Differential Equations Week **14**

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Problem 1

Show that:

$$\Gamma(x+1) = x\Gamma(x), \tag{1.1}$$

for all $x \in \mathbb{R}^+$.

Solution 1.1 — Note that:

$$\begin{align}
\Gamma(x) &= \int_0^\infty t^{x-1}e^{-x}dx \tag{1.2}\\
&= \mathscr{L}[t^{x-1}](1). \tag{1.3}
\end{align}$$

As:

$$\begin{align}
\Gamma(x+1) &= \mathscr{L}[xt^{x-1}] \tag{1.4}\\
&= \mathscr{L}[(t^x)'] \tag{1.5}\\
&= s \cdot \mathscr{L}[xt^{x-1}] + (t^x)'(0) \tag{1.6}\\
&= s \cdot \mathscr{L}[xt^{x-1}] + 0 \tag{1.7}\\
&= x\Gamma(x). \tag{1.8}
\end{align}$$

Problem 2

Compute the following Laplace transforms:

(a) $\mathscr{L}[t\cos bt]$

(b) $\mathscr{L}[t^2\cos bt]$

Solution 2.1 (Solution 2a) — Note that

$$\begin{aligned}
\mathscr{L}[t^n f(t)] &= (-D)^n F(s) && (2.1)\\
\mathscr{L}[\cos bt] &= \frac{s}{b^2+s^2} && (2.2)\\
\mathscr{L}[t\cos bt] &= -D\left(\frac{s}{b^2+s^2}\right) && (2.3)\\
&= -\frac{b^2+s^2-s(2s)}{(b^2+s^2)^2} && (2.4)\\
&= \frac{s^2-b^2}{(b^2+s^2)^2}. && (2.5)
\end{aligned}$$

Solution 2.2 (Solution 2b) — Note that

$$\begin{aligned}
\mathscr{L}[t^n f(t)] &= (-D)^n F(s) && (2.6)\\
\mathscr{L}[t\cos bt] &= \frac{s^2-b^2}{(b^2+s^2)^2} && (2.7)\\
\mathscr{L}[t^2\cos bt] &= -D\left(\frac{s^2-b^2}{(b^2+s^2)^2}\right) && (2.8)\\
&= -\frac{2s(b^2+s^2)^2-4s(s^2-b^2)(s^2+b^2)}{(b^2+s^2)^4} && (2.9)\\
&= -\frac{2sb^4+4s^3b^2+2s^5-4s^5+4sb^4}{(b^2+s^2)^4} && (2.10)\\
&= -\frac{-2s^5+4s^3b^2+6sb^4}{(b^2+s^2)^4} && (2.11)\\
&= \frac{2s(s^4-2s^2b^2-3b^4)}{(b^2+s^2)^4} && (2.12)\\
&= \frac{2s(s^2+b^2)(s^2-3b^2)}{(b^2+s^2)^4} && (2.13)\\
&= \frac{2s(s^2-3b^2)}{(b^2+s^2)^3}. && (2.14)
\end{aligned}$$

Problem 3

Show that the portion of the partial fraction expansion of $\dfrac{P(s)}{(s-r_i)(s-r_1)(s-r_2)\cdots} = \dfrac{P(x)}{Q(x)}$ is:

$$\frac{P(r)}{Q'(r)}. \tag{3.1}$$

and that:

$$\mathscr{L}^{-1}\left[\frac{P}{Q}\right] = \sum_{i=1}^{\deg Q} \frac{P(r_i)}{Q'(r_i)} e^{r_i t} \tag{3.2}$$

Solution 3.1 — Note that evaluating at $s = r_i + \varepsilon$, we have:

$$\frac{P(s)}{(s-r_i)(s-r_1)(s-r_2)\cdots} = \sum_{n=1}^{\deg Q} \frac{A_n}{s-r_n} \tag{3.3}$$

$$\frac{P(s)}{\varepsilon \cdot (s-r_1)(s-r_2)\cdots} = \frac{A_i}{\varepsilon} + \sum_{n=1, n\neq i}^{\deg Q} \frac{A_n}{r_i+\varepsilon-r_n} \tag{3.4}$$

$$\frac{P(s)}{Q(s)/(s-r_i)} = A_i + \varepsilon \cdot \sum_{n=1, n\neq i}^{\deg Q} \frac{A_n}{s-r_n}. \tag{3.5}$$

Now take $\lim\limits_{\varepsilon\to 0}$ to get:

$$\lim_{\varepsilon\to 0} \frac{P(s)}{Q(s)/(s-r_i)} = \lim_{\varepsilon\to 0} A_i + \varepsilon \cdot \sum_{n=1, n\neq i}^{\deg Q} \frac{A_n}{s-r_n} \tag{3.6}$$

$$\lim_{\varepsilon\to 0} \frac{P(s)}{Q(s)/(s-r_i)} = A_i \tag{3.7}$$

$$r = r_i \tag{3.8}$$

$$\lim_{s\to r} \frac{P(s)(s-r)}{Q(s)} = A_i, \tag{3.9}$$

and this is 0/0 L'Hopital. Taking derivitatives, we get:

$$\begin{aligned}
R(\alpha) &= P(r+\alpha) && (3.10)\\
\lim_{s\to r}\frac{P(s)(s-r)}{Q(s)} &= \lim_{\varepsilon\to 0}\frac{\varepsilon R(\varepsilon)}{Q(r+\varepsilon)} && (3.11)\\
&= \lim_{\varepsilon\to 0}\frac{R(\varepsilon)+\varepsilon R'(\varepsilon)}{Q'(r+\varepsilon)} && (3.12)\\
&= \lim_{\varepsilon\to 0}\frac{R(\varepsilon)}{Q'(r+\varepsilon)} + \lim_{\varepsilon\to 0}\frac{\varepsilon R'(\varepsilon)}{Q'(s)} && (3.13)\\
|Q'(r_i)| &> 0 \text{ (if it was 0, } r_i \text{ would be a double root)} && (3.14)\\
&= \frac{P(r_i)}{Q'(r_i)}. && (3.15)
\end{aligned}$$

