variable. The **CDF**(cumulative distribution function) F of X is defined by

$$F(x) = P(X \leq x) = P(X \in (-\infty, x]))$$
$$X : \text{random variable}, x : \text{number}$$

Properties of cdf:

1. F is non-decreasing. $F(x_1) \leq F(x_2), x_1 < x_2$
2. limits
   - $\lim_{x \to -\infty} F(x) = 0$
   - $\lim_{x \to \infty} F(x) = 1$
3. $F(x)$ is right continuous
   - $lim_{x \downarrow a} F(x) = F(a)$: x decreases to a(approaching from the right)
   - Hint: $\{x \leq a\} = \bigcap_{i=1}^{\infty} \{X \leq a_i\}$ for $a_i \downarrow a$

2.2 Discrete random variables and distributions

A random variable X is called **discrete** if it only takes values in an **at most countable** set $\{x_1, x_2, \ldots\}$ (finite or countable).

The distribution of a discrete random variable is fully characterized by its **probability mass function**(p.m.f)

$$p(x) := P(X = x); x = x_1, x_2, \ldots$$

Properties of pmf:

1. $p(x) \geq 0, \quad \forall x$
2. $\sum_i p(x_i) = 1$

Q: what does the cdf of a discrete random variable look like?

2.2.1 Examples of discrete distributions

1. Bemoulli distribution

$$\begin{aligned} &p(1) = P(X = 1) = p \\ &p(c) = P(X = c) = 1 - p \\ &p(x) = 0 \quad otherwise \end{aligned}$$

Denote $X \sim Ber(p)$

2.Binomial distribution

$$p(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

- $X \sim Bin(n, p)$ to choose k successes.
- Binomial distribution is the distribution of number of successes in n independent trials; each having probability p of success.

3.Geometric distribution

$$p(k) = P(X = k) = (1-p)^{k-1} p$$

$$(1-p)^{k-1} : \text{the first k-1 trials are all failures}, \quad p : \text{success in } k^{th} \text{ trial}$$

- $X \sim Geo(p)$
- X is the number of trials needed to get the first success in n independent trials with probability p of success each
- X has the memoryless property $P(X > n+m|X > m) = P(x > n) \qquad n, m = 0, 1, \ldots$

Memoryless property:

$$p(X > n+m|X > m) = P(X > n)$$

Proof:

$$\begin{aligned} P(X > k) &= \sum_{j=k+1}^{\infty} P(X = j) \\ &= \sum_{j=k+1}^{\infty} (1-p)^{j-1}p \\ &= (1-p)^k p \cdot \frac{1}{1-(1-p)} \\ &= (1-p)^k \end{aligned}$$

$$\begin{aligned} P(X > n+m|x > m) &= \frac{P(X > n+m \bigcap X > m)}{P(X > m)} \\ &= \frac{P(X > n+m)}{P(X > m)} = \frac{(1-p)^{n+m}}{(1-p)^m} = (1-p)^n = P(X > n) \end{aligned}$$

Intuition: The failures in the past have no influence on how long we still need to wait to get the first success in the future

4. Poisson distribution

$$p(k) = P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, \ldots, \lambda > 0$$

Other discrete distributions:

- negative binomial
- discrete uniform

2.3 Continuous random variables and distributions

Definition: A random variable X is called **continuous** if there exists a non-negative function f, such that for any interval $[a, b]$, (a, b) or $[a, b)$:

$$P(X \in [a, b]) = \int_a^b f(x)dx$$

The function f is called the *probability density function(pdf)* of X

Remark: probability density function(pdf) is not probability. $P(X = x) = 0$ if X is continuous. The probability density function f only gives probability when it is integrated.

If X is continuous, then we can get cdf by:

$$F(a) = P(X \in (-\infty, a]) = \int_{-\infty}^a f(x)dx$$

hence, $F(x)$ is continuous, and differentiable "almost everywhere".

We can take $f(x) = F'(x)$ when the derivative exists, and $f(x) =$arbitrary number otherwise often to choose a value to make f have some continuity.

Property of pdf:

1. $f(x) \leq 0, x \in R$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. For $A \subseteq R, P(X \in A) = \int_A f(x)dx$

2.3.1 Example of continuous distribution

Exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ,x \geq 0 \\ 0 & ,x \leq 0 \end{cases}$$
$$X \sim Exp(x)$$

Other continuous distributions:

- Normal distribution
- Uniform distribution

Exercises:

1. Find the cdf of $X \sim Exp(x)$

$$\begin{aligned} F(k) = P(X \leq k) &= \int_{-\infty}^{k} f(x)dx \\ &= \int_{0}^{k} \lambda e^{-\lambda x} dx \\ &= -e^{-\lambda x}\Big|_0^k \\ &= -e^{-\lambda k} - (-e^0) \\ &= 1 - e^{-\lambda k} \end{aligned}$$

2. Show that the exponential distribution has the memoryless property:

$$P(X > t+s|X > t) = P(X > s)$$

2.4 Joint distribution of r.v's

Let X and Y be two r.v's. defined on the same probability space (Ω, ξ, P)

For each $\omega \in \Omega$, we have at the same time $X(\omega)$ and $Y(\omega)$. Then we can talk about the joint behavior of X and Y

Two joint distribution of r.v's is characterized by joint cdf, joint pmf(discrete case) or joint pdf(continuous case).

- Joint cdf:
  - $F(x,y) = P(X < x, Y < y)$
- Joint pmf:
  - $f(x,y) = P(X = x, Y = y)$
- joint pdf $f(x,y)$ such that for $a < b, c < d$
  - $P(X,Y) \in (a,b] \times (c,d] = P(X \in (a,b], Y \in (c,d]) = \int_a^b \int_c^d f(x,y)dydx$
  - Equivalently:
    1. $F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)dtds$
       and
       $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$
    2. $P((X,Y) \in A) = \int\int_A f(x,y)dxdy$ for $A \subseteq R^2$

Definition: Two r.v's X and Y are called independent, if for all sets $A, B \subseteq R$,

$$P(X < A, Y < B) = P(X \in A) \cdot P(Y \in B)$$

($\{X \in A\}$ and $\{Y \in B\}$ are independent events)

Theorem: Two r.v's X and Y are

1. independent, if and only if
2. $F(x,y) = F_x(x)F_y(y); x, y \in R$; where F_x: cdf of x; F_y: cdf of y
3. $f(x,y) = f_x(x)f_y(y); x, y \in R$; where f is the joint pmf/pdf of X and Y; f_x, f_y are marginal pmf/pdf of X and Y, respectively

Proof:

1.$\Rightarrow$ 2.

If $X \perp\!\!\!\perp Y$, then by definition,

$$F(x,y)=P(X\in(-\infty,x],Y\in(-\infty,y]))=P(X\in(-\infty,x])\cdot P(Y\in(-\infty,y]))=F_x(x)F_y(y)$$

2.$\Rightarrow$ 3.

Assume $F(x,y)=F_x(x)\cdot F_y(y)$,

$$\begin{aligned}f(x,y)=\frac{\partial^2}{\partial x\partial y}F(x,y)=\frac{\partial^2}{\partial x\partial y}F_x(x)F_y(y)\\ =(\frac{\partial}{\partial x}F_x(x))(\frac{\partial}{\partial y}F_y(y))\\ =f_x(x)f_y(y)\end{aligned}$$

3.$\Rightarrow$ 1.

