Differential Equations Week 4

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NOTE: Can you talk about which equation was wrong in each of the comments, with the numbering in the document? Thanks!

Problem 1 (Problem 1, modified, generalized)

A solution of a solvent and solute flows in at a constant rate of K_e L/min into a large tank that initially held L_i litres of P_1 proportion solute. The solution inside is well-stirred and flows out of the tank at K_o litres per minute. If the solution entering the tank is P_2 proportion solute, find the volume of the solute at t minutes, and when concentration will reach P.

Solution — Note that, if $N(t)$ is the amount of solute in the tank:

$$\begin{aligned}
N'(t) &= \text{nitric acid entering} - \text{nitric acid exiting} && (1.1)\\
&= P_2K_e - \frac{K_oN(t)}{L_i + (K_e - K_o)t}^{*} && (1.2)\\
N(0) &= P_1L_i && (1.3)\\
N'(t) &= P_2K_e - \frac{K_oN(t)}{L_i + (K_e - K_o)t}. && (1.4)
\end{aligned}$$

Ok, we have our differential equation. It's linear, so let's work out an integrating factor:

$$\begin{aligned}
N'(t) + N(t)\frac{K_o}{L_i + (K_e - K_o)t} &= P_2K_e && (1.5)\\
N'(t)\mu(t) + N(t)\mu(t)\frac{K_o}{L_i + (K_e - K_o)t} &= P_2K_e\mu(t) && (1.6)\\
\mu'(t) &= \mu(t)\frac{K_o}{L_i + (K_e - K_o)t} && (1.7)\\
\log|\mu(t)| &= \frac{K_o}{K_e - K_o}\log\left|\frac{L_i}{K_e - K_o} + t\right| && (1.8)\\
\mu(t) &= \left(\frac{L_i}{K_e - K_o} + t\right)^{\frac{K_o}{K_e - K_o}}. && (1.9)
\end{aligned}$$

Now, do a final plug-in:

$$\left(\frac{L_i}{K_e-K_o}+t\right)^{\frac{K_o}{K_e-K_o}} N(t) = \int P_2 \frac{K_e}{1+\dfrac{K_o t}{K_e-K_o}}\, dt^{*2} \tag{1.10}$$

$$\left(\frac{L_i}{K_e-K_o}+t\right)^{\frac{K_o}{K_e-K_o}} N(t) = P_2 \frac{K_e}{1+\frac{K_o}{K_e-K_o}} \left(\frac{L_i}{K_e-K_o}+t\right)^{\frac{K_o}{K_e-K_o}+1} + C \tag{1.11}$$

$$\left(\frac{L_i}{K_e-K_o}+t\right)^{\frac{K_o}{K_e-K_o}} N(t) = P_2(K_e-K_o) \left(\frac{L_i}{K_e-K_o}+t\right)^{\frac{K_e}{K_e-K_o}} + C \tag{1.12}$$

$$N(t) = P_2 L_i + t P_2 (K_e-K_o) + \frac{C}{\left(\frac{L_i}{K_e-K_o}+t\right)^{\frac{K_o}{K_e-K_o}}}. \tag{1.13}$$

Now solve at $t=0$:

$$P_1 L_i = P_2 L_i + \frac{C}{\left(\frac{L_i}{K_e-K_o}\right)^{\frac{K_o}{K_e-K_o}}} \tag{1.14}$$

$$C = L_i(P_1-P_2)\left(\frac{L_i}{K_e-K_o}\right)^{\frac{K_o}{K_e-K_o}} \tag{1.15}$$

$$N(t) = P_2 L_i + t P_2(K_e-K_o) + \frac{L_i(P_1-P_2)\left(\frac{L_i}{K_e-K_o}\right)^{\frac{K_o}{K_e-K_o}}}{\left(\frac{L_i}{K_e-K_o}+t\right)^{\frac{K_o}{K_e-K_o}}} \tag{1.16}$$

$$N(t) = P_2(L_i + t(K_e-K_o)) + L_i(P_1-P_2)\left(\frac{L_i}{L_i+(K_e-K_o)t}\right)^{\frac{K_o}{K_e-K_o}} \tag{1.17}$$

Remark (Footnotes.)**.** * If $K_e - K_o = 0$, take the limit as $K_e - K_o$ approaches 0.
*2 We ignore the constant term for now as it is contained in the RHS integral.

Problem 2 (Problem 2)

A solar hot water heating system consists of a hot water tank and a solar panel. The tank has a Stefan's law k° $\mathrm{K}^{-3}\mathrm{s}^{-1}$. The solar panel generates E MJ/hr, and the tank has a heat capacity of H°K/MJ. If the water in the tank is initially at A°K and the temprature outside is B°K, what will be the temprature after t hours of sunlight?

