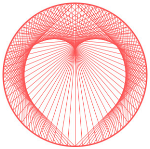
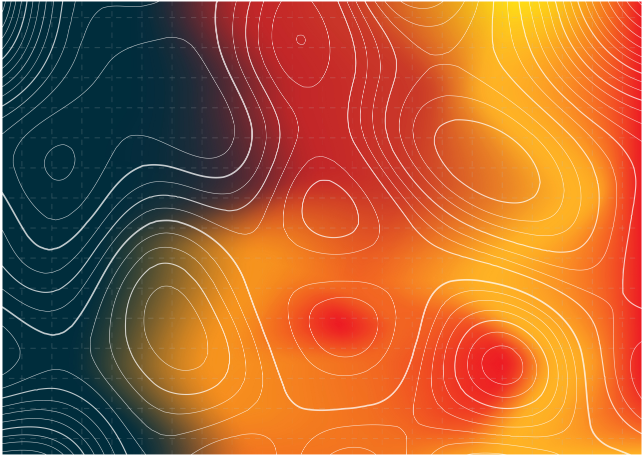
Differential Equations

Fundamentals





Every differential equation can be named by whether or not it is linear, it's order, and whether or not it has only one independent variable.

↳ ordinary differential eqn (ODE)

nonlinear

$y(x)\frac{dy}{dx} + 1 = 0$

↳ largest order of derivative in the equation

First Order

Linear

Newton's Law of Cooling:

$$\frac{dT}{dt} = -k(T - A)$$

(temperature change over time)

Circuit Modeling:

$$L\frac{dI}{dt} + RI = E(t)$$

(current due to voltage source)

Advection Equation:

$$\frac{\partial \psi}{\partial t} + \nabla \cdot (\psi \vec{u}) = 0$$

(transport via velocity field)

Nonlinear

Bernoulli Equations:

nonlinearity

$$y' + p(x)y = q(x)y^n$$

(special case: logistic equation)

Ricatti Equations:

nonlinearity

$$y' = q(x) + p(x)y + r(x)y^2$$

(control theory, falling body with air resistance)

Burger's Nonlinear Wave Equation:

nonlinearity

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

(size dependent wave motion)

Second Order

Linear

Bessel Equations:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

(*many* applications)

The Heat equation:

$$\frac{\partial f}{\partial t} = \frac{\partial f^2}{\partial x^2}$$

Schrödinger's equation:

$$ih\frac{\partial \psi}{\partial t} = -\frac{h^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + v\psi$$

(quantum mechanics)

Nonlinear

Incompressible Navier-Stokes equation:

$$\frac{\partial u}{\partial t} + \underbrace{u \cdot \nabla u}_{\text{nonlinearity}} - \nu \Delta u = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}f$$

(fluid flow: weather, ocean currents)

Emden-Chandrasekhar Eq'n:

$$\underbrace{\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{df}{dz}\right)}_{\text{linear}} = e^{-f}$$

(astrophysics)

nonlinearity

Third Order

Linear

The Airy Equation:

$$y'''(x) = xy(x)$$

(optics)

The Moore-Gibson-Thompson Equation:

$$\tau \frac{\partial^3 u}{\partial t^3} + \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u - \beta \Delta \left(\frac{\partial u}{\partial t}\right) = 0$$

(thermoelasticity)

Nonlinear

Blasius Eq'n:

$$\frac{d^3 y}{dx^3} + y \frac{d^2 y}{dx^2} = 0$$

(fluid boundary layer)

Benjamin-Bona-Mahony:

$$u_t + u_x + u u_x - u_{xxt} = 0$$

(wave propagation)

Fourth Order

Linear

The Beam Equation:

$$\frac{d^2}{dx^2}\left(EI \frac{d^2 w}{dx^2}\right) = q$$

(deflection of a uniform static beam)

Nonlinear

Kuramato-Sivashinksy Eq'n:

$$u_t + u_{xx} + u_{xxxx} + \frac{1}{2} u_x^2 = 0$$

(flame front instabilities)

And so many more!!

https://en.wikipedia.org/wiki/List_of_named_differential_equations

https://en.wikipedia.org/wiki/List_of_nonlinear_ordinary_differential_equations

Independent/Dependent Variable(s) and Explicit/Implicit Solutions

- When we have a function, the notion of dependent versus independent variables is generally straightforward: writing $\boxed{y(x) = x^2 + 3,}$ for example makes it clear that we intend for x to be independent and y to be dependent upon x because of the **explicit** dependence of the value of y upon the choice of x.

  x is independent , y dependent upon x!

