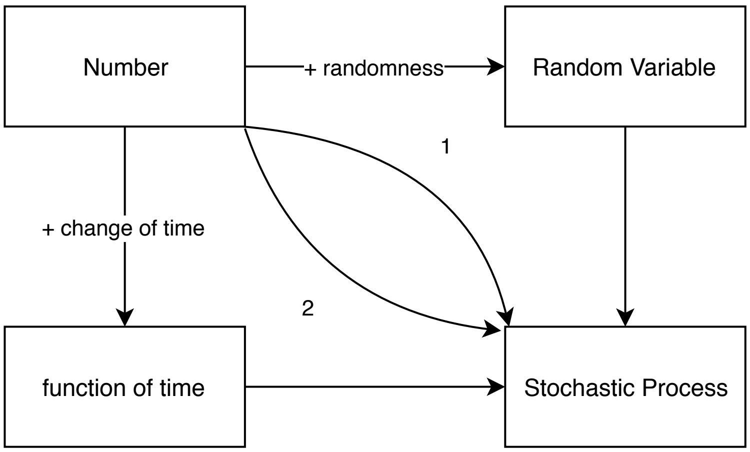
Note 08 - Jan 31

4. Stochastic Processes



1. sequence / family of random variables
2. a random function (hard to formulate)

Definition: A **stochastic process** $\{X_t\}_{t\in T}$ is a collection of random variables, defined on a common probability space.

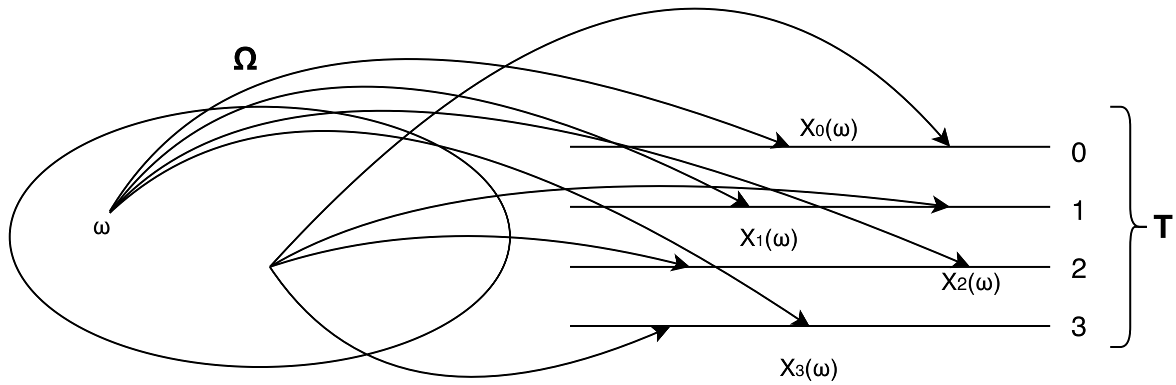
T: index set. In most cases, T corresponds to time, and is either discrete $\{0, 1, 2, \cdots\}$ or continuous $[0, \infty)$

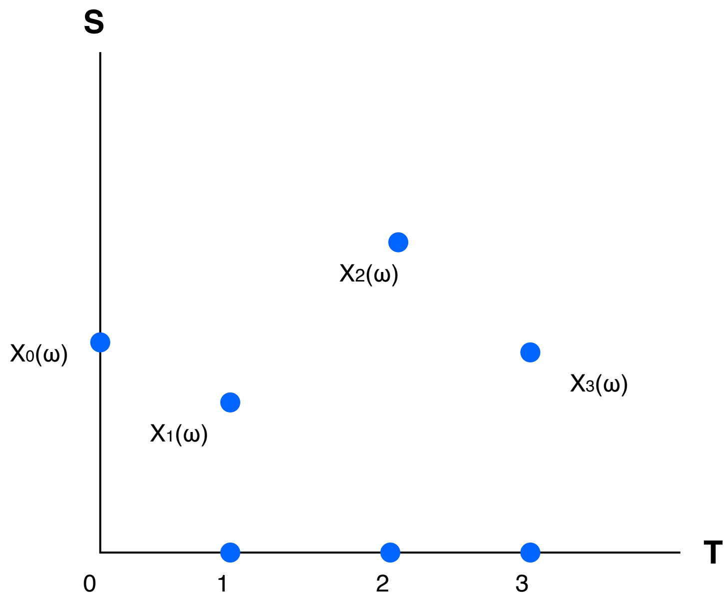
In discrete case, we writes $\{X_n\}_{n=0,1,2,...}$

This **state space** S os a stochastic process is the set of all possible value of $X_t, t \in T$

S can also be either discrete or continuous. In this course, we typically deal with **discrete** stat space. Then we relabel the stats so that $S = \{0, 1, 2, \cdots\}$ (countable state space) or $S = \{0, 1, 2, \cdots, M\}$ (finite state space)

