CMSE 820 HW11

This HW is due on Dec 5th at 11:59pm.

Question 1: Let Ω be a non-empty set and $\mathcal{A}$ be a collection of events. Define

$$\mathcal{I}(\mathcal{A}) = \{\mathcal{F} |\ \mathcal{F} \text{ is a } \sigma\text{-algebra and } \mathcal{F} \supseteq \mathcal{A}\}.$$

Then define

$$\sigma(\mathcal{A}) = \cap_{\mathcal{F}\in\mathcal{I}(\mathcal{A})}\mathcal{F}.$$

Prove $\sigma(\mathcal{A})$ is a $\sigma-$algebra. Note that $\sigma(\mathcal{A})$ is often called the $\sigma-$algebra generated by $\mathcal{A}$.

Question 2: Let $\Omega = \mathbb{R}$. Define

$$\mathcal{A} = \{(a, b) \text{ for } a, b \in \mathbb{R} \text{ and } a \le b\},$$

then $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$ is the Borel $\sigma-$algebra on $\mathbb{R}$. Let

$$\mathcal{D} = \{(-\infty, a) : a \in \mathbb{R}\},$$

Prove that $\sigma(\mathcal{D}) = \mathcal{B}(\mathbb{R})$.

Question 3: Pick a point $X \in [-1, 1]^d$ uniformly at random. Namely, $X = (X_[i], \ldots, X_[d])$. Prove that $\|X\|$ is tightly concentrated around $\sqrt{d/3}$. Namely,

$$\Pr\Big(\Big| \|X\|^2 - \frac{d}{3} \Big| > \epsilon d \Big) \to 0,$$

as $d \to \infty$. (You need to figure out the rate that the probability shrinks to zero as d goes to infinity.)

Question 4:

a. A random varable X with mean $\mu = \mathbb{E}[X]$ is sub-Gaussian if there is a positive number σ such that

$$\mathbb{E}(e^{\lambda(X-\mu)}) \le e^{\sigma^2\lambda^2/2}, \quad \text{for all } \lambda \in \mathbb{R}$$

where the constant σ is referred to as the sub-Gaussian parameter.

If X is a sub-Gaussian random variable with σ as its sub-Gaussian parameter, prove that

$$\mathbb{P}[|X - \mu| \ge t] \le 2e^{-\frac{t^2}{2\sigma^2}} \ \text{ for all } t \in \mathbb{R}.$$

b. Let's introduce the Hoeffding bound for sub-Gaussian random variables. Suppose taht the random variables X_i, $i = 1, \ldots, n$ are indepdendent, and X_i has mean μ_i and sub-Gaussian parameter σ_i. Prove that for all $t \geq 0$ we have

$$\mathbb{P}\left[\sum_{i=1}^{n}(X_i - \mu_i) \geq t\right] \leq \exp\left\{-\frac{t^2}{2\sum_{i=1}^{n}\sigma_i^2}\right\}.$$

Hint: you may need to use the fact that for two indepdent random variables X and Y, we have $\mathbb{E}[e^{XY}] = \mathbb{E}[e^X]\mathbb{E}[e^Y]$.