Note 09 - Feb 05

4. Stochastic Processes (cont'd)

4.2 Markov Chain

4.2.1 Discrete-time Markov Chain

4.2.1.1 Definition and Examples

Definition: A discrete-time Stochastic process $\{X_n\}_{n=0,1,...}$ is called a **discrete-time Markov Chain (DTMC)**, if its state space S is discrete, and it has the Markov property:

$$\begin{aligned} &P(X_{n+1} = x_{n+1} | X_n = x_n, ..., X_o = x_o) \\ &= P(X_{n+1} = x_{n+1} | X_n = x_n) \end{aligned}$$

for all $n, x_0, ..., x_n, x_{n+1} \in S$

If $X_{n+1} | \{x_n = i\}$ does not change over time, $P(X_{n+1} = j | N_n = i) = P(X_1 = j | X_0 = i)$, then we call this Markov chain **time-homogeneous** (default setting for this course).

$$\begin{aligned} &P(X_{n+1} = x_{n+1} | X_n = x_n, ..., X_0 = x_0) & X_{n+1} = x_{n+1}\text{: future; } X_n = x_n\text{: present(state)} \\ &= P(X_{n+1} = x_{x+1} | X_n = x_n) & X_{n-1} = x_{n-1}, ..., X_0 = x_0\text{: past(history)} \end{aligned}$$

Intuition: Given the present state, the past and the future are independent. In other words, the future depends on the previous results only through the current state.

Example: simple random walk

The simple random walk $\{S_n\}_{n=0,1,...}$ is a Markov chain

Proof:

Recall that $S_{n+1} = S_n + X_{n+1}$

$$\begin{aligned} &P(S_{n+1} = s_{n+1} | S_n = s_n, ..., S_0 = s_0) \\ &= 0 \\ &= P(S_{n+1} = s_{n+1} | S_n = s_n)s \end{aligned}$$

if $s_{n+1} \neq s_n \pm$

$$\begin{aligned}
&P(S_{n+1} = s_n + 1 | S_n = s_n, ..., s_0 = 0) \\
&= P(X_{n+1} | S_n = s_n, ..., S_0 = 0) \\
&= P(X_{n+1} = 1) \qquad X_{n+1} \perp (X_1, ..., X_n) \text{ hence also } (S_0, ..., S_n)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&P(S_{n+1} = s_n + 1 | S_n = s_n) \\
&= P(X_{n+1} = 1 | S_n = s_n) \\
&= P(X_{n+1} = 1) \\
&\Rightarrow P(S_{n+1} | S_n = s_n, ..., S_0 = s_0)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&P(S_{n+1} = s_n - 1 | S_n = s_n, ..., S_0 = 0) \\
&= P(S_{n+1} = s_n - 1 | S_n = s_n) \\
&= P(X_{n+1} = -1) \\
&\Rightarrow \{S_n\}_{n=0,1,...} \text{ is a DTMC} \qquad \blacksquare
\end{aligned}$$

4.3 One-step transition probability matrix

For a time-homogeneous DTMC, define

$$\begin{aligned}
P_{ij} &= P(X_1 = j | X_0 = i) \\
&= P(X_{n+1} = j | X_n = i) \qquad n = 0, 1, ...
\end{aligned}$$

P_{ij}: one step transition probability

The collection of $P_{ij}, i, j \in S$ governs all the one-step transitions of the DTMC. Since it has two indices i and j; it naturally forms a matrix $P = \{P_{ij}\}_{i,k \in S}$, called the **(one-setp) transition (probability) matrix** or **transition matrix**

Property of a transition matrix $P = \{P_{ij}\}_{i,j \in S}$:

$$\begin{aligned}
&P_{ij} \geq 0 \quad \forall i, j \in S \\
&\sum_{j \in S} P_{ij} = 1 \quad \forall i \in S \quad \rightarrow \text{ the row some of } P \text{are all 1}
\end{aligned}$$

