Quarterly Exam 1 (LoM DE Winter/Spring 2025)

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Problem 1 (Problem 1)
Consider

$$xy' - 2y = 0. \tag{1.0.1}$$

(a) Find the general solution.

(b) Graph five solutions.

(c) What does the existence and uniqueness theorem have to say

Solution 1.1 (Solution 1a) — We add $2y$ to both sides to get:

$$xy' = 2y. \tag{1.1.1}$$

As this is seperable, we perform the method for solving seperable equations. Thus:

$$xy' = 2y \tag{1.1.2}$$
$$\frac{y'}{2y} = \frac{1}{x} \tag{1.1.3}$$
$$\int \frac{dy}{2y} = \int \frac{dx}{x} \tag{1.1.4}$$
$$\frac{1}{2}\log|y| = \log|x| + C \tag{1.1.5}$$
$$|y| = A|x|^2,\ A > 0 \tag{1.1.6}$$
$$y = Ax^2,\ A \neq 0 \tag{1.1.7}$$
$$y = Ax^2,\ A \in \mathbb{R}. \tag{1.1.8}$$

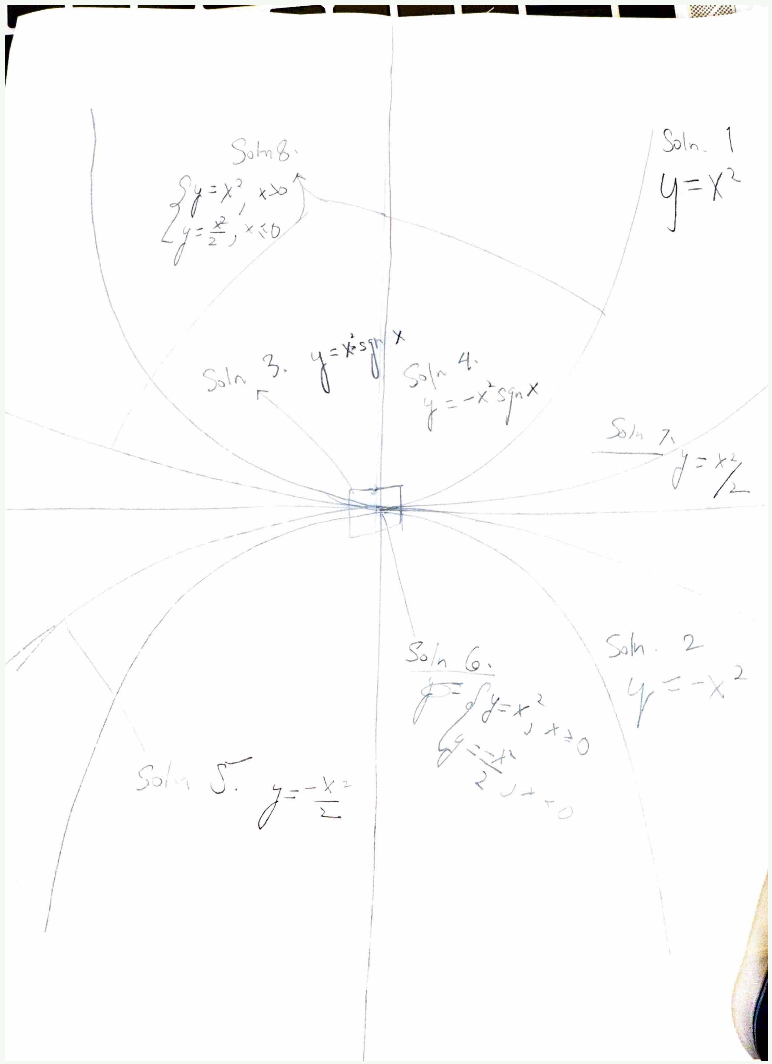
However when we removed the absolute value this gave us branches, thus the general solution is:

$$y(x) = \begin{cases} A_1x^2, & y \geq 0 \\ A_2x^2, & y < 0. \end{cases}, A_1, A_2 \in \mathbb{R} \tag{1.1.9}$$

Remark (Footnotes)**.**

1. Converting Eqn (1.1.2) to (1.1.3) drops the zero solution.
2. (1.1.7) to (1.1.8) *looks* like we added a solution out of nowhere, but we just added the solution we dropped in Footnote 1.1.1.

