Differential Equations Week 2

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January 27, 2025

Problem 1 (Problem 1a)

Consider the differential equation $y' = 3x^2y^2$.

I. Prove that there is a unique solution on some interval around all initial values.

II. Draw at least 5 level curves of $y'(x, y)$.

III. Solve the differential equation.

IV. Draw a direction field for the differential equation.

V. For the inital value problem $y(0) = 1$:

1. Draw the solutions of the differential equation with this intial value onto the direction field.
2. Approximate the solutions of the differential equation with Euler's Method, with a delta-value of Δ.
3. Graph the solutions in 1aV2 for
   (a) $\Delta = 0.4$
   (b) $\Delta = 0.2$
   (c) $\Delta = 0.01$.

VI. For the inital value problem $y(0) = 0$:

1. Draw the solutions of the differential equation with this intial value onto the direction field.
2. Approximate the solutions of the differential equation with Euler's Method, with a delta-value of Δ.
3. Graph the solutions in 1aVI2 for
   (a) $\Delta = 0.4$
   (b) $\Delta = 0.2$
   (c) $\Delta = 0.01$.

Problem 2 (Problem 1b)

Consider the differential equation $y' = \sqrt[3]{y}$.

I. Prove that there is a unique solution on some interval around all initial values if $y_0 \neq 0$.

II. Draw at least 5 level curves of $y'(x, y)$.

III. Solve the differential equation.

IV. Draw a direction field for the differential equation.

V. For the inital value problem $y(0) = 1$:

1. Draw the solutions of the differential equation with this intial value onto the direction field.
2. Approximate the solutions of the differential equation with Euler's Method, with a delta-value of Δ.
3. Graph the solutions in 1bV2 for
   (a) $\Delta = 0.4$
   (b) $\Delta = 0.2$
   (c) $\Delta = 0.01$.

VI. For the inital value problem $y(0) = 0$:

1. Draw the solutions of the differential equation with this intial value onto the direction field.
2. Approximate the solutions of the differential equation with Euler's Method, with a delta-value of Δ.
3. Graph the solutions in 1bVI2 for
   (a) $\Delta = 0.4$
   (b) $\Delta = 0.2$
   (c) $\Delta = 0.01$.

Solution 2.1 (Problem 1aI) — Notice the following:

- $f(x, y)$ is continuous everywhere.
- $\dfrac{\partial f}{\partial y} = 6x^2 y$ is also continous everywhere.
- By existence and uniqueness don.

Solution 2.2 (Problem 1aII) — BLAHS

Solution 2.3 (Problem 1bI) — Notice the following:

- $f(x, y)$ is continuous everywhere.

- $\frac{\partial f}{\partial y} = \frac{1}{3y^{2/3}}$ is continous everywhere except $y = 0$.
- By existence and uniqueness done.

Problem 3 (Problem 3)

Solution —

Problem 4 (Problem 5 (Equivilant to 2.3 39))
Consider the following initial value problem:

$$T(0) = 0; \tag{4.1}$$

$$\frac{dT}{dt} = -T + \begin{cases} 0, \lfloor t \rfloor \text{ is odd} \\ 1, \lfloor t \rfloor \text{ is even.} \end{cases} \tag{4.2}$$

(a) Find $100T(3)$ rounded to the nearest whole number.

(b) Find $100T(8)$ rounded to the nearest whole number.

Lemma 4.1
If t is an odd whole number:

$$T(t) = \sum_{i=0}^{t} \frac{(-1)^i}{e^i}.$$

First, we solve the relavent differential equationL

$$\begin{aligned} \frac{dT}{dt} &= 1 - T \\ \frac{T'}{1-T} &= 1 \\ \int \frac{1}{1-T} dT &= t \\ -\log|1-T| &= t + C \\ 1 - T &= ce^{-t} \\ T &= 1 - ce^{-t}. \end{aligned}$$

Then, proceed by induction.

Inductive Hypothesis. $t-1$ case of (Lemma 5.2)/$t-2$ case of this lemma (equivilant as noted in the lemma, ** of 5.2 provides us our base case)

Inductive Step. Note that the recurrence is $T(t) = 1 + \frac{1}{e}(T(t-1) - 1)$.

Working out the algebra, we get

$$\begin{aligned} &1-\frac{1}{e}+\sum_{i=2}^{t}\frac{(-1)^i}{e^i} \\ &=\sum_{i=0}^{t}\frac{(-1)^i}{e^i}, \end{aligned}$$

and we are done with this lemma.

*note the case where $T(t) = 1$: however this gets factored back in at the end as c can be any value □

Lemma 4.2

If t is an even whole number:

$$T(t) = \sum_{i=1}^{t}\frac{(-1)^{i+1}}{e^i}.$$

When $\lfloor t \rfloor$ is ODD, notice that:

$$\begin{aligned} \frac{dT}{dt} &= -T \\ \frac{T'}{T}* &= -1 \\ \int \frac{1}{T}dT &= -t \\ -\log \mid T \mid &= -t + C \\ T &= ce^{-t}*, \end{aligned}$$

for $t-1 \leq T \leq t$. This implies $T(t) = \dfrac{T(t-1)}{e}$ which** is the above formula given that (Lemma 5.1) holds for $t-1$.
*note the case where $T(t) = 0$: however this gets factored back in at the end as c can be any value **unless $t = 0$, when this statement trivally holds as empty sums are zero □