# Note 19 - Mar 21

# 6. Continuous-Time Markov Chain (cont'd)

## 6.1. Definitions and Structures (cont'd)

### Definition 6.1.1. Continuous-time Stochastic Process (cont'd)

**Example 6.1.1.1.**

We have seen that the Poisson process satisfy the continuous-time Markov property $\Rightarrow$ it is a CTMC

$$\lambda_i = \lambda \qquad i \in S = \{0, 1, 2, \cdots\}$$

(interarrival times do not depend on current state. $\Rightarrow$ time spent in different states are i.i.d.)

$$q_{i,i+i} = 1 \qquad q_{ij} = 0 \text{ otherwise} (j \not= i+1)$$

## 6.2. Generator Matrix

Similar to the discrete-time case, we have the transition probability at time $t$.

$$\begin{aligned} P_{ij}(t) &= \mathbb{P}(X(t) = j | X(0) = i) \\ &= \mathbb{P}(X(t+s) = j | X(s) = i) \qquad \text{assume the MC is time-homogeneous} \end{aligned}$$

and matrix

$$P(t) = \{P_{ij}(t)\}_{i,j \in S}$$

$$P(t) = \begin{pmatrix} p_{00}(t) & p_{01}(t) & \cdots \\ p_{10}(t) & p_{11}(t) & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

The C-K equation still holds

$$p(t+s) = P(t) \cdot P(s)$$

**Proof**: $\forall i, j \in S$

$$\begin{aligned}
P_{ij}(t+s) &= \mathbb{P}(X(t+s)=j|X(0)=i) \\
&= \sum_{k\in S}\mathbb{P}(X(t+s)=j|X(t)=k, \cancel{X(0)=i})\cdot\mathbb{P}(X(t)=k|X(0)=i) \\
&= \sum_{k\in S}\mathbb{P}(X(s)+j|X(o)=k)\cdot\mathbb{P}(X(t)=k|X(0)=i) \\
&= \sum_{k\in S}P_{kj}(s)\cdot P_{ik}(t) \\
&= (P(t)\cdot P(s))_{ij} \qquad \blacksquare
\end{aligned}$$

Note that we have

$$P(0)=I$$

$$(P_{ii}(0)=\mathbb{P}(X(0)=i|X(0)=i)=1, P_{ij}(0)=0 \text{ for } j\not=i)$$

and

$$\lim_{t\to 0^+}P(t)=I$$

Actually, we have the following stronger result:

$$R := \lim_{h\to 0^+}\frac{P(h)-P(0)}{h}=\lim_{h\to 0^+}\frac{P(h)-I}{h}$$

exists, and is called the **(infinitesimal) *generator* matrix of** $\{X(t)\}_{t\geq 0}$

Entry-wise:

$$R_{ij}=\lim_{h\to 0^+}\frac{P_{ij}(h)-P_{ij}(0)}{h}=\begin{cases}\lim_{h\to 0^+}\frac{P_{ii}(n)-1}{h}\leq 0 & j=i\\ \lim_{h\to 0^+}\frac{P_{ii}(n)}{h}\geq 0 & j\not=i\end{cases}$$

Relation between $R$ and $\{\lambda_i\}_{i\in S}$ and $Q=\{q_{ij}\}_{i,j\in S}$

$$R_{ii}=-\lambda_i \quad , \quad R_{ij}=\lambda_i q_{ij} \qquad j\not=i$$

**Reason**:

$$R_{ii}=\lim_{h\to 0^+}\frac{P_{ii}(h)-1}{h}=\lim_{h\to 0^+}\frac{\mathbb{P}(T_i>h)-1}{h}$$

Where $T_i$ is the random time the process stays in $i$. The equality holds because when $h$ is very small, the probability of having two or more jumps in time $h$ is negligible.

$$\mathbb{P}(X(h)=i|X(0)=i)=\mathbb{P}(T_i>h)+o(h)\leftarrow \text{having at least 2 jumps and back to } i$$

$$*a(h)=o(h) \text{ if } \lim_{h\to 0}\frac{a(h)}{h}=0$$