Differential Equations Week **12**

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May 16, 2025

Problem 1

Solve the following differential equation:

$$\begin{cases} P' &= P(c - r(N)) \\ Q' &= Pr(N) \\ N &= P + Q \end{cases} \tag{1.1}$$

for

(a) $r(N) = b(1 + \log N)$, $P(0) = 1$, $Q(0) = 0$, $N(0) = 1$

(b) $r(N) = (2N - 1)s$.

Solution 1.1 — Note that $Q(t)$ is very annoying, let's turn this into an P, N differential equation system.

$$\begin{cases} P' &= P(c - r(N)) \\ N' &= cP \end{cases}. \tag{1.2}$$

Now we solve the phase plane differential equation,

$$\frac{dP}{dN} = \frac{c - r(N)}{c} \tag{1.3}$$

$$P = N - \frac{1}{c}\int_{dontcare}^{N} r(x)\,dx + \alpha. \tag{1.4}$$

Now, plug back in:

$$N' = c\left(N - \frac{1}{c}\int_{dontcare}^{N} r(x)\,dx + \alpha\right) \tag{1.5}$$

$$t = \int \frac{1}{cN - \int_{dontcare}^{N} r(x)\,dx + c\alpha}\,dN \tag{1.6}$$

Solution 1.2 — Plug in (set dontcare to 1):

$$t = \int \frac{1}{cN - bN\log N}\,dN \tag{1.7}$$

$$= \int \frac{1}{N}\cdot\frac{1}{c - b\log N}\,dN \tag{1.8}$$

$$= \int \frac{1}{c - bu}\,du \tag{1.9}$$

$$= -\frac{1}{b}\log|c - b\log|N|| + C \tag{1.10}$$

$$c - ae^{-bt} = b\log|N| \tag{1.11}$$

$$N = e^{\frac{1}{b}(c - ae^{-bt})} \tag{1.12}$$

$$P = e^{\frac{1}{b}(c - ae^{-bt})} - \frac{1}{c}N\log N \tag{1.13}$$

$$= e^{\frac{1}{b}(c - ae^{-bt})}\left(1 - \frac{1}{bc}(c - ae^{-bt})\right) \tag{1.14}$$

Solution 1.3 — Plug in (set dontcare to 0):

$$t = \int \frac{1}{cN - sN + sN^2}\,dN \tag{1.15}$$

$$= \int \frac{1}{N(c - s + sN)}\,dN \tag{1.16}$$

$$= \frac{1}{s}\int \frac{1}{N(N - 1 + c/s)}\,dN \tag{1.17}$$

$$= \frac{1}{c - s}\log\left|\frac{N}{N - 1 + c/s}\right| \tag{1.18}$$

$$e^{t(c-s)} = \frac{N}{N - 1 + c/s} \tag{1.19}$$

$$N(1 - e^{t(c-s)}) \tag{1.20}$$