Oregon ARML PoTDs - Spring 2024

PoTD Season 7, Question 3

Garud Shah

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1 Abundancy Of Multiples Of Abundant Numbers

Definition 1.1. Define the divisor set of n to be $\mathcal{D}(n)$.

Definition 1.2. Let $A = kS$ for a set S be, if the elements of S are S_i:

$$kS_i = A_i.$$

Definition 1.3. Let $s(S)$ for a set S be the sum of all elements of S.

Lemma 1.4L. $k\mathcal{D}(n) \cup 1 \subset \mathcal{D}(kn)$ when $k > 1$.
Proof 1.4P. Notice that $k\mathcal{D}(n)$ is all $k \cdot f$ where f is a factor of n, now let $f = \dfrac{n}{a}$ for integer a. Then

$$k \cdot f = \frac{nk}{a},$$

so $kf \in \mathcal{D}(kn)$. Also since 1 is the trivial divisor it is in $\mathcal{D}(kn)$.

Lemma 1.5L. $s(A \cup B) = s(A) + s(B)$ if A, B disjoint.
Proof 1.5P. This is a corrolary of the dditive principle.

Lemma 1.6L. $A \subset B \implies s(A) \leq s(B)$ if A and B only contain positive integers.
Proof 1.6P. Notice that $s(B) - s(A) = s(B \setminus A) - s(A \setminus B)$.
Since $A \subset B$, $A \setminus B = \emptyset$,

$$s(B) - s(A) = s(B \setminus A) - s(\emptyset).$$

Since empty sums are zero, we have $s(\emptyset) = 0$, so $s(B) - s(A) = s(B \setminus A)$. Since A and B only contain positive integers, $s(B \setminus A) \geq 0$. So, $s(B) - s(A) \geq 0$, and $s(B) \geq s(A)$.

Lemma 1.7L. $A \subset B \implies s(B) \geq s(A)$ if A and B only contain positive integers.
Proof 1.7P. Notice that $s(B) - s(A) = s(B \cup A \setminus A) - s(A \setminus B)$ by Lemma 1.5.
Since $A \subset B$, $B \cup A = B$, and $A \setminus B = \emptyset$,

$$s(B) - s(A) = s(B \setminus A) - s(\emptyset).$$

Since empty sums are zero, we have $s(\emptyset) = 0$, so $s(B) - s(A) = s(B \setminus A)$. Since A and B only contain positive integers, $s(B \setminus A) \geq 0$. So, $s(B) - s(A) \geq 0$, and $s(B) \geq s(A)$.

Lemma 1.8L. $s(kS) = ks(S)$.
Proof 1.8P. We sum by element and this is equivilant to moving a constant out of a summation.

Lemma 1.9L. (Main Theorem of Section) If p is perfect or abundant, and $k > 1$, pk is abundant.
Proof 1.9P. The definition of abundancy is equivilant to saying that $s(\mathcal{D}(n)) \geq 2n$.

Plug in kn. We get that, by Lemma 1.6, that $s(\mathcal{D}(kn)) \geq s(A)$ for some $A \subset \mathcal{D}(kn)$.

By Lemma 1.4, we set $A = k\mathcal{D}(n) \cup 1 \subset \mathcal{D}(kn)$.

Since 1 and $k\mathcal{D}(n)$ are disjoint, by Lemmas 1.5 and 1.8, we have that $s(\mathcal{D}(kn)) \geq ks(\mathcal{D}(n))+1$. By our initial assumption, $s(\mathcal{D}(n)) \geq 2n$, so $s(\mathcal{D}(kn)) \geq ks(\mathcal{D}(n))+1 \geq 2kn + 1$ so $s(\mathcal{D}(kn)) > 2kn$. So, kn is abundant, and the lemma is proved.

2 Finding Perfect Numbers Modulo 6

Lemma 2.1L. If there exist abundant numbers modulo 6 with residue 1, 2, 3, 4, and 5, we are finished.
Proof 2.1P. Notice that 6 is perfect. So, by Lemma 1.9, we can add multiples of 6 for our result, and 12 plus any other multiple of 6 gives 0 mod 6.

Notice that **28** is perfect, so by Lemma 1.9L, we can get 56 and 112 abundant, meaning we only need residues 1, 3, 5.
Now, **945** is abundant, so we have residue 3 down as well.

Definition 2.2. The *abundancy index* $a(n)$ is $\frac{s(\mathcal{D}(n))}{n}$.

Now, if we only include factors of $5, 7, 11, \cdots$ we will get an abundant number. We notice that by the formula for $\sigma(n) = s(\mathcal{D}(n))$, we can break this up into primes. Now, notice that the maximum we can get for the term of a single prime that is:

$$\lim_{k\to\infty} \frac{1 + p + p^2 + \cdots + p^k}{p^k} \tag{1}$$

$$= \lim_{k\to\infty} 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^k} \tag{2}$$

$$\tag{3}$$

Now, multiply this for all primes to get an inf-reachable maximum for abundancy index. Notice that we get the harmonic series, which diverges, and if we divide by just the $p = 2, 3$ terms it still diverges. So, plug in an arbitrary number of primes, use arbitrarily large powers, and we clear one of residue 1 or 5. Now multiply by 5, and the other residue of 1 or 5 is done by mods. We are done.