# Note 22 - April 2

# 6. Continuous-Time Markov Chain (cont'd)

## 6.5. Birth and Death Processes (cont'd)

For a birth and death process, we use a new set of parameters:

Denote

$$
\begin{aligned}
\lambda_i &= R_{i,i+1} \qquad i = 0, 1, \cdots \\
\mu_i &= R_{i,i-1} \qquad i = 1, 2, \cdots
\end{aligned}
$$

$$
R = \begin{pmatrix}
-\lambda_0 & \lambda_0 & & & \\
\mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & & \\
 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \\
 & & \mu_3 & -(\lambda_3+\mu_3) & \lambda_3 \\
 & & & & \\
 & & \ddots & \ddots & \ddots
\end{pmatrix}
$$

$$
\Rightarrow \begin{aligned}
R_{ii} &= -(\lambda_i + \mu_i) \quad i \geq 1 \\
R_{00} &= \lambda_0
\end{aligned}
$$

$\lambda_i$'s are called the "birth rates" ($population + 1$)

$\mu_i$'s are called "death rates" ($population - 1$)

### Example 6.5.1. Previous Example of Restaurant

$$
R = \begin{pmatrix}
-\lambda & \lambda & & \\
\mu & -(\mu+\lambda) & \lambda & \\
 & 2\mu & -(2\mu+\lambda) & \lambda \\
 & & 3\mu & -3\mu
\end{pmatrix}
$$

This is a birth and death process.

- Birth rates: $\lambda_i = \lambda \qquad i = 0, 1, 2, 3$
- Death rates: $\mu_i = i \cdot \mu \qquad i = 1, 2, 3$

In general, consider a M/M/S queueing system.

- M: exponential interarrival time
- M: service time is exponential $Exp(\mu)$ (each server)
- S: number of servers

$X(t) =$ # of customers in the system at time $t$

- Birth rate:
  - birth $\Leftrightarrow$ arrival
  - $\Rightarrow \lambda_1 = \lambda$
- Death rate:
  - death $\Leftrightarrow$ service done
  - When there are $i$ customers:
    - Case $1: i \leq s$
      - $i$ servers are busy. Each $\sim Exp(\mu)$
      - $\Rightarrow$ total death rate $\mu_i = i \cdot \mu$
    - Case $2: i > s$
      - $s$ servers are busy, $\mu_i = s \cdot \mu$

Thus, for M/M/S queue,

$$\lambda_i = \lambda$$
$$\mu_i = \begin{cases} i \cdot \mu & i \leq s \\ s \cdot \mu & i > s \end{cases}$$

## Example 6.5.2. A Population Model

Each individual gives birth to an offspring with exponential rate $\lambda$. (i.e. the waiting time until it gets the next offspring $\sim Exp(\lambda)$ ). Each individual dies with exponential rate $\mu$. Let $X(t)$ be the population size at time $t$.

The time until the next (potential) birth is the smallest amount the $i$ i.i.d. birth time. $\sim Exp(i \cdot \lambda)$

Thus the birth rate $\lambda_i = i \cdot \lambda \quad i = 0, 1, 2, \cdots$

Similarly, the death rate $\mu_i = i\mu \quad i = 1, 2, \cdots$

$$R = \begin{pmatrix} 0 & 0 & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & 2\mu & -2(\lambda+\mu) & 2\lambda & \\ & & \ddots & \ddots & \end{pmatrix}$$
$$* : 0 \text{ is absorbing}$$

If we do need to know $\{\lambda_i\}$ and $Q$

$$\lambda_i = -R_{ii} = i(\lambda+\mu) \quad i \geq 0$$
$$q_{i,i+1} = \frac{i\lambda}{i(\lambda+\mu)} = \frac{\lambda}{\lambda+\mu} \quad i \geq 1 \quad (*)$$
$$q_{i,i-1} = \frac{i\mu}{i(\lambda+\mu)} = \frac{\mu}{\lambda+\mu} \quad 1 \geq 1$$

$$(*) q_{0,1} = 1. \text{ arbitrary since } \lambda_0 = 0$$

$$Q = \begin{pmatrix} 0 & 1 & & & \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & & \\ & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \\ & & \ddots & & \ddots \end{pmatrix}$$

We can further add immigration to the system. Individuals are added to the population according to a Poisson process with intensity $\alpha$.

This will not change the "rate" from $i$ to $i-1$

The time until the next increase in population is now the minimum of two r.v's following $Exp(i\lambda)$ (births) and $Exp(\alpha)$ (immigration).

$\Rightarrow$ rate from $i$ to $i+1$ becomes $i\lambda + \alpha$

$$R = \begin{pmatrix} -\alpha & \alpha & & & \\ \mu & -(\lambda+\mu+\alpha) & \lambda+\alpha & & \\ & 2\mu & -(\lambda+2\mu+\alpha) & \lambda+\alpha & \\ & & \ddots & & \ddots \end{pmatrix}$$

$$\begin{aligned} (*) \quad \lambda_i &= R_{i,i+1} = i\lambda + \alpha \qquad & i = 0, 1, \cdots \\ \mu_i &= R_{i,i-1} = i\mu \qquad & i = 1, 2, \cdots \end{aligned}$$

$(*)$ not the "biological" birth rate, but the total rate by which the process goes from $i$ to $i+1$

$\{\lambda_i\}$ and $Q$ will change accordingly.

Note that state 0 is no longer absorbing due to the immigration.