q theory

Garry Morrison*
(Dated: August 17, 2022)

In this note we explore q theory applied to Maxwell equations, the Dirac equation and differential forms, for the case q = 3. We will show that the resulting Dirac matrices can be used to compactly describe the Maxwell equations, the exterior derivative and more generally, differential forms. Further, the new Dirac equation predicts new particle and antiparticle types, but they are only of theoretical interest, since even if they exist, they are unlikely to be detectable. The ideas outlined extend to higher q, but the complexity increases rapidly with q, making calculations difficult even with the help of computer algebra systems.

I. INTRODUCTION

The mathematical language used to describe modern physics is of course, and rightly, founded on explaining results from physical experiments. But the current authors wondered what if the universe was slightly different? What if with small modifications of current physics theory there existed self consistent systems that are almost like known physics, but not quite.

The motivation for this train of thought is the energy-momentum relation [1]

$$E^2 = (mc^2)^2 + (pc)^2 \tag{1}$$

and the simple observation of "isn't it strange, the repeated 2's in this equation". So we decided to investigate the consequences of changing that 2 to other positive integers, denoted by q. Hence the question, can we find a self consistent system where

$$E^q = (mc^2)^q + (pc)^q \tag{2}$$

The answer turns out to be yes, almost. The exploration of this question is what we call q theory. In a previous unpublished note we considered q theory applied to the simplest possible quantum system, the $q = 3$ Schrödinger equation for a particle in a triangular box. In this note we derive Maxwell's equations, the Dirac equation, differential forms, and the exterior derivative, for the case $q = 3$. The resulting Maxwell's equations are significantly more complicated than their $q = 2$ sibling. The resulting Dirac equation predicts particles with a new type of spin, charge, and new types of antiparticles.

II. THE UNPHYSICAL NATURE OF OUR PROPOSAL

It is rapidly obvious that (2) contradicts known theory for $q \neq 2$. In particular it is not Lorentz invariant, and the low energy expansion contradicts the equation for kinetic energy

$$E = mc^2 + \frac{p^q}{qm^{q-1}c^{q-2}} + \cdots \tag{3}$$

If we assume that 'work is force times distance' still holds for arbitrary q, then even something as fundamental as the definition of $F = ma$ has q dependence, and needs to be modified for higher q. At this point it is not looking good for our baby theory. Further, if we rewrite (2) as

$$E^q - (pc)^q = (mc^2)^q \tag{4}$$

then mathematically the left hand side can only be invariant under a hyperbolic rotation for $q = 2$. There is no continuous symmetry of $E \to E'$ and $p \to p'$ such that

$$(E')^q - (p'c)^q = E^q - (pc)^q \tag{5}$$

This is a significant problem if we want the general concept of Lorentz transformation to hold for $q > 2$, and we do. However, there is a little wriggle room hiding in the mathematics.

III. THE J_q AND I_q POLYNOMIALS

The J_q and I_q polynomials are central to making any progress with q theory whatsoever. We start with another observation of well known equations

$$\begin{aligned} \cosh^2(t) - \sinh^2(t) = 1 \\ \cos^2(t) + \sin^2(t) = 1 \end{aligned} \tag{6}$$

Then extract out the corresponding polynomials, and give them a name

$$J_2(x_0, x_1) = x_0^2 - x_1^2 \tag{7}$$

$$I_2(x_0, x_1) = x_0^2 + x_1^2 \tag{8}$$

Noting that these can then be factorized in the standard way

$$J_2(x_0, x_1) = (x_0 + x_1)(x_0 - x_1) \tag{9}$$

$$I_2(x_0, x_1) = (x_0 + ix_1)(x_0 - ix_1) \tag{10}$$

The key observation is that there exist equivalent polynomials for higher values of q, and that if an equation from $q = 2$ is expressible in terms of J_2 or I_2 then the likely q theory equivalent is expressible in terms of J_q and I_q.

* garry.morrison@gmail.com

Here are the polynomials for $q = 3$

$$\begin{aligned} J_3(x_0, x_1, x_2) &= \prod_{i=1}^{3}(x_0 + \omega_i x_1 + \omega_i^2 x_2) \\ I_3(x_0, x_1, x_2) &= \prod_{i=1}^{3}(x_0 + \nu_i x_1 + \nu_i^2 x_2) \end{aligned} \tag{11}$$

where ω_i are the third roots of unity, and ν_i are the third roots of negative one. These expand out to

$$J_3(x_0, x_1, x_2) = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 \tag{12}$$

$$I_3(x_0, x_1, x_2) = x_0^3 - x_1^3 + x_2^3 + 3x_0x_1x_2 \tag{13}$$

Noting that while $q = 2$ is free of cross-terms, for all higher q these polynomials have cross terms. The presence of these cross-terms increase the complexity of our Maxwell and Dirac equations.

In practice we only need to calculate J_q since we can obtain I_q by swapping some signs. Alternatively, we can use this identity

$$J_q(x_0, \nu_i x_1, \cdots, \nu_i^{q-1} x_{q-1}) = I_q(x_0, x_1, \cdots, x_{q-1}) \tag{14}$$

Equations (11) also hold for general q

$$J_q(x_0, x_1, \cdots, x_{q-1}) = \prod_{i=1}^{q}\sum_{k=0}^{q-1}\omega_i^k x_k \text{ for } \omega_i^q = +1 \tag{15}$$

$$I_q(x_0, x_1, \cdots, x_{q-1}) = \prod_{i=1}^{q}\sum_{k=0}^{q-1}\nu_i^k x_k \text{ for } \nu_i^q = -1 \tag{16}$$

In the general case J_q and I_q can be more simply calculated using the determinant of circulant matrices in a computer algebra system, here we give the case $q = 4$, but other q follow in a similar manner

$$J_4(x_0, x_1, x_2, x_3) = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_3 & x_0 & x_1 & x_2 \\ x_2 & x_3 & x_0 & x_1 \\ x_1 & x_2 & x_3 & x_0 \end{vmatrix} \tag{17}$$

$$I_4(x_0, x_1, x_2, x_3) = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_3 & x_0 & x_1 & x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_1 & -x_2 & -x_3 & x_0 \end{vmatrix} \tag{18}$$

The number of terms in these polynomials grow rapidly with q. Here we give the number of terms $|J_q|$ for the first few values of q

$$2, 4, 10, 26, 68, 246, 810, \cdots \tag{19}$$

It should also be briefly noted that for arbitrary α

$$J_q(\alpha, \alpha, \cdots, \alpha) = 0 \tag{20}$$

$$I_q(\alpha, \alpha, \cdots, \alpha) = 2^{q-1}\alpha^q \tag{21}$$

The final set of J_q and I_q identities we need are

$$J_2(ax + by, ay + bx) = J_2(a, b)J_2(x, y) \tag{22}$$

$$I_2(ax + by, -ay + bx) = I_2(a, b)I_2(x, y) \tag{23}$$

and

$$\begin{aligned} &J_3(ax + by + cz, az + bx + cy, ay + bz + cx) \\ &\quad = J_3(a, b, c)J_3(x, y, z) \end{aligned} \tag{24}$$

$$\begin{aligned} &I_3(ax + by + cz, -az + bx + cy, -ay - bz + cx) \\ &\quad = I_3(a, b, c)I_3(x, y, z) \end{aligned} \tag{25}$$

which are true for arbitrary a, b, c, x, y, z, with similar identities for higher q. This property combined with the $f_{qk}(t)$ and $g_{qk}(t)$ functions introduced in the next section, provide us with a continuous transformation, $x_i \mapsto x'_i$ and $y_i \mapsto y'_i$ such that

$$J_q(x'_0, x'_1, \cdots, x'_{q-1}) = J_q(x_0, x_1, \cdots, x_{q-1}) \tag{26}$$

$$I_q(y'_0, y'_1, \cdots, y'_{q-1}) = I_q(y_0, y_1, \cdots, y_{q-1}) \tag{27}$$

In analogue with $q = 2$ we call the first of these a Lorentz transform, and the second one a rotation.