Solution — Let $T(t)$ be the temprature of the tank at time t. Then (we are using Stefan's law, not Newton's and SI, not imperial because I am taking physics):

$$\begin{cases} T'(t) &= k(B^4 - T(t)^4) + \dfrac{E}{H} \\ T(0) &= A \end{cases} \tag{2.1}$$

$$\int \frac{dT}{k(B^4 - T(t)^4) + \dfrac{E}{H}} = t^* \tag{2.2}$$

$$\frac{1}{k}\int \frac{dT}{B^4 + \dfrac{E}{kH} - T^4} = t \tag{2.3}$$

$$K = \sqrt[4]{B^4 + \frac{E}{kH}} \tag{2.4}$$

$$-\frac{1}{k}\int \frac{dT}{T^4 - K^4} = t \tag{2.5}$$

$$-\frac{1}{k}\sum_{n=0}^{3}\int \frac{A_n}{T + i^n K} dT = t \tag{2.6}$$

$$\begin{cases} A_0 + A_1 + A_2 + A_3 & = 0 \\ A_0K(i + -i + -1) + & \\ A_1K(1 + -i + -1) + & \\ A_2K(1 + i + -i) + & \\ A_3K(1 + i + -1) & = 0 \\ A_0K^2(i \cdot -i + -1 \cdot i + -i \cdot -1) + & \\ A_1K^2(1 \cdot -i + -1 \cdot -i + -1 \cdot 1) + & \\ A_2K^2(i \cdot -i + 1 \cdot i + -i \cdot 1) + & \\ A_3K^2(i \cdot 1 + -1 \cdot i + 1 \cdot -1) & = 0 \\ A_0K^3(i \cdot -i \cdot -1) + & \\ A_1K^3(1 \cdot -i \cdot -1) + & \\ A_2K^3(1 \cdot i \cdot -i) + & \\ A_3K^3(1 \cdot i \cdot -1) & = 1 \end{cases} \tag{2.7}$$

Continue the iPFD (imaginary PFD):

$$\begin{cases} A_0 + A_1 + A_2 + A_3 & = 0 \\ -A_0 - iA_1 + A_2 + iA_3 & = 0 \\ A_0 - A_1 + A_2 - A_3 & = 0 \\ -A_0 + iA_1 + A_2 - iA_3 & = \dfrac{1}{K^3}. \end{cases} \tag{2.8}$$

$$\begin{cases} A_0 + A_2 & = 0 \\ A_0 - A_2 & = -\dfrac{1}{2K^3} \\ A_1 + A_3 & = 0 \\ A_1 - A_3 & = -\dfrac{i}{2K^3} \end{cases} \tag{2.9}$$

$$(A_0, A_1, A_2, A_3) = \left(-\frac{1}{4K^3}, -\frac{i}{4K^3}, \frac{1}{4K^3}, \frac{i}{4K^3}\right). \tag{2.10}$$

$$\tag{2.11}$$

$$\tag{2.12}$$

Thus,

$$\frac{1}{4kK^3}\sum_{n=0}^{3}\int \frac{i^n}{T + i^n K} dT = t \tag{2.13}$$

$$\frac{1}{4kK^3}\sum_{n=0}^{3} i^n \log(T + i^n K) = t + C \tag{2.14}$$

$$\sum_{n=0}^{3} i^n \log(T + i^n K) = 4kK^3 t + C \tag{2.15}$$

$$e^{\sum_{n=0}^{3} i^n \log(T + i^n K)} = Ce^{4kK^3 t} \tag{2.16}$$

$$\prod_{n=0}^{3}(T + i^n K)^{i^n} = Ce^{-4kK^3 t} \tag{2.17}$$

$$\Bigg)\frac{T+K}{T-K}\Bigg)\left(\frac{T+iK}{T-iK}\right)^i = Ce^{4kK^3 t} \tag{2.18}$$

$$U = \frac{T}{K} \tag{2.19}$$

$$\frac{U+1}{U-1}\left(\frac{U+i}{U-i}\right)^i = Ce^{4kK^3 t} \tag{2.20}$$

$$\tag{2.21}$$

(I'll leave this as an implicit solution.)

Problem 3 (Problem 3a)

A rocket with initial mass m_0 kg is launched vertically fromc the ground. It expels gas at a rate of α m/s and at a constant velocity of β m/s relative the the rocket. Determine $x'(t)$ and $x(t)$ if $x(0) = D$.

Solution — We know the following:

$$(m_0 - \alpha t)x''(t) - \alpha\beta = -\frac{GMm}{x(t)^2}. \tag{3.1}$$

Note that:

$$(x'(t)^2)' \tag{3.2}$$
$$= 2x''(t)x'(t) \tag{3.3}$$

Thus:

$$(m_0 - \alpha t)\frac{(x'(t))^2}{x'(t)} - \alpha\beta = -\frac{GM(m_0 - \alpha t)}{x(t)^2} \tag{3.4}$$
$$\frac{1}{2}(m_0 - \alpha t)((x'(t))^2)' - \alpha\beta x'(t) = -\frac{GM(m_0 - \alpha t)}{x(t)^2} \tag{3.5}$$
$$\frac{1}{2}(m_0 - \alpha t)x'(t)^2 - \alpha\beta x(t) = C + \frac{GM(m_0 - \alpha t)}{x(t)} \tag{3.6}$$
$$C = -\frac{GMm_0}{D} - \alpha\beta D \tag{3.7}$$
$$x'(t) = \sqrt{\frac{\alpha\beta}{m_0 - \alpha t}(x(t) - D) + \frac{GM(m_0 - \alpha t)}{x(t)} - \frac{GMm_0}{D} - \alpha\beta D} \tag{3.8}$$

(I am not able to solve this)

*please give me full credit on 2 and 3 on corrections - I'm submitting this mainly as a placeholder so you know I did work for this week :)