Solution 1.2 (Solution 1b) —



Extrapolating, there are infinitely many solutions through any point.

Solution 1.3 (Solution 1c) — The existence and uniqueness theorem acts on:

$$xy' = 2y$$
$$y' = \frac{2y}{x}.$$

Thus we can use the existence and uniqueness theorem when $x \neq 0$. It does hold in this scneario in some local window, as all of our 'distinct' solutions are actually the same when we restrict to one side of the origin.

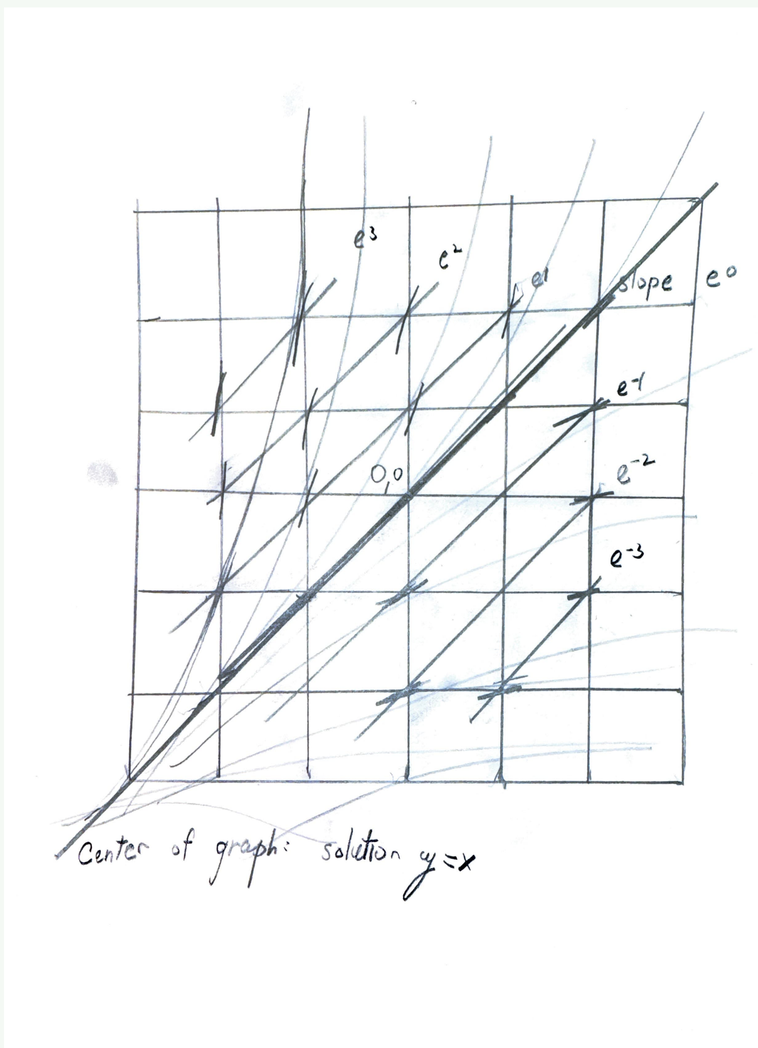
Problem 2

Consider the differential equation:

$$y' = e^{y-x}. \tag{2.0.1}$$

1. Sketch a direction field of the differential equation.
2. Show that $y = x$ is a solution to the DE.
3. What are the possible behavious of the solutions as $y \to \infty$?

Solution 2.1 (Solution 2a) —



Solution 2.2 (Solution 2b) — Note that as $x = y$, $e^{y-x} = 1$ which is the slope everywhere.