- When we have a relation that cannot be reorganized to yield a function, the notion of which variable(s) are independent and which is dependent is muddier. Consider $\boxed{x^2 + y^2 = 4,}$ for example. We could consider this relation as giving a description of how x depends on y or vice-versa, so it is a matter of choice as to which variable we would consider independent and which dependent.

- Note that in the equation $\boxed{x^2 + y^2 = 4}$ the dependence of one variable's value upon the other is given **implicitly**, rather than explicitly, and in this case given a value of x we have more than one associated value of y that satisfies the relation.

Example:

Show that any function of the form $\boxed{y(x) = c_1 \cos(x) + c_2 \sin(x)}$ for any real numbers c_1, c_2 is an (explicit) solution of the second order linear ODE

$$\underline{\frac{d^2 y}{dx^2}} + y = 0$$

for all $x \in \mathbb{R}$.

↳ this describes a change in y with respect to a change in x

Note: The expression $\frac{dy}{dx}$ implies that we are considering y to be dependent upon x, with x playing the role of our independent variable.

if $y(x) = c_1 \cos(x) + c_2 \sin(x) \Rightarrow y'(x) = -c_1 \sin(x) + c_2 \cos(x)$

$\Rightarrow y''(x) = -c_1 \cos(x) - c_2 \sin(x)$

$+ \quad y(x) = c_1 \cos(x) + c_2 \sin(x)$

$y''(x) + y(x) = 0$ for $\underline{\text{all}}$ x and any $c_1, c_2 \in \mathbb{R}$.

More generally, we define a solution to a differential equation by

Definition

A continuous function $\phi(x)$ is an **explicit solution on an interval I** to the differential equation $F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0$ if all of the derivatives $\phi', \phi'', \ldots, \phi^{(n)}$ exist on I and it satisfies the differential equation upon substitution for all $x \in I$.

$\hookrightarrow$ $F\left(x, \phi(x), \frac{d\phi}{dx}, \ldots, \frac{d^n\phi}{dx^n}\right) = 0$ for all $x \in I$.

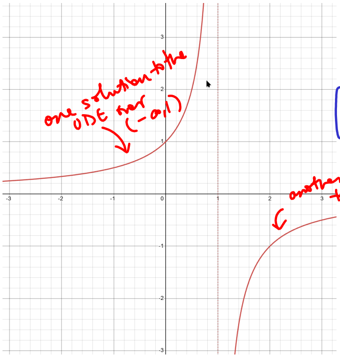
Example: For each real C, $\boxed{y(x) = \frac{1}{C-x}}$ defines two solutions of the differential equation

DNE if $x = C$!

$$y' = y^2 :$$

one on the interval (C, ∞) and one on the interval $(-\infty, C)$.

$$y' = \frac{1}{(C-x)^2} = y^2$$



let $C=1$

$$\boxed{y(x) = \frac{1}{1-x}}$$

Definition

A relation $G(x, y) = 0$ is an **implicit solution on an interval** I to the differential equation $F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0$ if there exists some explicit solution $y(x)$ of the differential equation that satisfies $G(x, y) = 0$ on some interval I.

Example: Consider the differential equation

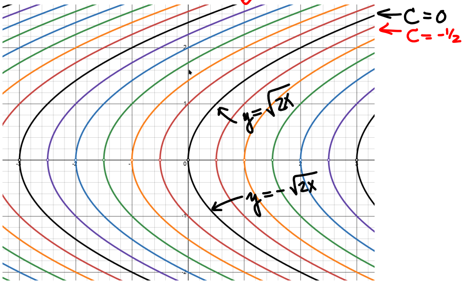
$$y\frac{dy}{dx} = 1.$$

$\boxed{\text{if}}$ x_0 is s.t. $y(x_0) = 0$ then $y(x_0)\cdot\frac{dy}{dx}(x_0) \neq 1$

It turns out that all solutions to this first order nonlinear equation are given **implicitly** by the family of relations

$$\boxed{y^2 = 2x + C.}$$

$y^2 = 2x + C$ for various values of C!



