Note 16 - Mar 12

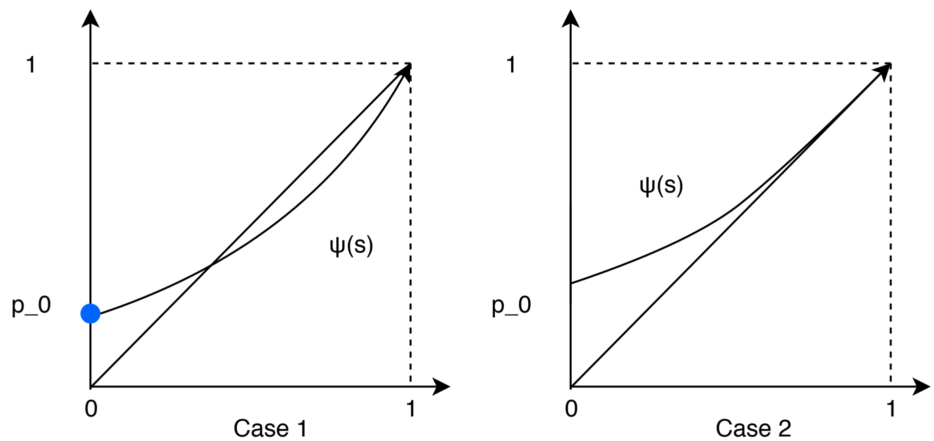
4. Stochastic Processes (cont'd)

4.6. Generating function and branching processes (cont'd)

4.6.1. Branching Process (cont'd)

4.6.1.2. Extinction Probability (cont'd)

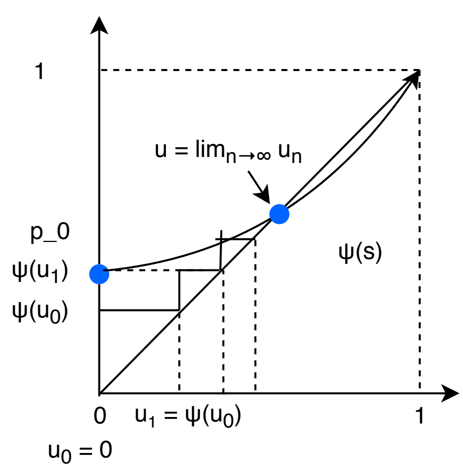
Draw $\psi(s)$ and function $f(s) = s$ between 0 and 1, we have two cases:

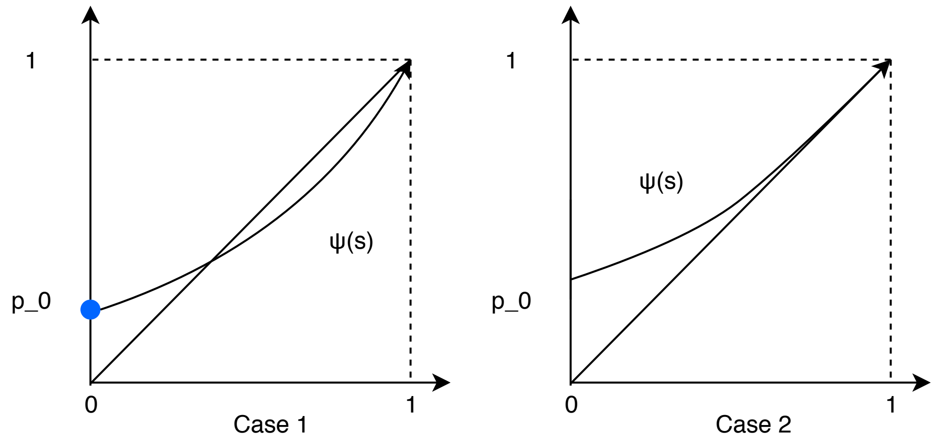


The extinction probability u will be the smallest intersection of $\psi(s)$ and $f(s)$. Equivalently, it is the smallest solution of the equation $\psi(s) = s$ between 0 and 1.