A. The f_{qk} and g_{qk} functions

Euler's formula defines $\cos(t)$ and $\sin(t)$ in terms of $\exp(t)$ and the imaginary unit i

$$\exp(it) = \cos(t) + i\sin(t) \tag{28}$$

However, this formula is extendible to the general case where we replace i with either roots of unity $\omega_k = \exp(i2\pi k/q)$ or roots of negative one $\nu_k = \exp(i\pi(2k + 1)/q)$

$$\exp(\omega_i t) = \sum_{k=1}^{q}\omega_i^{k-1} f_{qk}(t) \text{ for } \omega_i^q = +1 \tag{29}$$

$$\exp(\nu_i t) = \sum_{k=1}^{q}\nu_i^{k-1} g_{qk}(t) \text{ for } \nu_i^q = -1 \tag{30}$$

This formula defines a family of functions which extend the familiar $\cosh(t)$, $\sinh(t)$ and $\cos(t)$, $\sin(t)$. Re-writing Euler's formula, we can provide a more direct definition of these functions

$$f_{qk}(t) = \frac{1}{q}\sum_{i=1}^{q}\omega_i^{q-k+1}\exp(\omega_i t) \text{ for } \nu_i^q = +1 \tag{31}$$

$$g_{qk}(t) = -\frac{1}{q}\sum_{i=1}^{q}\nu_i^{q-k+1}\exp(\nu_i t) \text{ for } \nu_i^q = -1 \tag{32}$$

Or by considering the Taylor series for $\exp(t)$ and Euler's formula we find their respective Taylor series

$$f_{qk}(t) = \sum_{n=0}^{\infty}\frac{t^{qn+k-1}}{(qn + k - 1)!} \tag{33}$$

$$g_{qk}(t) = \sum_{n=0}^{\infty}(-1)^n\frac{t^{qn+k-1}}{(qn + k - 1)!} \tag{34}$$

From which it is clear that $\cosh(t) = f_{21}(t)$, $\sinh(t) = f_{22}(t)$, $\cos(t) = g_{21}(t)$ and $\sin(t) = g_{22}(t)$. From the Taylor series it is also clear that

$$f_{q1}(0) = g_{q1}(0) = 1 \tag{35}$$
$$f_{qk}(0) = g_{qk}(0) = 0 \text{ for } k > 1 \tag{36}$$

We can also extend the concept of even and odd functions, since, for roots of unity $\omega_i^q = +1$, we have

$$f_{qk}(\omega_i t) = \omega_i^{k-1} f_{qk}(t) \tag{37}$$
$$g_{qk}(\omega_i t) = \omega_i^{k-1} g_{qk}(t) \tag{38}$$

In particular, for $q = 2$

$$f_{21}(-t) = f_{21}(t) \tag{39}$$
$$f_{22}(-t) = -f_{22}(t) \tag{40}$$

which is the standard definition for even and odd functions.

Meanwhile roots of negative one $\nu_i^q = -1$ swap between the two functions

$$f_{qk}(\nu_i t) = \nu_i^{k-1} g_{qk}(t) \tag{41}$$
$$g_{qk}(\nu_i t) = \nu_i^{k-1} f_{qk}(t) \tag{42}$$

and the derivatives are

$$\frac{d}{dt} f_{q1}(t) = f_{qq}(t) \tag{43}$$
$$\frac{d}{dt} f_{qk}(t) = f_{qk-1}(t) \text{ for } k > 1 \tag{44}$$
$$\frac{d}{dt} g_{q1}(t) = -g_{qq}(t) \tag{45}$$
$$\frac{d}{dt} g_{qk}(t) = g_{qk-1}(t) \text{ for } k > 1 \tag{46}$$

So the derivatives cycle through the given family of functions.

Using Euler's formula, (15), (16), $\sum_{i=1}^q \omega_i = 0$ and $\sum_{i=1}^q \nu_i = 0$ we can derive the general identities

$$J_q(f_{q1}(t), f_{q2}(t), \cdots, f_{qq}(t)) = 1 \tag{47}$$
$$I_q(g_{q1}(t), g_{q2}(t), \cdots, g_{qq}(t)) = 1 \tag{48}$$

which for $q = 2$ are our starting identities (6). Combined with (24) and (25) results in (26) and (27). These identities are key to extending the concept of Lorentz invariance and rotation to $q > 2$. Note that because of numerical errors when using computer algebra systems to calculate the Taylor series, these identities only hold for small values of t.

IV. LORENTZ INVARIANT FOR GENERAL Q

Now we are familiar with J_q and f_{qk} we observe that for an equation to be Lorentz invariant in the $q = 2$ world, then it has the property

$$J_2(a', b') = J_2(a, b) \tag{49}$$

for a continuous symmetry that maps a, b to a', b'. That symmetry is parametrized by $\cosh(t)$ and $\sinh(t)$, or, to make the q dependence of our functions clear, $f_{21}(t)$ and $f_{22}(t)$. We can write that symmetry using the matrix version of Euler's formula

$$\begin{aligned}\begin{bmatrix} a' \\ b' \end{bmatrix} &= \exp\left(\phi \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} f_{21}(\phi) & f_{22}(\phi) \\ f_{22}(\phi) & f_{21}(\phi) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}\end{aligned} \tag{50}$$

where ϕ is a continuous parameter of this symmetry, called the rapidity. The proposal then is that for $q = 3$ a 'Lorentz transform' has the property

$$J_3(a', b', c') = J_3(a, b, c) \tag{51}$$

which also has a continuous symmetry if we consider the matrix version of Euler's formula, and use a permutation matrix with property $P^3 = I$

$$\begin{aligned}\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} &= \exp\left(\phi \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\right) \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} f_{31}(\phi) & f_{32}(\phi) & f_{33}(\phi) \\ f_{33}(\phi) & f_{31}(\phi) & f_{32}(\phi) \\ f_{32}(\phi) & f_{33}(\phi) & f_{31}(\phi) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}\end{aligned} \tag{52}$$

It is easily shown that (51) is satisfied by this transformation using (24) and (47)

$$\begin{aligned} J_3(a', b', c') &= J_3(f_{31}(\phi), f_{32}(\phi), f_{33}(\phi)) J_3(a, b, c) \\ &= J_3(a, b, c) \end{aligned} \tag{53}$$

With rearrangement, we can rewrite this in a more familiar form

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \gamma \begin{bmatrix} 1 & \beta_1 & \beta_2 \\ \beta_2 & 1 & \beta_1 \\ \beta_1 & \beta_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{54}$$

where

$$\gamma = f_{31}(\phi) \tag{55}$$
$$\beta_1 = \frac{f_{32}(\phi)}{f_{31}(\phi)} = \frac{v_a}{c} \tag{56}$$
$$\beta_2 = \frac{f_{33}(\phi)}{f_{31}(\phi)} = \frac{v_b}{c} \tag{57}$$

We can now rewrite (53) as

$$J_3(a', b', c') = \gamma^3 J_3(1, \beta_1, \beta_2) J_3(a, b, c) \tag{58}$$

Hence $\gamma^3 J_3(1, \beta_1, \beta_2) = 1$ and so γ can be written as

$$\gamma = \left(1 + \beta_1^3 + \beta_2^3 - 3\beta_1\beta_2\right)^{-\frac{1}{3}} \tag{59}$$

which has the property as $\beta_i \to 1$ then $\gamma \to \infty$. With further property that under $\phi \to \nu_i \phi$ then $\gamma \to \gamma$, $\beta_1 \to$

$\nu_i\beta_1$ and $\beta_2 \to \nu_i^2\beta_2$, where $\nu_i^3 = +1$. This contrasts with γ for $q = 2$

$$\gamma = \left(1 - \beta_1^2\right)^{-\frac{1}{2}} \tag{60}$$

For general q we have

$$\gamma = J_q(1, \beta_1, \beta_2, \cdots, \beta_{q-1})^{-\frac{1}{q}} \tag{61}$$

$$\beta_i = \frac{f_{q(i+1)}(\phi)}{f_{q1}(\phi)} = \frac{v_i}{c} \tag{62}$$

with property as $\beta_i \to 1$ then $\gamma \to \infty$, which is a result of (20), and under $\phi \to \nu_i\phi$ then $\gamma \to \gamma$, $\beta_k \to \nu_i^k\beta_k$, where $\nu_i^q = +1$, which is a result of (37). We are now ready to state the Lorentz invariant $q = 3$ energy-momentum relation

$$J_3(E/c, p_a, p_b) = (mc)^3 \tag{63}$$

Or expanded out

$$E^3 + (p_ac)^3 + (p_bc)^3 - 3Ep_ap_bc^2 = (mc^2)^3 \tag{64}$$

Which if we choose a frame in which $p_b = 0$ and $p_a < 0$ then we have our original q-theory equation (2)

$$E^3 - (|p_a|c)^3 = (mc^2)^3 \tag{65}$$

For general q the actual q-theory equation should not be (2), which we can't make Lorentz invariant in the general case, but rather

$$J_q(E/c, p_a, p_b, \cdots) = (mc)^q \tag{66}$$

This equation is then the basis for our proposed Schrödinger equation using the low energy limit, and the new Dirac equation which we will see later.

In summary the general case for Lorentz invariance of the space-time interval and wave equation are

$$J_q(ct', x'_a, x'_b, \cdots) = J_q(ct, x_a, x_b, \cdots) \tag{67}$$

$$J_q(\partial_0, \partial_a, \partial_b, \cdots)f = 0 \tag{68}$$

V. RECAP OF THE Q = 2 CASE

We motivate our $q = 3$ theory by first briefly examining the foundations of the standard $q = 2$ case. It quickly becomes obvious that the 2×2 Pauli matrices are fundamental to the $q = 2$ case, and so to proceed we need to generalize them to $q = 3$. These proposed 3×3 matrices, amongst other things, describe a new type of spin, that does not correspond to either integer or half-integer spin. Similar $q \times q$ matrices exist for higher q but will not be examined here.

A. Maxwell equations

1. Vector representation

The most common representation of the Maxwell equations is the vector representation, here in Gaussian units

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \end{aligned} \tag{69}$$

2. Matrix representation

However, we can rewrite this in terms of Pauli matrices, which is key to extending our theory to $q > 2$, by way of this identity

$$a \cdot i\sigma \; b \cdot i\sigma = -a \cdot b \; I_2 - (a \times b) \cdot i\sigma \tag{70}$$

Which produces the following matrix representation of the Maxwell equations

$$\begin{bmatrix} \partial_0 & \nabla \cdot i\sigma \\ \nabla \cdot i\sigma & -\partial_0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \cdot i\sigma \\ \mathbf{B} \cdot i\sigma \end{bmatrix} = -4\pi \begin{bmatrix} \mathbf{J} \cdot i\sigma \\ \rho I_2 \end{bmatrix} \tag{71}$$

This apparently complicates things, but is now in a form that is much easier for a computer algebra system such as SageMath to handle. For $q = 2$ the equations are simple enough to be manipulated by hand, but for $q > 2$, this is not the case, and so a computer algebra system becomes the only way to work with our equations.