Reason:

$$\begin{aligned}
\sum_{j \in S} P_{ij} &= \sum_{j \in S} P(X_1 = j | X_0 = i) \\
&= P(X_1 \in S | X_o = i) \\
&= 1
\end{aligned}$$

Example 1 : simple random walk

There will be 3 cases:

$$\begin{aligned}
&P_{i,i+1} = P(S_1 = i+1|S_0 = i) = P(X_1 = 1) = p \\
&P_{i,i-1} = P(S_1 = i-1|S_0 = i) = P(X_1 = -1) = 1-p =: q \\
&P_{i,j} = 0 \qquad\qquad \text{for } j \neq i \pm 1
\end{aligned}$$

$$\Rightarrow (\text{infinite dimension})p = \begin{Bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & p & 0 & \dots & \dots & \dots \\ \dots & q & 0 & p & \dots & \dots & \dots \\ \dots & \dots & q & 0 & p & \dots & \dots \\ \dots & \dots & \dots & q & 0 & p & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{Bmatrix}$$

Example 2: Ehrenfest's urn

Two urns A, B, total M balls. Each time, pick one ball randomly(uniformly), and move it to the opposite urn.

$$X_n : \# \text{ of balls in } A \text{after step } n$$

$$S = \{0, 1, ..., M\}$$

$$\begin{aligned}
P_{ij} &= P(X_1 = j|X_0 = j) \qquad (i \text{ balls in } A,\, M-i \text{ balls in } B) \\
&= \begin{cases} i/M & j = i-1 \\ (M-i)/M & j = i+1 \\ 0 & j \neq i \pm 1 \end{cases}
\end{aligned}$$

$$p = \begin{Bmatrix} 0 & 1 & & & & & \\ 1/M & 0 & (M-1)/M & & & & \\ & 1/M & 0 & (M-1)/M & & & \\ & & 2/M & 0 & (M-2)/M & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & (M-1)/M & 0 & 1/M \\ & & & & & 1 & 0 \end{Bmatrix}$$

Example 3: Gambler's ruin

A gambler, each time wins 1 with probability p, losses 1 with probability $1-p=q$. Initial wealth $S_0 = a$; wealth at time n: S_n. The gambler leaves if $S_n = 0$ (loses all money) or $S_n = M > a$ (wins certain amount of money and gets satisfied)

This is a variant of the simple random walk, where we have absorbing barriers($P_{ii} = 1$) at 0 and M

$$S = \{0, ..., M\}$$

$$P_{ij} = \begin{cases} p & j = i+1, i = 1, ..., M-1 \\ q & j = i-1, i = 1, ..., M-1 \\ 1 & i = j = 0 \text{ or } i = j = M \\ 0 & \text{otherwise} \end{cases}$$

$$p = \left\{ \begin{matrix} 1 & 0 & ... & & & & \\ q & 0 & p & ... & & & \\ ... & q & 0 & p & ... & & \\ & ... & q & 0 & p & ... & \\ ... & ... & ... & ... & ... & ... & ... \\ & & & ... & q & 0 & p \\ & & & & ... & 0 & 1 \end{matrix} \right\}$$

Example 4: Bonus-Malus system

Insurance company has 4 premium levels: 1, 2, 3, 4

Let $X_n \in \{1, 2, 3, 4\}$ be the premium level for a customer at year n

$$Y_n \overset{iid}{\sim} Poi(\lambda): \ \#\text{ of claims at year } n$$

- If $Y_n = 0$ (no claims)
  - $X_{n+1} = max(X_{n-1}, 1)$
- If $Y_n > 0$
  - $X_{n+1} = min(X_n + Y_n, 4)$

Denote $a_k = P(Y_n = k), k = 0, 1, ...$

$$p = \left\{ \begin{matrix} a_0 & a_1 & a_2 & (1 - a_0 - a_1 - a_2) \\ a_0 & 0 & a_1 & (1 - a_0 - a_1) \\ 0 & a_0 & 0 & (1 - a_0) \\ 0 & 0 & a_0 & (1 - a_0) \end{matrix} \right\}$$