Note 14 - Mar 05

Review

Period is a class property : $i \leftrightarrow \Rightarrow d_i = d_j$. Irreducible $\rightarrow $ period of the MC

Basic Limit Theorem

$\{X_n\}_{n=0,1,\dots}$ irreducible, aperiodic, positive recurrent DTMC; then an unique stationary distribution $\underline{\pi} = (\pi_0, \pi_1, \dots)$ exists. Moreover,

$$(*) \quad \underbrace{lim_{n-\to\infty} P_{ij}^{(n)}}_{\substack{\text{limiting distribution} \\ \text{(does not depend on the initial state i)}}} = lim_{n\to\infty} \underbrace{\frac{\sum_{k=1}^{n} \mathbb{1}_{\{X_k=j\}}}{n}}_{\text{long-run fraction of time spent in j}} = \frac{1}{\underbrace{\mathbb{E}(T_j|X_0=j)}_{\substack{T_j = min\{n>0: X_n = j\} \\ \text{expected revisit time}}}} = \pi_j \qquad , i, j \in S$$

Periodic extension:

$$\frac{\lim_{n\to\infty} P_{jj}^{(nd)}}{d} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mathbb{1}_{\{X_k=j\}}}{n} = \frac{1}{\mathbb{E}(T_j|X_0=j)} = \pi_j$$

Examples show:

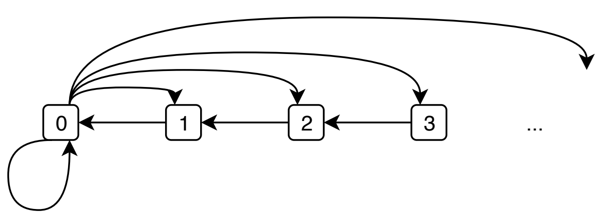
- Irreducibility is related to the uniqueness of the stationary distribution;
- Aperiodicity is related to the existence of the limiting distribution.

4. Stochastic Processes (cont'd)

4.5 Limiting Distribution (cont'd)

Example 4.5.3

$$P_{o,j} = p_j, j = 0, 1, \cdots, p_0 > 0$$
$$P_{i,i-1} = 1, i \geq 1$$



Given $X_0 = 0, T_0 = n + 1$ if and only if $X_1 = n$. his happens with prob p_n.

$$\Rightarrow \mathbb{E}(T_0|X_0 = 0) = \sum_{n=0}^{\infty}(n+1)p_n$$
$$= 1 + \sum_{n=0}^{\infty} np_n$$

We can construct p_n such that $\sum_{n=0}^{\infty} np_n = \infty$. (For example, $p_0 = \frac{1}{2}, p_2 = \frac{1}{4}, p_4 = \frac{1}{4}, \cdots$)

In this case, the chain is **null recurrent**. It is irreducible and aperiodic ($P_{00} = p_0 > 0$)

A stationary distribution does not exist. Reason:

$$p = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_i & \cdots \\ 1 & & 0 & & & \\ & 1 & & & & \\ \cdots & & & & & \\ & & & & 1 & \end{pmatrix}$$
$$\underline{\pi} \cdot P = \underline{\pi} \Rightarrow$$

$$\begin{aligned} p_0\pi_0 + \pi_1 &= \pi_0 \\ p_1\pi_0 + \pi_2 &= \pi_1 \\ &\vdots \\ p_{i-1}\pi_0 + \pi_i &= \pi_{i-1} \\ p_i\pi_0 + \pi_{i+1} &= \pi_I \end{aligned}$$

Add the first i equations:

$$(p_0 + \cdots + p_{i-1})\pi_0 + (\cancel{\pi_1} + \cancel{\pi_2} + \cancel{\cdots} + \pi_i) = \pi_0 + \cancel{\cdots} + \cancel{\pi_{i-1}}$$
$$(p_0 + \cdots + p_{i-1})\pi_0 + \pi_i = \pi_0$$
$$\Rightarrow \pi_i = (1 - (p_o + \cdots + p_{i-1}))\pi_0 = \sum_{k=i}^{\infty} p_k\pi_0$$

Try to normalize:

$$\begin{aligned} 1 &= \sum_{i=1}^{\infty} \pi_i \\ &= \sum_{i=0}^{\infty} \sum_{k_i}^{\infty} p_k\pi_0 \\ &= \sum_{k_i}^{\infty} \sum_{i=0}^{\infty} p_k\pi_0 \\ &= \sum_{k_i}^{\infty} p_k \sum_{i=0}^{\infty} \pi_0 \\ &= \underbrace{(\sum_{k_i}^{\infty}(k+1)p_k)}_{\infty}\pi_0 \end{aligned}$$
$$\Rightarrow \pi_0 = 0 \quad , \quad pi_i = 0 \quad \forall i$$

This is not a distribution. Thus, a stationary distribution does not exist.

positive recurrence is related to the existence of the stationary distribution

Example 4.5.4 : Electron

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \quad \alpha, \beta \in (0,1)$$

Irreducible, aperiodic, positive recurrence.

