Note 20 - Mar 26

6. Continuous-Time Markov Chain (cont'd)

6.1. Definitions and Structures (cont'd)

6.2. Generator Matrix (cont'd)

$$
\begin{aligned}
R_{ii} &= \lim_{h\to 0^+} \frac{\mathbb{P}(\overbrace{T_i}^{Exp(\lambda_i)} > h)}{h} \\
&= \lim_{h\to 0^+} \frac{e^{-\lambda_i h} - 1}{j} \\
&= \lim_{h\to 0^+} \frac{e^{-\lambda_i h} - e^{-\lambda_i \cdot 0}}{j} \\
&= \frac{de^{-\lambda_i}}{dh}\Big|_{h=0} \\
&= -\lambda_i e^{-\lambda_i h}|_{h=0} = -\lambda_i \\
R_{ij} = \lim_{h\to 0^+} \frac{P_{ij}(h)}{h} &= \lim_{h\to 0^+} \frac{\mathbb{P}(X(h) = j | X(0) = i)}{h} \\
\text{Only one jump happens two } &= \lim_{h\to 0^+} (\frac{\mathbb{P}(T_i < h, X(T_i) = j)}{h} \\
\text{jumps happen} \to \text{ uegligeable} &= q_{ij} \lim_{h\to 0^+} \frac{\mathbb{P}(T_i < h)}{h} \\
&= q_{ij} \lim_{h\to 0^+} \frac{1 - e^{-\lambda_i h}}{h} \\
&= q_{ij} \lim_{h\to 0^+} \frac{e^{-\lambda_i 0} - e^{-\lambda_i h}}{h} \\
&= q_{ij} (\frac{-de^{-\lambda_i h}}{dh}\Big|_{h=0}) \\
&= q_{ij}\lambda_i
\end{aligned}
$$

Thus, we conclude that

$$R_{ii} = -\lambda_i, \quad R_{ij} = \lambda_i q_{ij]}$$

Note that $R_{ii} \leq 0$, $R_{ij} \geq 0$, $j \neq i$

$$
\begin{aligned}
\sum_{i\in S} R_{ij} &= R_{ii} + \sum_{i\in S, j\neq i} R_{ij} \\
&= -\lambda_i + \sum_{i\in S, j\neq i} \lambda_i q_{ij} \quad \leftarrow \sum_{j\in S, j\neq i} q_{ij} = 1 \\
&= -\lambda_i + \lambda_i \\
&= 0
\end{aligned}
$$

The row sums of R are 0

R=\begin{pmatrix} -\lambda_0 & \lambda_0q_{01} & \lambda_0q_{02} & \cdots \\ -\lambda_1q_{10} & -\lambda_1 & \lambda_1q_{12} & \cdots \\ -\lambda_2q_{20} & \lambda_2q_{21} & -\lambda_2 & \cdots \\ \vdots &\vdots&\vdots&\ddots \end{pmatrix} $$-\lambda_0 $-\lambda_i$: probability flow / rate going out of state i λ_iq_{ij} : probability flow / rate going from i to j From R to $\{\lambda_i\}$ and Q: $\quad$ If we know R, then $$ \lambda_i=-R_{ii}

$$q_{ij} = \frac{-R_{ij}}{R_{ii}} \quad j\neq i$$

Thus, there is a $1-1$ relation between $\{\lambda_i\}_{i\in S} + \{q_{ij}\}_{i,j\in S}$ and $R = \{R_{ij}\}_{i,j\in S}$

$\Rightarrow$ the generator R itself also **fully characterizes** the transitional behaviour of the CDMC.

Conclusion: $\{\lambda_i\} + \{q_{ij}\}_{i,j\in S}$ and $\{R_{ij}\}_{i,j\in S}$ are two sets of parameters that can be used to specify a CDMC.