The left most matrix is particularly interesting, as it is essentially a matrix representation for the exterior derivative, a Dirac matrix, and provides a hint on how to represent differential forms using matrices. Again, useful for a computer algebra system.

3. Differential forms representation

Perhaps the third most common representation of the Maxwell equations, after the vector and integral forms, is in terms of the exterior derivative and the Hodge star operator

$$F = dA \tag{72}$$

$$dF = 0 \tag{73}$$

$$*d*F = J \tag{74}$$

where A is a 1-form, d is the exterior derivative, $*$ is the Hodge star operator, F is the Faraday tensor and J is the 4-current. In components, F is given by

$$\begin{aligned} F = &-E_x dt \wedge dx - E_y dt \wedge dy - E_z dt \wedge dz \\ &B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \end{aligned} \tag{75}$$

where three key properties of differential forms are

$$dd\alpha = 0, \text{ for any k-form } \alpha \tag{76}$$
$$dx \wedge dx = 0 \tag{77}$$
$$dx \wedge dy = -dy \wedge dx \tag{78}$$

Note that none of these properties are true for $q > 2$, more details later.

B. Dirac equation

In contrast to the Maxwell equation, the most common representation of the Dirac equation is in terms of 4×4 matrices, with the following properties

$$\hat{P} = \hat{E}\gamma_0 + \hat{p}_i\gamma_i \tag{79}$$
$$\hat{P}\psi = m\psi \tag{80}$$
$$\hat{P}^2\psi = J_2(E, \mathbf{p})\psi = m^2\psi \tag{81}$$

with the γ_μ chosen so that (81) follows immediately. This is achieved with the following properties

$$\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu \tag{82}$$
$$\gamma_0^2 = I_4 \tag{83}$$
$$\gamma_i^2 = -I_4 \tag{84}$$

C. Matrix representation

Finally, we are ready to propose that we can use matrices as a representation for differential forms basis elements, providing they satisfy (78), and we take care to handle (76) and (77). In this scheme, the wedge product reduces to standard matrix multiplication.

The Pauli matrices are effectively a 3 dimensional representation of the basis elements of differential forms, and the Dirac gamma matrices are a 4 dimensional representation of the same, since both sets of matrices anti-commute

$$\sigma_i\sigma_j = -\sigma_j\sigma_i \tag{85}$$
$$\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu \tag{86}$$

In this scheme, a 4 dimensional 1-form is written

$$A = A_0\gamma_0 + A_1\gamma_1 + A_2\gamma_2 + A_3\gamma_3 \tag{87}$$

and the exterior derivative is

$$d = \partial_0\gamma_0 + \partial_1\gamma_1 + \partial_2\gamma_2 + \partial_3\gamma_3 \tag{88}$$

If we choose our γ matrices as

$$\gamma_0 = \sigma_3 \otimes I_2 \tag{89}$$
$$\gamma_j = \sigma_1 \otimes i\sigma_j \tag{90}$$

then the exterior derivative in matrix representation is

$$\begin{bmatrix} \partial_0 & \nabla \cdot i\sigma \\ \nabla \cdot i\sigma & -\partial_0 \end{bmatrix} \tag{91}$$

which is the same matrix as we have in (71). Indeed, we can now write the Faraday 2-form (75) in matrix representation using

$$dt \wedge dx^i = \gamma_0\gamma_i \tag{92}$$
$$dy \wedge dz = \gamma_2\gamma_3 \tag{93}$$
$$dz \wedge dx = \gamma_3\gamma_1 \tag{94}$$
$$dx \wedge dy = \gamma_1\gamma_2 \tag{95}$$

which expand out to

$$\gamma_0\gamma_i = \sigma_3\sigma_1 \otimes i\sigma_i \tag{96}$$
$$\gamma_2\gamma_3 = -I_2 \otimes i\sigma_1 \tag{97}$$
$$\gamma_3\gamma_1 = -I_2 \otimes i\sigma_2 \tag{98}$$
$$\gamma_1\gamma_2 = -I_2 \otimes i\sigma_3 \tag{99}$$

Hence in matrix representation

$$F = \begin{bmatrix} -\mathbf{B} \cdot i\sigma & -\mathbf{E} \cdot i\sigma \\ \mathbf{E} \cdot i\sigma & -\mathbf{B} \cdot i\sigma \end{bmatrix} \tag{100}$$

And the 4-current J is

$$J = -\rho\gamma_0 + j_1\gamma_1 + j_2\gamma_2 + j_3\gamma_3 \tag{101}$$
$$= \begin{bmatrix} -\rho & \mathbf{j} \cdot i\sigma \\ \mathbf{j} \cdot i\sigma & \rho \end{bmatrix} \tag{102}$$

Putting it all together, we have the Maxwell's equations in matrix representation $dF = J$

$$\begin{bmatrix} \partial_0 & \nabla \cdot i\sigma \\ \nabla \cdot i\sigma & -\partial_0 \end{bmatrix} \begin{bmatrix} -\mathbf{B} \cdot i\sigma & -\mathbf{E} \cdot i\sigma \\ \mathbf{E} \cdot i\sigma & -\mathbf{B} \cdot i\sigma \end{bmatrix} = \begin{bmatrix} -\rho & \mathbf{j} \cdot i\sigma \\ \mathbf{j} \cdot i\sigma & \rho \end{bmatrix} \tag{103}$$

If we wish to verify this produces the original Maxwell's equations, it is simply a matter of expanding it out

$$-\partial_0\ \mathbf{B} \cdot i\sigma + \nabla \cdot i\sigma\ \mathbf{E} \cdot i\sigma = -\rho I_2 \tag{104}$$
$$-\partial_0\ \mathbf{E} \cdot i\sigma - \nabla \cdot i\sigma\ \mathbf{B} \cdot i\sigma = \mathbf{j} \cdot i\sigma \tag{105}$$
$$-\nabla \cdot i\sigma\ \mathbf{B} \cdot i\sigma - \partial_0\ \mathbf{E} \cdot i\sigma = \mathbf{j} \cdot i\sigma \tag{106}$$
$$-\nabla \cdot i\sigma\ \mathbf{E} \cdot i\sigma + \partial_0\ \mathbf{B} \cdot i\sigma = \rho I_2 \tag{107}$$

and using the identity (70)

$$\nabla \cdot i\sigma\ \mathbf{E} \cdot i\sigma = -\nabla \cdot \mathbf{E} I_2 - \nabla \times \mathbf{E} \cdot i\sigma \tag{108}$$

From which we recover two copies of the original Maxwell equations in vector form (69).

D. SageMath and Jupyter notebook

Given the above theory, it is straightforward to implement it in a computer algebra system, such as SageMath [4] and the Jupyter notebook. Full code for this note is available on github [2]. First we define our matrices

```
# The Pauli matrices:
I2 = matrix([[1,0],[0,1]])
sigma1 = matrix([[0,1],[1,0]])
sigma2 = matrix([[0,-I],[I,0]])
sigma3 = matrix([[1,0],[0,-1]])

# The Dirac gamma matrices:
gamma0 = sigma3.tensor_product(I2)
gamma1 = I*sigma1.tensor_product(sigma1)
gamma2 = I*sigma1.tensor_product(sigma2)
gamma3 = I*sigma1.tensor_product(sigma3)
```

Next we run our `dim-table` function, producing this k-form dimension table where we use an abbreviated notation for our basis elements. For example $dz \wedge dx$ is written as $D31$, and $dt \wedge dx \wedge dy \wedge dz$ is written as $D0123$, and so on. Noting that in the matrix representation, wedge product becomes a matrix product.