In order to find of P^n; we use the diagonalization technique.

$$P = Q\Lambda Q^{-1} \quad \text{where } \Lambda \text{ is diagonal}$$
$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{pmatrix} \quad Q = \begin{pmatrix} 1 & \alpha \\ 1 & 1-\beta \end{pmatrix} \quad Q^{-1} = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix}$$

Then

$$\begin{aligned} P^n &= (Q\Lambda \cancel{Q}^{-1})(\cancel{Q}\Lambda\cancel{Q}^{-1})\cdot(\cancel{Q}\Lambda Q^{-1}) \\ &= Q\Lambda^n Q^{n-1} \\ &= \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix}\begin{pmatrix} 1 & \\ & (1-\alpha-\beta)^n \end{pmatrix}\frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{\alpha+\beta}\begin{pmatrix} \beta+\alpha(1-\alpha-\beta)^n & \alpha-\alpha(1-\alpha-\beta)^n \\ \beta-\beta(1-\alpha-\beta)^n & \alpha+\beta(1-\alpha-\beta)^n \end{pmatrix} \\ &\Rightarrow \lim_{n\to\infty} P^n = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix} \end{aligned}$$

Note that $\lim_{n\to\infty} P^n$ has identical rows. This corresponds to the result that $\lim_{n\to\infty} P_{ij}^{(n)}$ does not depend on i

We saw that the stationary distribution $\underline{\pi} = (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$. So we verity that $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$

Also, given $X_0 = 0$, $\mathbb{P}(T_0 = 1 | X_0 = 0) = 1-\alpha$.

For $k = 2, 3, \cdots$

$$
\begin{aligned}
\mathbb{P}(T_0=k|X_0=0) &= \mathbb{P}(X_k=0, X_{k-1}=1,\cdots,X_1=1|X_0=0)\\
&= \alpha(1-\beta)^{k-2}\beta\\
&\Rightarrow \mathbb{E}(T_0|X_0=0)\\
&= 1\cdot(1-\alpha)+\sum_{k=2}^{\infty}\alpha(1-\beta)^{k-2}\beta k\\
&= 1-\alpha+\sum_{k=1}^{\infty}\underbrace{\alpha(1-\beta)^{k-2}\beta(k-1)}_{\mathbb{E}(Geo(\beta))}+\sum_{k=2}^{\infty}\alpha\underbrace{(1-\beta)^{k-2}\beta}_{\text{pmf of Geo}(\beta)}\\
&= 1-\alpha+\alpha\sum_{k=1}^{\infty}(1-\beta)^{k-2}\beta(k-1)+\sum_{k=2}^{\infty}\alpha(1-\beta)^{k-2}\beta\\
&= 1\alpha+\alpha\cdot\frac{1}{\beta}+\alpha\cdot 1\\
&= 1-\cancel{\alpha}+\frac{\alpha}{\beta}+\cancel{\alpha}\\
&= \frac{\alpha+\beta}{\beta}
\end{aligned}
$$

Hence we verify that $\mathbb{E}(T_0|X_0=0)=\frac{1}{\pi_0}$

4.6 Generating function and branching processes

Definition 4.6.1

Let $\underline{p}=(p_0,p_1,\cdots)$ be a distribution on $\{0,1,2,\cdots\}$. Let ξ be a r.v. following distribution $\underline{p}$. That is $\mathbb{P}(\xi=i)=p_i$. Then the generating function of ξ, or of $\underline{p}$, is defined by

$$
\begin{aligned}
\psi(s) &= \mathbb{E}(s^{\xi})\\
&= \sum_{k=0}^{\infty}p_k s^k \qquad for 0\le s\le 1
\end{aligned}
$$

Properties of generating function

1. $\psi(0)=p_0, \quad \psi(1)=\sum_{k=0}^{\infty}p_k=1$
2. Generating function determines the distribution

$$p_k=\frac{1}{k!}\frac{d^k\psi(s)}{ds^k}|_{s=0}$$

Reason:

$$\psi(s)=p_0+p_1s^1+\cdots+p_{k-1}s^{k-1}+p_ks^k+p_{k+1}s^{k+1}+\cdots$$

$$\frac{d^k\psi(s)}{ds^k}=k!p_k+(\cdots)s+(\cdots)s^2+\cdots$$

$$\frac{d^k\psi(s)}{ds^k}|_{s=0}=k!p_k$$

In particular, $p_1\ge 0\Rightarrow\psi(s)$ is increasing. $p_2\ge 0\Rightarrow\psi(s)$ is climax