Differential Equations Week 14

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Problem 1

Show that:

$$\Gamma(x + 1) = x\Gamma(x), \tag{1.1}$$

for all $x \in \mathbb{R}^+$.

Solution 1.1 — Note that:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-x} dx \tag{1.2}$$

$$= \mathscr{L}[t^{x-1}](1). \tag{1.3}$$

As:

$$\Gamma(x + 1) = \mathscr{L}[xt^{x-1}] \tag{1.4}$$

$$= \mathscr{L}[(t^x)'] \tag{1.5}$$

$$= s \cdot \mathscr{L}[xt^{x-1}] + (t^x)'(0) \tag{1.6}$$

$$= s \cdot \mathscr{L}[xt^{x-1}] + 0 \tag{1.7}$$

$$= x\Gamma(x). \tag{1.8}$$

Problem 2

Compute the following Laplace transforms:

(a) $\mathscr{L}[t \cos bt]$

(b) $\mathscr{L}[t^2 \cos bt]$

Solution 2.1 — Note that

$$\mathscr{L}[t^n f(t)] = (-D)^n F(s) \tag{2.1}$$

$$\mathscr{L}[\cos bt] = \frac{s}{b^2 + s^2} \tag{2.2}$$

$$\mathscr{L}[t \cos bt] = -D\left(\frac{s}{b^2 + s^2}\right) \tag{2.3}$$

$$= -\frac{b^2 + s^2 - s(2s)}{(b^2 + s^2)^2} \tag{2.4}$$

$$= \frac{s^2 - b^2}{(b^2 + s^2)^2}. \tag{2.5}$$

Solution 2.2 — Note that

$$\mathscr{L}[t^n f(t)] = (-D)^n F(s) \tag{2.6}$$

$$\mathscr{L}[t \cos bt] = \frac{s^2 - b^2}{(b^2 + s^2)^2} \tag{2.7}$$

$$\mathscr{L}[t^2 \cos bt] = -D\left(\frac{s^2 - b^2}{(b^2 + s^2)^2}\right) \tag{2.8}$$

$$= -\frac{2s(b^2 + s^2)^2 - 4s(s^2 - b^2)(s^2 + b^2)}{(b^2 + s^2)^4} \tag{2.9}$$

$$= -\frac{2sb^4 + 4s^3b^2 + 2s^5 - 4s^5 + 4sb^4}{(b^2 + s^2)^4} \tag{2.10}$$

$$= -\frac{-2s^5 + 4s^3b^2 + 6sb^4}{(b^2 + s^2)^4} \tag{2.11}$$

$$= \frac{2s(s^4 - 2s^2b^2 - 3b^4)}{(b^2 + s^2)^4} \tag{2.12}$$

$$= \frac{2s(s^2 + b^2)(s^2 - 3b^2)}{(b^2 + s^2)^4} \tag{2.13}$$

$$= \frac{2s(s^2 - 3b^2)}{(b^2 + s^2)^3}. \tag{2.14}$$

Problem 3

Show that the portion of the partial fraction expansion of
$$\frac{P(s)}{(s - r_i)(s - r_1)(s - r_2)\cdots} = \frac{P(x)}{Q(x)} \text{ is:}$$

$$\frac{P(r)}{Q'(r)}. \tag{3.1}$$

and that:

$$\mathscr{L}^{-1}\left[\frac{P}{Q}\right] = \sum_{i=1}^{\deg Q} \frac{P(r_i)}{Q'(r_i)} e^{r_i t} \tag{3.2}$$

