Differential Equations Week **7**

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Problem 1 (Problem 1)

Find the general solution to the differential equation

$$y'' - 6y' + 13y = xe^3x\sin(2x). \tag{1.1}$$

Solution 1.0 (Framework for solving this problem.) —

1. Find the solution to the corresponding homogenous equation.
2. Obtain a particular solution.
3. Combine to get our final result.

Solution 1.1 (Solving corresponding homogenous equation.) — First, homogenize (1.0):

$$y'' - 6y + 13y = 0. \tag{1.2}$$

Now, what's the characteristic polynomial? It's $r^2 - 6r + 13$, which produces solutions:

$$f(x) = c_1e^{3x}\cos(2x) + c_2e^{3x}\sin(2x). \tag{1.3}$$

Solution 1.2 (Obtaining particular solution.) — Let's take derivatives of $cx^ke^{3x}\sin(2x)$ and plug in! We get $ck(k-1)x^{k-2}e^{3x}\sin(2x)$ as if f is annihalted by the nth order constant coefficent differential operator I, $I[fg] = fg^{(n)}$ (we can prove this by decomposing into a bunch of linear expressions and then annihaling each part, but I won't do that here... probably has some special name I don't know)

So $k = 3$, and we have $6cxe^{3x}\sin(2x) = xe^{3x}\sin(2x)$ and:

$$f_p(x) = \frac{1}{6}xe^{3x}\sin(2x). \tag{1.4}$$

Solution 1.3 (Finish.) — Add the two things to get:

$$\boxed{f(x) = c_1e^{3x}\cos(2x) + c_2e^{3x}\sin(2x) + \frac{1}{6}xe^{3x}\sin(2x).} \tag{1.5}$$

Remark. The solutions for the corresponding homogenous equation are well-known to be linearly independent.
Please tell me what the name of that theorem in comments!

Problem 2 (Problem 2)

Solve the differential equation

$$y'' + 4y' = \sin^4(x). \tag{2.1}$$

Solution 2.1 — Substitute double angle identites:

$$\begin{aligned}
&\sin^4(x) && (2.2)\\
&= \left(\frac{1-\cos(2x)}{2}\right)^2 && (2.3)\\
&= 1/4 \cdot \big(1 - 2\cos(2x) + \cos^2(2x)\big) && (2.4)\\
&= \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{\cos(4x)-1}{2} && (2.5)\\
&= -\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{2}\cos(4x) && (2.6)\\
& && (2.7)
\end{aligned}$$

and using method of undetermined coefficents (the $-1/4$ immediately becomes $-1/16x$):

$$\begin{aligned}
\cos(kx) &= -ak^2\sin(kx) - bk^2\cos(kx) && (2.8)\\
&\quad + ak\cos(kx) - bk\sin(kx) && (2.9)\\
&\begin{cases} -ak^2 - bk &= 0\\ -bk^2 + ak &= 1. \end{cases} && (2.10)\\
b &= -ak && (2.11)\\
a(k^3+k) &= 1 && (2.12)\\
a &= \frac{1}{k^3+k} && (2.13)\\
b &= -\frac{1}{k^2+1} && (2.14)\\
f_p(x) &= -\frac{1}{16}x + \frac{1}{10}\cos(2x) - \frac{1}{20}\cos(2x) && (2.15)\\
&\quad - \frac{1}{17}\cos(4x) + \frac{1}{34}\cos(4x) && (2.16)\\
f(x) &= c_1 + c_2e^{4x} - \frac{1}{16}x + \frac{1}{10}\cos(2x) - \frac{1}{20}\cos(2x) && (2.17)\\
&\quad - \frac{1}{17}\cos(4x) + \frac{1}{34}\cos(4x) && (2.18)
\end{aligned}$$

Problem 3 (Problem 3)

Find the general solution to

$$y'' + 4y' + 4y = e^{-2t} \log t. \tag{3.1}$$

Solution 3.1 — Note that $v(t)e^{-2t}$ in our differential operator is $v''(t)e^{-2t}$. So plugging in our particular solution the answer is:

$$f(t) = t \log te^{-2t} + ae^{-2t} + bxe^{-2t} \tag{3.2}$$

Problem 4 (Problem 4)

Solve the initial value problem

$$y'' - y = \frac{1}{x}, y(1) = 0, y'(1) = 2 \tag{4.1}$$

Solution 4.1 — Let's use variation of parameters. The solutions to the corresponding homogenous equation are:

$$e^x, e^{-x} \tag{4.2}$$

so:

$$\begin{align} v_1' e^x + v_2' e^{-x} &= 0 \tag{4.3}\\ v_1' e^x - v_2' e^{-x} &= \frac{1}{x} \tag{4.4}\\ v_1' &= \frac{e^{-x}}{2x} \tag{4.5}\\ v_2' &= -\frac{e^x}{2x} \tag{4.6}\\ f(x) &= c_1 e^x + c_2 e^{-x} + e^x \int_1^x \frac{e^{-t}}{2t} dt + e^{-x} \int_1^x \frac{e^t}{2t} dt \tag{4.7} \end{align}$$

Plugging in at 1, we get:

$$\begin{align} ec_1 + e^{-1} c_2 &= 0 \tag{4.8}\\ ec_1 - e^{-1} c_2 &= 1 \tag{4.9}\\ c_1 &= \frac{1}{2e} \tag{4.10}\\ c_2 &= \frac{e}{2}. \tag{4.11} \end{align}$$

Thus the answer is

$$\boxed{f(x) = \frac{1}{2}\left(e^{x-1} + c_2 e^{-(x-1)}\right) + e^x \int_1^x \frac{e^{-t}}{2t} dt + e^{-x} \int_1^x \frac{e^t}{2t} dt.} \tag{4.12}$$

Problem 5 (Problem 5)

Solve the following differential equation:

$$t^2 y'' - 4ty' + 6y = t^3 + 1. \tag{5.1}$$

Solution 5.1 — Note that t^2, t^3 are solutions. Ignore the $+1$, just add $1/6$ to our final solution in the end. Applying variation of parameters,

$$v_1' t^2 + v_2' t^3 = 0 \tag{5.2}$$

$$2v_1' t + 3v_2' t^2 = t^2 \tag{5.3}$$

$$v_2' = 1, v_2(t) = t \tag{5.4}$$

$$v_1' = -t \tag{5.5}$$

$$v_1(t) = \frac{-t^2}{2} \tag{5.6}$$

$$f(t) = \frac{t^4}{2} + c_1 t^3 + c_2 t^2. \tag{5.7}$$