↳ do we have explicit solutions to $y\frac{dy}{dx} = 1$ that satisfies this?

try $y = \pm\sqrt{2x + C}$

if $C = 0$: $y = \pm\sqrt{2x}$

Check: $\boxed{y(x) = \sqrt{2x + C}}$ on $(-C/2, \infty)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{2x+C}} \Rightarrow y\frac{dy}{dx} = \frac{\sqrt{2x+C}}{\sqrt{2x+C}} = 1$

unless $\boxed{2x + C \leq 0}$

So $y(x) = \sqrt{2x + C}$ is a sol'n on the interval $I = (-\frac{C}{2}, \infty)$

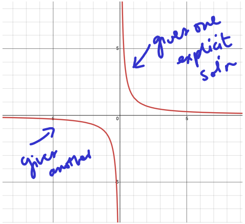
When Does a Relation Imply Existence of a Function?

A natural question with regards to solutions to relations $G(x, y) = 0$ is whether or not we can expect the variable y to be expressible as a function of x.

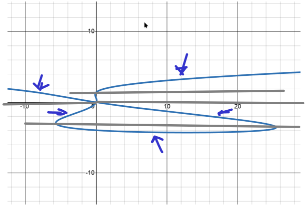
We know that sometimes $G(x, y) = 0$ can be easily rearranged to a form $y = f(x)$ regardless of the value of x or y, so in these cases the answer is obviously yes. But when this isn't apparently doable by hand, this can be a difficult question to answer.

Examples of Relations that Arise as Potential Implicit Solutions to Differential Equations:

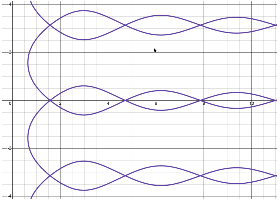
$x^3y^3 + xy = 4$



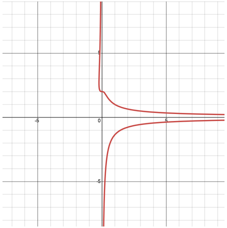
$y^5 + 2y^4 - 7y^3 + 3y^2 - 6xy - x^2 = 0$



$x\sin^2(y) = \cos^2(x)$



$x^3y^4 + y = 2$



Even if the relation isn't equivalent to a function globally, **locally** (on some interval containing each given point on the graph) we can certainly think of each of these relations as describing functions $y(x)$, except possibly on intervals containing self-intersection points and on intervals containing a vertical tangent line.

Note that at these potentially problematic points, the slope of the tangent to the graph of the solution set is not well-defined.

So, to get some clarity about on which intervals an implicit equation $G(x, y) = 0$ is guaranteed to define an explicit function $y(x)$, we need to get information about the structure of the multivariable function $G(x, y)$ – in particular, we want to understand the slopes of tangent lines to solutions to $G(x, y) = 0$. To do this, we need to know how to find partial derivatives.

Because G has two variables, x and y, it's value changes when we change either variable. As a result we can consider how G changes with respect to a change in x or with respect to a change in y, and we define the partial derivative of G with respect to x at a point (x_0, y_0) as

$$\frac{\partial G}{\partial x}\bigg|_{(x_0,y_0)} = \lim_{h\to 0} \frac{G(x_0 + h, y_0) - G(x_0, y_0)}{h}$$

pretend y is constant...

and the partial derivative of G with respect to y at a point (x_0, y_0) as

$$\frac{\partial G}{\partial y}\bigg|_{(x_0,y_0)} = \lim_{h\to 0} \frac{G(x_0, y_0 + h) - G(x_0, y_0)}{h}.$$

pretend x is constant

Find the partial derivatives of the following functions:

$G(x, y) = \sin(xy) + x^3y^5$

$$\frac{\partial G}{\partial x} = y\cos(xy) + 3x^2y^5$$

$$\frac{\partial G}{\partial y} = x\cos(xy) + 5x^3y^4$$

$G(x, y) = \frac{xy}{y^2+x^2}$

$$\frac{\partial G}{\partial x} = \frac{(x^2+y^2)(y) - (xy)(2x)}{(y^2+x^2)^2}$$

$$\frac{\partial G}{\partial y} = \frac{(x^2+y^2)(x) - (xy)(2y)}{(y^2+x^2)^2}$$

The Implicit Function Theorem

Given the relation $G(x, y) = 0$ we find the slope of the tangent line at a point (x_0, y_0) (if it exists) via

if $G(x, y(x)) = 0$

$$\frac{dG}{dx} = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial y}}$$

$$\left.\frac{dy}{dx}\right|_{(x_0, y_0)} = -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}$$

exists when $\frac{\partial G}{\partial y} \neq 0$

So assuming each of $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$ always individually exist over our interval of interest, we can simply rule out the points where $G_y(x_0, y_0) \neq 0$ in order to eliminate the places where the slope of the tangent does not exist.