Solution 3.1 — Note that evaluating at $s = r_i + \varepsilon$, we have:

$$\frac{P(s)}{(s - r_i)(s - r_1)(s - r_2)\cdots} = \sum_{n=1}^{\deg Q} \frac{A_n}{s - r_n} \tag{3.3}$$

$$\frac{P(s)}{\varepsilon \cdot (s - r_1)(s - r_2)\cdots} = \frac{A_i}{\varepsilon} + \sum_{n=1, n\neq i}^{\deg Q} \frac{A_n}{r_i + \varepsilon - r_n} \tag{3.4}$$

$$\frac{P(s)}{Q(s)/(s - r_i)} = A_i + \varepsilon \cdot \sum_{n=1, n\neq i}^{\deg Q} \frac{A_n}{s - r_n}. \tag{3.5}$$

Now take $\lim_{\varepsilon \to 0}$ to get:

$$\lim_{\varepsilon \to 0} \frac{P(s)}{Q(s)/(s - r_i)} = \lim_{\varepsilon \to 0} A_i + \varepsilon \cdot \sum_{n=1, n\neq i}^{\deg Q} \frac{A_n}{s - r_n} \tag{3.6}$$

$$\lim_{\varepsilon \to 0} \frac{P(s)}{Q(s)/(s - r_i)} = A_i \tag{3.7}$$

$$r = r_i \tag{3.8}$$

$$\lim_{s \to r} \frac{P(s)(s - r)}{Q(s)} = A_i, \tag{3.9}$$

and this is 0/0 L'Hopital. Taking derivitatives, we get:

$$R(\alpha) = P(r + \alpha) \tag{3.10}$$

$$\lim_{s \to r} \frac{P(s)(s - r)}{Q(s)} = \lim_{\varepsilon \to 0} \frac{\varepsilon R(\varepsilon)}{Q(r + \varepsilon)} \tag{3.11}$$

$$= \lim_{\varepsilon \to 0} \frac{R(\varepsilon) + \varepsilon R'(\varepsilon)}{Q'(r + \varepsilon)} \tag{3.12}$$

$$= \lim_{\varepsilon \to 0} \frac{R(\varepsilon)}{Q'(r + \varepsilon)} + \lim_{\varepsilon \to 0} \frac{\varepsilon R'(\varepsilon)}{Q'(s)} \tag{3.13}$$

$$|Q'(r_i)| > 0 \text{ (if it was 0, } r_i \text{ would be a double root)} \tag{3.14}$$

$$= \frac{P(r_i)}{Q'(r_i)}. \tag{3.15}$$

Now, noting that:

$$\mathscr{L}^{-1}\left[\frac{1}{s - r_i}\right] = e^{r_i t}, \tag{3.16}$$

using the previous finding, and summing over all r_i, and using linearity of $\mathscr{L}^{-1}$, we get:

$$\boxed{\frac{P}{Q} = \sum_{i=1}^{\deg Q} \frac{P(r_i)}{Q'(r_i)(s - r_i)}} \tag{3.17}$$

$$\boxed{\mathscr{L}^{-1}\left[\frac{P}{Q}\right] = \sum_{i=1}^{\deg Q} \frac{P(r_i)}{Q'(r_i)} e^{r_i t}} \tag{3.18}$$

Problem 4

Find the partial fraction decomposition of:

$$\frac{2s+1}{s(s-1)(s+2)} \tag{4.1}$$

Solution 4.1 — Use (3.17).
There are two things to start with:

- Find Q'.

- Find roots of Q.

Note that roots of Q are $-2, 0, 1$. Now also note that $Q' = (s^3+s^2-2s)' = 3s^2 + 2s - 2$ and plugging in:

$$\frac{2s+1}{s(s-1)(s+2)} = \frac{P(-2)}{Q'(-2)} \cdot \frac{1}{s+2} + \frac{P(0)}{Q'(0)} \cdot \frac{1}{s} + \frac{P(1)}{Q'(1)} \cdot \frac{1}{s-1} \tag{4.2}$$

$$= -\frac{1}{2} \cdot \frac{1}{s+2} + \frac{1}{3} \cdot \frac{1}{s} + \frac{1}{s-1}. \tag{4.3}$$

Problem 5

Find the inverse Laplace transform of:

$$\frac{3s^2 - 16s + 5}{(s+1)(s-2)(s-3)} \tag{5.1}$$

Solution 5.1 — Use (3.18).
There are two things to start with:

- Find Q'.

