Differential Equations Week 1

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Bold indicates corrections.

Problem 1

Let $c > 0$. Show that $\phi(x) = \dfrac{1}{c^2 - x^2}$ is a solution to the inital value problem $y' = 2xy^2$, $y(0) = \dfrac{1}{c^2}$ on the interval $-c < x < c$.

Solution — Plugging in, we get:

$$\begin{aligned} &\left(\frac{1}{c^2 - x^2}\right)' && \text{start} \\ &= -2x \cdot -\left(\frac{1}{c^2 - x^2}\right)^2 && \text{chain rule} \\ &= 2xy^2. && \text{finish } \checkmark \end{aligned}$$

Problem 2

Consider the nonlinear differential equation $\dfrac{dy}{dx} = 3y^{2/3}$ for all x.

(a) Verify that all functions of the form $f(x) = (x - c)^3$ are explicit solutions to the equation for all x.

(b) Verify that the identically zero function $f(x) = 0$ also satisifies the equation for all x.

(c) Verify that the function defined by $f(x) = \begin{cases} (x-c)^3, & x > c \\ 0, & x \le c \end{cases}$ satisfies the equation for all x, and that $\begin{cases} (x-c)^3, & x \le c \\ 0, & x > c \end{cases}$ does too.

(d) Is there any point (a, b) in the plane such that there is no solution to the differential equation that passes through that point?

(e) Is there any point (a, b) in the plane such that there is a unique solution to the differential equation that passes through that point?

Solution 2.1 (Problem 2a) — Plugging in:

$$\begin{aligned} &f'(x) && \text{inital} \\ &= 3(x-c)^2 && \text{chain rule} \\ &= 3f(x)^{2/3} && \text{manipulating } \checkmark. \end{aligned}$$

Solution 2.2 (Problem 2b) — Plugging in:

$$\begin{aligned} &f'(x) && \text{inital} \\ &= 0 && \text{duh} \\ &= 3f(x)^{2/3} && \text{duh } \checkmark. \end{aligned}$$

Solution 2.3 (Problem 2c) — As this is an first order differential equation, we only need to look at the function and its derivative. The only place where we could have a problem is where we switch functions (as in (a) and (b) looking at a section of those functions and combining do the other parts for us) but notice that here the value and slope are both always zero and $0 = 3 \cdot 0^{2/3}$ ✓

Solution 2.4 (Problem 2d) — **No.** Take a point a, b. We will only consider solutions of the form $(x-c)^3$. Then we have: $b = (a-c)^3$, and $c = a - \sqrt[3]{b}$, thus there is at least one of this form. ✓

Solution 2.5 (Problem 2e) — **No.** The solution derived in problem 2d can be duplicated as each solution generates another solution with $y \geq 0$:

$$f(x) = \begin{cases} (x-c)^3, & x > c \\ 0, & x \leq c \end{cases}$$

and one with $y \leq 0$

$$f(x) = \begin{cases} 0, & x \geq c \\ (x-c)^3, & x < c \end{cases}$$

as well from problem 2c. Thus, if $y > 0$, we can add the $(x-c)^3$ solution and the $y \geq 0$ solution above, and a similar thing for $y < 0$. At $y = 0$ we have $(x - x_0)^3$ and $y = 0$ so we are done ✓

Problem 3

Consider the differential equation

$$\frac{dy}{dx} = -\frac{1 + ye^{xy}}{1 + xe^{xy}},$$

(a) Show that $x + y + e^{xy} = 0$ defines y as an implicit function of x on some interval containing $(-1, 0)$.

(b) Given that any relation of the form $x + y + e^{xy} = C$ for any real number C satisfies the given differential equation, when and around which points will we not have a guarentee by the implicit function theorem not guarentee a solution for us?

Solution 3.1 (Problem 3a) — Notice that if we let $G(x,y) = x + y + e^{xy}$,

$$\frac{dy}{dx} = -\frac{\partial G/\partial x}{\partial G/\partial y}.$$

By the implicit function theorem, as $1 \neq 0$ and $G(x,y) = 0$, we are done with (a).

Solution 3.2 (Problem 3b) — Notice that with $G(x,y) = x + y + e^{xy} - C$, we'd only fail if $1 + xe^{xy} = 0$, by the implicit function theorem, or $xy = -\log x$, $y = -\frac{\log x}{x}$. This would mean $x - \frac{\log x}{x} + e^{-\log x} = x - \frac{\log x - 1}{x} = C$, which is possible for all $x > 0$ so we are done.

Problem 4

Consider $\frac{dy}{dx} = 3y^{2/3}$ again.

(a) Where does the existence and uniqueness theorem guarentee a unique solution to this differential equation?

(b) How do you reconsile this with the result of problem 2?

Solution 4.1 (Problem 4a) — Notice that $f = 3y^{2/3}$ is continuous everywhere, and $\frac{\partial f}{\partial y} = 2y^{-1/3}$ is continuous everywhere except $y = 0$. So we should be guarenteed unique solutions everywhere except $y = 0$.

Solution 4.2 (Problem 4b) — **It only guarentees on a small interval around the point for $y \neq 0$, which IS true.**

Problem 5 (Problem 5)

Use the conversion of the initial value problem $P = (x_0, y_0)$ and $\frac{dy}{dx} = f(x,y)$ into the integral operator

$$\begin{cases} O\hat{_{f,P}}[y(x)] = y_0 + \int_{x_0}^{x} f(t, y(t))dt \\ O\hat{_{f,P}}[y(x)] = y(x). \end{cases} \tag{5.1}$$

(a) Prove that if $f(x,y) = y$, and $P = (0,1)$, then

$$\hat{O}^n_{f,P}[1] = \sum_{i=0}^{n} \frac{x^i}{i!}.$$

(b) If $f(x,y) = 3x - y^2$, and $P = (0,0)$, find:

$$\hat{O}^3_{f,P}[0].$$

(c) If $f(x,y) = 3y^{2/3}$, and $P =$

Solution 5.1 (Problem 5a) — Note that we just need:

$$1 + \int_0^x \sum_{i=0}^{n} \frac{x^i}{i!} dt = \sum_{i=0}^{n+1} \frac{x^i}{i!},$$

which is the case termwise integrating.

Solution 5.2 (Problem 5b) — Notice that we need to find:

$$\int_0^x 3x - \left(\int_0^x 3x - \left(\int_0^x 3x dx \right)^2 dx \right)^2 dx.$$

Note that this is:

$$\begin{aligned}
&\int_0^x 3x - \left(\int_0^x 3x - \left(\int_0^x 3x dx \right)^2 dx \right)^2 dx \\
&= \int_0^x 3x - \left(\int_0^x 3x - \left(\frac{3x^2}{2} \right)^2 dx \right)^2 dx \\
&= \int_0^x 3x - \left(\int_0^x 3x - \frac{9x^4}{4} dx \right)^2 dx \\
&= \int_0^x 3x - \left(\frac{3x^2}{2} - \frac{9x^5}{20} dx \right)^2 dx \\
&= \int_0^x 3x - \left(\frac{9x^4}{4} - \frac{27x^7}{20} + \frac{81x^{10}}{400} \right) dx \\
&= \frac{3x^2}{2} - \frac{9x^5}{20} + \frac{27x^8}{160} - \frac{81x^{11}}{4400}.
\end{aligned}$$