Assume $f(x,y)=f_x(x)f_y(y)$; For $A,B\subseteq R$,

$$\begin{aligned}P(X\in A,Y\in B)&=\int_{y\in B}\int_{x\in A}f(x,y)dxdy\\ &=\int_{y\in B}\int_{x\in A}f_x(x)f_y(y)dxdy\\ &=(\int_{x\in A}f_x(x)dx)(\int_{y\in B}f_y(y)dy)\\ &=P(X\in A)P(Y\in B)\end{aligned}$$

2.5 Expectation

Definition: For a r.v X, the expectation of $g(x)$ is defined as

$$\mathbb{E}(g(X))=\begin{cases}\sum_{i=1}^{\infty}g(x_i)P(X=x_i) & \text{for discrete } X\\ \int_{-\infty}^{\infty}g(x)f(x)dx & \text{for continuous} X\end{cases}$$

Let X,Y be two r.v's; then the expectation of $g(X,Y)$ is defined in a similar way.

$$\mathbb{E}(g(X,Y))=\begin{cases}\sum_i\sum_j g(x_i,y_j)P(X=x_i,Y=y_j)\\ \int\int g(x_i,y_j)f(x,y)dxdy\end{cases}$$

2.5.1 Properties of expectation

1. Linearity:expectation of X: $\mathbb{E}(X)=\begin{cases}\sum x_iP(X=x_i)\\ \int_{-\infty}^{\infty}xf(x)dx\end{cases}$, $g(X)=x$
    - $\mathbb{E}(ax+b)=a\mathbb{E}(x)+b$
    - $\mathbb{E}(X+Y)=\mathbb{E}(X)+\mathbb{E}(Y)$
2. If $X\perp\!\!\!\perp Y$, then $\mathbb{E}(g(X)h(Y))=\mathbb{E}(g(X))\cdot\mathbb{E}(h(Y))$
    - **proof**: (continuous case)
    - $$\begin{aligned}\mathbb{E}(g(X)h(Y))&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x)h(y)f(x,y)dxdy\\ &=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x)h(y)f_X(x)f_Y(y)dxdy\\ &=\int_{-\infty}^{\infty}g(x)f_X(x)\cdot\int_{-\infty}^{\infty}h(y)f_Y(y)dy\end{aligned}$$
    - In particular, $\mathbb{E}(XY)=\mathbb{E}(X)\mathbb{E}(Y)$ if $X\perp\!\!\!\perp Y$

2.5.2 Definitions

Definition: The expectation $\mathbb{E}(X^n)$ is called the n-th moment of X:

- 1st moment: $\mathbb{E}(X)$

- 2st moment: $\mathbb{E}(X^2)$

Definition: The variance of a r.v X is defined as:

$$Var(x) = \mathbb{E}((X - \mathbb{E}(X))^2) \text{ also denoted as } \sigma^2, \sigma_x^2$$

Definition: the covariance of the r.v's X and Y is defined as:

$$Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]$$

Thus $Var(X) = Cov(X, X)$

Definition: the correlation between X and Y is defined as:

$$Cor(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

Fact: $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

Proof:

$$\begin{aligned} Var(X) &= \mathbb{E}((X - \mathbb{E}(X))^2) \\ &= \mathbb{E}(X^2 - 2X\mathbb{E}(X) + (\mathbb{E}(X))^2) \\ &= \mathbb{E}(X^2) - 2\mathbb{E}(X\mathbb{E}(X)) + (\mathbb{E}(X))^2 \\ &= \mathbb{E}(X^2) - 2(\mathbb{E}(X))^2 + (\mathbb{E}(X))^2 \\ &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \qquad \blacksquare \end{aligned}$$

Fact: $Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

Proof:

$$\begin{aligned} Cov(X, Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\ &= \mathbb{E}[XY - X\mathbb{E}[Y] - Y\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y]] \\ &= \mathbb{E}[XY] - \mathbb{E}[X\mathbb{E}[Y]] - \mathbb{E}[Y\mathbb{E}[X]] + \mathbb{E}[E]X]\mathbb{E}[Y]) \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \qquad \blacksquare \end{aligned}$$

Variance and covariance are **translation invariant**. Variance is quadratic, covariance is bilinear.

$$Var(aX + b) = a^2 \cdot Var(X)$$

$$Cov(aX + b, cY + d) = ac \cdot Cov(X, Y)$$

Proof: $Var(aX + b) = a^2 \cdot Var(X)$

$$\begin{aligned} Var(aX + b) &= \mathbb{E}((aX + b)^2) - (\mathbb{E}(aX + b))^2 \\ &= \mathbb{E}(a^2X^2 + 2abX + b^2) - (a\mathbb{E}(X) + b)^2 \\ &= a^2\mathbb{E}(X^2) + 2ab\mathbb{E}(X) + b^2 - a^2\mathbb{E}^2(X) - ab\mathbb{E}(X) - b^2 \\ &= a^2\mathbb{E}(X^2) - a^2\mathbb{E}^2(X) \\ &= a^2Var(X) \qquad \blacksquare \end{aligned}$$

Proof: $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

$$\begin{aligned} Var(X + Y) &= \mathbb{E}[(X + Y)^2] - E^2[X + Y] \\ &= \mathbb{E}[X^2 + XY + Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 \\ &= \mathbb{E}[X^2] + \mathbb{E}[XY] + \mathbb{E}[Y^2] - E^2[X] - 2\mathbb{E}[X]\mathbb{E}[Y] - E^2[Y] \\ &= (\mathbb{E}[X^2] - E^2[X]) + (\mathbb{E}[Y^2] - E^2[Y]) + (\mathbb{E}[XY] - 2\mathbb{E}[X]\mathbb{E}[Y]) \\ &= Var(X) + Var(Y) + 2Cov(X, Y) \qquad \blacksquare \end{aligned}$$

If $X \perp\!\!\!\perp Y$, then $Cov(X, Y) = 0$ and $Var(X + Y) = Var(X) + Var(Y)$

Proof:

$$
\begin{aligned}
&Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \\
&\text{we know:} \\
&X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \\
&\text{Thus, } Cov(X,Y) = 0 \Rightarrow Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) \\
&\text{So we see independence} \Rightarrow \text{Covariance is 0: "uncorrelated"} \\
&\text{the converse is not true.} \\
&Cov(X,Y) = 0 \not\Rightarrow \text{independence}
\end{aligned}
$$

Remarks

We have $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.

If $X \perp\!\!\!\perp Y$, we also have:

- $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$, and
- $Var(X+Y) = Var(X) + Var(Y)$

It's important to remember that the first result and the other two results are of very different nature. While $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$ is a property of expectation and holds unconditionally;

the other two, $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ and $Var(X+Y) = Var(X) + Var(Y)$, only hold if $X \perp\!\!\!\perp Y$.

It is more appropriate to consider them as **properties of independence** rather than properties of expectation and variance

2.6 Indicator

A random variable I is called an indicator, if

$$
I(w) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}
$$

$$
E(I_A) = P(A)
$$

for some event A

For A given, I is also elevated as I_A

The most important property of indicator is its expectation gives the probability of the event $\mathbb{E}(I_A) = \mathbb{P}(A)$

Proof:

$$
\begin{aligned}
\mathbb{P}(I_A = 1) &= \mathbb{P}(\omega : I_A(\omega = 1)) \\
&= \mathbb{P}(\omega : \omega \in A) \\
&= \mathbb{P}(A)
\end{aligned}
$$

$$
\mathbb{P}(I_A = 0) = 1 - \mathbb{P}(A) \Rightarrow \mathbb{E}(I_A) = 1 \cdot \mathbb{P}(A) + c \cdot (1 - \mathbb{P}(A)) = \mathbb{P}(A)
$$

2.6.1 Example

we see $I_A \sim Ber(\mathbb{P}(A))$

Let $X \sim Bin(n,p)$, X is number of successes in n Bernoulli trials, each with probability p of success

$$
\Rightarrow X = I_1 + \cdots + I_n
$$

where $I_1, \cdots, I_n$ are indicators for independent events. $I_i = 1$ if th i the trial is a success. $I_i = 0$ if the i th trial is a failure.