$\rightarrow \infty$

Reason: See the dynamics on a graph





$$\Rightarrow \begin{array}{ll} \text{Case 1}: & u < 1 \\ \text{Case 2}: & u = 1 \text{ (extinction happens for sure.)} \end{array}.$$

Q: How to tell if we are in case 1 or in case ?

A: check $\psi'(1) = \mathbb{E}(Y)$

$$\begin{array}{ll} \psi'1(1) > 1 & \rightarrow \text{Case 1} \\ \psi'1(1) \leq 1 & \rightarrow \text{Case 2} \end{array}$$

Thus, we conclude:

- $\mathbb{E}(Y) > 1$: an average more than 1 offspring
    - $\Rightarrow$ extinction with certain probability smaller than 1. u is the smallest/unique solution between 0 and 1 of $\psi(s) = s$
- $\mathbb{E}(Y) \leq 1$: an average less than or equal to 1 offspring
    - $\Rightarrow$ extinction happens for sure (with probability 1)

5. Poisson Processes

5.1. Counting Process

DTMC is a discrete-time process. That is, the index set $T = \{0, 1, 2, ...\}$ and $\{X_n\}_{n=01,2,3,\cdots}$.

We also want to consider the cases where time can be continuous,

$$\text{Continuous-time processes: } T = [0, \infty\}$$

$$\{X_t\}_{t\geq 0} \text{ or } \{X_{(t)}\}_{t\geq 0}$$

The simplest type of continuous-time process is counting process, which counts the number of occurrence of certain event before time t.

Definition 5.1.1. Counting Process $N(t)$

Let $0 \le S_1 \le S_2 \le \cdots$ be the time of occurrence of some events. Then, the process

$$\begin{aligned} N(t) &:= \#\{n : S_n \le t\} \\ &= \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \le t\}} \end{aligned}$$

is called the counting process (of the events $\{S_n\}_{n=1,2,\ldots}$)

Equivalently, $N(t) = n \iff S_n \le t < S_{n+1}$

Example 5.1.1

Calls arrive at a call center.

- S_n : arrival time of the n-th call
- $N(t)$: the number of calls received before time t

Other examples: cars passing a speed reader, atoms having radioactive decay, ...

Properties of a counting process

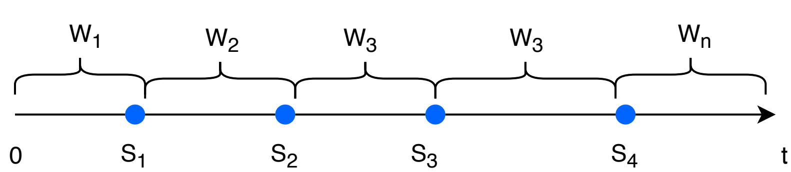
1. $N(t) \ge 0, t \ge 0$
2. $N(t)$ takes integer values
3. $N(t)$ is increasing.
    - $N(t_1) \le N(t_2)$ if $t_1 \le t_2$
4. $N(t)$ is right-continuous
    - $N(t) = \lim_{s \downarrow t} N(s)$

We also assume:

- $N(0) = 0$ (No event happens at time 0)
- $N(t)$ only has jumps at size 1.
    - (No two events happen at exactly the same time)

5.2. Definition of Poisson Process

Interarrival Times



- $W_1, W_2, \ldots$
- $W_1 = S_1$
- $W_n = S_n - S_{n-1}$: interarrival time between $n-1$-th and the n-th event

Definition 5.2.1. Renewal Process

A renewal process is a counting process for which the interarrival times $W_1, W_2, \ldots$ are independent and identical

ALl the three processes examples of counting processes can be reasonably modeled as renewal processes.

Definition 5.2.2. Poisson Process

Poisson Process $\{X_{(t)}\}_{t \geq 0}$ is a renewal process for which the interarrival times are exponentially distributed:

$$W_n \overset{i.i.d}{\sim} Exp(\lambda)$$

A Poisson process $\{N(t)\}_{t \geq 0}$ can be denoted as

$$\{N(t)\} \sim Poi(\underbrace{\lambda}_{intensity} t)$$