Differential Equations Week **11**

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May 5, 2025

Problem 1 (Problem 1)

For the differential equations:

$$x' = y \tag{1.0}$$
$$y' = -x + x^3, \tag{1.1}$$

(a) Sketch the direction field using computer software.

(b) Find whether the solution passing through each of the following points is periodic:

i. (0.25, 0.25)

ii. (2,2)

iii. (1,0)

Solution 1.1 (Phase planes are cool! - 1a) —
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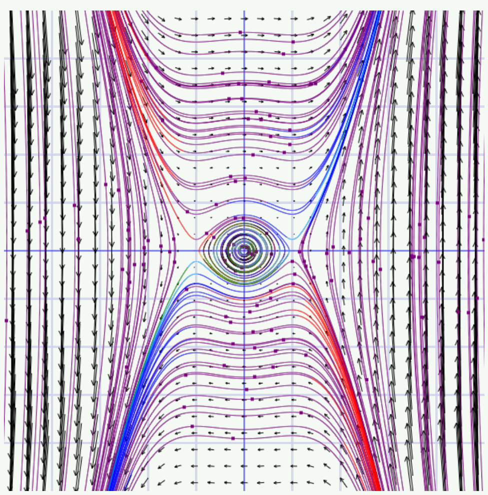

Figure 1.1: Phase plane of differential equation.

Example (Answers 1b). My answers are:

i. (0.25, 0.25): Why not?

ii. (2,2): No way.

iii. (1,0): Even better: critical point, you're never leaving.

Problem 2

For the differential equations

$$x' = y \tag{2.0}$$
$$y' = -x - x^3, \tag{2.1}$$

(a) Sketch the direction field using computer software.

(b) Conjecture whether all solutions are bounded.

(c) Solve the phase plane differential equation.

Solution 2.1 (Phase planes are cool! - 2a) — *Created with https://aeb019.hosted.uark.edu/pplane.html*

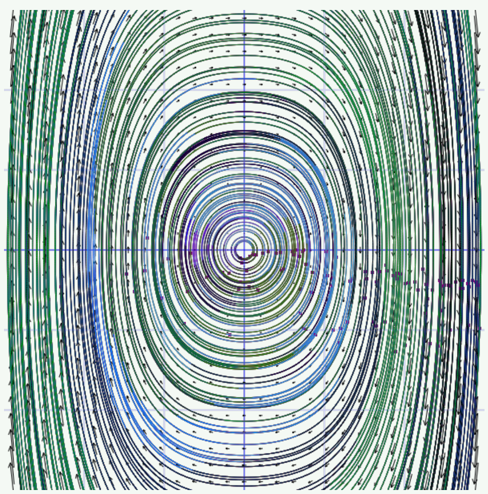

Figure 2.1: Phase plane of differential equation.

Solution 2.2 (Answer 2b) — Why not? Looks like it.

Solution 2.3 (Solution 2c) — We have:

$$\frac{dy}{dx} = \frac{-x - x^3}{y} \tag{2.2}$$
$$y^2 = C - x^2 - \frac{x^4}{2} \tag{2.3}$$
$$y = \sqrt{C - x^2 - \frac{x^4}{2}}. \tag{2.4}$$

Also this is literally a Duffing Equation in disguise so of course it's periodic.

Problem 3

For the differential equations:

$$x' = (x-1)(y-1) \tag{3.0}$$
$$y' = y(y-1), \tag{3.1}$$

(a) Construct the phase plane analysis by hand, and graph, without using computer software, the asymptotic behaviour of trajectories (as $t \to \infty$).

(b) Sketch the direction field using computer software.

(c) Solve the differential equation.

Solution 3.1 —

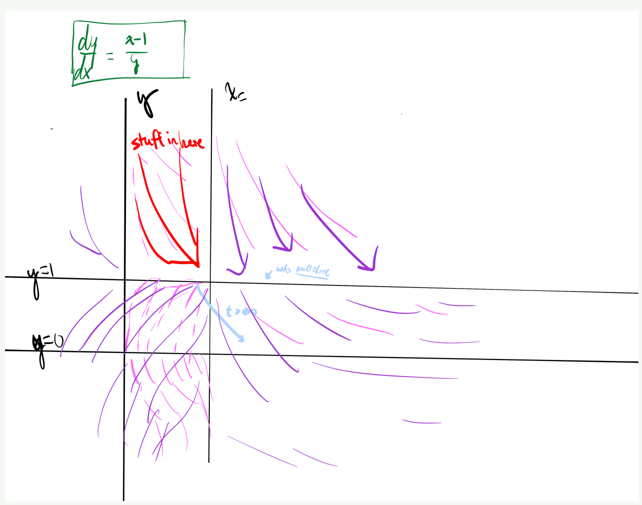

Figure 3.1: Hand-drawn phase plane of differential equation.

