CMSE 820 HW7

This HW is due on Nov 3rd at 11:59 pm.

Question 1: (Closedness of null space). Let L be a bounded linear functional on a Hilbert space $\mathcal{H}$. Show that its null space $\text{null}(L) = \{f \in \mathcal{H} : L(f) = 0\}$ is closed.

Question 2: Show that the kernel function associated with any reproducing kernel Hilbert space must be unique. Namely, given a RKHS $\mathcal{H}$, if there are two kernels, k_1 and k_2, assosciate with $\mathcal{H}$ (with the reproducing property), then $k_1(x, y) = k_2(x, y)$ for all $x, y \in \mathbb{X}$.

Question 3: Let $k : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ be a positive semidefinite kernel, and let $f : \mathbb{X} \to [0, \infty)$ be an arbitrary function. Show that $\widetilde{k}(x, y) = f(x)k(x, y)f(y)$ is also a positive semidefinite kernel.

Question 4: Show that the kernel $k : [0, 1][0, 1] \to \mathbb{R}$ given by $k(x, z) = min\{x, z\}$ is positive semi-definite.

Question 5: A function f over $[0, 1]$ is said to be absolutely continuous if its derivative f' satisfies $\int_0^1 f'(x)dx \leq \infty$, and we have $f(x) = f(0) + \int_0^x f'(z)dz$ for all $x \in [0, 1]$. Now consider the set of functions $\mathcal{H}^1[0, 1] = \{f : [0, 1] \to \mathbb{R} | f(0) = 0, \text{ and } f \text{is absolutely continuous with } f' \in L_2([0, 1])\}$. We define an inner product on this space via $\langle f, g\rangle^1_{\mathcal{H}} = \int_0^1 f'(z)g'(z)dz$, and claim that the resulting Hilbert space is an RKHS. (Hint: the evaluation functional for $\mathcal{H}$ is $\delta_x(z) = \min\{x, z\}$).