

Differential Equations Week **11**

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Problem 1 (Problem 1)

For the differential equations:

$$x' = y \quad (1.0)$$

$$y' = -x + x^3, \quad (1.1)$$

- (a) Sketch the direction field using computer software.
- (b) Find whether the solution passing through each of the following points is periodic:
 - i. (0.25, 0.25)
 - ii. (2,2)
 - iii. (1,0)

Solution 1.1 (Phase planes are cool! - 1a) —

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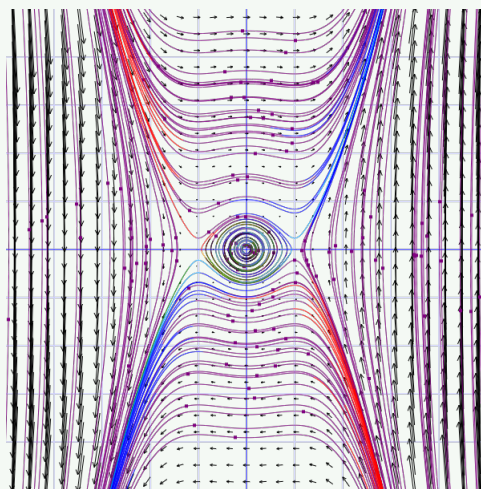


Figure 1.1: Phase plane of differential equation.

Example (Answers 1b). My answers are:

- i. (0.25, 0.25): Why not?
- ii. (2,2): No way.
- iii. (1,0): Even better: critical point, you're never leaving.

Problem 2

For the differential equations

$$x' = y \quad (2.0)$$

$$y' = -x - x^3, \quad (2.1)$$

- (a) Sketch the direction field using computer software.
- (b) Conjecture whether all solutions are bounded.
- (c) Solve the phase plane differential equation.

Solution 2.1 (Phase planes are cool! - 2a) —

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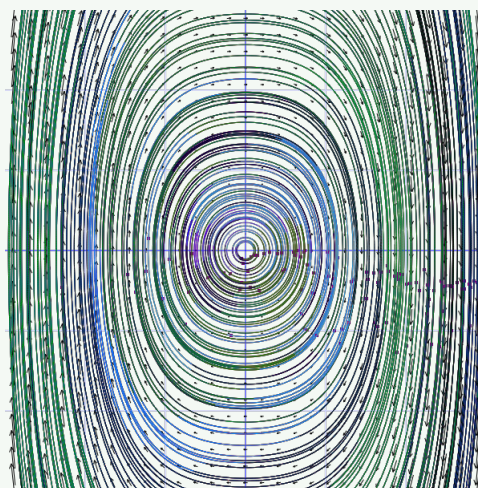


Figure 2.1: Phase plane of differential equation.

Solution 2.2 (Answer 2b) — Why not? Looks like it.**Solution 2.3** (Solution 2c) — We have:

$$\frac{dy}{dx} = \frac{-x - x^3}{y} \quad (2.2)$$

$$y^2 = C - x^2 - \frac{x^4}{2} \quad (2.3)$$

$$y = \sqrt{C - x^2 - \frac{x^4}{2}}. \quad (2.4)$$

Also this is literally a Duffing Equation in disguise so of course it's

periodic.

Problem 3

For the differential equations:

$$x' = (x - 1)(y - 1) \quad (3.0)$$

$$y' = y(y - 1), \quad (3.1)$$

- Construct the phase plane analysis by hand, and graph, without using computer software, the asymptotic behaviour of trajectories (as $t \rightarrow \infty$).
- Sketch the direction field using computer software.
- Solve the differential equation.

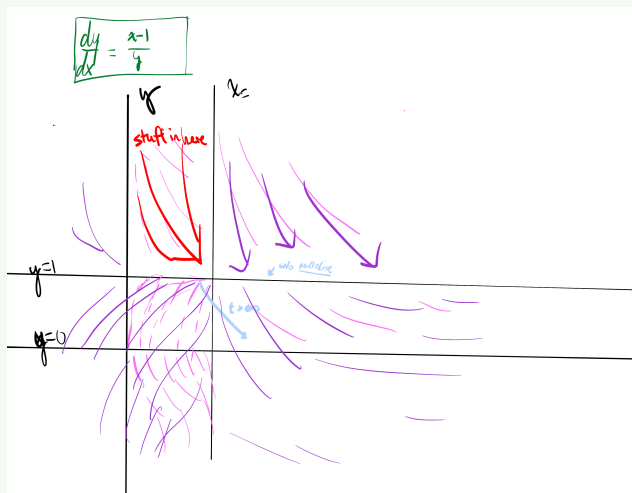
Solution 3.1 —

Figure 3.1: Hand-drawn phase plane of differential equation.