Hence I_i are **idd**(independent and identically distributed) r.v's

$$
\begin{aligned}
\Rightarrow \mathbb{E}(X) &= \mathbb{E}(I_1 + \cdots + I_N) \\
&= \mathbb{E}(I_1) + \cdots + \mathbb{E}(I_n) \\
&= p + \cdots + p = n \cdot p
\end{aligned}
$$

$$\begin{aligned}Var(X) &= Var(I_1 + \cdots + I_n)\\ &= Var(I_1) + \cdots + Var(I_n)\\ &= n \cdot Var(I_i)\\ &= n \cdot p(1-p)\end{aligned}$$

$$Var(I_1) = \mathbb{E}(I_1^2) - (\mathbb{E}(I_1))^2 = \mathbb{E}(I_1) - (\mathbb{E}(I_1))^2 = p - p^2 = p(1-p)$$

2.6.1 Example 3

Let X be a r.v. taking values in non-negative integers, then

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} P(X > n)$$

Proof:

Note that $X = \sum_{n=0}^{\infty} I_n$ where $I_n = I_{x>n}$. ($x > n$ is an event)

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}(\sum_{n=0}^{\infty} I_n)\\ &= \sum_{n=0}^{\infty} \mathbb{E}(I_n)\\ &= \sum_{n=0}^{\infty} P(X > n)\end{aligned}$$

In particular, let $X \sim Geo(p)$. As we have seen, $P(X > n) = (1-p)^n \Rightarrow$

$$\begin{aligned}\mathbb{E}(X) &= \sum_{n=0}^{\infty} P(X > n)\\ &= \sum_{n=0}^{\infty} (1-p)^n\\ &= \frac{1}{1-(1-p)} = \frac{1}{p}\end{aligned}$$

2.7 Moment generating function

Definition: Let X be a r.v. Then the function $M(t) = \mathbb{E}(e^{tX})$ is called the *moment generating function(mgf)* of X, if the expectation exists for all $t \in (-h, h)$ for some $h > 0$.

Remark: The mgf is not always well-defined. It is important to check the existence of the expectation.

2.7.1 Properties of mgf

1. Moment Generating Function generates moments
   - **Theorem**:
     - $M(0) = 1$
     - $M^{(k)}(0) = \mathbb{E}(X^k), k = 1, 2, \ldots$ ($M^{(k)} = \frac{d^k}{dt^k} M(t)|_{t=0}$)
       - **Proof**:

$$\begin{aligned}M(0) &= \mathbb{E}(e^{0 \cdot X}) = \mathbb{E}(1) = 1\\ M^{(k)}(0) &= \frac{d^k}{dt^k} \mathbb{E}(e^{t \cdot X})|_{t=0}\\ &= \mathbb{E}(\frac{d^k}{dt^k} e^{tX}|_{t=0})\\ &= \mathbb{E}(X^k)\end{aligned}$$

     - As a result, we have: $M(t) = \sum_{k=0}^{\infty} \frac{M^{(k)}(0)}{k!} t^k = \sum_{k=0}^{\infty} \frac{E * X^k}{k!} t^k$ (a method to get moment of a r.v)
2. $X \perp\!\!\!\perp Y$, with mgf's M_x, M_y. Let M_{X+Y} be the mgf of $X + Y$. then

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

- **Proof**:

$$\begin{aligned} M_{X+Y}(t) &= \mathbb{E}(e^{t(X+Y)}) \\ &= \mathbb{E}(e^{tx}e^{ty}) \\ &= \mathbb{E}(e^{tx})\mathbb{E}(e^{ty}) \\ &= M_X(t)M_Y(t) \end{aligned}$$

3. The mgf completely determines the distribution of a r.v.
   - $M_X(t) = M_Y(t)$ for all $t \in (-h, h)$ for some $h > 0$, then $X \overset{d}{=} Y$. ($\overset{d}{=}$: have the same distribution)
   - Example: Let $X \sim Poi(\lambda_1), Y \sim Poi(\lambda_2)$. $X \perp\!\!\!\perp Y$. Find the distribution of $X + Y$
     - First, derive the mgf of a Poisson distribution.

$$\begin{aligned} M_X(t) &= \mathbb{E}(e^{tX}) \\ &= \sum_{n=0}^{\infty} e^{tn} \cdot P(X = n) \\ &= \sum_{n=0}^{\infty} e^{tn} \cdot \frac{\lambda_1^n}{n!} e^{-\lambda_1} \\ &= \sum_{n=0}^{\infty} \frac{(e^t \cdot \lambda_1)^n}{n!} \cdot e^{-\lambda_1} \end{aligned}$$

$$\text{we know that } \sum_{n=0}^{\infty} \frac{(e^t\lambda_1)^n}{n!} = e^{e^t \cdot \lambda_1}.(\text{Since } \frac{(e^t\lambda_1^n)}{n!} e^{-e^t\lambda_1} \text{ is the pmf of } Poi(e^t\lambda_1))$$
$$\Rightarrow M_X(t) = e^{e^t\lambda_1} e^{-\lambda_1} = e^{\lambda_1(e^t-1)}, t \in \mathbb{R}.(e^{\lambda_1(e^t-1)} \text{ is mgf of } Poi(\lambda_1))$$
$$\text{Similarly, } M_Y(t) = e^{\lambda_2(e^t-1)}.$$

We know that

$$\begin{aligned} M_{X+Y}(t) &= M_X(t)M_Y(t) \\ &= e^{\lambda_1(e^t-1)} e^{\lambda_2(e^-1)} \\ &= e^{(\lambda_1+\lambda_2)(e^t-1)} \end{aligned}$$

This is the mgf of $Poi(\lambda_1 + \lambda_2)$!

Since the mgf uniquely determines the distribution $X + Y \sim Poi(\lambda_1 + \lambda_2)$

In general, if $X_1, X_2, \ldots, X_n$ independent, $X_i \sim Poi(\lambda_i)$, then $\sum X_i \sim Poi(\sum \lambda_i$

2.7.2 Joint mgf

Definition: Let X, Y be r.v's. Then $M(t_1, t_2) := \mathbb{E}(e^{t_1X+t_2Y})$ is called the joint mgf of X and Y, if the expectation exists for all $t_1 \in (-h_1, h_1)$, $t_2 \in (-h_2, h_2)$ for some $h_1, h_2 > 0$.

More generally, we can define $M(t_1, \ldots, t_n) = \mathbb{E}(exp(\sum_{i=1}^{n} t_iX_i))$ for r.v's $X_1, \cdots, X_n$, if the expectation exists for $\{(t_1, \cdots, t_n) : t_i \in (-h_i, h_i), i = 1, \cdots, n\}$ for some $\{h_i > 0\}, i = 1, \cdots, n$

2.7.2.1 Properties of the joint mgf

1. $$\begin{aligned} M_X(t) &= \mathbb{E}(e^{tX}) \\ &= \mathbb{E}(e^{tX+0Y}) \\ &= M(t, 0) \\ M_Y(t) &= M(0, t) \end{aligned}$$

2. $$\frac{\partial^{m+n}}{\partial t_1^m \partial t_2^n} M(t_1, t_2)|_{(0,0)} = \mathbb{E}(X^mY^n)$$
$$\text{the proof is similar to the single r.v. case}$$

3. If $X \perp\!\!\!\perp Y$, then $M(t_1, t_2) = M_X(t_1)M_Y(t_2)$
   - **Proof**:

$$\begin{aligned} M(t_1,t_2) &= \mathbb{E}(e^{t_1X+t_2Y}) \\ (X \perp\!\!\!\perp Y) &= \mathbb{E}(e^{t_1X}e^{t_2Y}) \\ &= \mathbb{E}(e^{t_1X})\cdot\mathbb{E}(e^{t_2Y}) \\ &= M_X(t_1)\cdot M_Y(t_2) \end{aligned}$$

