# CMSE 820 HW6

This HW is due on Oct 26th at 11:59 pm.

**Question 1**: Implement the classic MDS we learned in the class with input $D_{n\times n}$ and the desired number of dimension $d$. You are only allowed to use SVD or Eigen decomposition functions from existing packages.

**Question 2**: MDS of cities: Go to the following website geobytes.com Perform the following experiment.

(1) Input the following cities: Boston, Chicago, DC, Denver, LA, Miami, New York City, Seattle, and San Francisco, and collect the pairwise air traveling distances shown on the website in to a matrix $D_1$; Also,

(2) Run your own codes of classic MDS and plot the normalized eigenvalues $\lambda_i/(\sum_i \lambda_i)$ in a descending order of magnitudes, analyze your observations (did you see any negative eigenvalues? if yes, why?);

(3) Make a scatter plot of those cities using top 2 or 3 eigenvectors, and analyze your observations.

(4) Now, go to google map, and replace the distance matrix $D$ with driving distance and repeat the analysis. Interpret your results.

(5) Now, go to google map, and replace the distance matrix $D$ with driving hours and repeat the analysis. Interpret your results.

**Question 3**: *Positive Semi-definiteness:* Recall that a $n$-by-$n$ real symmetric matrix $K$ is called positive semi-definite (p.s.d. or $K \succeq 0$) iff for every $x \in \mathbb{R}^n, x^T K x \geq 0$. Assume $A \succeq 0$ and $B \succeq 0$ ($A, B \in \mathbb{R}^{n\times n}$).

(1) Show that $K \succeq 0$ if and only if its eigenvalues are all nonnegative.

(2) Show that $d_{ij} = K_{ii} + K_{jj} - 2K_{ij}$ is a squared distance function, i.e. there exists vectors $u_i, u_j \in \mathbb{R}^n$ for $1 \leq i, j \leq n$ such that $d_{ij} = \|u_i - u_j\|^2$.

(3) Show that $A + B \succeq 0$ (elementwise sum), and $A \circ B = \{A_{ij}B_{ij}\} \succeq 0$ (Hadamard product or elementwise product). Show that the eigen values of $AB$ are all positive.

(4) If $A \succeq 0$ and $c \geq 0$, then $cA \succeq 0$

(5) If $A \succeq 0$ and $C$ can be written as $C = \{t_{[i]}A_{ij}t_{[j]}\}$ for $\forall t \in \mathbb{R}^n$, then $C \succeq$.

(6) Show that the Hadamard integral power $A^{\circ p} = \{A_{ij}^p\}$ with $p \in \mathbb{N}$ and Hadamard exponential $\exp(\circ A) = \{\exp(A_{ij})\}$ are p.s.d..

**Question 4**: Now, let's prove the following claims

1 **Infinitely Divisible Kernels** Let $C = \{c_{ij}\}$ be a symmetric matrix, and define $B = \exp(\circ - C)$ that is $b_{ij} = \exp(-c_{ij})$. hen $B$ is infinitely divisible if and only if $C$ is conditionally negative definite. (hint: for the '$\Rightarrow$' part, you can define $a_{ij}(\lambda) = (1 - b_{ij}^{\lambda})/\lambda$ and then take limit as $\lambda \to 0$. For the '$\Leftarrow$' part, you can define $F = -H_\alpha C H_\alpha'$ and write down the element-wise expression of $F$, and take Hadamard exponetial.)

2 **Gaussian Kernel** Let $D^{(2)}$ be a squared Euclidean distance. Then, for any $\lambda \geq 0$, $\exp(\circ - D^{(2)})$ is p.d., and $\tilde{D}_{\lambda}^{(2)} = \{\tilde{D}_{\lambda}^{(2)}[ij]\} = \{1 - \exp(-\lambda D^{(2)}[ij])\}$ is a squared Euclidean distance. Namely, $\tilde{D}_{\lambda}^{(2)}$ is c.n.d with a zero diagnoal.

More generally (you don't need to prove the following claim), any mixture of $\tilde{D}_{\lambda}^{(2)} = \{1 - \exp(-\lambda D^{(2)}[ij])\}$ over $\lambda \geq 0$ is still a squared Euclidean distance, leading to the central part of Schoenberg transformation.

$$\phi(D^{(2)}[ij]) = \int_0^\infty \frac{1 - \exp(-\lambda D^{(2)}[ij])}{\lambda} g(\lambda) d\lambda.$$