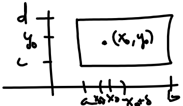
The following theorem gives conditions guaranteeing that we can locally express y as a function of x for all x in some interval I when (x, y) satisfies a given relation $G(x, y) = 0$ on I.

Theorem (The Implicit Function Theorem)

Let $G(x, y)$ have continuous first partial derivatives in the rectangle $R = \{(x, y) : a < x < b, c < y < d\}$ containing the point (x_0, y_0). If $G(x_0, y_0) = 0$ and the partial derivative $\boxed{G_y(x_0, y_0) \neq 0}$ then there exists a differentiable function $y = \phi(x)$ defined in some interval $\mathcal{I} = (x_0 - \delta, x_0 + \delta)$ that satisfies $G(x, \phi(x)) = 0$ for all $x \in \mathcal{I}$.

explicit sol'n

G has cts 1st partial derivs $\Rightarrow$ $\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$ exist and are continuous.



The relation $x\sin^2(y) = \cos^2(x)$, aka

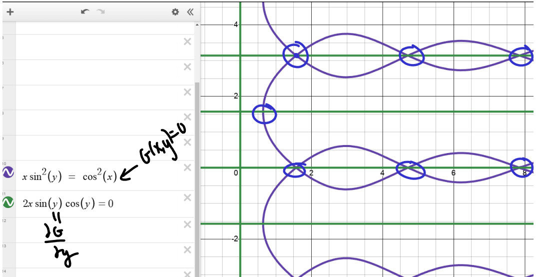
$$G(x,y) = \boxed{x\sin^2(y) - \cos^2(x) = 0,}$$

is continuous for all (x,y), and has partial derivatives

$$\frac{\partial G}{\partial x} = \sin^2(y) + 2\cos(x)\sin(x)$$

$$\boxed{\frac{\partial G}{\partial y} = 2x\sin(y)\cos(y) \neq 0}$$

that are also continuous for all (x,y). So the Implicit Function Theorem states that near every point $(x_0, y_0) \in \mathbb{R}^2$ we can express the solution set (x,y) via some differentiable function $y = \phi_{(x_0,y_0)}(x)$ except near those points satisfying $G(x,y) = 0$ for which



It turns out that the relation $x^3y^3 + xy = 4$ defines a function $y = y(x)$ on $x > 0$ and another on $x < 0$.

$$G(x, y) = x^3y^3 + xy - 4$$

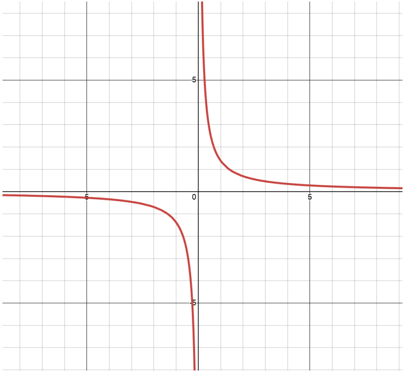
$$G_x(x, y) = 3x^2y^3 + y \leftarrow (\text{well-defined and cts on all of } \mathbb{R})$$

$$G_y(x, y) = 3x^3y^2 + x$$

$$3x^3y^2 + x = 0 \underset{\text{if } x \neq 0}{\Rightarrow} y^2 = -\frac{1}{3x^2} \quad (\text{impossible in } \mathbb{R})$$

$$\underline{\text{or}}\ x = 0 \Rightarrow G(0, y) = 0 \quad \text{for all } y!$$

So we will be guaranteed the relation has explicit solutions on every interval that does not contain zero.



But it is important to keep in mind that the implicit function theorem really can only give us local existence information!

