CMSE 820 HW 11

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1 Question 1

Let Ω be a non-empty set and $\mathcal{A}$ be a collection of events. Define

$$\mathcal{I}(\mathcal{A}) = \{\mathcal{F} | \mathcal{F} \text{ is a } \sigma\text{-algebra and } \mathcal{F} \supseteq \mathcal{A}\}.$$

and

$$\sigma(\mathcal{A}) = \cap_{\mathcal{F} \in \mathcal{I}(\mathcal{A})} \mathcal{F}.$$

Prove that $\sigma(\mathcal{A})$ is a σ-algebra. Note that $\sigma(\mathcal{A})$ is called the σ-algebra generated by $\mathcal{A}$.

Proof: Let $a \subseteq \mathcal{A}$ be any subset of $\mathcal{A}$. By the definition of $\sigma(\mathcal{A})$,

$$\forall\, \mathcal{F} \in \mathcal{I},\, a \in \mathcal{F}.$$

Following from the facts that $\mathcal{F}$ is a σ-algebra of $\mathcal{A}$, we have, for any sequence of subsets of $\mathcal{A}$, $\{a_i\}_{i=1}^{\infty}$,

$$\begin{aligned}
&[\forall\, \mathcal{F} \in \mathcal{I},\, (\mathcal{A} \in \mathcal{F} \implies \emptyset \in \mathcal{F})] \implies \emptyset \in \sigma(\mathcal{A}),\\
&\forall\, \mathcal{F} \in \mathcal{I},\, a_i^c \in \mathcal{F} \implies a_i^c \in \sigma(\mathcal{A}),\\
&\forall\, \mathcal{F} \in \mathcal{I},\, \cup_{i=1}^{\infty} a_i \in \mathcal{F} \implies \cup_{i=1}^{\infty} a_i \in \sigma(\mathcal{A}).
\end{aligned}$$

$\sigma(\mathcal{A})$ contains $\emptyset$, and is closed under taking complement and countable union. Thus, $\sigma(\mathcal{A})$ is a σ-algebra of $\mathcal{A}$.

2 Question 2

Let $\Omega = \mathbb{R}$. Define

$$\mathcal{A} = \{(a, b) | a, b \in \mathbb{R}, a \leq b\}.$$

Then $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. Let

$$\mathcal{D} = \{(-\infty, a) : a \in \mathbb{R}\}.$$

Prove that $\sigma(\mathcal{D}) = \mathcal{B}(\mathbb{R})$.

Proof:
To show that $\sigma(\mathcal{D}) \supseteq \sigma(\mathcal{A})$, it suffices to show that for any $a, b \in \mathbb{R}$ such that $a \leq b$,

$$(a, b) \in \sigma(\mathcal{D}).$$

Note that σ-algebra is closed under countable intersection (which amounts to the complement of the countable union of the complements). So

$$\cap_{n=1}^{\infty}(-\infty, a + \frac{1}{n}) \in \sigma(\mathcal{D}).$$

In fact, we claim that $\cap_{n=1}^{\infty}(-\infty, a + \frac{1}{n}) = (-\infty, a]$, because

$$\forall\, x \in (-\infty, a],\ x \in (-\infty, a + \frac{1}{n}) \implies (-\infty, a] \subseteq \cap_{n=1}^{\infty}(-\infty, a + \frac{1}{n}),$$

and

$$\forall\, x > a,\ \exists\, n \text{ such that } x \notin (-\infty, a + \frac{1}{n}) \implies x \notin \cap_{n=1}^{\infty}(-\infty, a + \frac{1}{n}).$$

Now we can simply express (a, b) as

$$(a, b) = (a, \infty) \cap (-\infty, b) = \left(\cap_{n=1}^{\infty} \left(-\infty, a + \frac{1}{n} \right) \right)^c \cap (-\infty, b) \in \sigma(\mathcal{D}).$$

So $\sigma(\mathcal{A}) \subseteq \sigma(\mathcal{D})$.

Next, we will need to show that $\sigma(\mathcal{A}) \supseteq \sigma(\mathcal{D})$. That is, for any $a \in \mathbb{R}$, we need to show that $(-\infty, a) \in \sigma(\mathcal{A})$.

Consider a sequence of elements $\{A_n\}_{n=1}^{\infty}$ in $\sigma(\mathcal{A})$ defined by $A_n = (a - n, a)$. It is easy to see that

$$\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty}(a - n, a) = (-\infty, a) \subseteq \sigma(\mathcal{A}).$$

Thus, $\sigma(\mathcal{A}) = \sigma(\mathcal{D}) = \mathcal{B}(\mathbb{R})$.