TABLE I. Dimension table

k	dim	basis
0	1	
1	4	0 1 2 3
2	6	01 02 03 12 13 23
3	4	012 013 023 123
4	1	0123

With this in mind, we define our needed k-forms

```
d = d0*D0 + d1*D1 + d2*D2 + d3*D3
A = A0*D0 + A1*D1 + A2*D2 + A3*D3
F = - E1*D01 - E2*D02 - E3*D03
    + B1*D23 + B2*D31 + B3*D12
J = - rho*D0 + j1*D1 + j2*D2 + j3*D3
```

Then we introduce two new functions. The first one extracts the k-form terms from the given matrix m using the properties that the trace of Dirac matrices is 0, the trace of the 4×4 identity matrix is 4, and $\gamma_\mu^4 = I_4$. The second function applies the Hodge star operator to the given k-form m.

```
get_d_basis(k, m)
star_k(k, m)
```

With these functions in place it is a very simple matter to read off the full Maxwell equations.

dF in terms of 1-forms, `get_d_basis(1, d*F)`

$$\texttt{D0:} \quad -\partial_1 E_1 - \partial_2 E_2 - \partial_3 E_3 \tag{109}$$
$$\texttt{D1:} \quad -\partial_0 E_1 + \partial_2 B_3 - \partial_3 B_2 \tag{110}$$
$$\texttt{D2:} \quad -\partial_0 E_2 - \partial_1 B_3 + \partial_3 B_1 \tag{111}$$
$$\texttt{D3:} \quad -\partial_0 E_3 + \partial_1 B_2 - \partial_2 B_1 \tag{112}$$

dF in terms of 3-forms, `get_d_basis(3, d*F)`

$$\texttt{D023:} \quad \partial_0 B_1 + \partial_2 E_3 - \partial_3 E_2 \tag{113}$$
$$\texttt{D031:} \quad \partial_0 B_2 - \partial_1 E_3 + \partial_3 E_1 \tag{114}$$
$$\texttt{D012:} \quad \partial_0 B_3 + \partial_1 E_2 - \partial_2 E_1 \tag{115}$$
$$\texttt{D123:} \quad \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 \tag{116}$$

$F = dA$ in terms of 2-forms, `get_d_basis(2, d*A)`

$$\texttt{D01:} \quad \partial_0 A_1 - \partial_1 A_0 \tag{117}$$
$$\texttt{D02:} \quad \partial_0 A_2 - \partial_2 A_0 \tag{118}$$
$$\texttt{D03:} \quad \partial_0 A_3 - \partial_3 A_0 \tag{119}$$
$$\texttt{D23:} \quad \partial_2 A_3 - \partial_3 A_2 \tag{120}$$
$$\texttt{D31:} \quad -\partial_1 A_3 + \partial_3 A_1 \tag{121}$$
$$\texttt{D12:} \quad \partial_1 A_2 - \partial_2 A_1 \tag{122}$$

$F \wedge *F$ in terms of 4-forms, `get_d_basis(4, F*star_k(2, F))`

$$\texttt{D0123:} \quad -B_1^2 - B_2^2 - B_3^2 + E_1^2 + E_2^2 + E_3^2 \tag{123}$$

$d * J$ in terms of 4-forms, `get_d_basis(4, d*star_k(1, J))`

$$\texttt{D0123:} \quad -\partial_1 J_1 - \partial_2 J_2 - \partial_3 J_3 - \partial_0 \rho \tag{124}$$

1. *Matrix representation and Differential forms generations*

Now we make an observation on the difference between the standard differential form notation, and our matrix representation. Differential forms have the property that $dx \wedge dx = 0$, while our matrices satisfy $\gamma_\mu^2 = \pm I_4$. So how do we reconcile this? The hint is in our above equations, where we obtained all of the Maxwell equations even though we didn't use the $*d * F = J$ equation, $dF = J$ was sufficient. The solution is to introduce the idea of 'generations'. For each term D^{ii} in a differential form expression we promote the expression to the next generation. Some examples

$dt \wedge dt \wedge dy = 0$ `in generation 1`
$dt \wedge dt \wedge dy = dy$ `in generation 2`

$dt \wedge dt \wedge dy \wedge dy \wedge dz = 0$ `in generation 1 and 2`
$dt \wedge dt \wedge dy \wedge dy \wedge dz = dz$ `in generation 3`

Then the matrix representation is the same as the differential forms representation, with the property that all the generations are wrapped on top of each other. Indeed, we see this above where we considered dF in terms of 1-forms, and saw terms that traditional differential forms would send to zero. For example in generation 1

$$\partial_x dx \wedge E_x dt \wedge dx = 0$$
$$\partial_y dy \wedge E_y dt \wedge dy = 0$$
$$\partial_z dz \wedge E_z dt \wedge dz = 0$$

But in generation 2

$$\partial_x dx \wedge E_x dt \wedge dx = \partial_x E_x dt$$
$$\partial_y dy \wedge E_y dt \wedge dy = \partial_y E_y dt$$
$$\partial_z dz \wedge E_z dt \wedge dz = \partial_z E_z dt$$

There is an apparent sign error, but that is not the case, because $\gamma_i\gamma_0 = -\gamma_0\gamma_i$ and $\gamma_i\gamma_i = -I_4$ hence $\gamma_i\gamma_0\gamma_i = \gamma_0$.

A question arises, why use the matrix representation at all, when the standard vector or differential forms notation is more than sufficient? The answer is that it serves as a foundation for $q > 2$ where they are not a practical option. We are forced to use the matrix representation and a computer algebra system.

VI. THE Q = 3 CASE

With the $q = 2$ case in place, we are ready to proceed to $q = 3$. We will not be considering higher q in this document, but we presume they follow similarly. We would now like to consider the Dirac equation, before moving on to the Maxwell equations, but first we need to extend the Pauli matrices to $q = 3$.

A. The Pauli matrices

The Pauli matrices are a set of 3 matrices that along with the identity matrix form a basis for 2×2 Hermitian matrices.

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{125}$$

For 3×3 matrices, the space has 9 dimensions, so there is some freedom in choosing our new Pauli matrices. We can choose T_1 and T_3 in analogy with σ_1 and σ_3, but we need further constraints to find T_2. Observe that our Pauli matrix eigenvalues are now third roots of unity, labelled w, instead of ± 1.

For $q = 2$ we have the property

$$\begin{aligned} &r = p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3 \\ &\implies \\ &r^2 = p_1^2 + p_2^2 + p_3^2 = I_2(p_1, p_2, p_3) \end{aligned}$$

For $q = 3$ there are three choices

$$\begin{aligned} &r = p_1T_1 + p_2T_2 + p_3T_3 \\ &\implies \\ &r^3 = p_1^3 + p_2^3 + p_3^3 && (126) \\ &r^3 = p_1^3 + p_2^3 + p_3^3 - 3p_1p_2p_3 = J_3(p_1, p_2, p_3) && (127) \\ &r^3 = p_1^3 - p_2^3 + p_3^3 + 3p_1p_2p_3 = I_3(p_1, p_2, p_3) && (128) \end{aligned}$$

The first of these looks the most like the $q = 2$ case, but has the problem that it has no continuous symmetry. The second of these produces the cleanest equations in terms of our Dirac and Maxwell equations, but is technically a Lorentz transform, and not a rotation. The third of these corresponds to $q = 3$ rotations, so in that regard is the best choice, but at the cost of the 'prettiness' of our equations. Given this is only a theoretical exercise, for simplicity sake we chose the second option. We briefly consider the first option later in section **??**. Hence we propose these as our new Pauli matrices

$$T_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} T_2 = \begin{bmatrix} 0 & 0 & w^2 \\ w & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^2 \end{bmatrix} \tag{129}$$

We can also construct a further set of 3 matrices, in some sense dual to the first set

$$\tau_1 = T_2T_3 = \begin{bmatrix} 0 & 0 & w \\ w & 0 & 0 \\ 0 & w & 0 \end{bmatrix} \tag{130}$$

$$\tau_2 = T_3T_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & w \\ w^2 & 0 & 0 \end{bmatrix} \tag{131}$$

$$\tau_3 = T_1T_2 = \begin{bmatrix} w & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w^2 \end{bmatrix} \tag{132}$$

with the property that

$$\begin{aligned} &r = q_1\tau_1 + q_2\tau_2 + q_3\tau_3 \\ &\implies \\ &r^3 = q_1^3 + q_2^3 + q_3^3 - 3q_1q_2q_3 = J_3(q_1, q_2, q_3) && (133) \end{aligned}$$

Given that the 3×3 space has dimension 9 and we currently only have 6 matrices, we require two new matrices, plus the identity matrix, to fill out the basis. They are given by

$$T_4 = \begin{bmatrix} 0 & 0 & w \\ w^2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{134}$$

$$\tau_4 = \begin{bmatrix} 0 & w^2 & 0 \\ 0 & 0 & w \\ 1 & 0 & 0 \end{bmatrix} \tag{135}$$

Which have these properties

$$\begin{aligned} &r = p_1T_1 + p_2T_2 + p_3T_3 + p_4T_4 \\ &\implies \\ &r^3 = J_3(p_1, p_2, p_3) + p_4^3 && (136) \end{aligned}$$

$$\begin{aligned} &r = q_1\tau_1 + q_2\tau_2 + q_3\tau_3 + q_4\tau_4 \\ &\implies \\ &r^3 = J_3(q_1, q_2, q_3) + q_4^3 && (137) \end{aligned}$$

These new Pauli matrices, T_1, T_2, T_3 and τ_1, τ_2, τ_3, clearly do not correspond to integer or half-integer spin particles. Perhaps they correspond to spin $\frac{1}{3}$ or $\frac{2}{3}$ particles.

B. The Dirac equation

The Dirac matrices can be considered an extension of the 3 dimensional Pauli matrices to 4 dimensional space-time. In natural units, the defining Dirac equations are

$$\hat{B}\Psi = m\Psi \tag{138}$$

$$\hat{B}^3\Psi = J_3(E,p,q)\Psi = m^3\Psi \tag{139}$$

where we use B to label our new matrices, to distinguish from the $q = 2$ γ matrices. With the comment that Lorentz invariance for $q = 3$ requires the addition of the extra component q in (139), which is in some way related to the momentum term p. A little more on that later.

