Differential Equations Quarterly **2**

Garud Shah

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Problem 1 (Problem 1)

Suppose we have a mass-spring system modeled by the equation

$$y'' + 2y' + 2y = \cos t. \tag{1.1}$$

(a) Find the general solution to the equation.

(b) Describe the behaviour as t goes to infinity.

Remark 1.1 (Method.)**.** Like any non-homogenous differential equation, we'll find the homogenous solution and then find a non-homogenous particular solution.

Solution 1.1 (Solution 1a) — The corresponding homogenous equation is

$$y'' + 2y' + 2y = 0. \tag{1.2}$$

Consider its characteristic equation:

$$r^2 + 2r + 2, \tag{1.3}$$

which gives us roots

$$-1 + i, -1 - i \tag{1.4}$$

so the general solution of the corresponding homogenous equation is:

$$e^{-t} \sin t + e^{-t} \cos t. \tag{1.5}$$

Solution 1.2 (Solution 1b) — Set

$$f_p = A\cos t + B\sin t. \tag{1.6}$$

Plug in to the differential equation:

$$\begin{align}
\cos t &= 2(A\cos t + B\sin t) \tag{1.7}\\
&\quad + 2(A\cos t + B\sin t)' \tag{1.8}\\
&\quad + (A\cos t + B\sin t)'' \tag{1.9}\\
&= (A+2B)\cos t + (B-2A)\sin t. \tag{1.10}\\
A + 2B &= 1 \tag{1.11}\\
B - 2A &= 0 \tag{1.12}\\
A = 1/5, B &= 2/5. \tag{1.13}\\
f_p(t) &= \frac{1}{5}\cos t + \frac{2}{5}\sin t \tag{1.14}\\
f(t) &= \frac{1}{5}\cos t + \frac{2}{5}\sin t + e^{-t}\cos t \tag{1.15}
\end{align}$$

Remark 1.2 (Footnotes.)**.** Do our Wronskian Sanity Check:

$$\begin{align}
0 &= e^{-t}\cos t(-e^{-t}\sin t + e^{-t}\cos t) - e^{-t}\sin t(-e^{-t}\cos t - e^{-t}\sin t) \tag{1.16}\\
&= e^{-2t}(\cos t(-\sin t + \cos t) - \sin t(\cos t - \sin t)) \tag{1.17}\\
&= e^{-2t}, \tag{1.18}
\end{align}$$

which is impossible so by E+U theorem, we have the unique solution for all initial value problems.

Now do the back-substitution check for our particular solution:

$$\begin{align}
\cos t &= 2\left(\frac{1}{5}\cos t + \frac{2}{5}\sin t\right) \tag{1.19}\\
&\quad + 2\left(\frac{1}{5}\cos t + \frac{2}{5}\sin t\right)' \tag{1.20}\\
&\quad + \left(\frac{1}{5}\cos t + \frac{2}{5}\sin t\right)'' \tag{1.21}\\
&= \left(\frac{2}{5} - \frac{1}{5} + \frac{4}{5}\right)\cos t + \left(\frac{4}{5} - \frac{2}{5} - \frac{2}{5}\right)\sin t \tag{1.22}\\
&= \cos t \tag{1.23}
\end{align}$$

Problem 2 (Problem 2a, harder)
Find the general solution to:

$$y'' - 4y = t^2 \cos(2t) \tag{2.1}$$

Remark 2.1 (Method.)**.** Like any non-homogenous differential equation, we'll find the homogenous solution and then find a non-homogenous particular solution.

Solution 2.1 (Solution 2ai) — The homogenous equation:

$$y'' = 4y, \tag{2.2}$$

roots of characteristic polynomial:

$$r^2 - 4 = 0, r = \pm 2, \tag{2.3}$$

general solution:

$$c_1 e^{2t} + c_2 e^{-2t}. \tag{2.4}$$

Solution 2.2 (Solution 2aii) — We use the Annihaltor Method to get the form:

$$t^2(A_2 \cos(2t) + B_2 \sin(2t)) + \tag{2.5}$$
$$t^1(A_1 \cos(2t) + B_1 \sin(2t)) + \tag{2.6}$$
$$t^0(A_0 \cos(2t) + B_0 \sin(2t)), \tag{2.7}$$

but it's *very* annoying to work out a particular solution so I'll stop here.

Remark 2.2 (Footnotes.)**.** Wronskian Sanity Check:

$$0 = -2e^{2t}e^{-2t} - 2e^{-2t}e^{2t} \tag{2.8}$$
$$= -4 \tag{2.9}$$

which is impossible so by E+U theorem, we have the unique solution for all initial value problems.