- **Remark**: Don't confuse this with the result $X \perp\!\!\!\perp Y \Rightarrow M_{X+Y}(t) = M_X(t)M_Y(t)$.
  - $M_{X+Y}(t) \to$ mgf of $X+Y$; single argument function t
  - $M(t_1,t_2) \to$ joint mgf of (X,Y); two arguments t_1,t_2

3. Conditional distribution and conditional expectation

3.1 Conditional distribution

3.1.1 Discrete case

Definition Let X and Y be discrete r.v's. The conditional distribution of X given Y is given by:

$$P(X=x|Y=y) = \frac{(P(X=x,Y=u))}{P(Y=y)}$$

$$P(X=x|Y=y) : f_{X|Y=y}(x). \qquad f_{X|Y}(x|y) \leftarrow \text{conditional probability mass function})$$

Conditional pmf is a legitimate pmf: given any y, $f_{X|Y=y}(x) \geq 0, \forall x$

$$\sum_x f_{X|Y=y}(x) = 1$$

Note that given $Y=y$, as x changes, the value of the function $f_{X|Y=y}(x)$ is proportional to the joint probability.

$$f_{X|Y=y}(x) \propto P(X=x,Y=y)$$

This is useful for solving problems where the denominator $P(Y=y)$ is hard to find.

3.1.1.1 Example

$X_1 \sim Poi(\lambda_1), X_2 \sim Poi(\lambda_2)$. $X_1 \perp\!\!\!\perp X_2$, $Y = X_1 + X_2$

Q: $P(X_1 = k|Y=n)$?

Note $P(X_1=k|Y=n) = f_{X_1|Y=n}(k)$

A: $P(X_1=k|Y=n)$ can only be non-zero for $k=0,\cdots,n$ in this case,

$$\begin{aligned} P(X_1=k|Y=n) &= \frac{P(X_1=k,Y=n)}{P(Y=n)} \\ &\propto P(X_1=k,Y=n) \\ &= P(X_1=k,X_2=n-k) \\ &= e^{-\lambda_1}\frac{\lambda_1^k}{k!}\cdot e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!} \\ &\propto (\frac{\lambda_1}{\lambda_2})^k/k!(n-k)! \end{aligned}$$

we can get $P(X=k|Y=n)$ by normalizing the above expression.

$$P(X_1=k,Y=n) = \frac{(\frac{\lambda_1}{\lambda_2})^k/k!(n-k)!}{\sum_{k=0}^n(\frac{\lambda_1}{\lambda_2})^k/k!(n-k)!}$$

but then we will need to fine $\sum_{k=0}^n(\frac{\lambda_1}{\lambda_2})^k/k!(n-k)!$

An easier way is to compare $\sum_{k=0}^n(\frac{\lambda_1}{\lambda_2})^k/k!(n-k)!$ with the known results for common distribution. In particular, if $X \sim Bin(n,p)$

$$\begin{aligned} P(X=k) &= \binom{n}{k}p^k(1-p)^{n-k} \\ &\propto (\frac{p}{1-p})^k/k!(n-k)! \end{aligned}$$

$\Rightarrow P(X_1 = k|Y = n)$ follows a binomial distributions with parameters n and p given by $\frac{p}{1-p} = \frac{\lambda_1}{\lambda_2} \Rightarrow p = \frac{\lambda_1}{\lambda_1+\lambda_2}$

Thus, given $Y = X_1 + X_2 = n$, the conditional distribution of X_1 is binomial with parameter n and $\frac{\lambda_1}{\lambda_1+\lambda_2}$

3.1.2 Continuous case

Definition: Let X and Y be continuous r.v's. The conditional distribution of X given Y is given by

$$f_{X|Y}(x|y) = f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

A conditional pdf is a legitimate pdf

$$\begin{aligned} f_{X|Y}(x|y) \geq 0 \qquad & x, y \in \mathbb{R} \\ \int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = 1, \quad & y \in \mathbb{R} \end{aligned}$$

3.1.2.1 Example

Suppose $X \sim Exp(\lambda), Y|X = x \sim Exp(x) = f_{Y|X}(y|x) = xe^{-xy}, y = e \leftarrow$ conditional distribution of Y given $X = x$

Q: Find the condition pdf $f_{X|Y}(x|y)$

A:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &\propto f(x,y) \\ &= f_{Y|X}(y|x) \cdot f_X(x) \\ &= xe^{-xy}\lambda e^{-\lambda x} \\ &\propto xe^{-x(y+\lambda)}, \qquad x > 0, y > 0 \end{aligned}$$

Normalization (make the total probability 1)

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{xe^{-x(y+\lambda)}}{\int_0^\infty xe^{-x(y+\lambda)}dx} \\ \int_0^\infty xe^{-x(y+\lambda)}dx &= (\frac{1}{\lambda+y})^2 \leftarrow \text{integration by parts} \end{aligned}$$

Thus, $f_{X|Y}(x|y) = (\lambda+y)^2 xe^{-x(y+\lambda)}$, $x > 0$.

This is a gamma distribution with parameters γ and $\lambda + y$

3.1.2.1. Example 2

Find the distribution of $Z = XY$.

Attention: the following method is wrong:

$$f_Z(z) = \int_0^\infty f_{Y|X}(\frac{z}{x}|x) \cdot f_X(x)dx$$

If we want to directly work with pdf's, we will need to use the change of variable formula for multi-variables. The right formula have turns out to be

$$\begin{aligned} f_Z(z) &= \int_0^\infty f_{X,Z}(x,z)dx = \int_0^\infty f_{Z|X}(z|x)f_X(x)dx \\ &= \int_0^\infty f(x, \frac{z}{x}) \cdot \frac{1}{x}dx \\ &= f_{Y|X}(\frac{z}{x}|x)f_X(x) \cdot \frac{1}{x}dx \end{aligned}$$

As an **easier way** is to use cdf, which gives probability rather than density:

$$
\begin{aligned}
P(Z<z) &= P(XY \le z) \\
&= \int_0^\infty P(XY \le z|X=x) f_X(x)dx \qquad \text{(law of total probability)} \\
&= \int_0^\infty P(Y \le \frac{z}{x}|X=x) \cdot f_X(x)dx \\
Y|X=x \sim Exp(x) \\
&= \int_0^\infty (1-e^{-x\cdot\frac{z}{x}}) \cdot \lambda e^{-\lambda x} dx \\
&= 1-e^{-z} \int_0^\infty \lambda e^{-\lambda x} dx \\
&= 1-e^{-z} \Rightarrow Z \sim Exp(1)
\end{aligned}
$$

Notation $X, Y|\{Z=k\} \overset{iid}{\sim} \cdots$ means that given $Z=k$, X and Y are *conditionally independent*, and they follow certain distribution.

(the conditional joint cdf/pmf/pdf equals the predict of the conditional cdf's/pmf's/pdf's)

3.2 Conditional expectation

We have seen that conditional pmf/pdf are legitimate pmf/pdf. Correspondingly, a conditional distribution is nothing else but a probability distributions. It is simply a (potentially) different distribution, since it takes more information into consideration.

As a result, we can define everything which are previously defined for unconditional distributions also for conditional distributions.

In particular, it is natural to define the conditional expectation.