- Find roots of Q.

Note that roots of Q are $-2, 0, 1$. Now also note that $Q' = (s^3 - 4s^2 + s - 6)' = 3s^2 - 8s + 1$ and plugging in:

$$\mathscr{L}^{-1}\left[\frac{3s^2 - 16s + 5}{(s+1)(s-2)(s-3)}\right] = \frac{P(-1)}{Q'(-1)} \cdot e^{-t} + \frac{P(2)}{Q'(2)} \cdot e^{2t} + \frac{P(3)}{Q'(3)} \cdot e^{3t} \tag{5.2}$$

$$= 2e^{-t} - 3e^{2t} - 13 \cdot e^{3t} \tag{5.3}$$

$$\tag{5.4}$$

Problem 6

Solve the intial value problem:

$$y''(t) + 4y = \begin{cases} 3\sin t, & 0 \leq t \leq 2\pi \\ 0, & 2\pi < t \end{cases} \tag{6.1}$$

$$\begin{cases} y(0) = 1 \\ y'(0) = 3 \end{cases} \tag{6.2}$$

Solution 6.1 — **Step 1.** Express the piecewise continuous forcing function in terms of heaviside functions.

Note that for any piecewise continuous function that is equal to f_k on intereval $I_k = (a_k, b_k)$, that function is equal to:

$$\sum_{I_k} f_k(x)u(x - a_k) - f_k(x)u(x - b_k), \tag{6.3}$$

and note that for our forcing function, that is:

$$\boxed{3\sin t - 3\sin t \cdot u(x - 2\pi).} \tag{6.4}$$

Solution 6.2 — **Step 2.** Take the Laplace Transform and solve for $\mathscr{L}[y]$.

Taking the Laplace transform, we get:

$$\mathscr{L}[y''(t) + 4y] = \mathscr{L}[3\sin t - 3\sin t \cdot u(x - 2\pi)] \tag{6.5}$$

$$(s^2 + 4)\mathscr{L}[y] - sy(0) - y'(0) = \frac{3(1 - e^{2\pi s})}{1 + s^2} \tag{6.6}$$

$$\boxed{\mathscr{L}[y] = \frac{3}{(1 + s^2)(4 + s^2)} - \frac{3e^{2\pi s}}{(1 + s^2)(4 + s^2)} + \frac{s - 3}{s^2 + 4}} \tag{6.7}$$

Solution 6.3 — **Step 3.** Take the Inverse Laplace Transform and solve for u.

Taking the inverse Laplace transform, we get:

$$\mathcal{L}[y] = \frac{3}{(1+s^2)(4+s^2)} - \frac{3e^{2\pi s}}{(1+s^2)(4+s^2)} + \frac{s-3}{s^2+4} \tag{6.8}$$

$$y = \mathcal{L}^{-1}\left[\frac{3}{(1+s^2)(4+s^2)} - \frac{3e^{2\pi s}}{(1+s^2)(4+s^2)} + \frac{s-3}{s^2+4}\right] \tag{6.9}$$

$$= \cos(2t) - \frac{3}{2}\sin(2t) + 3(1 - u(t - 2\pi))\mathcal{L}^{-1}\left[\frac{3}{(1+s^2)(4+s^2)}\right] \tag{6.10}$$

$$= \cos(2t) - \frac{3}{2}\sin(2t) \tag{6.11}$$

$$+ 3(1 - u(t - 2\pi))\mathcal{L}^{-1}\left[\frac{1}{(1+s^2)} - \frac{1}{(4+s^2)}\right] \tag{6.12}$$

$$= \boxed{3(1 - u(t - 2\pi))\left(\sin(t) - \frac{1}{2}\sin(2t)\right) + \cos(2t) - \frac{3}{2}\sin(2t).} \tag{6.13}$$