Back-substitution sanity check: none as I didn't work out a particular solution.

Problem 3 (Problem 2b, harder)

Find the general solution to:

$$y'' - 4y = e^{2t} \tag{3.1}$$

Remark 3.1 (Method.)**.** Like any non-homogenous differential equation, we'll find the homogenous solution and then find a non-homogenous particular solution.

Solution 3.1 (Solution 2bi) — The homogenous equation:

$$y'' = 4y, \tag{3.2}$$

roots of characteristic polynomial:

$$r^2 - 4 = 0, r = \pm 2, \tag{3.3}$$

general solution:

$$c_1 e^{2t} + c_2 e^{-2t}. \tag{3.4}$$

Solution 3.2 (Solution 2bii) — Try Cte^{2t}. Note that:

$$\begin{aligned} &(D+2)(D-2)Cte^2t && (3.5)\\ &= (D+2)Ce^{2t} && (3.6)\\ &= 4Ce^{2t}, && (3.7) \end{aligned}$$

so $C = 1/4$ and:

$$\begin{aligned} f_p(t) &= \frac{1}{4}te^{2t} && (3.8)\\ f(t) &= \frac{1}{4}te^{2t} + c_1 e^{2t} + c_2 e^{-2t}. && (3.9) \end{aligned}$$

Remark 3.2 (Footnotes.)**.** Wronskian Sanity Check:

$$\begin{aligned} 0 &= -2e^{2t}e^{-2t} - 2e^{-2t}e^{2t} && (3.10)\\ &= -4 && (3.11) \end{aligned}$$

which is impossible so by E+U theorem, we have the unique solution for all initial value problems.

Back-substitution sanity check:

$$e^{2x} = -4 \cdot \frac{1}{4} te^{2t} + \left(\frac{1}{4} te^{2t}\right)'' \tag{3.12}$$

$$= -te^{2t} + te^{2t} + \frac{1}{2} e^{2t} \cdot 2 \tag{3.13}$$

$$= e^{2t}. \tag{3.14}$$

Problem 4 (Problem 2c, harder)
Find the general solution to:

$$y'' + 3y' + 2y = t^2 e^{-t} \tag{4.1}$$

Remark 4.1 (Method.). Like any non-homogenous differential equation, we'll find the homogenous solution and then find a non-homogenous particular solution.

Solution 4.1 (Solution 2ci) — The corresponding homogenous equation:

$$y'' + 3y + 2 = 0 \tag{4.2}$$
$$(D+1)(D+2)y = 0 \tag{4.3}$$
$$y(t) = e^{-t} + e^{-2t}. \tag{4.4}$$

Solution 4.2 (Solution 2cii) — Guess cx^3e^{-t}. Plug in to get:

$$(D+2)(D+1)(ct^3e^{-t}) = t^2e^{-t} \tag{4.5}$$
$$3c(D+2)(t^2e^{-t}) = t^2e^{-t} \tag{4.6}$$
$$6cte^{-t} + 3ct^2e^{-t} = t^2e^{-t}. \tag{4.7}$$

Now let $c = 1/3$. OK, now we've reduced the inhomogenity to $2te^{-t}$. Then subtracting t^2e^{-t} reduce to $2e^{-t}$ and finally adding $2te^{-t}$ completes the redution.

This gives:

$$f_p(t) = \frac{1}{3}t^3e^{-t} - t^2e^{-t} + 2te^{-t} \tag{4.8}$$
$$f(t) = \frac{1}{3}t^3e^{-t} - t^2e^{-t} + 2te^{-t} + c_1e^{-t} + c_2e^{-2t}. \tag{4.9}$$

Remark 4.2 (Footnotes.). Wronskian Sanity Check:

$$0 = -2e^{-t}e^{-2t} + e^{-2t}e^{-t} \tag{4.10}$$
$$= e^{-3t} \tag{4.11}$$

which is impossible so by E+U theorem, we have the unique solution for all initial value problems.

Back-substitution sanity check:

$$t^2e^{-t} = (D+2)(D+1)\left(\frac{1}{3}t^3 - t^2 + 2t\right)e^{-t} \tag{4.12}$$

$$= \left(t^2 - 2t + 2 + 2t - 2\right)e^{-t} \tag{4.13}$$

$$= t^2e^{-t} \tag{4.14}$$

Problem 5 (Problem 2d. harder)

Find the general solution to:

$$y'' + 2y' + y = e^{-t} + t^2 \cos t \tag{5.1}$$

Remark 5.1 (Method.)**.** Like any non-homogenous differential equation, we'll find the homogenous solution and then find a non-homogenous particular solution.