For example, the relation $x^3y^4 + y = 2$ is such that

$G(x, y) = x^3y^4 + y - 2$

$G_x(x, y) = 3x^2y^4$

$G_y(x, y) = 4x^3y^3 + 1 \neq 0$

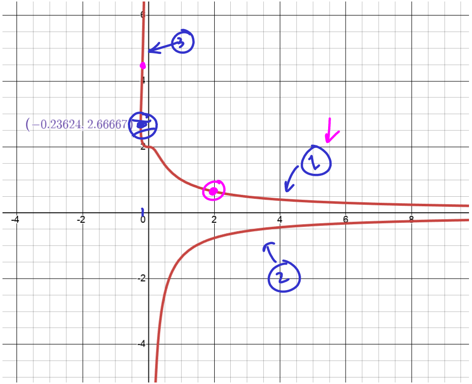
$x^3y^3 = -\frac{1}{4}$

$xy = -\frac{1}{\sqrt[3]{4}}$

$\Rightarrow \left(-\frac{1}{4}\right)y + y = 2$

$y = \frac{2}{3/4}$

$\Rightarrow \boxed{y = 8/3 \quad x = \frac{-3}{8\sqrt[3]{4}}}$



Notice that we can't actually express $y = y(x)$ for the entire curve, though! Near every point satisfying $x^3y^4 + y = 2$ except $P = (\frac{-3}{8\sqrt[3]{4}}, \frac{8}{3})$ the theorem guarantees we can locally express y as a function of x.

is implicit sol'n to

We can see that any function that satisfies $x^3y^4 + y = 2$ also satisfies the differential equation $\boxed{4x^3y^3\frac{dy}{dx} + 3x^2y^4 + 1 = 0.}$ Since we know that there exists at least one function $y(x)$ defined on some interval I such that $y(x)$ satisfies $x^3y^4 + y = 2$, we say that $x^3y^3 + y = 2$ is an implicit solution to the differential equation on I, so long as I does not contain $x = \frac{-3}{8\sqrt[3]{4}}$!

Initial Value Problems

Definition

An **initial value problem** is a system of equations formed by some ordinary differential equation

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0$$

defined for all $x \in \mathcal{I}$ for some interval $\mathcal{I}$ along with initial conditions

$$\begin{aligned} y(x_0) &= y_0 \\ \frac{dy}{dx}(x_0) &= y_1 \\ &\vdots \\ \frac{d^{n-1}y}{dx^{n-1}}(x_0) &= y_{n-1} \end{aligned}$$

where $x_0 \in \mathcal{I}$ and $y_0, y_1, \dots, y_{n-1}$ are given constants.

Given that the solutions to $\frac{d^2y}{dx^2} + y = 0$ all have the form $y(x) = c_1 \cos(x) + c_2 \sin(x)$ for real numbers c_1, c_2 and all $x \in \mathbb{R}$, find the solution to the IVP where $y(0) = -1$ and $y'(0) = 1$.

Question: Does every differential equation have at least one solution?

Question: Given an IVP if we know the differential equation is solvable, must there be a solution to the IVP for any choice of initial conditions?

Question: If an IVP has at least one solution, must that solution be unique?

When can we be certain unique solutions to IVPs exist?

- This is a tough question in general!! In fact there is a million dollar prize available to anyone that answers it for an equation we saw earlier called the Navier-Stokes equation.

  https://www.claymath.org/millennium/navier-stokes-equation/

- We'll focus on addressing this question for first order equations for now. Suppose we can put our first order differential equation in the form
  $$\frac{dy}{dx} = f(x, y)$$
  (in some texts referred to as the "normal form" of the differential equation) and we have initial condition $y(x_0) = y_0$.

- It turns out that we have the following theorem:

Theorem (Existence & Uniqueness Theorem)

Consider the IVP
$$\frac{dy}{dx} = f(x, y) , \qquad y(x_0) = y_0 .$$
If f and $\frac{\partial f}{\partial y}$ are continuous functions in some rectangle
$$R = \{(x, y) | x \in [a, b], y \in [c, d]\}$$
that contains (x_0, y_0), then the IVP has a unique solution $y^*(x)$ in some interval $\mathcal{I} = [x_0 - h, x_0 + h]$ for some $h > 0$.

But why on earth is this theorem true? To begin our foray into making sense of the foundations of this theorem, we recast the IVP

$$\frac{dy}{dx} = f(x, y) \qquad \text{for } y(x_0) = y_0$$

into an integral equation:

Example: What integral equation must a function $y(x)$ defined on some interval containing $x_0 = 0$ satisfy if we know that $\frac{dy}{dx} = 2y$ and $y(0) = 1$?