Solution 2.3 (Solution 2c) — Taking a look at the lightly drawn functions, they either:

- Going TO infinity they either
  - increase super fast
  - increase super slow
  - stay as $y = x$
- To negative infinity they have a asymptote of $y = x$ approacing negative infinity.

Problem 3 (Problem 3, modified/generalized)
Given the following differential equation:

$$T' = k(T_0 - T), \tag{3.0.1}$$

find:

(a) the general solution.

(b) If $T(0) = T_i$, find when $T(t) = T_f$ if $T_i < T_f < T_0$ or $T_0 < T_f < T_i$.

Solution 3.1 (Solution 3a) — Note that:

$$\begin{aligned} T' &= k(T_0 - T) && (3.1.1)\\ \frac{T'}{k(T_0 - T)} &= 1 && (3.1.2)\\ \int \frac{dT}{k(T_0 - T)} &= \int dt && (3.1.3)\\ -k \log |T - T_0| &= t + C && (3.1.4)\\ T - T_0 &= Ae^{-t/k} && (3.1.5)\\ T &= Ae^{-t/k} + T_0 && (3.1.6) \end{aligned}$$

(A similar note as Footnotes 1 can be made here with the lost and gained solution being $T \equiv T_0$).

Solution 3.2 (Solution 3b) — We have:

$$\begin{aligned} T_i &= Ae^{-0/k} + T_0 && (3.2.1)\\ A &= T_i - T_0 && (3.2.2)\\ T_f &= (T_i - T_0)e^{-t/k} + T_0 && (3.2.3)\\ T_f - T_0 &= (T_i - T_0)e^{-t/k} && (3.2.4)\\ e^{-t/k} &= \frac{T_f - T_0}{T_i - T_0} && (3.2.5)\\ t &= k \log \left| \frac{T_i - T_0}{T_f - T_0} \right|. && (3.2.6) \end{aligned}$$

Problem 4

Solve the following differential equations:

(a) $(2x - y)dx + (3y + x)dy = 0$

(b) $(3x^2 + y)dx + (x^2y - x)dy = 0$

(c) $(x + y + 4)dx + (x - 2y + 3)dy = 0$

(d) $(x + y + 4)dx + (2x + 2y + 3)dy = 0$

Solution 4.1 (Solution 4a) — This equation is HOMOGENOUS. Substitute $y = vx$, $\dfrac{dy}{dx} = v + v'x$:

$$(2x - y)dx + (3y + x)dy = 0 \tag{4.1.1}$$

$$-\frac{2x - y}{3y + x} = y' \tag{4.1.2}$$

$$-\frac{2 - v}{3v + 1} - v = v'x \tag{4.1.3}$$

$$\frac{v'}{\dfrac{2 - v}{3v - 1} + v} = -\frac{1}{x} \tag{4.1.4}$$

$$\int \frac{dv}{\dfrac{2 - v}{3v - 1} + v} = -\log x \tag{4.1.5}$$

$$\int \frac{3v - 1}{3v^2 - 2v + 2}\, dv = -\log x \tag{4.1.6}$$

$$u = v - 1/3 : \tag{4.1.7}$$

$$\int \frac{u}{u^2 + \frac{1}{3}}\, du = -\log x \tag{4.1.8}$$

$$k = u\sqrt{3} : \tag{4.1.9}$$

$$\int \frac{k}{k^2 + 1}\, dk = -\log x \tag{4.1.10}$$

$$\frac{1}{2} \log \left|k^2 + 1\right| = -\log |x| + C \tag{4.1.11}$$

$$\frac{1}{2} \log \left| 3\left(v - \frac{1}{3}\right)^2 + 1 \right| = -\log |x| + C \tag{4.1.12}$$

$$3\left(v - \frac{1}{3}\right)^2 - \left(\frac{C}{x}\right)^2 = -1. \tag{4.1.13}$$