Solution 3.2 —

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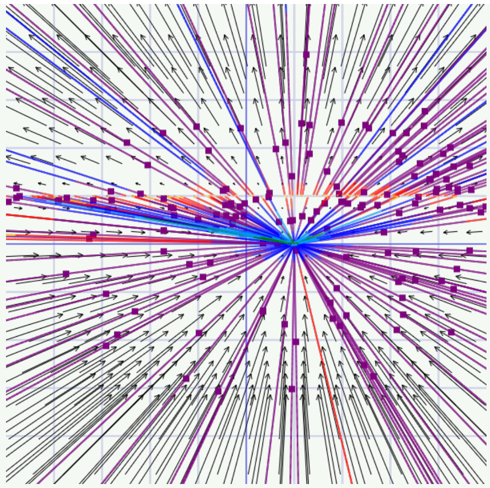

Figure 3.1: Phase plane of differential equation.

Solution 3.3 — We solve for y, then x.

Note that by seperation of variables

$$y' = y(y-1) \tag{3.2}$$

$$\frac{y'}{y(y-1)} = 1 \tag{3.3}$$

$$\int \frac{dy}{y(y-1)} = t \tag{3.4}$$

$$\log \left| \frac{y-1}{y} \right| = t + C \tag{3.5}$$

$$\frac{y-1}{y} = Ce^t \tag{3.6}$$

$$y(1 - Ce^t) = -1 \tag{3.7}$$

$$y = \frac{1}{Ce^t - 1} \tag{3.8}$$

Problem 4

Consider the system:

$$v' = -x + \frac{1}{\lambda - x} \tag{4.0}$$
$$x' = v. \tag{4.1}$$

(a) Solve this equation in the phase plane.

(b) Find the critical points of this system.

(c) Plot the phase plane diagrams for $\lambda = 1, 3$ and describe x for $\lambda = 1, 3$ under various initial conditions.

Solution 4.1 — We note that:

$$\frac{dv}{dx} = \frac{-x + 1/(\lambda - x)}{v} \tag{4.2}$$
$$\frac{v^2}{2} = \int -x + \frac{1}{\lambda - x}\, dx \tag{4.3}$$
$$= -\frac{x^2}{2} - \log|x - \lambda| + C.v^2 \qquad = C - x^2 - \log((x-\lambda)^2) \tag{4.4}$$
$$v = \pm\sqrt{C - x^2 - \log((x-\lambda)^2)} \tag{4.5}$$
$$\tag{4.6}$$

Solution 4.2 —

Observation 4.1 — $v = 0$ for any critical point. (obviously, bottom equation).

Now:

$$0 = -x + \frac{1}{\lambda - x} \tag{4.7}$$
$$x = \frac{1}{\lambda - x} \tag{4.8}$$
$$x^2 - \lambda x + 1 = 0 \tag{4.9}$$
$$x = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}. \tag{4.10}$$

Thus to have critical points, we need $\lambda < 2$.

Solution 4.3 —
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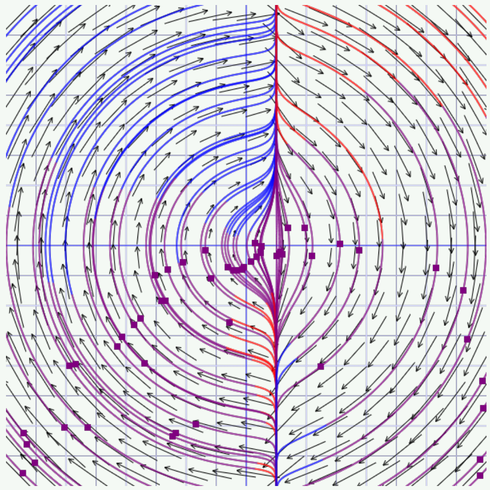

Figure 4.1: Phase plane of differential equation, $\lambda = 1$.

It acclerates, faster and faster, until it has reached the wire.

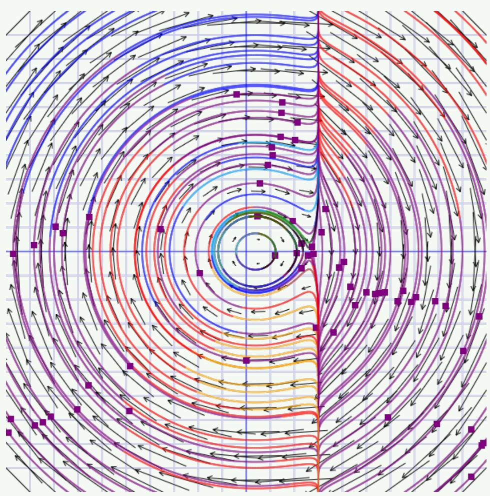

Figure 4.2: Phase plane of differential equation, $\lambda = 3$.

It will perform behaviour as earlier, but with a bit more of a detour around the zone of stability and closed paths.