- Notice this doesn't really get us any closer to a solution, but it does change the problem into a form where we can apply different tools to it.

- In particular, we can think of the right hand side as the end result of applying an operator T to the function y:

$$T[y](x) := y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

Example: For the integral operator $T[y]$ associated to the initial value problem $\frac{dy}{dx} = 2y$ where $y(0) = 1$, find $T[y]$ for various functions y.

- What general conditions on f will ensure the operator T is well-defined for all relevant x and y and equivalent to the original differential equation?

So we'll assume f is continuous on some $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$ and that y is continuous on some interval $\mathcal{I}$ containing x_0 and maps into $[c, d]$ so that the function g that results from

$$g(x) = T[y](x) := y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

is guaranteed to be well-defined (and differentiable) on $\mathcal{I}$ and $y = T[y]$ is equivalent to our original initial value problem.

- Solving the differential equation now means finding a function y so that $y = T[y]$, i.e. finding a fixed "point" of the operator T.

- Since we're looking for fixed points of the operator T, a common technique that *sometimes* works is iterative successive approximation: so let's let

$$y_{i+1}(x) := T[y_i](x) = y_0 + \int_{x_0}^{x} f(t, y_i(t))\, dt,$$

and start with the initial guess

$$y_0(x) \equiv y_0.$$

So we end up with

$$y_n(x) = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots + \frac{(2x)^n}{n!}.$$

Does this look familiar?

So our iterated approximation functions $y_n(x) \to e^{2x}$ which is in fact the unique solution to the given IVP!!!

Does this always work? Meaning, if we carry out this iteration process starting with some initial guess $y_0(x)$, will the sequence of functions we obtain converge to a solution of the IVP?

No... it does not.

But we can actually guarantee that it will converge to a solution to the IVP, at least for $x \in \mathcal{I}$ for some interval $\mathcal{I}$ containing x_0, under relatively simple assumptions on f!

Let's assume that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous on some rectangle $R = [a, b] \times [c, d]$ where $(x_0, y_0) \in R$.

- In order for successive approximations $y_{i+1}(x) = T[y_i](x) := y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$ to make sense, we need the integration to be well defined, and to ensure we don't end up with values of $y_n(x)$ running off to ∞ as $n \to \infty$, assuming that $f(x, y)$ is continuous helps with both!

- Note that we'll automatically have $y_i(x_0) = y_0$ for all i, so our initial condition is satisfied for every approximation in the iteration.

- We can also note that we'd like to require that the y_i to be continuous for all i and to be such that $y_i : \mathcal{I} \subset [a, b] \to [c, d]$ with $x_0 \in \mathcal{I}$ so that the composition $f(t, y_i(t))$ is guaranteed to be continuous over the region of integration. The question is whether or not we can be sure that such an interval $\mathcal{I}$ exists...

$$|y_i(x) - y_0| =$$

- We want the sequence of functions y_n created in this way to converge to some function y^*: this is where requiring $\frac{\partial f}{\partial y}$ to be continuous on R is helpful

$$|y_{i+1}(x) - y_i(x)| =$$

- Since $y_n = y_0 + (y_1 - y_0) + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) = y_0 + \sum_{i=0}^{n-1}(y_{i+1} - y_i)$, we can show that $y_n(x)$ converges for each $x \in \mathcal{I}$:

- For y^* defined as $y_n \to y^*$, we can see quickly now that y^* must be the unique fixed point of T overall all continuous function on $\mathcal{I}$ that map into $[c, d]$:

So we just gave a quick overview for why we can use a successive approximation approach to prove that

Theorem (Existence & Uniqueness Theorem)

Consider the IVP

$$\frac{dy}{dx} = f(x,y) \ , \qquad y(x_0) = y_0.$$

If f *and* $\frac{\partial f}{\partial y}$ *are continuous functions in some rectangle*

$$R = \{(x,y) | x \in [a,b], y \in [c,d]\}$$

that contains (x_0, y_0), *then the IVP has a unique solution* $y^*(x)$ *in some interval* $\mathcal{I} = [x_0 - h, x_0 + h]$ *for some* $h > 0$.

That's it for today!

Any questions?

Homework Tip