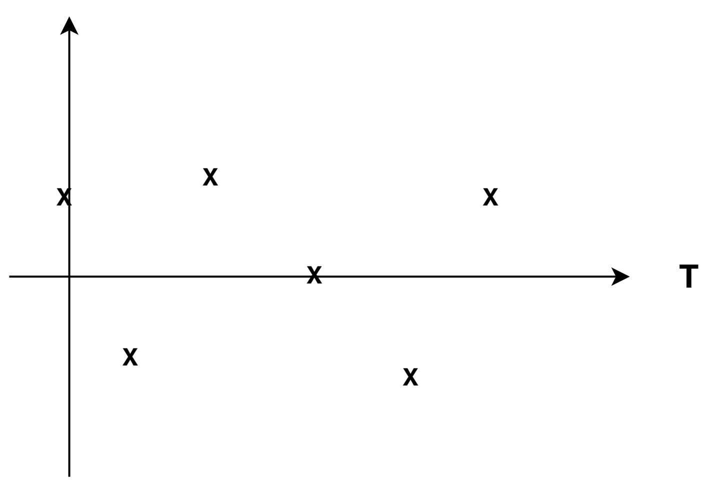
Remark: As in the case of the joint distribution, we need the r.v's in a stochastic process to be defined on a common probability space, because we want to discuss their joint behaviours, i.t, how things change over time.





Thus, we can identify each point ω in the sample space Ω with a function defined on T and taking value in S. Each function is called a **path** of this stochastic process

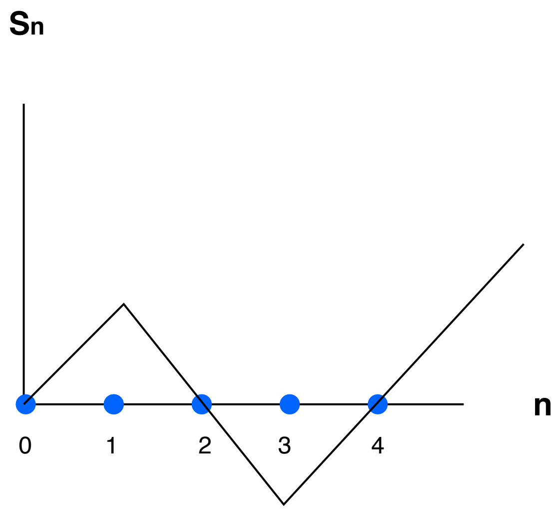
Example Let $X_0, X_1, \cdots$ be independent and identical r.v's following some distribution. Then $\{X_n\}_{n=0,1,2,\ldots}$ is a stochastic process



Example Let $X_1, X_2, ...$ be independent and identical r.v.'s. $P(X_1 = 1) = p$, and $P(X_1 = -1) = 1 - p$. Define $S_0 = 0, S_n = \sum_{i=1}^{n} X_i, n \leq 1$, e.g.

- $S_0 = 0$
- $S_1 = X_1$
- $S_2 = X_1 + X_2$
- $\cdots\cdots$

Then $\{S_n\}_{n=0,1,...}$ is a stochastic process, with state space $S = \mathbb{Z}$ (integer)



$\{S_n\}_{n=0,1,...}$ is called a **"simple random walk"**. ($S_n = S_{n-1} + X_n$)

$$S_n = \begin{cases} S_{n-1} + 1 \\ S_{n-1} - 1 \end{cases}$$

Remark: Why we need the concept of "stochastic process"? Why don't we just look at the joint distribution of $(X_0, X_1, ..., X_n)$?

Answer: The joint distribution of a large number of r.v's is very complicated, because it does not take advantage of the special structure of T(time).

For example, simple random walk. The full distribution of $(S_0, S_1, ..., S_n$ is complicated or n large. However, the structure is actually simple if we focus on the adjacent times:

$$S_{n+1} = S_n + X_{n+1}$$
$$S_n: \text{ last value.} \quad X_{n+1}: \text{related to } Ber(p). \text{ They are independent}$$

By introducing time into the framework, we can greatly simplify many things.

More precisely, we fine that for simple random walk, $\{S_n\}_{n=0,1,...}$, if we know S_n the distribution of $S_n + 1$ will not depend on the history $(S_0, ..., S_n - 1$. THis is a very useful property

In general for a stochastic process $\{X_n\}_{n=0,1,...}$, at time n, we already know $X_0, X_1, ..., X_n$, S_0 our best estimate of the distribution of X_{n+1} should be the conditional distribution:

$$X_{n+1}|X_n, ..., X_n$$

given by:

$$P(X_{n+1} = x_{n+1}|X_n = x_n, ..., X_0 = x_0)$$

As time passes, the expression becomes more and more complicated $\rightarrow$ impossible to handle.

However, if we know that this conditional distribution is actually the same as the conditional distribution only given X_n, then the structure will remain simple for any time. This motivates the notion of *Markov chain*.