(*) Show that the VPA Eqn reproduces the analytic phase shifts for square well potential:

For square well potential, $V(r) = -V_0\,\theta(R-r)$.

For $r \le R$, the VPA Eqn reads

$$\frac{d\delta}{dr} = \frac{1}{k}\, 2MV_0 \sin^2(kr+\delta).$$

Change of variable $y = kr + \delta$ $\quad \left(\delta\big|_{k=0} = 0\right)$

$$\Rightarrow \quad \frac{dy}{dr} = k + \frac{d\delta}{dr} = \frac{1}{k}\, 2MV_0 \sin^2 y + k$$

$$k\,\frac{dy}{dr} = k^2 + 2MV_0 \sin^2 y$$

$$\int_0^{y'} \frac{dy}{k^2 + 2MV_0 \sin^2 y} = \frac{1}{k}\int_0^R dr = \frac{R}{k}$$

$$\cancel{\text{LHS} = \int_0^{y'} \frac{dy}{\left(k+\sqrt{2MV_0}\sin y\right)\left(k-\sqrt{2MV_0}\sin y\right)}}$$

$$\cancel{= \int_0^{y'}}$$

The integral on the LHS can be computed analytically by Mathematica.

$$\text{LHS} = \frac{\arctan\left[\frac{1}{k}\sqrt{k^2+2MV_0}\,\tan y'\right]}{\cancel{k}\sqrt{k^2+2MV_0}} = \text{RHS} = \frac{R}{\cancel{k}}$$

$$\Rightarrow \quad y' = kR + \delta = \tan^{-1}\left[\frac{k\tan\left(R\sqrt{k^2+2MV_0}\right)}{\sqrt{k^2+2MV_0}}\right]$$

$$\Rightarrow \quad \boxed{\delta = \tan^{-1}\left[\frac{k\tan\left(R\sqrt{k^2+2MV_0}\right)}{\sqrt{k^2+2MV_0}}\right] - kR}$$

analytic phase shift for potential

(*) Show that attractive potential $\Rightarrow$ positive phase shift
negative potential $\Rightarrow$ negative phase shift

$$\frac{d\delta}{dr} = -\frac{2M\sin^2[kr+\delta]}{k} \cdot V(r)$$

This is trivial if one notes that the factor in front of $V(r)$ is always non-positive.

1) $V(r)$ is attractive $\Rightarrow$ $V(r) < 0$ for all ~~r~~

$\Rightarrow \frac{d\delta}{dr} \geq 0 \Rightarrow \delta \geq 0$ for any M or k.

2) The argument for negative potentials is similar.

(*) VPA and Levinson's Thm

By integrating the VPA equation, we get

$$\delta(k) = -\frac{2M}{k}\int_0^\infty V(r)\sin^2[kr+\delta]\,dr$$

$$\leq -\frac{2M}{k}\int_0^\infty V(r)\,dr.$$

~~Note that $\delta(k=0) >$~~

Now let's consider an attractive potential $V(r)$ s.t. $\int_0^\infty V(r)\,dr$ is finite and negative.

It follows that $0 \leq \delta(k) \leq \delta(k=0) < \infty$ for all $k \in \mathbb{R}$

Since ~~$\delta(k=0)$~~ at $k=0$, there's no scattering process and that δ is well-defined up to a period of π, we conclude that $\delta(k=0) = n\pi$ for some $n \in \mathbb{N}$.

Up to this point. $\delta(k)$ is a bounded function defined for positive real-valued k only. By analytic continuation, we may extend the definition of $\delta(k)$ so that it's well-defined on the entire complex plane. It can be proven that $\underline{\delta(k) \leq \delta(k=0) = n\pi \text{ for } k \in \mathbb{C}}$.
(assumed)

For bound states, we're concerned with k on the positive imaginary axis, i.e., $k = i\kappa$ for some $\kappa > 0$. Bound states also correspond to poles of the S-matrix. Recall that

$$S_{l=0}(k) = 1 + \frac{2ik}{k[\cot\delta_0(k) - i]} .$$ (defined on $\mathbb{C}$-plane.)

We examine the denominator and consider $k = i\kappa$ ($\kappa > 0$),

$$k[\cot\delta(k) - i] \to i[\kappa \cot\delta_\kappa - 1].$$

Since $0 \leq \delta_\kappa \leq n\pi$ and $\cot\delta$ has a period of π, $\underline{\cot\delta_\kappa \text{ can "traverse" } (-\infty, \infty) \quad n \text{ times}}$ as κ (and hence δ_κ) varies. Thus "$\kappa\cot\delta_\kappa - 1$" has exactly n zeros. $\Rightarrow S_{l=0}$ has n poles on the positive imaginary axis

$\Rightarrow$ The attractive potential has n bound states with $l = 0$.