CMSE 820 HW 7

Hao Lin

November 2, 2018

1 Question 1

Let L be a bounded linear functional on a Hilbert space $\mathcal{H}$. Show that its null space $\text{null}(L) = \{f \in \mathcal{H} : L(f) = 0\}$ is bounded.
Proof: To show that $\text{null}(L)$ is closed, it suffices to show that the limit of every convergent sequence in $\text{null}(L)$ lies in $\text{null}(L)$.

Let $\{f_n\}$ be a sequence in $\text{null}(L)$ converging to some limit point f, i.e., $\forall\, n \in \mathbb{N}$, $L(f_n) = 0$ and $\lim_{n\to\infty} f_n = f$. The convergence of the sequence implies that

$$\forall\, \epsilon > 0,\, \exists\, N \in \mathbb{N},\, \forall n > N,\, \|f - f_n\| < \epsilon.$$

If follows that

$$|L(f)| = |L(f) - L(f_n)| = |L(f - f_n)| \leq \|L\|\|f - f_n\| < \|L\|\epsilon \implies L(f) = 0,$$

because $\|L\|$ is finite and ϵ can be an arbitrarily small positive number. Thus, $f \in \text{null}(L) = \{f \in \mathcal{H} : L(f) = 0\}$ and $\text{null}(L) = \{f \in \mathcal{H} : L(f) = 0\}$ is closed.

2 Question 2

Show that the kernel function associated with any reproducing kernel Hilbert space must be unique.
Proof: Let $k_1(\cdot,\cdot)$ and $k_2(\cdot,\cdot)$ be two reproducing kernels associated with an arbitrary RKHS $\mathcal{H}$ over $\mathcal{X}$. Note that $\forall x \in \mathcal{X}$, $k_1(\cdot,x) \in \mathcal{H}$ and $k_2(\cdot,x) \in \mathcal{H}$. Moreover, by the reproducing property, we have

$$\forall\, x,\, y\, \in \mathcal{X},\, k_1(x,y) = k_1(y,x) = \langle k_1(\cdot,x), k_2(\cdot,y)\rangle_{\mathcal{H}} = \langle k_2(\cdot,y), k_1(\cdot,x)\rangle_{\mathcal{H}} = k_2(x,y).$$

So the reproducing kernel is unique.

3 Question 3

Let $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a p.s.d. kernel and let $f : \mathcal{X} \to [0, \infty)$ be an arbitrary function. Show that $\tilde{k}(x, y) = f(x)k(x, y)f(y)$ is also a p.s.d. kernel.
Proof: $\forall\, N \in \mathbb{N}^+$, $\forall\, x_1, x_2, ..., x_N \in \mathcal{X}$, $\forall\, c_1, c_2, ..., c_N \in \mathbb{R}$

$$\sum_{i,j} c_i c_j f(x_i) k(x_i, x_j) f(x_j) = \sum_{i,j} [c_i f(x_i)][c_j f(x_j)] k(x_i, x_j) \geq 0,$$

which implies immediately the positive semi-definiteness of $\tilde{k}(x, y)$. (The last inequality follows from the positive semi-definiteness of $k(x, y)$.

4 Question 4

Show that the kernel $k : [0, 1] \times [0, 1] \to \mathbb{R}$ given by $k(x, y) = \min\{x, y\}$ is positive semi-definite.
Proof: Rewrite the kernel function as

$$k(x, y) = \min\{x, y\} = \int_0^1 \mathbb{1}_{[0,x]}(z) \mathbb{1}_{[0,y]}(z) dz.$$

$\forall\, N \in \mathbb{N}^+$, $\forall\, x_1, x_2, ..., x_N \in [0, 1]$, $\forall\, c_1, c_2, ..., c_N \in \mathbb{R}$,

$$\sum_{i,j} c_i c_j \min\{x_i, x_j\} = \sum_{i,j} \int_0^1 [c_i \mathbb{1}_{[0,x_i]}(z)][c_j \mathbb{1}_{[0,x_j]}(z)] dz = \int_0^1 \big[\sum_i c_i \mathbb{1}_{[0,x_i]}(z)\big]^2 dz \geq 0.$$

So k is positive semi-definite.

5 Question 5

Consider the set of functions $\mathcal{H}^1[0, 1] = \{f : [0, 1] \to \mathbb{R} | f(0) = 0$, f is absolutely continuous and $f' \in L_2([0, 1])\}$. We define an inner product in this space: $\langle f, g \rangle_{\mathcal{H}^1} = \int_0^1 f'(z) g'(z) dz$ and claim that the resulting Hilbert space is an RKHS.
Proof:

(1)$\forall x \in [0, 1]$, we can define a function $g_x(z) = \min\{x, z\} = \begin{cases} z, \ z \leq x \\ x, \ z > x \end{cases}$. It is obvious that $g'_x(z) = \begin{cases} 1, \ z < x \\ 0, \ z > x \end{cases}$ is bounded and continuous almost everywhere (except at $z = x$), and hence integrable on $[0, 1]$, which shows the absolute continuity of g_x. Moreover, $g_x(0) = 0$, so $g_x \in \mathcal{H}^1$.

Now consider the inner product $\langle f, g_x \rangle_{\mathcal{H}^1}$ for any arbitrary $x \in [0,1]$ and any $f \in \mathcal{H}^1$.

$$\langle f, g_x \rangle_{\mathcal{H}^1} = \int_0^x f'(z) \cdot 1 \, dz + \int_x^1 f'(z) \cdot 0 \, dz = \int_0^x f'(z) \, dz = f(x).$$

(2) The equation above allows us to rewrite the evaluational functional as $\delta_x(f) = f(x) = \langle f, g_x \rangle_{\mathcal{H}^1}$. We'll show that δ_x is bounded.

$$\forall \, f \in \mathcal{H}^1, \, |\delta_x(f)| = |\langle f, g_x \rangle_{\mathcal{H}^1}| \leq \|g_x\|_{\mathcal{H}^1} \|f\|_{\mathcal{H}^1}.$$

The operator norm of δ_x

$$\|\delta_x\| \leq \|g_x\|_{\mathcal{H}^1} = (\langle g_x, g_x \rangle_{\mathcal{H}^1})^{\frac{1}{2}} = \sqrt{x} \leq 1.$$

Thus, δ_x is bounded and $\mathcal{H}^1$ is an RKHS.