Note 17 - Mar 14

Review

Branching Processes

$\mathbb{E}(Y) \le 1 \quad \Rightarrow$ case 2. Expectation probability $= 1$

$\mathbb{E}(Y) > 1 \quad \Rightarrow$ case 1. Expectation probability is the smallest solution of $\psi(s) = s$ between 0 and 1

Poisson Processes

Definition

$$0 \le S_1 \le S_2 \le \cdots$$

$$N(t) = \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \le t\}} \text{ is the counting process of events } \{S_n\}_{n=1,2,\ldots}$$

Properties of a counting process

1. $N(t) \ge 0, t \ge 0$
2. $N(t)$ takes integer values
3. $N(t)$ is increasing.
    - $N(t_1) \le N(t_2)$ if $t_1 \le t_2$
4. $N(t)$ is right-continuous
    - $N(t) = \lim_{s \downarrow t} N(s)$

Additional asumption: $N(0) = 0, N(t)$ only has jumps with size 1

5 Poisson Processes (cont'd)

Recall : Properties of the Exponential Distributions

$X \sim Exp(\lambda)$

- Basic properties
    - pdf: $f(x) = \lambda e^{-\lambda x} \quad (x > 0)$
    - cdf: $F(x) = 1 - e^{-\lambda x}$
    - $\mathbb{E}(x) = \frac{1}{\lambda}$
    - $Var(X) = \frac{1}{\lambda^2}$
- Memoryless property

- $\mathbb{P}(X > s+t | x > s) = \mathbb{P}x > t$
- Min ofexponentials
  - $X_1, ..., X_n$ independent, $X_i \sim Exp(\lambda_i)$, then
    1. $min(X_1, \cdots, X_n) \sim Exp(\lambda_1 + \cdots + \lambda_n)$
       Proof: it suffices to prove the result for $n = 2$

$$\text{Let } Z = min(X_1, X_2), \text{ then}$$

$$\begin{aligned}\mathbb{P}(Z > z) &= \mathbb{P}(X_1 > z, X_2 > z) \\ &= \mathbb{P}(X_1 > z) \cdot \mathbb{P}(X_2 > z) \\ &= e^{-\lambda_1 z} \cdot x^{-\lambda_2 z} \\ &= e^{-(\lambda_1 + \lambda_2) z}\end{aligned}$$

$$\Rightarrow \mathbb{P}(Z \leq z) = \underbrace{1 - e^{-(\lambda_1+\lambda_2)z}}_{\text{cdf of } Exp(\lambda_1+\lambda_2)} \qquad z > 0$$

$$Z \sim Exp(\lambda_1 + \lambda_2)$$

    2. $\mathbb{P}(X_i = min(X_1, \cdots, X_2)) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$
       Proof: (again for $n = 2$)

$$\begin{aligned}&\mathbb{P}(X_1 = min(X_1, X_2)) \\ &= \mathbb{P}(X_1 \leq X_2) \\ &= \mathbb{E}(\mathbb{P}(X_1 \leq X_2 | X_1)) \\ &= \mathbb{E}(e^{-\lambda_2 X_1}) \\ &= \int_0^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx \\ &= \lambda_1 \int_0^\infty e^{-(\lambda_1+\lambda_2)x} dx \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2}\end{aligned}$$

5.3. Properties of Poisson Processes

5.3.1. Continuous-time Markov Property

$$\begin{aligned}&\mathbb{P}(N(t_m) = j | N(t_{m-1}) = i, N(t_{m-2}) = i_{m-2}, \cdots, N(t_1) = i_1) \\ &\mathbb{P}(N(t_m) = j | N(t_{m-1} = i))\end{aligned}$$

for any m, $t_1 < \cdots < t_m$, $i_1, i_2, \cdots, i_{m-2}, i, j \in S$

Fact 5.3.1.1

The Poisson Process is the only renewal process having the Markov Property

Reason:

Since the exponential distribution is memoryless, the future arrival times will not depend on how long we have waited $\Rightarrow$ The future of the counting process only depends on its current value.

In fact,

$$\begin{aligned} & \mathbb{P}(N(t+s)=j|N(s)+j) \\ \text{time homogeneity} &= \mathbb{P}(N(t)=j|N(0)=i) \quad \text{only difference by which number we start ti ciybt} \\ &= \mathbb{P}(N(t)=j-i|N(0)=0) \end{aligned}$$

5.3.1.1. Independent Increments

The Poisson Process has independent increments

$$t_1 < t_2 < t_3 < t_4 \Rightarrow \underbrace{N(t_2)-N(t_1)}_{\text{increments}} \perp\!\!\!\perp \underbrace{N(t_4)-N(t_3)}_{\text{increments}}$$

Reasons:

Memoryless property of exponential distribution.

5.3.1.2. Poisson Increments

The Poisson Process has Poisson increments

$$N(t_2)-N(t_1) \sim Poi(\lambda(t_2-t_1))$$

Reason:

Let the arrival times between t_1 and t_2 be $S_1, cdots, S_N$, where $N = N(t_2) - N(t_1)$. Then $W_1 = S_1 - t$. $W_2 = S_2 - S_1, \cdots$ are i.i.d r.v's with distribution $Exp(\lambda)$

$$\begin{aligned} N = n \Leftrightarrow & W_1 + W_2 + \cdots + W_n \leq t_2 - t_1 \\ & W_1 + W_2 + \cdots + W_n + W_{n+1} > t_2 - t_1 \end{aligned}$$

Fact 5.3.1.2

If $W_1, \cdots, W_n$ are i.i.d. r.v's following $Exp(\lambda)$, then $W_1 + \cdots + W_n \sim Erlang(n, \lambda)$(a special type of $Gamma$)

$$c.d.f : F(x) = 1 \sum_{k=1}^{n-1} \frac{1}{k!} e^{-\lambda x} (\lambda x)^k$$

Thus,

$$\begin{aligned}&\mathbb{P}(W_1 + W_2 + \cdots + W_n \leq t_2 - t_1) \\ &= 1 - \sum_{k=0}^{n-1} \frac{1}{k!} e^{-\lambda(t_2 - t_1)} (\lambda(t_2 - t_1))^k\end{aligned}$$

$$\begin{aligned}&\mathbb{P}(W_1 + W_2 + \cdots + W_n + W_{n+1} \leq t_2 - t_1) \\ &= 1 - \sum_{k=0}^{n} \frac{1}{k!} e^{-\lambda(t_2 - t_1)} (\lambda(t_2 - t_1))^k\end{aligned}$$

$$\begin{aligned}\mathbb{P}(N = n) &= \mathbb{P}(W_1 + \cdots + W_n \leq t_2 + t_1) - \mathbb{P}(W_1 + \cdots + W_{n+1} \leq t_2 - t_1) \\ &= \frac{1}{n!} e^{-\lambda(t_2 - t_1)} (\lambda(t_2 - t_1))^n\end{aligned}$$

In particular, $N(t) = N(t) - N(0) \sim Poi(\lambda t)$

$$\mathbb{E}(N(1)) = \lambda \quad \leftarrow \text{intensity: expected number of arrivals in one unit of time}$$

5.3.1.3. Combining and Thining of Poisson Process

Theorem:

$$\{N_1(t)\} \sim Poi(\lambda_1 t)$$

$$\{N_2(t)\} \sim Poi(\lambda_2 t)$$

$\{N_1(t)\}$ and $\{N_2(t)\}$ are independent

Let $N(t) = N_1(t) + N_2(t)$, then $\{N(t)\} \sim Poi((\lambda_1 + \lambda_2)t)$