3 Question 3

Pick $X \in [-1, 1]^d$ uniformly at random. Prove that $\|X\|$ is tightly concentrated around $\sqrt{d/3}$. Namely,

$$\Pr\left(\left| \|X\|^2 - \frac{d}{3} \right| > \epsilon d \right) \to 0,$$

as $d \to \infty$.

Proof: First of all, we compute a few moments of a uniform random variable $x \in [-1, 1]$,

$$\begin{aligned} \mathrm{E}[x] &= 0, \\ \mathrm{E}[x^2] &= \frac{1}{3}, \\ \mathrm{E}[x^4] &= \int_{-1}^{1} \frac{1}{2} x^4 dx = \frac{1}{5}. \end{aligned}$$

Clearly, $\|X\|^2$ should concentrated around $\frac{d}{3}$, because $\mathrm{E}[\|X\|^2] = \mathrm{E}[\sum_{i=1}^{d} x_i^2] = \sum_{i=1}^{d} \mathrm{E}[x^2] = \frac{d}{3}$. We apply the Chebyshev's inequality to $\|X\|^2$,

$$\Pr\left(\left| \|X\|^2 - \frac{d}{3} \right| > \epsilon d \right) \le \frac{\mathrm{E}[(\|X\|^2 - d/3)^2]}{(\epsilon d)^2}.$$

We want to show that $\mathrm{E}[(\|X\|^2 - d/3)^2]$ is finite.

$$\begin{aligned} \mathrm{E}[(\|X\|^2 - d/3)^2] &= \mathrm{E}\left[[\sum_{i=1}^{d} (x_i^2 - \frac{1}{3})]^2 \right] \\ &= \sum_{i=1}^{d} \mathrm{E}[(x_i^2 - 1/3)^2] + \sum_{i=1, j>i}^{d} \mathrm{E}[(x_i^2 - 1/3)(x_j^2 - 1/3)] \\ &= \sum_{i=1}^{d} \mathrm{E}[(x^2 - 1/3)^2] = \frac{4d}{45}. \end{aligned}$$

So the probability goes to 0 as d approaches ∞ at a rate of $4/45\epsilon^2 d$.

4 Question 4

4.1 a

A random variable X with mean $\mathrm{E}[X] = \mu$ is sub-Gaussian if there is a positive number σ such that, for all $\lambda \in \mathbb{R}$,

$$\mathrm{E}[e^{\lambda(X-\mu)}] \le e^{\sigma^2 \lambda^2 / 2}.$$

If X is a sub-Gaussian random variable with σ as its sub-Gaussian parameter, prove that

$$\Pr(|X - \mu| \ge t) \le 2e^{-\frac{t^2}{2\sigma^2}}, \quad \text{for all } t \in \mathbb{R}.$$

Proof: For any $t \in \mathbb{R}$, we apply the Cramer's inequality,

$$\begin{aligned}
\Pr(|X-\mu| \geq t) &= 2\Pr[(X-\mu) \geq t] \leq 2\inf_{\lambda>0} e^{-\lambda t}\mathrm{E}[e^{\lambda(X-\mu)}] \\
&\leq 2\inf_{\lambda>0} \exp\left[\frac{\sigma^2}{2}\lambda^2 - t\lambda\right] \\
&\leq 2\exp\left(-\frac{t^2}{2\sigma^2}\right).
\end{aligned}$$

4.2 b

Suppose that the random variables X_i, $i = 1, ..., n$ are independent, and X_i has mean μ_i and sub-Gaussian parameter σ_i. Prove that for all $t \geq 0$ we have

$$\Pr\left[\sum_{i=1}^n (X_i - \mu_i) \geq t\right] \leq \exp\left\{-\frac{t^2}{2\sum_{i=1}^n \sigma_i^2}\right\}.$$

Proof: For any $t \in \mathbb{R}$, we again apply the Cramer's inequality,

$$\begin{aligned}
\Pr\left[\sum_{i=1}^n (X_i - \mu_i) \geq t\right] &\leq \inf_{\lambda>0} e^{-\lambda t}\mathrm{E}[e^{\lambda\sum_{i=1}^n (X_i-\mu_i)}] \\
&= \inf_{\lambda>0} \prod_{i=1}^n e^{-\lambda t/n}\mathrm{E}[e^{\lambda(X_i-\mu_i)}] \\
&\leq \inf_{\lambda>0} \prod_{i=1}^n \exp\left[\frac{\sigma_i^2}{2}\lambda^2 - \frac{t}{n}\lambda\right] \\
&= \inf_{\lambda>0} \exp\left[\frac{\sum_{i=1}^n \sigma_i^2}{2}\lambda^2 - t\lambda\right] \\
&= \exp\left(-\frac{t^2}{2\sum_{i=1}^n \sigma_i^2}\right).
\end{aligned}$$