Note 11 - Feb 26

4. Stochastic Processes (cont'd)

4.4. Classification of States (cont'd)

4.4.3 Equivalent classes and irreducibility (cont'd)

Definition 4.4.3.5 : Proposition

If an irreducible MC has a finite state space, then it is recurrent

Idea of proof

If the MC is transient, then with probability 1, each state has a last visit time. Finite states $\Rightarrow \exists$ a last visit time for all the states. As a result, the MC has nowhere to go after that time. $\Rightarrow$ Contradiction.

Remark 4.4.3.1

We can actually prove that the MC must be positive recurrent, if the state space is finite and the MC is irreducible.

Theorem 4.4.3.1

Periodicity is a class property: $i \leftrightarrow j \Rightarrow d_i = d_j$.

For an irreducible MC, its period is defined as the period of any state.

4.5 Limiting Distribution

In this part, we are interested in $lim_{n\to\infty} P_{ij}^{(n)}$ and $lim_{n\to\infty} P(X_u = i)$

To make things simple, we focus on the irreducible case.

Theorem 4.5.1 : Basic Limit Theorem

Let $\{X_n\}_{n=0,1,...}$ be an **irreducible, aperiodic, positive recurrent** DTMC. Then a unique stationary distribution:

$$\underline{\pi} = (\pi_0, \pi_1, \ldots) \text{ exits}$$

Moreover:

$$(*) \quad \underbrace{lim_{n-\to\infty} P_{ij}^{(n)}}_{\substack{\text{limiting distribution} \\ \text{(does not depend on the initial state i)}}} = lim_{n\to\infty} \underbrace{\frac{\sum_{k=1}^{n} 1\!\!\mathbb{L}_{\{X_k=j\}}}{n}}_{\text{long-run fraction of time spent in j}} = \underbrace{\frac{1}{\mathbb{E}(T_j|X_0=j)}}_{\substack{T_j = min\{n>0 : X_n = j\} \\ \text{expected revisite time}}} = \pi_j \qquad , i,j \in S$$

Limiting distribution =

- long-run fraction of time
- $1/$ expected revisit time
- stationary distribution

Remark 4.5.1

The result $(*)$ is still true if the MC is null recurrent, where all the terms are **0**, and $\underline{\pi}$ is no longer a distribution. (in other words, there does not exist a stationary distribution)

Remark 4.5.2

If $\{X_n\}_{n=0,1,...}$ has a period $d > 1$:

$$\frac{\lim_{n\to\infty} P_{jj}^{(nd)}}{d} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mathbb{1}_{\{X_k=j\}}}{n} = \frac{1}{\mathbb{E}(T_j|X_0=j)} = \pi_j$$

Back to the aperiodic case. SInce the limit $\lim_{n\to\infty} P_{ij}^{(n)}$ does not depend on i, $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$ is also the limiting(marginal) distribution at state j:

$$\lim_{n\to\infty} \alpha_{n,j} = \lim_{n\to\infty} P(X_n = j) = \pi_j$$

regardless of the initial distribution α_0

Detail:

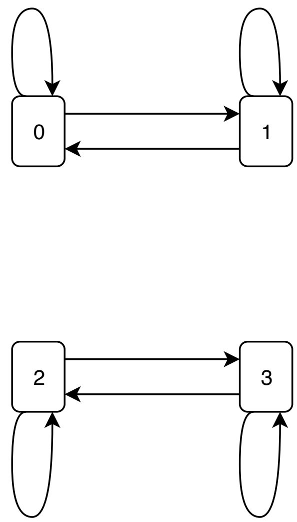
$$\begin{aligned}
\lim_{n\to\infty} \alpha_{n,j} &= \lim_{n\to\infty} (\alpha_0 \cdot p^{(n)})_j \\
&= \lim_{n\to\infty} \sum_{i\in S} \alpha_{0,i} \cdot P_{ij}^{(n)} \\
&= \sum_{i\in S} \lim_{n\to\infty} \alpha_{0,i} \cdot P_{ij}^{(n)} \\
&= \sum_{i\in S} \alpha_{0,i} \lim_{n\to\infty} \cdot P_{ij}^{(n)} \\
&= (\sum_{i\in S} \alpha_{0,i})\pi_j \\
&= \pi_j
\end{aligned}$$

Why are the conditions in the *Basic Limit Theorem* necessary?

Example 4.5.1

Consider a MC with

$$p = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{2} & \frac{1}{2} \\ & & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



Two classes: $\{0,1\},\{2,3\} \Rightarrow$ it is **not** irreducible. All the states are still aperiodic, positive recurrent

This MC can be decomposed into two MC's:

State $0, 1$, with

$$p_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \text{irreducible}$$

State $2, 3$, with

$$p_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \text{irreducible}$$

And

$$p = \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}$$

Note that both $(\frac{1}{2},\frac{1}{2},0,0)$ and $(0,0,\frac{1}{2},\frac{1}{2})$ are stationary distributions. Consequently, any convex combination of these two distributions, of the form:

$$a(\frac{1}{2},\frac{1}{2},0,0)+(1-a)(0,0,\frac{1}{2},\frac{1}{2}) \quad ,a \in \{0,1\}$$

is also a stationary distribution

Thus, irreducibility is related to the uniqueness of the stationary distribution.

Correspondingly, the limiting transition probability will depend on i:

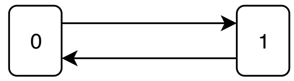
$$\lim_{n\to\infty} P_{00}^{(n)} = (\lim_{n\to\infty} P_1^n)_{00} = \frac{1}{2}$$

but $\lim_{n\to\infty} P_{20}^{(n)} = 0$

Example 4.5.2

Consider a MC with

$$p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



Irreducible, positive recurrent, but not aperiodic: $d=2$

Note that $p^2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = I \Rightarrow p^{2n} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, p^{2n+1} = p = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$p_{00}^{(n)} = 1$ for n even, 0 for n odd $\Rightarrow \lim_{n\to\infty} P_{00}^{(n)}$ does not exist.

Aperiodicity is related to the existence of the limit $\lim_{n\to\infty} P_{ij}^{(n)}$