Example 6.2.1. Poisson Process

$$
\begin{gathered}
\lambda_i = \lambda \\
Q = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & & \\ & & & \ddots & \ddots \end{pmatrix}
\end{gathered}
$$

$$
\begin{aligned}
\Rightarrow R_{ii} &= -\lambda_i = -\lambda \qquad i = 0, 1, \ldots \\
R_{ij} &= \lambda_i q_{ij} = \begin{cases} \lambda & j = i+1 \\ 0 & j\neq i, i+1 \end{cases} \\
R &= \begin{pmatrix} -\lambda & \lambda & & \\ & -\lambda & \lambda & \\ & & & \\ & & \ddots & \ddots \end{pmatrix}
\end{aligned}
$$

Example 6.2.2. 3-tables in a restaurant

Parties of customers arrive according to a Poisson Process with intensity λ.

If there are free tables $\rightarrow$ the party is served, and spend an exponential amount of time with average $\frac{1}{\mu}$ (parameter μ)

If there is no free table $\rightarrow$ the party leaves immediately

Let $X(t)$ be the number of occupied tables at time $t \Rightarrow S = \{0, 1, 2, 3\}$

Since all the interarrival times and service times are exponential and independent, the process $\{X(t)\}$ is a CTMC.

Find λ_i and q_{ij}:

For $i = 0$:

- $q_{01} = 1$ no customer $\rightarrow$ one table occupied
- $\lambda_0 = \lambda$ leave state 0 $\Leftrightarrow$ one party arrives
    - $q_{02} = q_{03} = 0$

For $i = 1$:

- Potential change of states $\begin{cases} 1 \rightarrow 2 & \text{if one party arrives first} & \sim Exp(\lambda) \sim T \\ 1 \rightarrow 0 & \text{if a service is completed first} \sim Exp(\mu) \sim S \end{cases}$
- Which one actually happens depends on which time is smaller.

Recall a property of exponential

$$min(\underbrace{T}_{Exp(\lambda)}, \underbrace{S}_{Exp(\mu)}) \sim Exp(\lambda + \mu) \leftarrow \text{the distribution of the time spent in the current state}$$

$$\begin{aligned} \mathbb{P}(T < S) &= \frac{\lambda}{\lambda + \mu} \\ q_{12} = \mathbb{P}(T < S)) &= \frac{\lambda}{\lambda + \mu} \\ q_{10} = 1 - q_{12} &= \frac{\mu}{\lambda + \mu} \\ q_{13} &= 0 \end{aligned}$$

Thus,

- $\lambda_1 = \lambda + \mu$
- $q_{12} = \mathbb{P}(T < S) = \frac{\lambda}{\lambda + \mu}$
- $q_{10} = 1 - q_{12} = \frac{\mu}{\lambda + \mu}$
- $q_{13} = 0$

Similarly, when 2 tables are occupied.

- T: time until next arrival $\sim Exp(\lambda)$
- S_1: service time for table 1 $\sim Exp(\mu)$
- S_2: service time for table 2 $\sim Exp(\mu)$

$\Rightarrow min(T, S_1, S_2) \sim Exp(\lambda + 2\mu)$

$$\mathbb{P}(\underbrace{T < S_1, S_2}_{2\rightarrow 3}) = \frac{\lambda}{\lambda + 2\mu}$$

Thus,

- $\lambda_2 = \lambda + 2\mu$
- $q_{23} = \frac{\lambda}{\lambda+\mu}, \quad q_{21} = 1 - q_{23} = \frac{2\mu}{\lambda+2\mu}$
- $q_{20} = 0$

Finally,

- $\lambda_3 = 3\mu \rightarrow$ one service is completed.
- $q_{32} = 1$
- $q_{31} = q_{30} = 0$

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 \\ 0 & \frac{2\mu}{\lambda+2\mu} & 0 & \frac{\lambda}{\lambda+2\mu} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- $\lambda_0 = \lambda$
- $\lambda_1 = \lambda + \mu$
- $\lambda_2 = \lambda + 2\mu$
- $\lambda_3 = 3\mu$

$$R_{ii} = -\lambda_1, \quad R_{ij} = \lambda_i q_{ij}$$

$$R = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \mu \\ 0 & 0 & 3\mu & -3\mu \end{pmatrix}$$

We see that R was simpler form than Q. Actually $R_{ij} \quad i \neq j$ directly corresponds to the "rate" by which the process moves from state i to j. Thus, in practice, we often directly model R rather than getting $\{\lambda_i\}_{i \in S}$ and $Q = \{q_{ij}\}_{i,j \in S}$