Differential Equations Week 4

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Problem 1 (Problem 1)
Consider the following initial value problem:

$$\begin{cases} y' + \lambda y & = 0 \\ y(0) & = 1 \end{cases} \tag{1.1}$$

(a) Show that Euler's Method produces the approximation f_n of $(nh, (1-\lambda h)^n)$.

(b) Show that the trapezoid scheme produces the approximation f_n of $\left(nh, \left(\frac{1-\lambda h/2}{1+\lambda h/2}\right)^n\right)$ which goes to 0 as x goes to infinity.

(c) Show that iff $0 < \lambda h < 2$, Imporved Euler goes to 0 as x goes to infinity.

Solution 1.1 (Solution 1a) — We proceed by induction.

Base Case. $n = 0$. Euler produces 1 as this is the initial value, trivial.

Inductive Hypothesis. Assume true for $n-1$.

Inductive Step. Note that $y' = -\lambda y$. With step size h, and value for $(n-1)h$ as y, Euler will give:

$$\begin{aligned} y_n &= y_{n-1} - \lambda h y_{n-1} && (1.2) \\ &= (1-\lambda h) y_{n-1} && (1.3) \\ &= (1-\lambda h)^n, && (1.4) \end{aligned}$$

and we are done.

Solution 1.2 (Solution 1b) — We follow similar inductive reasoning as 1a.

Base Case. $n = 0$. Trapezoid scheme produces 1 as this is the initial value, trivial.

Inductive Hypothesis. Assume true for $n - 1$.

Inductive Step. Note that $y' = -\lambda y$. With step size h, and value for $(n-1)h$ as y, guessing the given value will give:

$$\left(\tfrac{1-\lambda h/2}{1+\lambda h/2}\right)^n = \left(\tfrac{1-\lambda h/2}{1+\lambda h/2}\right)^{n-1} - \tfrac{\lambda h}{2}\left(\left(\tfrac{1-\lambda h/2}{1+\lambda h/2}\right)^{n-1} + \left(\tfrac{1-\lambda h/2}{1+\lambda h/2}\right)^n\right) \tag{1.5}$$

$$\frac{1-\lambda h/2}{1+\lambda h/2} = 1 - \frac{\lambda h}{2}\left(1 + \left(\frac{1-\lambda h/2}{1+\lambda h/2}\right)\right) \tag{1.6}$$

$$\frac{1-\lambda h/2}{1+\lambda h/2} = 1 - \frac{\lambda h}{1+\lambda h/2}, \tag{1.7}$$

and we are done.

Solution 1.3 (Solution 1c) — Note that $y' = -\lambda y$. With step size h, and value for $(n-1)h$ as y, Improved Euler will give:

$$y_n = y_{n-1} - \frac{h}{2}\left(\lambda y_{n-1} + \lambda y_{n-1}(1-\lambda h)\right) \tag{1.8}$$

$$= y_{n-1}\left(1 - \lambda h + \frac{(\lambda h)^2}{2}\right), \tag{1.9}$$

which means that $|y_n| < |y_{n-1}|$ iff $1 - \lambda h + \dfrac{(\lambda h)^2}{2} < 1$, or $0 < \lambda h < 2$, so we are done.

Problem 2 (Problem 2)

For the intial value problem:

$$p' = 10p(1-p), p(0) = 0.1 \tag{2.1}$$

(a) Solve the intial value problem and note that $p(t)$ goes to 1 as t goes to infinity.

(b) Show that Euler's method gives:

$$p_{n+1} = p_n + h \cdot 10p_n(1-p_n). \tag{2.2}$$

(c) Calculate 60 values of Euler's Method for:

(i) $h = 0.18$.
(ii) $h = 0.23$.
(iii) $h = 0.25$.
(iv) $h = 0.3$.

(d) Calculate 60 values of RK4 for:

(i) $h = 0.3$.
(ii) $h = 0.325$.
(iii) $h = 0.35$.

(None of these convergee to 1, the correct value.)

