Oregon ARML PoTDs - Spring 2024

PoTD Problem 6

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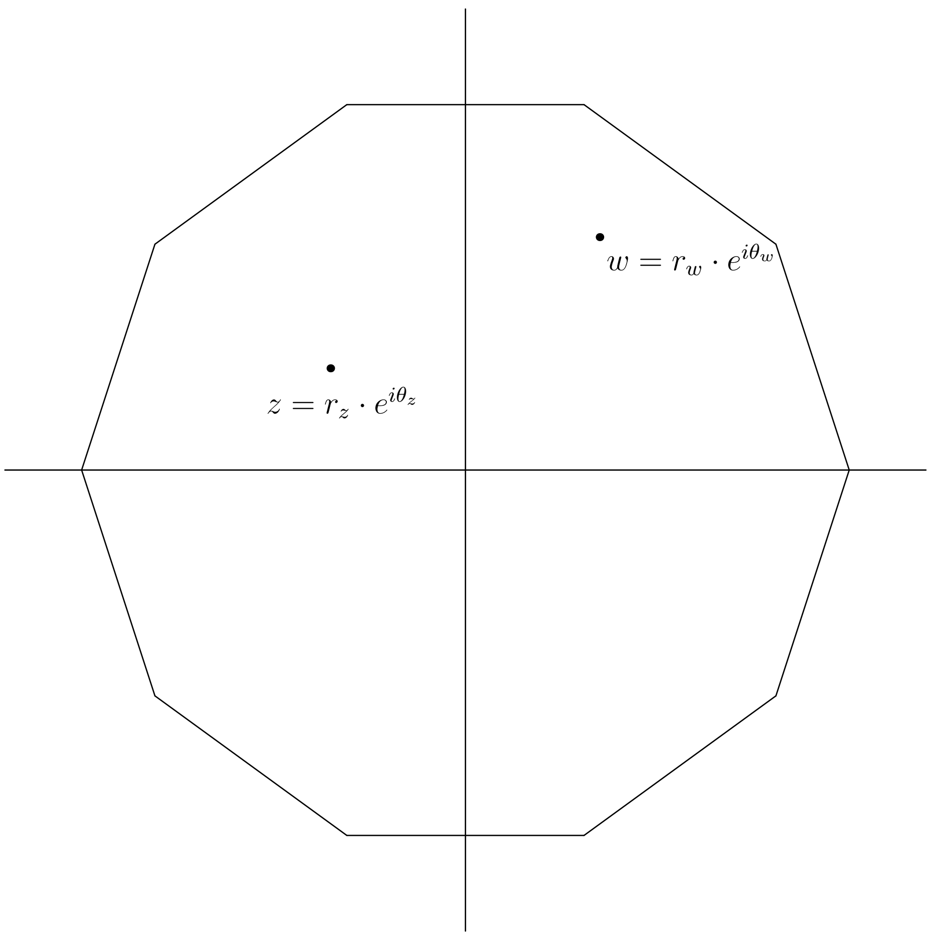
1 Problem

Let $n \geq 3$ be a positive integer. Consider the polygon formed by the nth roots of unity. If complex z, w are in this polygon, prove zw is too.

2 Modulus

Put z and w in exponential form. Let $w = r_w \cdot e^{i\theta_w}$ and $z = r_z \cdot e^{i\theta_z}$. Now, notice that the polygon is symmetric with rotation symmetry of angle $\frac{2\pi}{n}$. Thus, we only consider $\theta \pmod{\frac{2\pi}{n}}$.