Using a computer algebra system, we find the following set of Dirac matrices, constructed using tensor products of our new Pauli matrices

$$\begin{aligned} B_0 = T_3 \otimes I_3 \quad & B_1 = wT_2 \otimes T_1 \quad B_4 = wT_1 \otimes \tau_1 \\ & B_2 = wT_2 \otimes T_2 \quad B_5 = wT_1 \otimes \tau_2 \\ & B_3 = wT_2 \otimes T_3 \quad B_6 = wT_1 \otimes \tau_3 \end{aligned} \tag{140}$$

and the new Dirac operator

$$\begin{aligned} \hat{B} &= \hat{E}B_0 \\ &+= \hat{p_1}B_1 + \hat{p_2}B_2 + \hat{p_3}B_3 \\ &+= \hat{q_1}B_4 + \hat{q_2}B_5 + \hat{q_3}B_6 \end{aligned}$$

Which we can simplify if we define

$$\hat{p} = \hat{p_1}T_1 + \hat{p_2}T_2 + \hat{p_3}T_3 \tag{141}$$

$$\hat{q} = \hat{q_1}\tau_1 + \hat{q_2}\tau_2 + \hat{q_3}\tau_3 \tag{142}$$

Giving

$$\hat{B} = \begin{bmatrix} \hat{E} & w\hat{q} & \hat{p} \\ w^2\hat{p} & w\hat{E} & w\hat{q} \\ w\hat{q} & w\hat{p} & w^2\hat{E} \end{bmatrix} \tag{143}$$

However, the p_i and q_i are not fully independent. If we define

$$p \cdot q = p_1q_1 + p_2q_2 + p_3q_3 \tag{144}$$

$$p \cdot\cdot q = q \cdot\cdot\cdot p = p_1q_2 + p_2q_3 + p_3q_1 \tag{145}$$

$$p \cdot\cdot\cdot q = q \cdot\cdot p = p_1q_3 + p_2q_1 + p_3q_2 \tag{146}$$

then the requirement for $\hat{B}^3\Psi = J_3(E,p,q)\Psi$, and for B^3 to be diagonal, is that $p \cdot\cdot\cdot q = 0$. Further, if we require $[\hat{p},\hat{q}]\Psi = 0$ then we also require $p \cdot\cdot q = 0$. One solution to these requirements is

$$q_1 = p_1^2 - p_2p_3 \tag{147}$$

$$q_2 = p_2^2 - p_1p_3 \tag{148}$$

$$q_3 = p_3^2 - p_1p_2 \tag{149}$$

in which case

$$p \cdot q = J_3(p_1,p_2,p_3) \tag{150}$$

$$p \cdot\cdot q = 0 \tag{151}$$

$$p \cdot\cdot\cdot q = 0 \tag{152}$$

1. Dirac equation solutions

Labelling the components of Ψ

$$\Psi = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{bmatrix} \tag{153}$$

it is straight-forward to write down the solutions

$$(E-m)\psi_0 = -p \cdot T\psi_2 - wq \cdot \tau\psi_1 \tag{154}$$

$$(wE-m)\psi_1 = -w^2p \cdot T\psi_0 - wq \cdot \tau\psi_2 \tag{155}$$

$$(w^2E-m)\psi_2 = -wp \cdot T\psi_1 - wq \cdot \tau\psi_0 \tag{156}$$

We identify ψ_0 as the particle solution, and ψ_1 and ψ_2 as the two anti-particle solutions, and gloss over the fact that w is a complex number, and the interpretation of complex energies. This is a general pattern that for q there are q particles that mutually annihilate, which generalizes $q = 2$ where there is only a particle and an anti-particle solution.

To simplify our equations, and highlight the structure a little more clearly, we define

$$\alpha_0 = E - m \tag{157}$$

$$\alpha_1 = wE - m \tag{158}$$

$$\alpha_2 = w^2E - m \tag{159}$$

Then, with a little rearrangement, we can extract out the solutions. First, the particle solution, in terms of ψ_0

$$\psi_1 = \frac{1}{\alpha_1\alpha_2 - p \cdot q}\left(w^2(q \cdot \tau)^2 - w^2\alpha_2 p \cdot T\right)\psi_0 \tag{160}$$

$$\psi_2 = \frac{1}{\alpha_1\alpha_2 - p \cdot q}\left((p \cdot T)^2 - w\alpha_1 q \cdot \tau\right)\psi_0 \tag{161}$$

Then the first anti-particle solution, in terms of ψ_1

$$\psi_0 = \frac{1}{\alpha_0\alpha_2 - w^2p \cdot q}\left(w(p \cdot T)^2 - w\alpha_2 q \cdot \tau\right)\psi_1 \tag{162}$$

$$\psi_2 = \frac{1}{\alpha_0\alpha_2 - w^2p \cdot q}\left(w^2(q \cdot \tau)^2 - w\alpha_0 p \cdot T\right)\psi_1 \tag{163}$$

Then the second anti-particle solution, in terms of ψ_2

$$\psi_0 = \frac{1}{\alpha_0\alpha_1 - wp \cdot q}\left(w^2(q \cdot \tau)^2 - \alpha_1 p \cdot T\right)\psi_2 \tag{164}$$

$$\psi_1 = \frac{1}{\alpha_0\alpha_1 - wp \cdot q}\left(w^2(p \cdot T)^2 - w\alpha_0 q \cdot \tau\right)\psi_2 \tag{165}$$

Where we have used

$$(p \cdot T)(q \cdot \tau) = (q \cdot \tau)(p \cdot T) = wp \cdot qI_3 \tag{166}$$

which holds only if $p \cdot\cdot q = p \cdot\cdot\cdot q = 0$.

2. U(1) gauge invariance

In this section we briefly consider the $q = 3$ version of $U(1)$ local gauge invariance of the Dirac Lagrangian. Thus we need a Lagrangian invariant under the mapping

$$\Psi \mapsto \Psi = e^{w\theta(x)}\Psi \tag{167}$$

Which we can easily construct since the $q = 3$ inner-product $\langle|\Psi\rangle|\Psi\rangle|\Psi\rangle$ is invariant under this mapping, and the 'complex conjugation' of w is given by

$$w^\star = w^2 \tag{168}$$
$$w^{\star\star} = 1 \tag{169}$$

and $1 + w + w^2 = 0$.

It is straight-forward to show that

$$(wB^\mu D_\mu - m)\Psi = 0 \tag{170}$$

is invariant under the local gauge transformation

$$\Psi \mapsto \Psi = e^{w\theta(x)}\Psi \tag{171}$$
$$A_\mu \mapsto A_\mu - \frac{1}{q}\partial_\mu\theta \tag{172}$$

where the covariant derivative is given by

$$D_\mu \equiv \partial_\mu + wqA_\mu \tag{173}$$

3. Anti-particles

Using the minimal coupling of the Dirac equation to the electrodynamic field, as seen in the previous section

$$\left(wB^\mu(\partial_\mu + weA_\mu) - m\right)\Psi = 0 \tag{174}$$

we wish to consider the $q = 3$ charge conjugation of (174). Following Zee [5], charge conjugation gives

$$\left(w(wB^{\mu\star})(\partial_\mu + w(we)A_\mu) - m\right)\Psi^\star = 0 \tag{175}$$

Then we assume there exists a matrix C such that

$$wB^{\mu\star} = C^{-1}B^\mu C \tag{176}$$

and we set $\Psi_c = C\Psi^\star$ then we have

$$\left(wB^\mu(\partial_\mu + w(we)A_\mu) - m\right)\Psi_c = 0 \tag{177}$$

So we have another solution to the Dirac equation, Ψ_c, but with charge $e \mapsto we$. We can repeat this process

$$\Psi_{cc} = C\Psi_c^\star = CC^\star\Psi^{\star\star} \qquad e \mapsto w^2 e \tag{178}$$
$$\Psi_{ccc} = C\Psi_{cc}^\star = CC^\star C^{\star\star}\Psi^{\star\star\star} \qquad e \mapsto w^3 e = e \tag{179}$$

where $CC^\star C^{\star\star}\Psi^{\star\star\star} = \Psi$. Hence our $q = 3$ set of particles have charges that correspond to roots of unity, which is what we expected all along. Only for $q = 2$ does the Dirac equation predict real valued charges. The interpretation of complex valued charges is left for another discussion.

C. The exterior derivative and differential forms

At this point we have a choice. There are two approaches to defining a matrix representation of differential forms. We have the so called 'naive' $q = 3$ differential forms, and the Dirac matrix based differential forms. For $q = 2$ these two are identical, which traces back to $J_2(E, p)$ having no cross terms. For $q = 3$, $J_3(E, p, q)$ has several cross terms, resulting in what we call 'basis matrix collapse'. Given we want our Maxwell equations to be $q = 3$ Lorentz invariant, it seems sensible to choose our differential forms to use the Dirac matrices rather than the naive matrices.