Definition. The conditional expectation of $g(X)$ given $Y=y$ is defined as

$$
\mathbb{E}(g(X)|Y=y) = \begin{cases} \sum_{i_1}^\infty g(x_i)P(X=x_u|Y=y) & \text{if } X|Y=y \text{ is discrete} \\ \int_{-\infty}^\infty g(x) f_{X|Y}(x|y)dx & \text{if } X|X=y \text{ is continuous} \end{cases}
$$

Fix y, the conditional expectation is nothing but the expectation taken under the conditional distribution.

3.2.1 What is $\mathbb{E}(X|Y)$?

Different ways to understand *conditional expectation*

1. Fix a value y, $\mathbb{E}(g(X)|Y=y)$ is a number
2. As y changes $\mathbb{E}(g(x)|Y=y)$ becomes a function of y (that each y gives a value): $h(y) =: \mathbb{E}(g(x)|Y=y)$
3. since y is actually random, we can define $\mathbb{E}(g(x)|Y) = h(Y)$. This is a random variable

$$
\mathbb{E}(g(x)|Y))_{(\omega)} = \mathbb{E}(g(x)|Y=Y(\omega)
$$

$\omega \in \Omega$ this random variable takes value $\mathbb{E}(g(x)|Y=y)$ When $Y=y$

$$
\begin{gathered}
\Omega \to \mathbb{R} \\
h(Y)_{(\omega)} = h(Y(\omega))
\end{gathered}
$$

3.2.2 Properties of conditional expectation

1. Linearity (inherited from expectation)

$$
\mathbb{E}(aX+b|Y=y) = a\mathbb{E}(X|Y=y)+b
$$

$$
\mathbb{E}(X+Z|Y=y) = \mathbb{E}(X|Y=y)+\mathbb{E}(Z|Y=y)
$$

2. $\mathbb{E}(g(X,Y)|Y=y) = \mathbb{E}(g(X,y)|Y=y) \neq \mathbb{E}(g(X,y))$ when X and Y are not independent

Proof (Discrete):

$$
\mathbb{E}(g(X,Y)|Y=y) = \sum_{x_i}\sum_{y_j} g(x_i,y_j) \cdot P(X=x_i, Y=y_j|Y=y)
$$

$$
P(X=x_i, Y=y_j|Y=y) = \begin{cases} 0 & \text{if } y_j \neq y \\ \\ P(X=x_1, Y=y_j)/P(Y+y) = P(X=x_i|Y=y) & \text{if } y_j = y \end{cases}
$$

$$\Rightarrow \mathbb{E}(g(X,Y)|Y=y) = \sum_{x_i} g(x_i,y)\cdot P(X=x_i|Y=y)$$
$$= \mathbb{E}(g(X,y)|Y=y) \qquad g(X,y) \text{ regarded as a function of } x$$

In particular,

$$\mathbb{E}(g(X)\cdot h(Y)|Y=y) = h(y)\mathbb{E}(g(X)|Y=y)$$
$$\mathbb{E}(g(X)\cdot h(Y)|Y) = h(Y)\mathbb{E}(g(X)|Y)$$

3. If $X \perp Y$, then $\mathbb{E}(g(X)|Y=y) = \mathbb{E}(g(X))$

   Fact: If $X \perp Y$, then conditional distribution of X given $Y=y$ is the same as the unconditional distribution of X

   Proof(Discrete):

$$\begin{aligned}
&\text{if } X \perp Y \\
&P(X=x_i|Y=y_j) \\
&\quad = P(X=x_i|Y=y_j)/P(Y=y_j) \\
&\quad = P(X=x_i)P(Y=y_j)/P(Y=y_j) \\
&\quad = P(X=x_i)
\end{aligned}$$

4. Law of iterated expectation (or double expectation): Expectation of conditionally expectation is its unconditional expectation

$$\mathbb{E}(\mathbb{E}(X|Y))) = \mathbb{E}(X)$$

   $\mathbb{E}(X|Y)$ is a r.v, a function of Y.
   Proof(Discrete):

   When $Y=y_j$, the r.v. $\mathbb{E}(X|Y) = \mathbb{E}(X|Y=y_j) = \sum_{x_i} x_i P(X=x_i|Y=y_j)$. This happens with probability $P(Y=y_j)$

$$\Rightarrow \begin{aligned}
\mathbb{E}(\mathbb{E}(X|Y)) &= \sum_{y_j}(\sum_{x_i} x_i P(X=x_i|Y=y_j))P(Y=y_j) \\
&= \sum_{x_i}\sum_{y_j} x_i P(X=x_i|Y=y_j)P(Y=y_j) \\
&= \sum_{x_i} x_i \sum_{y_j} P(X=x_i|Y=y_j)P(Y=y_j) \qquad \text{law of total probability} \\
&= \sum_{x_i} x_i P(X=x_i) = \mathbb{E}(X)
\end{aligned}$$

   Alternatively,

$$\begin{aligned}
&\sum_{x_i}\sum_{y_j} x_i P(X=x_i|Y=y_j)P(Y=y_j) \\
&\quad = \sum_{x_i}\sum_{y_j} x_i P(X=x_i, Y=y_j) \qquad g(X,Y)=X \text{ at } (x_i,y_j) \\
&\quad = \mathbb{E}(X)
\end{aligned}$$

   Example:

   Y: # of claims received by insurance company
   X: some random parameter

$$Y|X \sim Poi(X), X \sim Exp(\lambda)$$

   a) $\mathbb{E}(Y)$?
   b) $P(Y=n)$?

   a)

$$Y|X \sim Poi(X) \Rightarrow \mathbb{E}(Y|X=x) = x \Rightarrow \mathbb{E}(Y|X) = X$$

$$\therefore \begin{aligned}\mathbb{E}(Y) &= \mathbb{E}(\mathbb{E}(Y|X)) \\ &= \mathbb{E}(X) = \frac{1}{\lambda}\end{aligned}$$

b)

$$\begin{aligned} P(Y=n) &= \int_0^\infty P(Y=n|X=x) f_X(x) dx \\ &= \int_o^\infty \frac{e^{-x}x^n}{n!} \cdot \lambda e^{-\lambda x} dx \\ &= \frac{\lambda}{n!}\int_0^\infty x^n e^{-(\lambda+1)x} dx \\ &= \frac{\lambda}{(\lambda+1)^{n+1} n!}\int_0^\infty ((\lambda+1)x)^n e^{-(\lambda+1)x} d(\lambda+1)x \\ &= \frac{\lambda}{(\lambda+1)^{n+1} n!}\Gamma(n+1) \qquad\qquad \Gamma(n+1) = n! \text{ ; formula for gamma function or integration by parts} \\ &= \frac{\lambda}{(\lambda+1)^{n+1}} = (\frac{1}{\lambda+1})^n \cdot \frac{1}{\lambda+1} \\ &\Rightarrow Y+1 \sim Geo(\lambda/(\lambda+1)) \end{aligned}$$

We verify that $\mathbb{E}(Y) = \frac{\lambda+1}{\lambda} - 1 = \frac{1}{\lambda}$

3.3 Decomposition of variance (EVVE's low)

Definition: The conditional variance of Y given $X = x$ is defined as

$$Var(Y|X=x) = \mathbb{E}((Y - \mathbb{E}(Y|X=x))^2 | X=x)$$

$$Var(Y|X)_{(\omega)} = Var(Y|X = X_{(\omega)}) \qquad Var(Y|X)_{(\omega)}\text{: a r.v, a function of } X$$