Solution 4.2 (Solution 4b) — Mutiply by an integrating factor $\mu(x, y)$ and get using test for exactness:

$$\mu(x,y) + \mu_y(x,y)(3x^2 + y) = \mu_x(x,y)(x^2y - x) + (2xy - 1)\mu(x,y), \tag{4.2.1}$$

so let μ be a function of x:

$$\mu(x)(2 - 2xy) = \mu'(x)'(x^2y - x) \tag{4.2.2}$$

$$\mu = \mu' \cdot x \cdot \frac{-1}{2} \tag{4.2.3}$$

$$\frac{\mu'}{\mu} = -\frac{2}{x} \tag{4.2.4}$$

$$\log|\mu| = -2\log|x| + C \tag{4.2.5}$$

$$\mu = x^{-2}. \tag{4.2.6}$$

Thus:

$$\left(3 + \frac{y}{x^2}\right) dx + \left(y - \frac{1}{x}\right) dy = 0 \tag{4.2.7}$$

$$F(x,y) = 3x - \frac{y}{x} + g(y) = \frac{y^2}{2} - \frac{y}{x} + h(x) = 3x + \frac{y^2}{2} - \frac{y}{x} \tag{4.2.8}$$

$$3x + \frac{y^2}{2} - \frac{y}{x} = C. \tag{4.2.9}$$

Solution 4.3 (Solution 4c) — This is an equation with linear coeffiecents. Let $y = y_0 - \frac{1}{3}$, $x = x_0 - \frac{11}{3}$. Then:

$$(x_0 + y_0)dx + (x_0 - 2y_0)dy = 0 \tag{4.3.1}$$

$$y' = -\frac{x_0 + y_0}{x_0 - 2y_0}, \tag{4.3.2}$$

and now let $y_0 = x_0 v$:

$$xv' + v = -\frac{1+v}{1-2v} \tag{4.3.3}$$

$$xv' = -\left(\frac{1+2v^2}{1-2v}\right) \tag{4.3.4}$$

$$\log x = \int -\frac{1-2v}{1+2v^2}\,dv \tag{4.3.5}$$

$$u = \sqrt{2}v \tag{4.3.6}$$

$$\int \frac{1-2v}{1+2v^2}\,dv = \frac{1}{\sqrt{2}}\int -\frac{1}{1+u^2}\,du + \sqrt{2}\int \frac{u}{1+u^2}\,du \tag{4.3.7}$$

$$\log x + C = \frac{1}{2}\log|2v^2+1| - \sqrt{2}\arctan\left(\sqrt{2}v\right) \tag{4.3.8}$$

$$C = \frac{1}{2}\log(2y^2+x^2) - 2\log|x| - \sqrt{2}\arctan\left(\frac{3y+1}{3x+11}\right). \tag{4.3.9}$$

Solution 4.4 (Solution 4d) — Note that:

$$(x+y+4)dx + (2x+2y+3)dy = 0 \tag{4.4.1}$$

$$y' = -\frac{2(x+y+2)-1}{x+y+1}, \tag{4.4.2}$$

and letting $k = x+y+1$,

$$k' = 3 - \frac{1}{[}k] \tag{4.4.3}$$

$$\int \frac{k}{3k-1} dk = x \tag{4.4.4}$$

$$\frac{1}{3}\int 1 + \frac{1}{k-1/3} dk = x \tag{4.4.5}$$

$$k + \log|3k-1| - 3x = C \tag{4.4.6}$$

$$\log|3x+3y+2| + y - 2x = C. \tag{4.4.7}$$

Problem 5 (Problem 5, parts removed similar to AoPS)

