Oregon ARML PoTDs - Spring 2024

PoTD Season 8, Question 6

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Note that

$$a^2 + b^2 = (a + bi)(a - bi).$$

Define $z = a + bi$. Then:

$$\begin{aligned}&a^2 + b^2\\&= (a + bi)(a - bi)\\&= z \cdot \overline{z}.\end{aligned}$$

Now, notice that:

$$\begin{aligned}&(a^2 + b^2)^n\\&= (z \cdot \overline{z})^n\\&= z^n \cdot \overline{z}^n\\&= z^n \cdot \overline{z^n}.\end{aligned}$$

Now, for some integer c, d, notice that $z^n = c + di$ by Binomial Theorem.

So, we can let $(x, y) = (c, d)$ and we are done.

Now, for the bonus, notice that the only potential problems occour in perfect square scenarios: so, either n is even, or $a^2 + b^2 = c^2$ for some c. However, we only deal with the first case, as it is well known that $c = \alpha^2 + \beta^2$ for integer α, β in the second case.

We need $(a^2 + b^2)^{2n} = c^2 + d^2$, or show that $(a^2 + b^2)^n = \alpha^2 + \beta^2$ for non-zero α, β and $\alpha \neq \beta$ to avoid $d = 0$.
Now we prove that if an $\alpha = \beta$ solution exists, we show that some other solution must also exist. Cut because I don't have time to write a proof.

Note: There need to be more conditions on this bonus. $n = 8$, $a = 1$, $b = 1$ is a counterexample and I think this is true for all $n = 2k$ - I have not proved this yet, though, but I am certain for $k = 4$.