Now, noting that:

$$\mathscr{L}^{-1}\left[\frac{1}{s-r_i}\right] = e^{r_i t}, \tag{3.16}$$

using the previous finding, and summing over all r_i, and using linearity of $\mathscr{L}^{-1}$, we get:

$$\boxed{\frac{P}{Q} = \sum_{i=1}^{\deg Q}\frac{P(r_i)}{Q'(r_i)(s-r_i)}} \tag{3.17}$$

$$\boxed{\mathscr{L}^{-1}\left[\frac{P}{Q}\right] = \sum_{i=1}^{\deg Q}\frac{P(r_i)}{Q'(r_i)}e^{r_i t}} \tag{3.18}$$

Problem 4

Find the partial fraction decomposition of:

$$\frac{2s+1}{s(s-1)(s+2)} \tag{4.1}$$

Solution 4.1 — Use (3.17).
There are two things to start with:

- Find Q'.
- Find roots of Q.

Note that roots of Q are $-2, 0, 1$. Now also note that $Q' = (s^3+s^2-2s)' = 3s^2+2s-2$ and plugging in:

$$\frac{2s+1}{s(s-1)(s+2)} = \frac{P(-2)}{Q'(-2)} \cdot \frac{1}{s+2} + \frac{P(0)}{Q'(0)} \cdot \frac{1}{s} + \frac{P(1)}{Q'(1)} \cdot \frac{1}{s-1} \tag{4.2}$$

$$= -\frac{1}{2} \cdot \frac{1}{s+2} + \frac{1}{3} \cdot \frac{1}{s} + \frac{1}{s-1}. \tag{4.3}$$

Problem 5

Find the inverse Laplace transform of:

$$\frac{3s^2-16s+5}{(s+1)(s-2)(s-3)} \tag{5.1}$$

Solution 5.1 — Use (3.18).
There are two things to start with:

- Find Q'.
- Find roots of Q.

Note that roots of Q are $-2, 0, 1$. Now also note that $Q' = (s^3-4s^2+s-6)' = 3s^2-8s+1$ and plugging in:

$$\mathscr{L}^{-1}\left[\frac{3s^2-16s+5}{(s+1)(s-2)(s-3)}\right] = \frac{P(-1)}{Q'(-1)} \cdot e^{-t} + \frac{P(2)}{Q'(2)} \cdot e^{2t} + \frac{P(3)}{Q'(3)} \cdot e^{3t} \tag{5.2}$$

$$= 2e^{-t} - 3e^{2t} - 13 \cdot e^{3t} \tag{5.3}$$

$$\tag{5.4}$$

Problem 6

Solve the intial value problem:

$$y''(t) + 4y = \begin{cases} 3\sin t, & 0 \le t \le 2\pi \\ 0, & 2\pi < t \end{cases} \tag{6.1}$$

$$\begin{cases} y(0) = 1 \\ y'(0) = 3 \end{cases} \tag{6.2}$$

Solution 6.1 — Step 1. Express the piecewise continuous forcing function in terms of heaviside functions.
Note that for any piecewise continuous function that is equal to f_k on intereval $I_k = (a_k, b_k)$, that function is equal to:

$$\sum_{I_k} f_k(x)u(x - a_k) - f_k(x)u(x - b_k), \tag{6.3}$$

and note that for our forcing function, that is:

$$\boxed{3\sin t - 3\sin t u(x - 2\pi).} \tag{6.4}$$

Step 2. Take the Laplace Transform and solve for $\mathscr{L}[y]$.
Taking the Laplace transform, we get:

$$\mathscr{L}[y''(t) + 4y] = \mathscr{L}[3\sin t - 3\sin t u(x - 2\pi)] \tag{6.5}$$

$$(s^2 + 4)\mathscr{L}[y] - sy(0) - y'(0) = \frac{3(1 - e^{2\pi})}{1 + s^2} \tag{6.6}$$

$$\boxed{\mathscr{L}[y] = \frac{3}{(1 + s^2)(4 + s^2)} - \frac{3e^{2\pi s}}{(1 + s^2)(4 + s^2)} + \frac{s - 3}{s^2 + 4}} \tag{6.7}$$

Step 2. Take the Inverse Laplace Transform and solve for u.
Taking the inverse Laplace transform, we get:

$$\boxed{\mathscr{L}[y] = \frac{3}{(1 + s^2)(4 + s^2)} - \frac{3e^{2\pi s}}{(1 + s^2)(4 + s^2)} + \frac{s - 3}{s^2 + 4}} \tag{6.8}$$