The conditional variance is simply the variance taken under the conditional distribution

$$\Rightarrow V(Y|X=x) = \mathbb{E}(Y^2|X=x) - (\mathbb{E}(Y|X=x))^2$$

Thus

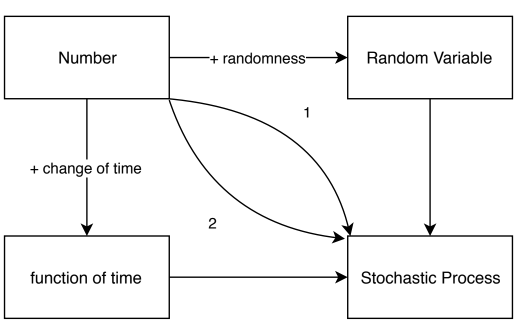
$$Var(Y) = \mathbb{E}(Var(Y|X)) + Var(\mathbb{E}(Y|X))$$

$$\mathbb{E}(Var(Y|X)) : \text{"intra-group variance"} \qquad Var(\mathbb{E}(Y|X)) : \text{"inter-group variance"}$$

Proof:

$$\begin{aligned} RHS &= E(E(Y^2|X) - (E(Y|X))^2) + E((E(Y|X))^2) - (E(E(Y|X)))^2 \\ &= E(E(Y^2|X)) - \cancel{E((E(Y|X))^2)} + \cancel{E((E(Y|X))^2)} - (E(E(Y|X)))^2 \\ &= E(Y^2) - (E(Y))^2 \\ &= Var(Y) \end{aligned}$$

4. Stochastic Processes



1. sequence / family of random variables
2. a random function (hard to formulate)

Definition: A **stochastic process** $\{X_t\}_{t\in T}$ is a collection of random variables, defined on a common probability space.

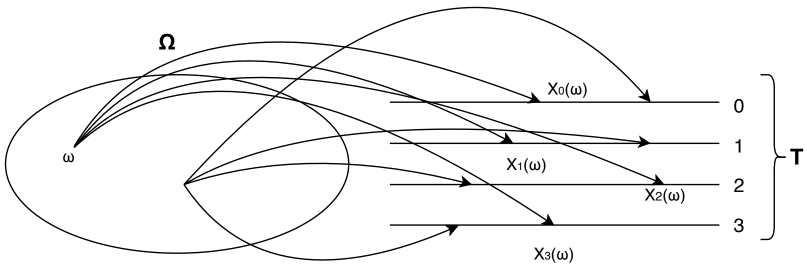
T: index set. In most cases, T corresponds to time, and is either discrete $\{0,1,2,\cdots\}$ or continuous $[0,\infty)$

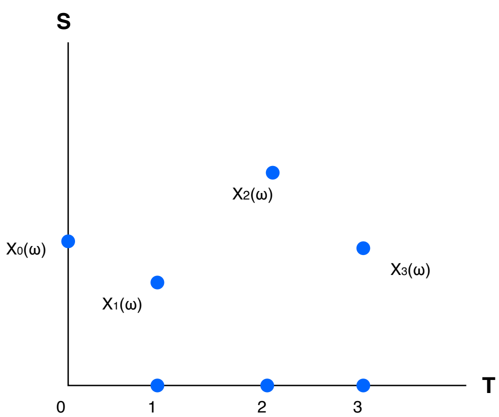
In discrete case, we writes $\{X_n\}_{n=0,1,2,...}$

This **state space** S os a stochastic process is the set of all possible value of $X_t, t\in T$

S can also be either discrete or continuous. In this course, we typically deal with **discrete** state space. Then we relabel the states so that $S = \{0,1,2,\cdots\}$ (countable state space) or $S=\{0,1,2,\cdots,M\}$ (finite state space)

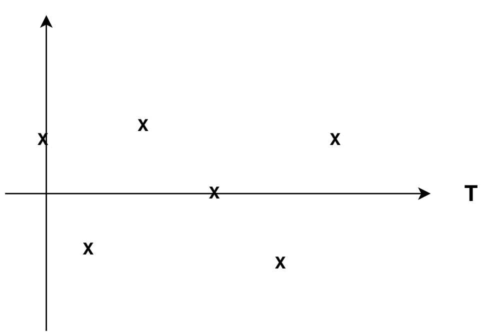
Remark: As in the case of the joint distribution, we need the r.v's in a stochastic process to be defined on a common probability space, because we want to discuss their joint behaviours, i.t, how things change over time.





Thus, we can identify each point ω in the sample space Ω with a function defined on T and taking value in S. Each function is called a **path** of this stochastic process

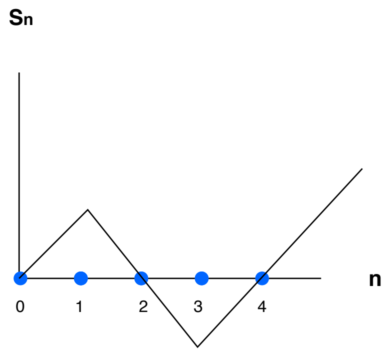
Example Let $X_0, X_1, \cdots$ be independent and identical r.v's following some distribution. Then $\{X_n\}_{n=0,1,2,...}$ is a stochastic process



Example Let $X_1, X_2, ...$ be independent and identical r.v.'s. $P(X_1 = 1) = p$, and $P(X_1=-1)=1-p$. Define $S_0 = 0, S_n = \sum_{i=1}^n X_i, n \le 1$, e.g.

- $S_0 = 0$
- $S_1 = X_1$
- $S_2 = X_1 + X_2$
- $\cdots\cdots$

Then $\{S_n\}_{n=0,1,...}$ is a stochastic process, with state space $S=\mathbb{Z}$ (integer)



4.1 Markov Chain

4.1.1 Simple Random Walk

$\{S_n\}_{n=0,1,\ldots}$ is called a **"simple random walk"**. ($S_n = S_{n-1} + X_n$)

$$S_n = \begin{cases} S_{n-1}+1 \\ S_{n-1}-1 \end{cases}$$

Remark: Why we need the concept of "stochastic process"? Why don't we just look at the joint distribution of $(X_0, X_1, ..., X_n)$?

Answer: The joint distribution of a large number of r.v's is very complicated, because it does not take advantage of the special structure of T(time).

For example, simple random walk. The full distribution of $(S_0, S_1, ..., S_n)$ is complicated for n large. However, the structure is actually simple if we focus on the adjacent times:

$$S_{n+1} = S_n + X_{n+1}$$

$$S_n: \text{ last value.} \quad X_{n+1}: \text{related to } Ber(p)\text{. They are independent}$$

By introducing time into the framework, we can greatly simplify many things.

More precisely, we find that for simple random walk, $\{S_n\}_{n=0,1,\ldots}$, if we know S_n the distribution of S_n+1 will not depend on the history $(S_0, ..., S_{n-1})$. This is a very useful property

In general for a stochastic process $\{X_n\}_{n=0,1,\ldots}$, at time n, we already know $X_0, X_1, ..., X_n$, S_0; our best estimate of the distribution of X_{n+1} should be the conditional distribution:

$$X_{n+1}|X_n, ..., X_n$$

given by:

$$P(X_{n+1} = x_{n+1}|X_n = x_n, ..., X_0 = x_0)$$

As time passes, the expression becomes more and more complicated $\rightarrow$ impossible to handle.

However, if we know that this conditional distribution is actually the same as the conditional distribution only given X_n, then the structure will remain simple for any time. This motivates the notion of *Markov chain*.

