Differential Equations Week **12**

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May 16, 2025

Problem 1

Solve the following differential equation:

$$\begin{cases} P' & = P(c - r(N)) \\ Q' & = Pr(N) \\ N & = P + Q \end{cases} \tag{1.1}$$

for

(a) $r(N) = b(1 + \log N)$, $P(0) = 1$, $Q(0) = 0$, $N(0) = 1$

(b) $r(N) = (2N - 1)s$.

Solution 1.1 — Note that $Q(t)$ is very annoying, let's turn this into an P, N differential equation system.

$$\begin{cases} P' & = P(c - r(N)) \\ N' & = cP \end{cases}. \tag{1.2}$$

Now we solve the phase plane differential equation,

$$\frac{dP}{dN} = \frac{c - r(N)}{c} \tag{1.3}$$

$$P = N - \frac{1}{c} \int_{dontcare}^{N} r(x)\, dx + \alpha. \tag{1.4}$$

Now, plug back in:

$$N' = c\left(N - \frac{1}{c} \int_{dontcare}^{N} r(x)\, dx + \alpha\right) \tag{1.5}$$

$$t = \int \frac{1}{cN - \int_{dontcare}^{N} r(x)\, dx + c\alpha}\, dN \tag{1.6}$$

Solution 1.2 — Plug in (set dontcare to 1):

$$t = \int \frac{1}{cN - bN \log N} \, dN \tag{1.7}$$

$$= \int \frac{1}{N} \cdot \frac{1}{c - b \log N} \, dN \tag{1.8}$$

$$= \int \frac{1}{c - bu} \, du \tag{1.9}$$

$$= -\frac{1}{b} \log |c - b \log |N|| + C \tag{1.10}$$

$$c - ae^{-bt} = b \log |N| \tag{1.11}$$

$$N = e^{\frac{1}{b}(c - ae^{-bt})} \tag{1.12}$$

$$P = e^{\frac{1}{b}(c - ae^{-bt})} - \frac{1}{c} N \log N \tag{1.13}$$

$$= e^{\frac{1}{b}(c - ae^{-bt})} \left(1 - \frac{1}{bc}(c - ae^{-bt})\right) \tag{1.14}$$