1. Naive q = 3 differential forms

Before we move on to our Dirac based differential forms, we will briefly outline the naive version. They have these properties:

1. df is the standard derivative of the 0-form f
2. $d^3\alpha = 0$, for any k-form α
3. $d^j \wedge d^i = wd^i \wedge d^j$, for $j > i$
4. $d^i \wedge d^i \wedge d^i \equiv d^{iii} = 0$, for any index i
5. $\alpha \wedge \beta \wedge \star\star\gamma = \langle\alpha, \beta, \gamma\rangle\, d^I$, for k-forms α, β and γ
6. $\int_\Omega dd\alpha = \int_{\partial_3\Omega} d\alpha = \int_{\partial_3\partial_3\Omega} \alpha$

where

- for compactness we use $dx^i \wedge dx^j \equiv d^i \wedge d^j \equiv d^{ij}$
- $\star\star$ is the extension of the Hodge dual operator to $q = 3$, and is defined by the property (5)
- $\langle\alpha, \beta, \gamma\rangle$ is the $q = 3$ inner product, usually either based on $J_3()$ or $I_3()$
- d^I is the $q = 3$ volume element, given by $d^I \equiv \bigwedge_\mu d^{\mu\mu}$
- (6) is the $q = 3$ extension to the general Stoke's Theorem
- $\partial_3\Omega$ is the $q = 3$ boundary of Ω
- $\partial_3\partial_3\Omega$ is the $q = 3$ boundary of the boundary of Ω

One possible $q = 3$ inner product is given by

$$\langle\alpha, \beta, \gamma\rangle = h^{\mu\nu\rho}\alpha_\mu\beta_\nu\gamma_\rho \tag{180}$$

where for dim = 3

$$h^{\mu\nu\rho} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & w\\ 0 & w^2 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & w^2\\ 0 & 1 & 0\\ w & 0 & 0\end{bmatrix}\begin{bmatrix}0 & w & 0\\ w^2 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{181}$$

and

$$\langle\alpha,\beta,\gamma\rangle = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 + \alpha_3\beta_3\gamma_3 \quad (182)$$
$$+= w\alpha_1\beta_2\gamma_3 + w\alpha_2\beta_3\gamma_1 + w\alpha_3\beta_1\gamma_2 \quad (183)$$
$$+= w^2\alpha_1\beta_3\gamma_2 + w^2\alpha_2\beta_1\gamma_3 + w^2\alpha_3\beta_2\gamma_1 \quad (184)$$

Hence

$$\langle\alpha,\alpha,\alpha\rangle = J_3(\alpha_1,\alpha_2,\alpha_3) \quad (185)$$

using $1 + w + w^2 = 0$.

The exact meaning of the $q = 3$ boundary, $\partial_3\Omega$, is not entirely clear, but must be distinct from the $q = 2$ definition of a boundary since we require $\partial_3\partial_3\Omega$ to not be the empty set. Though we do expect $\partial_3\partial_3\partial_3\Omega$ to be the empty set, in line with (2) and (6).

a. matrix representation For dim = 3 it is easy to find a matrix representation for our naive differential forms, that satisfy property (3). One set of such matrices is given by

$$d^1 = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} d^2 = \begin{bmatrix}0&1&0\\0&0&w^2\\w&0&0\end{bmatrix} d^3 = \begin{bmatrix}1&0&0\\0&w^2&0\\0&0&w\end{bmatrix} \quad (186)$$

These matrices have the property of being a matrix cube root

$$\left(\sum_{i=1}^{3} x_i d^i\right)^3 = \sum_{i=1}^{3} x_i^3 \quad (187)$$

but they are not $q = 3$ Lorentz or rotation invariant, and hence why we didn't choose them for our proposed Pauli matrices.

For dim = 4 case, we reuse the dim = 3 matrices, and find that

$$N^0 = d^1 \otimes I_3 \quad (188)$$
$$N^1 = d^3 \otimes d^1 \quad (189)$$
$$N^2 = d^3 \otimes d^2 \quad (190)$$
$$N^3 = d^3 \otimes d^3 \quad (191)$$

also satisfy property (3), and are again a matrix cube root

$$\left(\sum_{i=0}^{3} x_i N^i\right)^3 = \sum_{i=0}^{3} x_i^3 \quad (192)$$

However, they are again not $q = 3$ Lorentz or rotation invariant so we can't use them for our Dirac or Maxwell equations.

b. matrix properties One complication of $q > 2$ is that the commutator structure is more involved than just commutators and anti-commutators. Instead we have the slightly more general equation

$$B^\mu B^\nu = w^{k_{\mu\nu}} B^\nu B^\mu \quad (193)$$

TABLE II. Naive matrices commutation table

	D0	D1	D2	D3
D0	0	1	1	1
D1	2	0	1	1
D2	2	2	0	1
D3	2	2	2	0

where w is the appropriate principle root of unity, and $k_{\mu\nu} \in \{0, 1, \cdots, q-1\}$. In particular, for $q = 2$ we have $w = -1$ and $k_{\mu\nu} = 1 - \delta_{\mu\nu}$. In contrast the commutator table for $q = 3$ and our naive matrices is given by table II. The k-form dimension table is given by table III. This table is quite different from the Dirac matrices version, table V, which has the basis collapse property. In contrast

TABLE III. Naive matrices k-form dimension table

k	dim	unique dim
0	1	1
1	4	4
2	10	10
3	16	16
4	19	19
5	16	16
6	10	10
7	4	4
8	1	1

to the later Dirac matrix version (225), here is the exterior derivative applied to a 1-form `get_basis4(2, d*A)`

$$\texttt{D00:} \quad \partial_0 A_0 \quad (194)$$
$$\texttt{D11:} \quad \partial_1 A_1 \quad (195)$$
$$\texttt{D22:} \quad \partial_2 A_2 \quad (196)$$
$$\texttt{D33:} \quad \partial_3 A_3 \quad (197)$$

$$\texttt{D01:} \quad w\partial_1 A_0 + \partial_0 A_1 \quad (198)$$
$$\texttt{D02:} \quad w\partial_2 A_0 + \partial_0 A_2 \quad (199)$$
$$\texttt{D03:} \quad w\partial_3 A_0 + \partial_0 A_3 \quad (200)$$

$$\texttt{D23:} \quad w\partial_3 A_2 + \partial_2 A_3 \quad (201)$$
$$\texttt{D13:} \quad w\partial_3 A_1 + \partial_1 A_3 \quad (202)$$
$$\texttt{D12:} \quad w\partial_2 A_1 + \partial_1 A_2 \quad (203)$$

Here is the exterior derivative applied twice to a 1-form `get_basis4(3, d*d*A)`

$$\texttt{D001:}\quad \partial_0^2 A_1 - \partial_0\partial_1 A_0 \tag{204}$$
$$\texttt{D002:}\quad \partial_0^2 A_2 - \partial_0\partial_2 A_0 \tag{205}$$
$$\texttt{D003:}\quad \partial_0^2 A_3 - \partial_0\partial_3 A_0 \tag{206}$$

$$\texttt{D011:}\quad -w^2\partial_0\partial_1 A_1 + w^2\partial_1^2 A_0 \tag{207}$$
$$\texttt{D022:}\quad -w^2\partial_0\partial_2 A_2 + w^2\partial_2^2 A_0 \tag{208}$$
$$\texttt{D033:}\quad -w^2\partial_0\partial_3 A_3 + w^2\partial_3^2 A_0 \tag{209}$$

$$\texttt{D012:}\quad -w^2\partial_0\partial_1 A_2 - w\partial_1\partial_2 A_0 - \partial_0\partial_2 A_1 \tag{210}$$
$$\texttt{D013:}\quad -w^2\partial_0\partial_1 A_3 - w\partial_1\partial_3 A_0 - \partial_0\partial_3 A_1 \tag{211}$$
$$\texttt{D023:}\quad -w^2\partial_0\partial_2 A_3 - w\partial_2\partial_3 A_0 - \partial_0\partial_3 A_2 \tag{212}$$

$$\texttt{D112:}\quad \partial_1^2 A_2 - \partial_1\partial_2 A_1 \tag{213}$$
$$\texttt{D113:}\quad \partial_1^2 A_3 - \partial_1\partial_3 A_1 \tag{214}$$
$$\texttt{D223:}\quad \partial_2^2 A_3 - \partial_2\partial_3 A_2 \tag{215}$$

$$\texttt{D122:}\quad -w^2\partial_1\partial_2 A_2 + w^2\partial_2^2 A_1 \tag{216}$$
$$\texttt{D133:}\quad -w^2\partial_1\partial_3 A_3 + w^2\partial_3^2 A_1 \tag{217}$$
$$\texttt{D233:}\quad -w^2\partial_2\partial_3 A_3 + w^2\partial_3^2 A_2 \tag{218}$$

$$\texttt{D123:}\quad -w^2\partial_1\partial_2 A_3 - w\partial_2\partial_3 A_1 - \partial_1\partial_3 A_2 \tag{219}$$

Finally, $ddd A = 0$, and so `get_basis4(4, d*d*d*A)` returns no equations.

2. *Dirac $q = 3$ differential forms*

Given the naive matrix representation, we now move on to the Dirac matrix representation. It is an easy matter to write down our new exterior derivative in terms of the Dirac matrices

$$d = \partial_\mu B^\mu \tag{220}$$

In matrix form

$$d = \begin{bmatrix} \partial_0 & w\bar{\nabla}\cdot\tau & \nabla\cdot T \\ w^2\nabla\cdot T & w\partial_0 & w\bar{\nabla}\cdot\tau \\ w\bar{\nabla}\cdot\tau & w\nabla\cdot T & w^2\partial_0 \end{bmatrix} \tag{221}$$

where

$$\bar{\nabla}_1 \equiv \partial_4 \equiv \partial_1^2 - \partial_2\partial_3 \tag{222}$$
$$\bar{\nabla}_2 \equiv \partial_5 \equiv \partial_2^2 - \partial_1\partial_3 \tag{223}$$
$$\bar{\nabla}_3 \equiv \partial_6 \equiv \partial_3^2 - \partial_1\partial_2 \tag{224}$$

The corresponding commutation table is given by table IV. This commutation structure is more involved than that of table II, hence in practice one must use a computer algebra system to do the calculations. $D^{\mu\mu} \neq 0$, and effectively 7 basis elements rather than 4, further complicate the algebra.

