Note 23 - April 4

Review

$$\text{Birth rates: } \lambda_i = R_{i,i+1} \qquad i = 0, 1, \cdots$$
$$\text{Death rates: } \mu_i = R_{i,i-1} \qquad i = 1, 2, \cdots$$

$$R = \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \\ & & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} R_{ii} = -(\lambda_i + \mu_i) \quad i \geq 1 \\ R_{00} = \lambda_0 \end{array}$$

6. Continuous-Time Markov Chain (cont'd)

6.5. Birth and Death Processes (cont'd)

As we see, there are two main types of birth and death processes: **queueing system** and **population model**. THe key difference between them is that the birth rate in the queueing system is typically a constant (does not depend on the current state i), while the birth rate in population model increases as i increases.

6.5.1. Stationary Distribution of a Birth and Death Process

$$\begin{cases} \underline{\pi} \cdot R = \underline{0} & (1) \\ \underline{\pi} \cdot \mathbb{1} = \underline{1} & (2) \end{cases}$$

$$(\pi_0, \pi_1, \cdots) \cdot \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$(1) \Rightarrow \begin{array}{c} -\lambda_0\pi_0 + \mu_1\pi_1 = 0 \Rightarrow \pi_1 = \dfrac{\lambda_0}{\mu_1}\pi_0 \\ \lambda_0\pi_0 - (\lambda_1+\mu_1)\mu_1 + \mu_2\pi_2 = 0 \end{array}$$

Add this to the first equation, we have

$$-\lambda_1\pi_1 + \mu_2\pi_2 = 0 \Rightarrow \pi_2 = \frac{\lambda_1}{\mu_2}\pi_1$$

In general, adding the first i equations, we have

$$\begin{gathered}
-\lambda_0\pi_0 + \mu_1\pi_1 = 0 \\
\lambda_0\pi_0 - (\lambda_1 + \mu_1)\pi_1 + \mu_2\pi_2 = 0 \\
\vdots \\
\lambda_{i-2}\pi_{i-2} - (\lambda_{i-1} + \mu_{i-1} + \mu_i\pi_i) = 0 \\
-\lambda_{i-1}\pi_{i-1} + \mu_i\pi_i = 0
\end{gathered}$$

$$\begin{aligned}
\Rightarrow \pi_i &= \frac{\lambda_{i-1}}{\mu_i}\pi_{i-1} \\
&= \cdots \\
&= \frac{\lambda_0\lambda_1\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i}\pi_0
\end{aligned}$$

Use (2) to normalize

$$\begin{aligned}
1 = \sum_{n=0}^{\infty}\pi_n &= (1 + \sum_{n=1}^{\infty}\Pi_{j=1}^{n}\frac{\lambda_{j-1}}{\mu_j} \\
\Rightarrow \pi_0 &= \frac{1}{1 + \sum_{n=1}^{\infty}\Pi_{j=1}^{n}\frac{\lambda_{j-1}}{\mu_j}} \\
\pi_i &= \frac{\Pi_{j=1}^{i}\frac{\lambda_{j-1}}{\mu_j}}{1 + \sum_{n=1}^{\infty}\Pi_{j=1}^{n}\frac{\lambda_{j-1}}{\mu_j}}
\end{aligned}$$

Thus, a stationary distribution exists(the MC is positive recurrent, assuming irreducible) if and only if

$$\sum_{n=1}^{\infty}\Pi_{j=1}^{n}\frac{\lambda_{j-1}}{\mu_j} < \infty$$

Example 6.5.1.1. M/M/S Queue (cont'd)

$$\lambda_i = \lambda \qquad \mu_i = \begin{cases} i\mu & i \le s \\ s\mu & i > s \end{cases}$$

$$\sum_{n=1}^{\infty} \Pi_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j}$$
$$= \underbrace{\frac{\lambda}{\mu} + \frac{\lambda}{\mu} \cdot \frac{\lambda}{2\mu} + \cdots + \frac{\lambda}{\mu}\frac{\lambda}{2\mu} \cdots \frac{\lambda}{s\mu} + \frac{\lambda}{\mu}\frac{\lambda}{2\mu} \cdots (\frac{\lambda}{s\mu})^2 + \frac{\lambda}{\mu}\frac{\lambda}{2\mu} \cdots (\frac{\lambda}{s\mu})^3 + \cdots}_{\text{geometric series with ration } \frac{\lambda}{s\mu}}$$

$\Rightarrow$ The sum is finite if and only if $\lambda < s\mu$

$\Rightarrow$ the process $\{X(t)\}_{t\geq 0}$ is positive recurrent if and only if $\underbrace{\lambda}_{\text{arrival rate}} < \underbrace{s\mu}_{\text{maximal (total) service rate}}$

Example 6.5.1.2. Population Model (with immigration)

$$\lambda_i = i\lambda + \alpha \qquad \mu_i = i\mu$$

$$\sum_{n=1}^{\infty} \Pi_{j=1}^{n} \frac{\lambda_{j-1}}{\mu_j} = \sum_{n=1}^{\infty} \Pi_{j=1}^{n} \frac{(j-1)\lambda + \alpha}{j\mu}$$

$$\lim_{j\to\infty} \frac{(j-1)\lambda + \alpha}{j\mu} = \frac{\lambda}{\mu}$$

If $\lambda < \mu$, then $\quad \sum_{n=1}^{\infty} \Pi_{j=1}^{n} \frac{(j-1)\lambda+\alpha}{j\mu} < \infty$ by ratio test.

If $\lambda > \mu$, then $\quad \sum_{n=1}^{\infty} \Pi_{j=1}^{n} \frac{(j-1)\lambda+\alpha}{j\mu} = \infty$.

If $\lambda = \mu$, then $\quad \alpha \geq \lambda = \mu$, the ratio $\frac{(j-1)\lambda+\alpha}{j\mu} \geq 1$ for all j

$\Rightarrow$ the terms in the summation is non-decreasing

$\Rightarrow$ the $sum = \infty$

If $\lambda = \mu, \quad \alpha < \lambda = \mu$:

Raabe-Duhamel's test: (not required content)

$$L := \lim_{n\to\infty} n(\frac{a_n}{a_{n+1}} - 1) \begin{cases} > 1 & \text{converge} \\ < 1 & \text{diverge} \\ = 1 & \text{inconclusive} \end{cases}$$

Here:

$$
\begin{aligned}
L &= \lim_{n\to\infty} n(\frac{n\mu}{(n-1)\lambda+\alpha}) \\
&= \lim_{n\to\infty} n(\frac{n\lambda-(n-1)\lambda-\alpha}{(n-1)\lambda+\alpha}) \\
&= \lim_{n\to\infty} n(\frac{\lambda-\alpha}{(n-1)\lambda+\alpha}) \\
&= \frac{\lambda-\alpha}{\lambda} < 1
\end{aligned}
$$

$\Rightarrow$ the sum $= \infty$

Conclusion

To sum up, the CTMC is positive recurrent if and only if $\lambda < \mu$

Q: What happens if $\lambda_0 = 0$? (0 is absorbing)

A:

The chain is not irreducible; typically two classes:

- $\{0\}$ positive recurrent
- $\{1, 2, \cdots\}$ transient

But the chain does not necessarily end up with being in state 0, because it can also have $X(t) \to \infty$. Whether this is a possibility depends on the relation between $\{\lambda_i\}$ and $\{\mu_i\}$.