3 Law Of Sines

Going forward, let θ_w are θ_z have the Modulus automatically applied.
Now, let some complex number w have the modulus applied. We want to check whether it is within the polygon.
Let its angle be theta. Now, notice that we now must make a judgement solely based on magnitude. Now, by drawing a line segment from the origin to the polygon, we find the magnitude cutoff. Notice that this cutoff is the length of this segment. By Law Of Sines, the length of this segment is, if the length is k,

$$\frac{k}{\sin\left(\frac{\pi}{2}-\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{\pi}{2}-\frac{\pi}{n}+\theta\right)} \tag{1}$$

$$\iff \frac{k}{\cos\left(\frac{\pi}{n}\right)} = \frac{1}{\cos\left(\frac{\pi}{n}-\theta\right)} \tag{2}$$

$$\iff k = \frac{\cos\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}-\theta\right)}. \tag{3}$$

4 Inequalities I

We consider Case 1.
Case 1: $\frac{2\pi}{n} \leq \theta_w + \theta_z < \frac{2\pi}{n}$
Notice that to bring us back into the proper range, we must subtract an extra $\frac{2\pi}{n}$.
So, since $\theta_w + \theta_z - \frac{3\pi}{n}$ and $\theta_w + \theta_z - \frac{3\pi}{n}$ are both less than π, by monotonicity of cosine:

$$\frac{\cos\left(\frac{\pi}{n}\right)}{\cos\left(\theta_w + \theta_z - \frac{3\pi}{n}\right)} \tag{4}$$

$$\geq \frac{\cos^2\left(\frac{\pi}{n}\right)}{\cos\left(\theta_w + \theta_z - \frac{2\pi}{n}\right)} \tag{5}$$

$$= \frac{\cos^2\left(\frac{\pi}{n}\right)}{\cos\left(\theta_w - \frac{\pi}{n}\right) \cdot \cos\left(\theta_w - \frac{\pi}{n}\right) + \sin\left(\theta_w - \frac{\pi}{n}\right) \cdot \sin\left(\theta_z - \frac{\pi}{n}\right)} \tag{6}$$

$$\geq \frac{\cos^2\left(\frac{\pi}{n}\right)}{\cos\left(\theta_w - \frac{\pi}{n}\right) \cdot \cos\left(\theta_w - \frac{\pi}{n}\right)} \tag{7}$$

$$\geq r_w r_z, \tag{8}$$

so we are done with this case. $\square$

5 Inequalities II

We consider Case 2.

Case 2: $\theta_w + \theta_z < \dfrac{2\pi}{n}$

Notice that if

$$\frac{\cos^2\left(\frac{\pi}{n}\right)}{\cos\left(\theta_w - \frac{\pi}{n}\right)\cdot\cos\left(\theta_w - \frac{\pi}{n}\right)} \leq \frac{\cos\left(\frac{\pi}{n}\right)}{\cos\left(\theta_w + \theta_z - \frac{\pi}{n}\right)}, \tag{9}$$

we'd be done. Rearranging, this is, by product-to-sum

$$\begin{align}
\cos\left(\frac{\pi}{n}\right)\cos\left(\theta_w + \theta_z - \frac{\pi}{n}\right) \leq& \cos\left(\theta_w - \frac{\pi}{n}\right)\cos\left(\theta_w - \frac{\pi}{n}\right) \tag{10}\\
\iff \frac{1}{2}\left(\cos\left(\theta_w + \theta_z\right) + \cos\left(\frac{\theta_w + \theta_z}{2} - \frac{\pi}{n}\right)\right)& \tag{11}\\
\leq& \frac{1}{2}\left(\cos\left(\theta_w - \theta_z\right) + \cos\left(\theta_w + \theta_z - \frac{2\pi}{n}\right)\right) \tag{12}\\
\iff \cos\left(\theta_w + \theta_z\right) \leq& \cos\left(\theta_w - \theta_z\right) \tag{13}\\
\iff \cos\theta_w\cos\theta_z - \sin\theta_w\sin\theta_z \leq& \cos\theta_w\cos\theta_z + \sin\theta_w\sin\theta_z \tag{14}\\
\iff 2\sin\theta_w\sin\theta_z \geq& 0 \tag{15}\\
\iff \sin\theta_w\sin\theta_z \geq& 0 \tag{16}\\
& \tag{17}
\end{align}$$

Since $\sin\theta_w > 0$ and $\sin\theta_z > 0$, we are done with this case, and have finished the proof. $\square$