TABLE IV. q = 3, Dirac matrices commutation table

	D0	D1	D2	D3	D4	D5	D6
D0	0	2	2	2	1	1	1
D1	1	0	1	2	2	1	0
D2	1	2	0	1	0	2	1
D3	1	1	2	0	1	0	2
D4	2	1	0	2	0	1	2
D5	2	2	1	0	2	0	1
D6	2	0	2	1	1	2	0

a. basis matrix collapse Finally, a note on 'basis matrix collapse'. It was initially unexpected that there are less unique matrices for a given k-form, see table V, when using Dirac matrices than one would naively assume. This does not happen for the naive matrix representation, see table III, and so must be a result of the presence of cross terms in $J_3(E, p, q)$ in contrast to (192).

TABLE V. q = 3, Dirac k-form, unique dimension table

k	dim	unique dim
0	1	1
1	7	7
2	28	9
3	77	9
4	161	9
5	266	9
6	357	9
7	393	9
8	357	9
9	266	9
10	161	9
11	77	9
12	28	9
13	7	7
14	1	1

One consequence is that the exterior derivative applied to a 1-form, `get_basis4(2, d*A)`, is less 'clean' than that for the naive case

$$\texttt{D00:}\quad w^2\partial_1 A_4 + w^2\partial_2 A_5 + w^2\partial_3 A_6 + w\partial_4 A_1 + w\partial_5 A_2 + w\partial_6 A_3 + \partial_0 A_0 \tag{225}$$
$$\texttt{D15:}\quad w\partial_4 A_3 + w\partial_5 A_1 + w\partial_6 A_2 + \partial_1 A_5 + \partial_2 A_6 + \partial_3 A_4 \tag{226}$$
$$\texttt{D16:}\quad \partial_1 A_6 + \partial_2 A_4 + \partial_3 A_5 + \partial_4 A_2 + \partial_5 A_3 + \partial_6 A_1 \tag{227}$$

$$\texttt{D01:}\quad w^2\partial_1 A_0 + w^2\partial_5 A_6 + w\partial_4 A_4 + \partial_0 A_1 + \partial_6 A_5 \tag{228}$$
$$\texttt{D02:}\quad w^2\partial_2 A_0 + w^2\partial_6 A_4 + w\partial_5 A_5 + \partial_0 A_2 + \partial_4 A_6 \tag{229}$$
$$\texttt{D03:}\quad w^2\partial_3 A_0 + w^2\partial_4 A_5 + w\partial_6 A_6 + \partial_0 A_3 + \partial_5 A_4 \tag{230}$$

$$\texttt{D04:}\quad w^2\partial_1 A_1 + w\partial_3 A_2 + w\partial_4 A_0 + \partial_0 A_4 + \partial_2 A_3 \tag{231}$$
$$\texttt{D05:}\quad w^2\partial_2 A_2 + w\partial_1 A_3 + w\partial_5 A_0 + \partial_0 A_5 + \partial_3 A_1 \tag{232}$$
$$\texttt{D06:}\quad w^2\partial_3 A_3 + w\partial_2 A_1 + w\partial_6 A_0 + \partial_0 A_6 + \partial_1 A_2 \tag{233}$$

D. Maxwell's equations

With the above preparation, we are ready to give our new matrix representation of the Maxwell equations

$$G = dA \tag{234}$$
$$ddG = J \tag{235}$$

where A is the electromagnetic 1-form, J is the electromagnetic source 1-form, and G is the Faraday matrix. Because of the dd operator, the expanded out Maxwell equations are significantly more complicated than the $q = 2$ case. So all further calculations have been done using the SageMath/Jupyter notebook computer algebra system.

The first step is to label the components of G. In $q = 2$ we label the components of F as E_i and B_i, so for $q = 3$ we will follow similarly. To proceed, we need to know which components of dA belong in the same class, and to do that we label the quadrants of a matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{236}$$

Then assign quadrants to classes

$$\texttt{Class 1:}\quad [1, 5, 9] \tag{237}$$
$$\texttt{Class 2:}\quad [3, 4, 8] \tag{238}$$
$$\texttt{Class 3:}\quad [2, 6, 7] \tag{239}$$

Then using the permutation matrix P and the 'quadrant' matrices Q_i

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{240}$$

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{241}$$

we extract out the quadrants of A using $Q_i P^j A Q_i$, for some matrix of interest A, and $i, j \in \{1, 2, 3\}$. For 2-forms, this produces the following classes of basis matrices `classify_basis_matrices(2)`

$$\texttt{Class 1:}\quad [D00, D15, D16] \tag{242}$$
$$\texttt{Class 2:}\quad [D01, D02, D03] \tag{243}$$
$$\texttt{Class 3:}\quad [D04, D05, D06] \tag{244}$$

Hence we can label the components of G, up to factors of w

$$G = E_1 D01 + E_2 D02 + E_3 D03 \tag{245}$$
$$+ B_1 D04 + B_2 D05 + B_3 D06 \tag{246}$$
$$+ C_1 D00 + C_2 D15 + C_3 D16 \tag{247}$$

where the sub-components E_i, B_i and C_i are given by the equations starting with (225).

Using `get_basis4(4, d*d*G)` we can extract out the Maxwell equations

$$\bar{Z}\cdot(C1,2C_2,-C_3)+we\cdot(E_1,E_2,E_3)+w^2\bar{e}\cdot(B_1,B_2,B_3)-w^2\partial_0\nabla\cdot(B_1,B_2,B_3)-w\partial_0\bar{\nabla}\cdot(E_1,E_2,E_3)$$
$$\bar{Z}\cdot(C2,2C_3,-C_1)+w^2e\cdot(E_3,E_1,E_2)+w\bar{e}\cdot(B_2,B_3,B_1)-w\partial_0\nabla\cdot(B_2,B_3,B_1)-w^2\partial_0\bar{\nabla}\cdot(E_3,E_1,E_2)$$
$$\bar{Z}\cdot(C3,2C_1,-C_2)+e\cdot(E_2,E_3,E_1)+\bar{e}\cdot(B_3,B_1,B_2)-\partial_0\nabla\cdot(B_3,B_1,B_2)-\partial_0\bar{\nabla}\cdot(E_2,E_3,E_1)$$

$$\bar{Z}\cdot(E1,2w^2E_3,-wE_2)+we\cdot(B_1,wB_3,w^2B_2)+w^2\bar{e}\cdot(C_1,w^2C_2,wC_3)-w^2\partial_0\nabla\cdot(C_1,w^2C_2,wC_3)-w\partial_0\bar{\nabla}\cdot(B_1,wB_3,w^2B_2)$$
$$\bar{Z}\cdot(E2,2w^2E_1,-wE_3)+w^2e\cdot(wB_3,w^2B_2,B_1)+w^2\bar{e}\cdot(wC_3,C_1,w^2C_2)-w^2\partial_0\nabla\cdot(wC_3,C_1,w^2C_2)-w^2\partial_0\bar{\nabla}\cdot(wB_3,w^2B_2,B_1)$$
$$\bar{Z}\cdot(E3,2w^2E_2,-wE_1)+e\cdot(w^2B_2,B_1,wB_3)+w^2\bar{e}\cdot(w^2C_2,wC_3,C_1)-w^2\partial_0\nabla\cdot(w^2C_2,wC_3,C_1)-\partial_0\bar{\nabla}\cdot(w^2B_2,B_1,wB_3)$$

$$\bar{Z}\cdot(B1,2wB_2,-w^2B_3)+we\cdot(C_1,w^2C_3,wC_2)+w^2\bar{e}\cdot(E_1,wE_3,w^2E_2)-w^2\partial_0\nabla\cdot(E_1,wE_3,w^2E_2)-w\partial_0\bar{\nabla}\cdot(C_1,w^2C_3,wC_2)$$
$$\bar{Z}\cdot(B2,2wB_3,-w^2B_1)+we\cdot(wC_2,C_1,w^2C_3)+\bar{e}\cdot(wE_3,w^2E_2,E_1)-\partial_0\nabla\cdot(wE_3,w^2E_2,E_1)-w\partial_0\bar{\nabla}\cdot(wC_2,C_1,w^2C_3)$$
$$\bar{Z}\cdot(B3,2wB_1,-w^2B_2)+we\cdot(w^2C_3,wC_2,C_1)+w\bar{e}\cdot(w^2E_2,E_1,wE_3)-w\partial_0\nabla\cdot(w^2E_2,E_1,wE_3)-w\partial_0\bar{\nabla}\cdot(w^2C_3,wC_2,C_1)$$

where we have used

$$\bar{Z}=(e_0,Z_1,Z_2) \tag{248}$$
$$e=(e_1,e_2,e_3) \tag{249}$$
$$\bar{e}=(e_4,e_5,e_6) \tag{250}$$
$$\nabla=(\partial_1,\partial_2,\partial_3) \tag{251}$$
$$\bar{\nabla}=(\partial_4,\partial_5,\partial_6) \tag{252}$$

And in turn

$$e_0=\partial_0^2-\partial_1\partial_4-\partial_2\partial_5-\partial_3\partial_6 \tag{253}$$
$$e_1=\partial_1^2-\partial_2\partial_3 \tag{254}$$
$$e_2=\partial_2^2-\partial_1\partial_3 \tag{255}$$
$$e_3=\partial_3^2-\partial_1\partial_2 \tag{256}$$
$$e_4=\partial_4^2-\partial_5\partial_6 \tag{257}$$
$$e_5=\partial_5^2-\partial_4\partial_6 \tag{258}$$
$$e_6=\partial_6^2-\partial_4\partial_5 \tag{259}$$
$$Z_1=\partial_1\partial_6+\partial_2\partial_4+\partial_3\partial_5 \tag{260}$$
$$Z_2=\partial_1\partial_5+\partial_2\partial_6+\partial_3\partial_4 \tag{261}$$

It is apparent these equations are going to be difficult to solve in practice.