Solution 5.1 (Solution 2di) — The homogenous equation:

$$y'' + 2y' + y = 0, \tag{5.2}$$

roots of characteristic polynomial:

$$(r+1)^2 = 0, r = -1\,(2), \tag{5.3}$$

general solution:

$$c_1 e^{-t} + c_2 x e^{-t}. \tag{5.4}$$

Solution 5.2 (Solution 2dii) — Note that e^{-t} will yield $\frac{1}{2}x^2 e^{-t}$ just plugging in with differential operators, and for $t^2 \cos t$ we should have something like:

$$t^2(A_2 \cos(2t) + B_2 \sin(2t)) + \tag{5.5}$$

$$t^1(A_1 \cos(2t) + B_1 \sin(2t)) + \tag{5.6}$$

$$t^0(A_0 \cos(2t) + B_0 \sin(2t)), \tag{5.7}$$

Remark 5.2 (Footnotes.)**.** Wronskian Sanity Check:

$$0 = -xe^{-t}e^{-t} - e^{-t}(e^{-t} - xe^{-t}) \tag{5.8}$$

$$= e^{-2t} \tag{5.9}$$

which is impossible so by E+U theorem, we have the unique solution for all initial value problems.

Back-substitution sanity check: none as no particular solution derived.

Problem 6 (Problem 3)

Find the general solution to:

$$x^2y'' + 4xy' + 2y = e^x \tag{6.1}$$

Remark 6.1 (Method.)**.** Like any non-homogenous differential equation, we'll find the homogenous solution and then find a non-homogenous particular solution.

Solution 6.1 (Solution 3a) — This is a Cauchy-Euler differential equation. The homogenized version is:

$$x^2y'' + 4xy' + 2y = 0, \tag{6.2}$$

and the characteristic equation:

$$r(r-1) + 4r + 2 = 0 \tag{6.3}$$

$$(r+1)(r+2) + 0, r = -1, -2. \tag{6.4}$$

Solution 6.2 (Solution 3b) — Now let's use Variation Of Parametrs! We have:

$$\frac{v_1'}{x} + \frac{v_2'}{x^2} = 0 \tag{6.5}$$

$$\frac{v_1'}{x^2} + \frac{2v_2'}{x^3} = -\frac{e^x}{x^2} \tag{6.6}$$

$$v_2' = -xe^x \tag{6.7}$$

$$v_2 = e^x - xe^x \tag{6.8}$$

$$v_1' = e^x \tag{6.9}$$

$$v_1 = e^x \tag{6.10}$$

$$f_p = \frac{e^x}{x^2} \tag{6.11}$$

$$y(x) = \frac{e^x + c_1x + c_2}{x^2}. \tag{6.12}$$

Remark 6.2 (Footnotes.)**.** Wronskian Sanity Check:

$$0 = \frac{1}{x^3}, \tag{6.13}$$

which is impossible so by E+U theorem, we have the unique solution for all initial value problems.

> Back-substitution sanity check: not doing as it's very annoying.

Problem 7 (Problem 4)
Find the general solution to:

$$x^2y'' - x(x+2)y' + (x+2)y = 0 \tag{7.1}$$

and over what intervals we are guarentted a unique solution.

Remark 7.1 (Method.). Get a solution, use reduction of order then use the Wronskian to finish.

Solution 7.1 (Solution 4a) — Note that x is a solution. Now let $y = fx$ and:

$$\begin{aligned}
0 &= x^2(f''x + f') - x(x+2)(f'x + f) + (x+2)fx &(7.2)\\
&= x^3 f'' + (x^3 + 3x^2)f' &(7.3)\\
f' &= e^{\int 1 + \dfrac{3}{x} dx} &(7.4)\\
&= x^3 e^x &(7.5)\\
f(x) &= x^3e^x - 3x^2e^x + 6xe^x - 6e^x &(7.6)\\
y(x) &= c_1x + c_2(x^3e^x - 3x^2e^x + 6xe^x - 6e^x) &(7.7)
\end{aligned}$$

Solution 7.2 (Solution 4b) — Take the Wronskian to get:

$$\begin{aligned}
0 &= x^4e^x - (x^3e^x - 3x^2e^x + 6xe^x - 6e^x) &(7.8)\\
&= x^4 - x^3 + 3x^2 - 6x + 6 &(7.9)
\end{aligned}$$

and the roots are the x-values where solutions may converge and diverge. The intervals between the roots and plus/minus infinity are the intervals we are guarenteed a unique solution.