Solution 3.2 —

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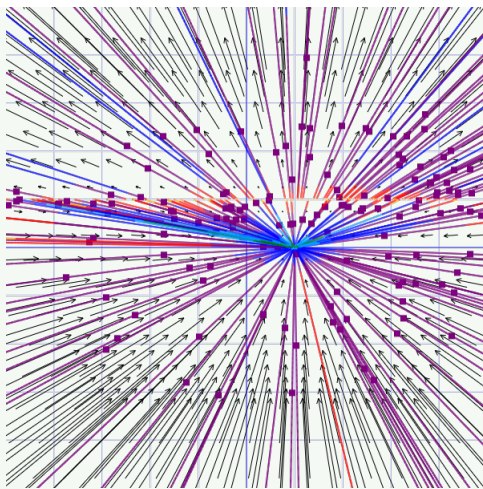


Figure 3.1: Phase plane of differential equation.

Solution 3.3 — We solve for y , then x .

Note that by separation of variables

$$y' = y(y - 1) \quad (3.2)$$

$$\frac{y'}{y(y - 1)} = 1 \quad (3.3)$$

$$\int \frac{dy}{y(y - 1)} = t \quad (3.4)$$

$$\log \left| \frac{y - 1}{y} \right| = t + C \quad (3.5)$$

$$\frac{y - 1}{y} = Ce^t \quad (3.6)$$

$$y(1 - Ce^t) = -1 \quad (3.7)$$

$$y = \frac{1}{Ce^t - 1} \quad (3.8)$$

Problem 4

Consider the system:

$$v' = -x + \frac{1}{\lambda - x} \quad (4.0)$$

$$x' = v. \quad (4.1)$$

- (a) Solve this equation in the phase plane.
- (b) Find the critical points of this system.
- (c) Plot the phase plane diagrams for $\lambda = 1, 3$ and describe x for $\lambda = 1, 3$ under various initial conditions.

Solution 4.1 — We note that:

$$\frac{dv}{dx} = \frac{-x + 1/(\lambda - x)}{v} \quad (4.2)$$

$$\frac{v^2}{2} = \int -x + \frac{1}{\lambda - x} dx \quad (4.3)$$

$$= -\frac{x^2}{2} - \log|x - \lambda| + C.v^2 = C - x^2 - \log((x - \lambda)^2) \quad (4.4)$$

$$v = \pm \sqrt{C - x^2 - \log((x - \lambda)^2)} \quad (4.5)$$

$$(4.6)$$

Solution 4.2 —

Observation 4.1 — $v = 0$ for any critical point. (obviously, bottom equation).

Now:

$$0 = -x + \frac{1}{\lambda - x} \quad (4.7)$$

$$x = \frac{1}{\lambda - x} \quad (4.8)$$

$$x^2 - \lambda x + 1 = 0 \quad (4.9)$$

$$x = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}. \quad (4.10)$$

Thus to have critical points, we need $\lambda < 2$.

Solution 4.3 —

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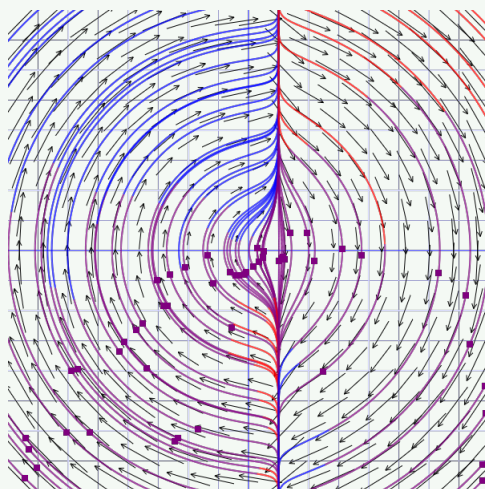


Figure 4.1: Phase plane of differential equation, $\lambda = 1$.

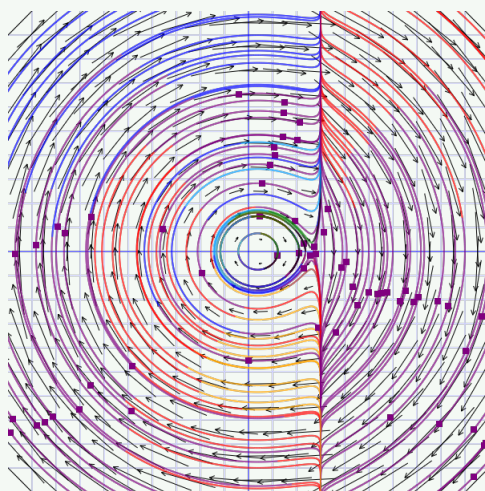


Figure 4.1: Phase plane of differential equation, $\lambda = 3$.