4.1.2 Markov Chain

4.1.2.1 Discrete-time Markov Chain

Definition and Examples

Definition: A discrete-time Stochastic process $\{X_n\}_{n=0,1,\ldots}$ is called a **discrete-time Markov Chain (DTMC)**, if its state space S is discrete, and it has the Markov property:

$$\begin{aligned} &P(X_{n+1} = x_{n+1}|X_n = x_n, ..., X_o = x_o) \\ &= P(X_{n+1} = x_{n+1}|X_n = x_n) \end{aligned}$$

for all $n, x_0, ..., x_n, x_{n+1} \in S$

If $X_{n+1}|\{x_n = i\}$ does not change over time, $P(X_{n+1} = j|N_n = i) = P(X_1 = j|X_0 = i)$, then we call this Markov chain **time-homogeneous** (default setting for this course).

$$\begin{aligned} & P(X_{n+1} = x_{n+1}|X_n = x_n, ..., X_0 = x_0) \qquad && X_{n+1} = x_{n+1}\text{: future; } X_n = x_n\text{: present(state)} \\ &= P(X_{n+1} = x_{x+1}|X_n = x_n) && X_{n-1} = x_{n-1}, ..., X_0 = x_0\text{: past(history)} \end{aligned}$$

Intuition: Given the present state, the past and the future are independent. In other words, the future depends on the previous results only through the current state.

Example: simple random walk

The simple random walk $\{S_n\}_{n=0,1,...}$ is a Markov chain

Proof:

Recall that $S_{n+1} = S_n + X_{n+1}$

$$\begin{aligned} & P(S_{n+1} = s_{n+1}|S_n = s_n, ..., S_0 = s_0) \\ &= 0 \\ &= P(S_{n+1} = s_{n+1}|S_n = s_n) \end{aligned}$$

if $s_{n+1} \neq s_n \pm$

$$\begin{aligned} & P(S_{n+1} = s_n + 1|S_n = s_n, ..., s_0 = 0) \\ &= P(X_{n+1}|S_n = s_n, ..., S_0 = 0) \\ &= P(X_{n+1} = 1) \qquad X_{n+1} \perp (X_1, ..., X_n) \text{ hence also } (S_0, ..., S_n) \end{aligned}$$

Similarly,

$$\begin{aligned} & P(S_{n+1} = s_n + 1|S_n = s_n) \\ &= P(X_{n+1} = 1|S_n = s_n) \\ &= P(X_{n+1} = 1) \\ &\Rightarrow P(S_{n+1}|S_n = s_n, ..., S_0 = s_0) \end{aligned}$$

Similarly,

$$\begin{aligned} & P(S_{n+1} = s_n - 1|S_n = s_n, ..., S_0 = 0) \\ &= P(S_{n+1} = s_n - 1|S_n = s_n) \\ &= P(X_{n+1} = -1) \\ &\Rightarrow \{S_n\}_{n=0,1,...} \text{ is a DTMC} \qquad \blacksquare \end{aligned}$$

4.1.3 One-step transition probability matrix

For a time-homogeneous DTMC, define

$$\begin{aligned} P_{ij} &= P(X_1 = j|X_0 = i) \\ &= P(X_{n+1} = j|X_n = i) \qquad n = 0, 1, ... \end{aligned}$$

P_{ij}: one step transition probability

The collection of $P_{ij}, i, j \in S$ governs all the one-step transitions of the DTMC. Since it has two indices i and j; it naturally forms a matrix $P = \{P_{ij}\}_{i,j \in S}$, called the **(one-setp) transition (probability) matrix** or **transition matrix**

Property of a transition matrix $P = \{P_{ij}\}_{i,j \in S}$:

$$\begin{aligned} & P_{ij} \geq 0 \quad \forall i, j \in S \\ & \sum_{j \in S} P_{ij} = 1 \quad \forall i \in S \quad \rightarrow \text{ the row some of } P \text{ are all 1} \end{aligned}$$

Reason:

$$\begin{aligned}\sum_{j\in S} P_{ij} &= \sum_{j\in S} P(X_1=j|X_0=i) \\ &= P(X_1 \in S|X_o=i) \\ &= 1\end{aligned}$$

Example 1 : simple random walk

There will be 3 cases:

$$\begin{aligned}&P_{i,i+1} = P(S_1=i+1|S_0=i) = P(X_1=1) = p \\ &P_{i,i-1} = P(S_1=i-1|S_0=i) = P(X_1=-1) = 1-p =: q \\ &P_{i,j} = 0 \qquad\qquad \text{for } j \neq i \pm 1 \\ &\Rightarrow (\text{infinite dimension}) p = \begin{Bmatrix} ... & ... & ... & ... & ... & ... & ... \\ ... & 0 & p & 0 & ... & ... & ... \\ ... & q & 0 & p & ... & ... & ... \\ ... & ... & q & 0 & p & ... & ... \\ ... & ... & ... & q & 0 & p & ... \\ ... & ... & ... & ... & ... & ... & ... \end{Bmatrix}\end{aligned}$$

Example 2: Ehrenfest's urn

Two urns A, B, total M balls. Each time, pick one ball randomly(uniformly), and move it to the opposite urn.

$$X_n : \#\text{ of balls in } A \text{ after step } n$$

$$S = \{0, 1, ..., M\}$$

$$\begin{aligned}P_{ij} &= P(X_1=j|X_0=j) \qquad (i \text{ balls in } A,\, M-i \text{ balls in } B) \\ &= \begin{cases} i/M & j=i-1 \\ (M-i)/M & j=i+1 \\ 0 & j \neq i\pm 1 \end{cases}\end{aligned}$$

$$p = \begin{Bmatrix} 0 & 1 & & & & & \\ 1/M & 0 & (M-1)/M & & & & \\ & 1/M & 0 & (M-1)/M & & & \\ & & 2/M & 0 & (M-2)/M & & \\ ... & ... & ... & ... & ... & ... & ... \\ & & & & (M-1)/M & 0 & 1/M \\ & & & & & 1 & 0 \end{Bmatrix}$$

Example 3: Gambler's ruin

A gambler, each time wins 1 with probability p, losses 1 with probability $1-p=q$. Initial wealth $S_0 = a$; wealth at time n: S_n. The gambler leaves if $S_n = 0$ (loses all money) or $S_n = M > a$ (wins certain amount of money and gets satisfied)

This is a variant of the simple random walk, where we have absorbing barriers($P_{ii} = 1$) at 0 and M

$$S = \{0, ..., M\}$$

$$P_{ij} = \begin{cases} p & j=i+1, i=1,...,M-1 \\ q & j=i-1, i=1,...,M-1 \\ 1 & i=j=0 \text{ or } i=j=M \\ 0 & \text{otherwise} \end{cases}$$

$$p = \begin{Bmatrix} 1 & 0 & ... & & & & \\ q & 0 & p & ... & & & \\ ... & q & 0 & p & ... & & \\ & ... & q & 0 & p & ... & \\ ... & ... & ... & ... & ... & ... & ... \\ & & & ... & q & 0 & p \\ & & & & ... & 0 & 1 \end{Bmatrix}$$

Example 4: Bonus-Malus system

Insurance company has 4 premium levels: 1, 2, 3, 4

Let $X_n \in \{1, 2, 3, 4\}$ be the premium level for a customer at year n

$$Y_n \overset{iid}{\sim} Poi(\lambda): \text{ \# of claims at year } n$$

- If $Y_n = 0$ (no claims)
  - $X_{n+1} = max(X_{n-1}, 1)$
- If $Y_n > 0$
  - $X_{n+1} = min(X_n + Y_n, 4)$

Denote $a_k = P(Y_n = k), k = 0, 1, ...$

$$p = \begin{Bmatrix} a_0 & a_1 & a_2 & (1 - a_0 - a_1 - a_2) \\ a_0 & 0 & a_1 & (1 - a_0 - a_1) \\ 0 & a_0 & 0 & (1 - a_0) \\ 0 & 0 & a_0 & (1 - a_0) \end{Bmatrix}$$

4.2 Chapman-Kolmogorov equations

Q: Given the (one-step) transition matrix, $P = \{P_{ij}\}_{i,j \in S}$, how can we decide the n-step transition probability

$$\begin{aligned} P_{ij}^{(n)} &:= P(X_n = j | X_0 = i) \\ &= P(X_{n+m} = j | X_m = i), \quad m = 0, 1, ... \end{aligned}$$

