Differential Equations Week **7**

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Can you please mark the equations you have problems with in the index of comments? I'm NOT asking for that to be the number, just when you make a comment, within the *text* of the comment (not the label) mark the bad equation (for all future weeks too, this is a clarification of last's week's!)

Problem 1 (Problem 1)
Solve the differential equation

$$y'' = ay, a >= 0. \tag{1.1}$$

Solution — If $a = 0$ trivial, $ax + b$
We have characteristic equation:

$$\lambda^2 = a \tag{1.2}$$
$$\lambda = \pm\sqrt{a}, \tag{1.3}$$

thus the answer is:

$$c_1 e^{\sqrt{a}} + c_2 e^{-\sqrt{a}}. \tag{1.4}$$

Problem 2 (Problem 2)
Solve the following IVP's:

1. $y'' + 16y = 0$, $y(0) = 2$, $y'(0) = 0$
2. $y'' + 100y' + y = 0$, $y(0) = 1$, $y'(0) = 0$.
3. $y'' - 6y' + 8y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution 2.1 (Solution 2a) — $\omega = \sqrt{16}$, the answer is $2\cos(4t)$.

Solution 2.2 (Solution 2b) — Roots of charactersitic equation are $-50 \pm 7\sqrt{51}$, which gives us solutions $c_1 e^{(-50+7\sqrt{51})t} + c_2 e^{(-50+7\sqrt{51})t}$, so $c_1 + c_2 = 1$ and $c_1 - c_2 = 0$, thus answer is $\frac{1}{2}\left(e^{(-50+7\sqrt{51})t} + e^{(-50+7\sqrt{51})t}\right)$.

Solution 2.3 (Solution 2c) — We have roots 2 and 4 which means we have $c_1 e^{2t} + c_2 e^{4t}$, or $c_1 + c_2 = 1$ and $c_1 - c_2 = 0$ thus answer is $\frac{1}{2}\left(e^{2t} + e^{4t}\right)$.

Problem 3 (Problem 3)
For the differential equation:

$$I\theta'' + b\theta' + k\theta = 0, \tag{3.1}$$

when will θ oscilate?

Solution 3.1 — $b^2 < 4Ik$ by criticality criteria...

Problem 4 (Problem 4)
Show that for

$$y'' + y = 0 \tag{4.1}$$
$$y(t) = c_1 \cos t + c_2 \sin t, \tag{4.2}$$

(a) There is a unique solution that satisifes $y(0) = 2$ and $y(\pi/2) = 0$.

(b) There is not a solution that satisifes $y(0) = 2$ and $y(\pi) = 0$.

(c) There is are infinite solutions that satisifes $y(0) = 2$ and $y(\pi) = -2$.

Solution 4.1 (Solution 4a) — Yes as the Wronskian for those two values is non-zero.

Solution 4.2 (Solution 4b) — As $y(0) = c_1$ and $y(\pi) = -c_1$ and $-2 \neq 0$ done.

Solution 4.3 (Solution 4c) — As $y(0) = c_1$ and $y(\pi) = -c_1$ and $-2 = 2$, y_1 can be anything done.

Problem 5 (Problem 5)

Solve the 2nd-Order Euler equation:

$$ax^2y'' + bxy' + cy = 0. \tag{5.1}$$

Solution 5.1 — Note that:

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}, \tag{5.2}$$

thus when substituting $v = \log x$, we need:

$$ay'' + by' + cy = 0. \tag{5.3}$$

Thus, if $ax^2 + bx + c$ has distinct roots r_1 and r_2 OR has a single root r,

$$y(v) = c_1e^{r_1 v} + c_2e^{r_2 v} = c_1x^{r_1} + c_2x^{r_2} \tag{5.4}$$

$$OR \tag{5.5}$$

$$y(v) = c_1e^{rv} + c_2ve^{r_2 v} = (c_1 + c_2 \log x)x^r. \tag{5.6}$$