Remark 7.2 (Footnotes.). NONE!

Problem 8 (Problem 5a, harder)

Derive Euler's method via the 1st order E+U operator formula (Picard's Method).

Remark 8.1 (Method.)**.** Plug in an integral approximation formula into Picard's Method.

Solution 8.1 (Solution 5a) — You use Picard Integral form of first-order differential equations:

$$y(x_1) = y_0 + \int_{x_0}^{x_1} f(x, y(x))dx, \tag{8.1}$$

and to get y_1 we plug in a zeroth order approximation (left rieman sum) for the RHS integral, which is a very simple but poor 1st order approximation that is numerically unstable for $h > 0.15$ and only produces decent results when $h < 10^{-5}$.

Remark 8.2 (Footnotes.)**.** None!

Problem 9 (Problem 5b, harder)

Derive Improved Euler via the 1st order E+U operator formula (Picard's Method).

Remark 9.1 (Method.)**.** Plug in an integral approximation formula into Picard's Method and predictor-corrector it out.

Solution 9.1 (Solution 5b) — You use Picard Integral form of first-order differential equations:

$$y(x_1) = y_0 + \int_{x_0}^{x_1} f(x, y(x))dx \tag{9.1}$$

and to get y_1 we plug in a 1st order approximation (left rieman sum) for the RHS integral to get:

$$y_1 = y_0 + h\left(\frac{f(x_0, y_0) + f(x_1, y_1)}{2}\right) \tag{9.2}$$

but now to make this explicit substitute euler for the implicit y_1:

$$y_1 = y_0 + h\left(\frac{f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))}{2}\right) \tag{9.3}$$

which is a less simple but much better 2nd order approximation that is numerically stable and produces good results when $h < 10^{-4}$.

Remark 9.2 (Footnotes.)**.** None!

Problem 10 (Problem 5c)

How is RK4 constructed?

Solution 10.1 (Answer 5c) — You use Picard Integral form of first-order differential equations:

$$y(x_1) = y_0 + \int_{x_0}^{x_1} f(x, y(x))dx, \tag{10.1}$$

and to get y_1 we plug in a second order approximation (Simpson's Rule) for the RHS integral, then use Euler/Improved Euler as predictor-correctors to get rid of implicitness on the RHS to end up with a very good 4th approximation that is complicated but still computationally very cheap and extremely accurate for $h < 10^{-3}$.

Remark 10.1 (Footnotes.)**.** None!

Problem 11 (Problem 6a)

Find the numerical approximation for Euler's method after n steps of step size h for the differential equation:

$$y' = -\lambda y. \tag{11.1}$$

Remark 11.1 (Method.)**.** Plug in to Euler's Method and use induction.

Solution 11.1 (Solution 6a) — We get:

$$\begin{align} y_1 &= y_0 - \lambda h y_0 \tag{11.2} \\ &= y_0(1 - \lambda h). \tag{11.3} \\ y_n &= (1 - \lambda h) y_{n-1} \tag{11.4} \\ &= (1 - \lambda h)^n y_0. \tag{11.5} \end{align}$$

Remark 11.2 (Footnotes.)**.** None!

Problem 12 (Problem 6b)

Find the numerical approximation for the trapezoidal method after n steps of step size h for the differential equation:

$$y' = -\lambda y. \tag{12.1}$$

Remark 12.1 (Method.)**.** Plug in to trapezoidal method and use induction.

Solution 12.1 (Solution 6b) — We have:

$$y_1 = y_0 + h\left(\frac{f(x_0, y_0) + f(x_1, y_1)}{2}\right) \tag{12.2}$$

$$y_1 = y_0 + h\left(\frac{-\lambda y_0 - \lambda y_1}{2}\right) \tag{12.3}$$

$$y_1\left(1 + \frac{\lambda h}{2}\right) = y_0\left(1 - \frac{\lambda h}{2}\right) \tag{12.4}$$

$$y_1 = y_0\left(\frac{1 - \lambda h/2}{1 + \lambda h/2}\right) \tag{12.5}$$

$$y_n = y_0\left(\frac{1 - \lambda h/2}{1 + \lambda h/2}\right)^n \tag{12.6}$$

Remark 12.2 (Footnotes.)**.** None!

This took me a bit under two and a half hours - three if you include document setup time.