Solution 2.1 (Solution 2a) — We solve the differential equation:

$$p' = 10p(1-p) \tag{2.3}$$

$$\int \frac{p'}{10p(1-p)} dt = t \tag{2.4}$$

$$\log|p| + \log|1-p| = 20t + C \tag{2.5}$$

$$\frac{p}{1-p} = Ce^{20t} \tag{2.6}$$

$$p = \frac{1}{1+Ce^{20t}} \tag{2.7}$$

$$0.1 = \frac{1}{1+C} \tag{2.8}$$

$$p = \frac{1}{1+9e^{20t}}. \tag{2.9}$$

Solution 2.2 (Solution 2b) — Trivial by defintion of Euler's Method.

Solution 2.3 (Answer 2c-i) — Iteration 0: (0,0.1)
Iteration 1: (0.18,0.262)
Iteration 2: (0.36,0.610041)
Iteration 3: (0.54,1.03824)
Iteration 4: (0.72,0.966772)
Iteration 10: (1.8,0.991664)
Iteration 20: (3.6,0.999114)
Iteration 30: (5.4,0.999905)
Iteration 40: (7.2,0.99999)
Iteration 50: (9,0.999999)
Iteration 55: (9.9,1)
Iteration 56: (10.08,1)
Iteration 57: (10.26,1)
Iteration 58: (10.44,1)
Iteration 59: (10.62,1)
Iteration 60: (10.8,1)

Solution 2.4 (Answer 2c-ii) — Iteration 0: (0,0.1)
Iteration 1: (0.23,0.307)
Iteration 2: (0.46,0.796327)
Iteration 3: (0.69,1.16936)
Iteration 4: (0.92,0.713852)
Iteration 5: (1.15,1.18367)
Iteration 56: (12.88,0.687874)
Iteration 57: (13.11,1.18169)
Iteration 58: (13.34,0.687874)
Iteration 59: (13.57,1.18169)
Iteration 60: (13.8,0.687874)

Solution 2.5 (Answer 2c-iii) — Iteration 0: (0,0.1)
Iteration 1: (0.25,0.325)
Iteration 2: (0.5,0.873438)
Iteration 3: (0.75,1.1498)
Iteration 4: (1,0.719203)
Iteration 53: (13.25,1.225)
Iteration 54: (13.5,0.535948)
Iteration 55: (13.75,1.15772)
Iteration 56: (14,0.701238)
Iteration 57: (14.25,1.225)
Iteration 58: (14.5,0.535948)
Iteration 59: (14.75,1.15772)
Iteration 60: (15,0.701238)

Solution 2.6 (Answer 2c-iv) — Iteration 0: (0,0.1)
Iteration 1: (0.3,0.37)
Iteration 2: (0.6,1.0693)
Iteration 3: (0.9,0.846993)
Iteration 4: (1.2,1.23578)
Iteration 10: (3,1.25115)
Iteration 20: (6,0.557382)
Iteration 30: (9,0.320003)
Iteration 40: (12,0.186425)
Iteration 50: (15,0.0918856)
Iteration 56: (16.8,1.23251)
Iteration 57: (17.1,0.372804)
Iteration 58: (17.4,1.07427)
Iteration 59: (17.7,0.834916)
Iteration 60: (18,1.24841)

Solution 2.7 (Answer 2d-i) — Iteration 0: (0,0.1)
Iteration 1: (0.3,0.638076)
Iteration 2: (0.6,0.789598)
Iteration 3: (0.9,0.765825)
Iteration 4: (1.2,0.761264)
Iteration 5: (1.5,0.76085)
Iteration 6: (1.8,0.76082)
Iteration 7: (2.1,0.760817)
Iteration 8: (2.4,0.760817) - stays static.

Solution 2.8 (Answer 2d-ii) — Iteration 0: (0,0.1)
Iteration 1: (0.325,0.666946)
Iteration 2: (0.65,0.67586)
Iteration 3: (0.975,0.670683)-converges to 0.6726.

Solution 2.9 (Answer 2d-iii) — Iteration 0: (0,0.1)
Iteration 1: (0.35,0.684621)
Iteration 2: (0.7,0.530325)
Iteration 3: (1.05,0.692029)
Iteration 4: (1.4,0.524791)
Iteration 57: (19.95,0.721933)
Iteration 58: (20.3,0.506152)
Iteration 59: (20.65,0.721933)
Iteration 60: (21,0.506152) - oscilatory motion continues! Period doubling bifurcation!