(a) Show that if y_1 is a particular solution to the Ricatti Equation:

$$y' = A(x)y^2 + B(x)y + C(x), \tag{5.0.1}$$

then the substitution $y = y_1 + \frac{1}{v}$ transforms our equation into

$$v' + (B(x) + 2A(x)y_1(x))v = -A(x). \tag{5.0.2}$$

(b) Solve the Ricatti Equation:

$$y' = y^2 + \frac{y}{x} - \frac{3}{x^2}. \tag{5.0.3}$$

Solution 5.1 (Problem 5a) — Plug in:

$$y_1' - \frac{v'}{v^2} = A(x)\left(\frac{1}{v} + y_1(x)\right)^2 + B(x)\left(\frac{1}{v} + y_1(x)\right) + C(x) \tag{5.1.1}$$

$$-v' = A(x) + 2vy_1(x)A(x) + B(x)v \tag{5.1.2}$$

$$-A(x) = v' + (B(x) + 2A(x)y_1(x))v \tag{5.1.3}$$

Solution 5.2 (Problem 5b) — Use the fact that $xy = 1$, or $y_1 = \dfrac{1}{x}$ is a solution (as $y' = -\dfrac{1}{x^2} = RHS$)and use 5a:

$$v' + (B(x) + 2A(x)y_1(x))v = -A(x) \tag{5.2.1}$$

$$v' + \frac{3v}{x} = -1. \tag{5.2.2}$$

Multiply by a integrating factor $\mu(x)$:

$$\mu'(x) = \frac{3}{x}\mu(x) \tag{5.2.3}$$

$$\frac{\mu'}{\mu} = \frac{3}{x} \tag{5.2.4}$$

$$\mu = x^3. \tag{5.2.5}$$

Then:

$$(x^3 v)' = -x^3 \tag{5.2.6}$$

$$vx^3 = -\frac{x^4}{4} + C \tag{5.2.7}$$

$$v = -\frac{x}{4} + \frac{C}{x^3}. \tag{5.2.8}$$

Thus:

$$y(x) = \frac{1}{x} + \frac{4x^3}{Cx - x^4}. \tag{5.2.9}$$

Problem 6 (Problem 6, modified, EXTRA CREDIT)
Consider the Bernoulli Equation:

$$y' + \left(-\frac{2}{x}\right) y = -\frac{y^3}{x^3}. \tag{6.0.1}$$

(including the solution $y = 0$)

(a) Find the general solution.

(b) For the solutions $\frac{x^4}{y^2} - x^2 = C$, when does the implicit function theorem guarentee a unique explicit solution?

(c) Are there any points with more than one explicit solution, and with none at all?

(d) What does existence and uniqueness say about this?

Solution 6.1 (Solution 6a) — We divide through by y^3:

$$\frac{y'}{y^3} + \left(-\frac{2}{x}\right) y^{-2} = \frac{-1}{x^3}. \tag{6.1.1}$$

Now substitute $y_1 = y^{-2}$. Then:

$$\frac{-y_1'}{2} + \left(-\frac{2}{x}\right) y_1 = \frac{-1}{x^3} \tag{6.1.2}$$

$$y_1' + \left(\frac{4}{x}\right) y_1 = \frac{2}{x^3}. \tag{6.1.3}$$

This is a linear. Use the integrating factor x^4 to get:

$$(y_1 x^4)' = 2x \tag{6.1.4}$$

$$\frac{x^4}{y^2} = x^2 + C. \tag{6.1.5}$$

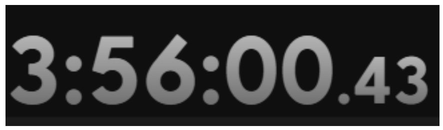
Solution 6.2 (Solution 6b) — Note that we need the function to be well-defined - or $y \neq 0$ - and it to have a partial derivative (same condition!)

Thus the points are those with $y \neq 0$.

Solution 6.3 (Solution 6c) — Not with none if $x \neq 0$: the explicit solutions $\pm\sqrt{\frac{x^4}{x^2 + C}}$ exist for all points $y \neq 0$ BUT we can mix and match at $(x, y) = 0$ thus every point has infinitely many solutions through it (unless $x = \pm y$!). But when $x = 0$, either $y = 0$ or nothing.

Solution 6.4 (Solution 6d) — E and U theorem works when $x \neq 0$, but it provides only *local* information which is compatible with our knowledge.

That's it! This took me almost four hours to write up, mainly because I can't remember integrals that I probably *should* remember.



(This is my timer...)