As a special case, let us start with $P_{ij}^{(2)}$ and their collection $p^{(2)} = \{P_{ij}^{(2)}\}_{i,j \in S}$ (also a square matrix, same dimension as P)

Condition on what happens at time 1:

$$\begin{aligned} P_{ij}^{(2)} &= P(X_2 = j | X_0 = i) \\ &= \sum_{j \in S} P(X_2 = j | X_0 = i, X_1 = k) \cdot P(X_1 = k | X_0 = i) \quad \text{conditional law of total probability} \end{aligned}$$

4.2.1 Conditional Law of total probability

$$\begin{aligned} & P(X_2 = j | X_0 = i) \\ &= \sum_{k \in S} P(X_2 = j, X_1 = k | X_0 = i) \\ &= \sum_{k \in S} \frac{P(X_2 = j, X_1 = k, X_0 = i)}{P(X_0 = i)} \\ &= \sum_{k \in S} \frac{P(X_2 = j, X_1 = k, X_0 = i)}{P(X_1 = k, X_0 = i)} \cdot \frac{P(X_1 = k, X_0 = i)}{P(X_0 = i)} \\ &= \sum_{k \in S} P(X_2 = j | X_0 = i, X_1 = k) \cdot P(X_1 = k | X_0 = i) \end{aligned}$$

continue on $P_{ij}^{(2)}$

$$\begin{aligned} P_{ij}^{(2)} &= P(X_2 = j | X_0 = i) \\ &= \sum_{j \in S} P(X_2 = j | X_0 = i, X_1 = k) \cdot P(X_1 = k | X_0 = i) \quad \text{conditional law of total probability} \\ &= \sum_{k \in S} P(X_2 = j | X_1 = k) \cdot P(X_1 = k | X_0 = i) \\ &= \sum_{k \in S} P(X_1 = j | X_0 = k) \cdot P(X_1 = k | X_0 = i) \\ &= \sum_{k \in S} P_{ik} \cdot P_{kj} \\ &= (P \cdot P)_{ij} \end{aligned}$$

Thus, $p^{(2)} = P \cdot P = p^2$

Using the smae idea, for $n, m = 0, 1, 2, 3...$:

$$
\begin{aligned}
P_{ij}^{(n+m)} &= P(X_{n+m} = j | X_0 = i) \\
&= \sum_{k \in S} P(X_{n+m} = j | X_0 = i, X_m = k) \cdot P(X_m = k | X_0 = i) \\
&= \sum_{k \in S} P(X_{n+m} = j | X_m = k) \cdot P(X_m = k | X_0 = i) \quad \text{Markov property} \\
&= \sum_{k \in S} P(X_n = j | X_0 = k) \cdot P(X_m = k | X_0 = i) \\
&= \sum_{k \in S} p_{ik}^{(m)} \cdot P_{kj}^{(n)} \\
&= (p^{(m)} \cdot p^{(n)})_{ij} \\
&\Rightarrow p^{(n+m)} = p^{(m)} \cdot p^{(n)} \qquad (*)
\end{aligned}
$$

By definition, $p^{(1)} = p$

- $\Rightarrow p^{(2)} = p^{(1)} \cdot p^{(1)} = p^2$
- $\Rightarrow p^{(3)} = p^{(2)} \cdot p^{(1)} = p^3$
- $\cdots\cdots\cdots$
- $\Rightarrow p^{(n)} = p^n$

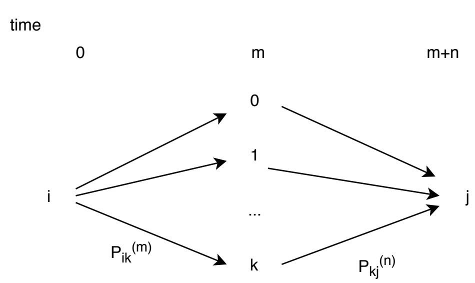
Note:

- n from $p^{(n)}$: n-step transition probability matrix
  - $p^{(n)} = \{p_{ij}^{(n)}\}_{i,j \in S}$
    $p_{ij}^{(n)} = P(X_n = j | X_0 = i)$
- n from p^n: n-th power of the (one-step) transition matrix
  - $p^n = p \cdot ... \cdot p$
    $p = \{P_{ij}\}_{i,j \in S}$
    $p_{ij} = P(X_1 = j | X_0 = i)$

$(*)$ is called the **Chapman-Kolmogorov equations** (c-k equation). Entry-wise:

$$
P_{ij}^{n+m} = \sum_{k \in S} P_{ik}^{(m)} P_{kj}^{(n)}
$$

Intuition:



"Condition at time m (on X_m) and sum p all the possibilities"

4.2.2 Distribution of X_n

So far, we have seen transition probability $P_{ij}^{(m)} = P(X_n = j | X_0 = i)$. This is not the probability $P(X_n = j)$. In order to get this distribution, we need the information about which state the Markov chain starts with.

Let $\alpha_{0,i} = P(X_0 = i)$. The row vector $\alpha_0 = (\alpha_0, 0, \alpha_0, 1, ...)$ is called the **initial distribution** of the Markov chain. This is the distribution of the initial state X_0

Similarly, we define distribution of X_n: $\alpha_n = (\alpha_n, 0, \alpha_n, 1, ...)$ where $\alpha_{n,i} = P(X_n = i)$

Fact: $\alpha_n = \alpha_0 \cdot p^n$

Proof:

$$
\forall j \in S
$$

$$\begin{aligned}
\alpha_{n,j} &= P(X_n = j) \\
&= \sum_{i \in S} P(X_n = j | X_0 = i) \cdot P(X_0 = i) \\
&= \sum_{i \in S} \alpha_{0,i} \cdot P_{ij}^{(n)} \\
&= (\alpha_0 \cdot P^{(n)})_j = (\alpha_0 \cdot p^n)_j
\end{aligned}$$

$$\Rightarrow \alpha_n = \alpha 0 \cdot p^n$$

- α_n: distribution of X_n
- α_0: initial distribution
- p^n: transition matrix

Remark: The distribution of a DTMC is completely determined by two things:

- the initial distribution α_0 (row vector), and
- the transition matrix p (square matrix)

4.3 Stationary distribution (invariant distribution)

Definition: A probability distribution $\pi = (\pi_0, \pi_1, ...)$ os ca;;ed a **stationary distribution**(invariant distribution) of the DTMC $\{X_n\}_{n=0,1,...}$ with transition matrix P, if :

1. $\underline{\pi} = \pi \cdot P$
2. $\sum_{i \in S} \pi_i = 1 (\Leftrightarrow \underleftarrow{\pi} \cdot \mathbb{1})$. (𝟙: a column of all 1's)

Why such $\underline{\pi}$ is called stationary/invariant distribution?

$$\begin{aligned}
\sum_{i \in S} \pi_i = 1, \pi_i \geq 0, i = 0, 1, ... &\Rightarrow \text{distribution} \\
\underline{\pi} = \pi \cdot P &\Rightarrow \text{invariant/stationary.}
\end{aligned}$$

Assume the MC starts from the initial distribution $\alpha_0 = \underline{\pi}$. hen the distribution of X_1 is

$$\alpha_1 = \alpha_0 \cot P = \underline{\pi} \cdot P = \underline{\pi} = \alpha_0$$

The distribution of X_2:

$$\alpha_2 = \alpha_0 \cdot P^2 = \underline{\pi} \cdot P \cdot P = \underline{\pi} \cdot P = \underline{(}\pi) = \alpha_0$$

$$\cdots\cdots$$

$$\alpha_n = \alpha_0$$

Thus, if the MC starts from a stationary distribution, then its distribution will not change over time.