In calculations of $G\wedge G\wedge\star\star G$ we find that we can only obtain a Lorentz invariant term if C_2 and C_3 are zero. Giving

$$G\wedge G\wedge\star\star G=J_3(C_1,\mathbf{E},\mathbf{B})d^I \tag{262}$$

where d^I is the $q=3$ volume element. Hence we propose the Maxwell action

$$S_{Maxwell}=\int_\Omega G\wedge G\wedge\star\star G+A\wedge A\wedge\star\star J \tag{263}$$

up to normalization coefficients. We then attempt to solve $\delta S=0$ using

$$\delta S=\lim_{\epsilon\to0}\frac{S[A+\epsilon\eta]-S[A]}{\epsilon} \tag{264}$$

where η is a 1-form, and ϵ is a small scalar. Resulting in

$$\delta S=\int_\Omega 3d\eta\wedge dA\wedge\star\star dA+\eta\wedge A\wedge\star\star J+A\wedge\eta\wedge\star\star J \tag{265}$$

by making use of the property

$$A\wedge A\wedge\star\star B=B\wedge A\wedge\star\star A=A\wedge B\wedge\star\star A \tag{266}$$

In contrast if we consider $\delta\delta S=0$

$$\delta\delta S=\lim_{\epsilon\to0}\frac{S[A+\epsilon\eta]-2S[A]+S[A-\epsilon\eta]}{\epsilon^2} \tag{267}$$

we obtain

$$\delta\delta S=\int_\Omega 3d\eta\wedge d\eta\wedge\star\star G+\eta\wedge\eta\wedge\star\star J \tag{268}$$

We would like to proceed using integration by parts, and Stoke's theorem, unfortunately, taking the exterior derivative of a wedged term is not so clean in $q=3$. For example in $q=2$ we have

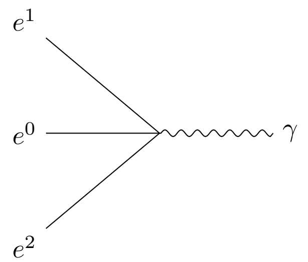
$$d(\alpha\wedge\beta)=(d\alpha)\wedge\beta+(-1)^p\alpha\wedge d\beta \tag{269}$$

where α is a p-form. Due to the complicated 'commutation structure' of $q=3$, the equivalent of (269) is not known, and may not exist. Hence the solution to $\delta S_{Maxwell}=0$ or $\delta\delta S_{Maxwell}=0$ is also not known.

Given that $G=dA$ it is clear that at least one set of Maxwell equations is ddG. Further, given that these equations produce equations for each of C_i, E_i and B_i, and $dF=J$ was sufficient in the matrix representation for $q=2$, then we assume $ddG=J$ is also sufficient in the $q=3$ case. However, this assumption may turn out to be wrong.

E. Future work

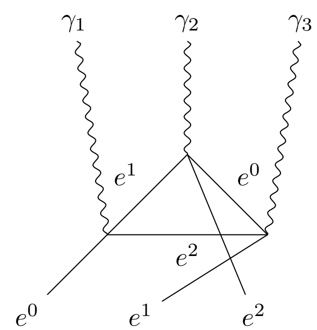
Eventually it would be desirable to consider quantum field theory in the case $q>2$, or even just $q=3$. To that end, we assume the $q = 3$ vertex takes the form



(270)

where e^0, e^1, e^2 are particle, and 1st and 2nd anti-particles respectively, as we saw in the Dirac equation, while γ is the $q = 3$ photon. From there we can construct the diagram for

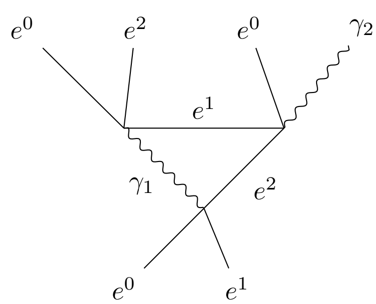
$$e^0 + e^1 + e^2 \to \gamma + \gamma + \gamma \tag{271}$$



(272)

and the diagram for

$$e^0 + e^1 \to e^0 + e^2 + e^0 + \gamma \tag{273}$$



(274)

Though it appears this second diagram does not satisfy $q = 3$ lepton number conservation, since the initial state has lepton number $1 + w$ and the final state has lepton number $1 + w^2 + 1 = 1 - w$.

Using the $q = 3$ extension to the bra-ket notation of an inner product $\langle|\phi\rangle|\chi\rangle|\psi\rangle$, we can write these down as

$$\begin{aligned}\mathcal{M} \sim h^{\alpha\beta\gamma} &\langle|\gamma_2\rangle|e^1 e^0\rangle H_I|e^2\rangle_\alpha \\ &\langle|e^0 e^2\rangle|\gamma_3\rangle H_I|e^1\rangle_\beta \\ &\langle|\gamma_1\rangle|e^1\rangle H_I|e^0 e^2\rangle_\gamma\end{aligned} \tag{275}$$

and

$$\begin{aligned}\mathcal{M} \sim h^{\alpha\beta\gamma} &\langle|\gamma_2\rangle|e^0\rangle H_I|e^1 e^2\rangle_\alpha \\ &\langle|e^1\rangle|e^0 e^2\rangle H_I|\gamma_1\rangle_\beta \\ &\langle|e^2\rangle|\gamma_1\rangle H_I|e^0 e^1\rangle_\gamma\end{aligned} \tag{276}$$

Interpretation of the physics and actual calculations of these are left for another note.

The extension of general relativity to $q > 2$ would be even more technical than that for quantum field theory, and will not be considered by the current authors. Given that $q > 2$ general relativity predictions almost certainly differ from the standard $q = 2$ case, as seen by the differing metric tensors $g^{\mu\nu}$ vs $h^{\alpha\beta\gamma}$ for special relativity, the probability that q particles correspond to dark matter is reduced.

F. Conclusion

In this note we have provided a brief introduction to q-theory for the Dirac and Maxwell equations, and that of exterior derivatives and differential forms. We hope to have shown that current physics and mathematics has a notable $q = 2$ bias, and that there is actually a parallel set of mathematics to be explored. Even though there are clear contradictions with known physics, we hope the current work is still of some theoretical interest, and forms the seed for a larger project. If it can be shown that $q > 2$ particles interact gravitationally with those of $q = 2$, then perhaps the above work will serve as a description of dark matter.

G. Open problems

The above is to be taken as a seed for a larger project, and in that spirit, here are some of the problems left untouched by the current authors (some of which are more technical than others)

- physical interpretation of it all, including
  - complex valued eigenvalues of the Pauli/spin matrices
  - complex valued energy solutions of the Dirac equation
  - complex valued charges of the anti-particles
  - complex values in the space-time metric $h^{\alpha\beta\gamma}$
  - $q = 3$ photons
- the development of the mathematics for $q = 3$ groups
  - what is the Lorentz invariant form for the $q = 3$ Dirac Lagrangian?
  - the derivation of Maxwell's equations from the $q = 3$ Maxwell action

- are there gauge bosons other than that generated by $q = 3$ $U(1)$?
- does $SU(N)$ exist for $q > 2$?
- are our proposed Pauli matrices a representation for $q = 3$ $SU(3)$?

- the development of the mathematics for $q = 3$ differential forms
  - whether to use 'naive' differential forms, or the Dirac matrix inspired version
  - the $q > 2$ product rule for an exterior derivative of a wedged term, eg $d(\alpha \wedge \beta)$
  - the $q > 2$ general Stokes Theorem
  - the $q > 2$ boundary of a region, eg $\partial_3\Omega$ and $\partial_3\partial_3\Omega$
- further development of the proposed Dirac and Maxwell equations
- the development of $q > 2$ Quantum Field Theory
- the development of $q > 2$ General Relativity
- whether there is a connection between q theory and dark matter

---

[1] Paul Adrien Maurice Dirac. The quantum theory of the electron. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 117(778):610–624, 1928.

[2] Paul Adrien Maurice Dirac. A new notation for quantum mechanics. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 35, pages 416–418. Cambridge University Press, 1939.

[3] Albert Einstein et al. On the electrodynamics of moving bodies. *Annalen der physik*, 17(10):891–921, 1905.

[4] G.J. Morrison. q-theory. `https://github.com/GarryMorrison/q-theory`, 2022.

[5] M.E. Peskin and D.V. Schroeder. *An Introduction To Quantum Field Theory*. Frontiers in Physics. Avalon Publishing, 1995.

[6] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.0)*, 2020-01-01. `https://www.sagemath.org`.

[7] A. Zee. *Quantum Field Theory in a Nutshell: Second Edition*. In a Nutshell. Princeton University Press, 2010.