Note 18 - Mar 19

5 Poisson Processes (cont'd)

5.3. Properties of Poisson Processes (cont'd)

5.3.1. Continuous-time Markov Property (cont'd)

5.3.1.3. Combining and Thinning of Poisson Process (cont'd)

The combined Poisson Process is still a Poisson Process, with intensity being the sum of intensities.

Reason: Memoryless property, and

$$
\begin{gathered}
min(W_1, W_2) \\
W1 \sim Exp(\lambda_1) \\
W2 \sim Exp(\lambda_2) \\
W_1 \perp\!\!\!\perp W_2
\end{gathered}
$$

$\Rightarrow$ the combined process is the counting process of events with interarrival time following $Exp(\lambda_1 + \lambda_2$

Thinning

Let $\{N(t)\} \sim Poi(\lambda t)$. Each arrival (customer) is labeled as type 1 or type 2, with probability p and $1-p$, independently from others.

Let $N_1(t)$ and $N_2(t)$ be the number of customers of type 1and type 2 respectively, who arrived before time t. Then

$$
\begin{aligned}
&\{N_1(t) \sim Poi(p\lambda t) \\
&\{N_2(t) \sim Poi((1-p)\lambda t) \\
\text{and } &\{N_1(t)\} \perp\!\!\!\perp \{N_2(t)\}
\end{aligned}
$$

Reason: This is the inverse procedure of combining two independent Poisson processes into one Poisson process

5.3.1.4 Order Statistics Property

Let $X_1, \ldots, X_n$ be i.i.d. r.v's. The order statistics of $X_1, \ldots, X_n$ are random variables defined as follows.

$$\begin{aligned}
X_{(1)} &= min\{X_1, \ldots, X_n\} \\
X_{(2)} &= \text{2nd smallest among} X_1, \ldots, X_n \\
&\vdots \\
X_{(n)} &= max\{X_1, \ldots, X_n\}
\end{aligned}$$

In orher words, $X_{(1)}, \ldots, X_{(n)}$ are such that $\{X_{(1)}, \ldots, X_{(n)}\} = \{X_1, \ldots, X_n\}$ and $X_{(n)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$

Thus, let $\{N(t(\} \sim Poi(\lambda t)$. Condition on $N(t) = n$, the points/arrivals of N in $[0, t]$ are distributed as the order statistics of n i.i.d. uniform r.v's on $[0, t]$

That is

$$(S_1, \ldots, S_n | N(t) = n) \overset{d}{=} (U_{(1)}, \ldots, U_{(n)})$$

where, $U_{(1)}, \ldots, U_{(n)}$ are the order statistics of $U_1, \ldots, U_n \overset{iid}{\sim} Unif[0, t]$

Reason:

$$\begin{aligned}
f_{S_1|\{N(t)=1\}}(s) &= \frac{f_{S_1}(s)\mathbb{P}(W_2 > t - s)}{\mathbb{P}(N(t) = 1)} \\
&\propto f_{S_1}(s)\mathbb{P}(\underbrace{W_2}_{Exp(\lambda)} > t - s) \\
&= \lambda e^{-\lambda s} e^{-\lambda(t-s)} \\
&= \underbrace{\lambda e^{-\lambda t}}_{\text{const w.r.t. s}}
\end{aligned}$$

$$\Rightarrow S_1 | \{N(t) = 1\} \sim Unif[0, t]$$

As a result of the order statistics property, we have preposition

$$N(s) | \{N(t) = n\} \sim Bin(n, \frac{s}{t}) \qquad \text{for } s \leq t$$

Reason: Given $N(t) = n$, then

$$\begin{aligned}
N(s) &= \#\{S_i : S_i \leq s, i = 1, 2, \ldots, n\} \\
\text{Since } \{U_{(i)}\} \text{ is a } &= \#\{U_{(i)} : U_{(i)} \leq s, i = 1, 2, \ldots, n\} \\
\text{permutation of } \{U_i\} &= \#\{U_i : U_i \leq s, i = 1, 2, \ldots, n\}
\end{aligned}$$

$$U_i \overset{iid}{\sim} Unif[0, t]$$

$$\mathbb{P}(U_i \leq s) = \frac{s}{t} \qquad i = 1, \dots, n$$

$$\Rightarrow N(s)|\{N(t) = n\} \sim Bin(n, \frac{s}{t})$$

6. Continuous-Time Markov Chain

6.1. Definitions and Structures

Definition 6.1.1. Continuous-time Stochastic Process

A continuous-time stochastic process $\{X(t)\}_{t\geq 0}$ is called a continuous-time Markov Chain (CTMC), if its state space is at most countable, and it satisfies the continuous-time Markov property:

$$\begin{aligned} &\mathbb{P}(X(t_m) = j|X(t_{m-1} = i, X(t_{m-2}) = i_{m-2}, \dots, X(t_1) = i_1 \\ =&\mathbb{P}(X(t_m) = j|X(t_{m-1} = i)) \\ &\text{for any } m, t_1 < t_2 < \dots < t_m, i_1, \dots, i_{m-2}, i, j \in S \end{aligned}$$

As DTMC, typically $S = \{0, \dots, m\}$ or $\{1, \dots, m\}$ or $\{0, \pm 1, \pm 2, ldots\}$

Time is continuous, but the state space is discrete $\Rightarrow$ Process will "jump" between states

It can be regarded as a random step function.

Therefore, we need to specify two things:

1. When the jumps happen? $\Leftrightarrow$ How long the process stays in a state?
   - Given the process is in state i, it will stay in this state for an exponential random time, with parameter denoted as λ_i
   - **Reason**:
     - Markov property $\Rightarrow$ when the process will jump in the future only depends on its current state, not on how long it has been in the current state $\Rightarrow$ memoryless property $\Rightarrow$ exponential.
   - Markov Property $\Rightarrow$ the parameter of the exponential can only depend on the current state i.
2. When it jumps, where it jumps to.
   - The continuous-time Markov Chain will jump according to a transition probability q_{ij}, which only depends on i and j
   - **Reason**:

- Markov property $\Rightarrow$ given i, $\underbrace{q_{ij}}_{future}$ can not depend on anything else.
- $q_{ij} = \mathbb{P}(X(t) \text{ jumps to } j | X(t) \text{ jumps from } j) \Rightarrow$
$\begin{cases} q_{ij} = 0 & q_{ij} \geq 0, j \not= i \\ \sum_{j \in S} q_{ij} = \sum_{j \in S, j \not= i} & q_{ij} = 1 \end{cases}$
- Define $Q = \{q_{ij}\}_{i,j \in S}$
$Q = \begin{bmatrix} 0 & q_{01} & q_{02} & \cdots \\ q_{10} & 0 & q_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$
- the row sums of Q are 1

A CTMC is fully characterized by $\{\lambda_i\}_{i \in S}$ and $Q = \{q_{ij}\}_{i,j \in S}$

To conclude, a CTMC stays in a state i for an exponential random time T_i; then jumps to another state j with probability q_{ij}, then stays in j for an exponential random time T_j, $\cdots$, all the jumps and times spent in different states are independent.