Differential Equations Week 3

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March 3, 2025

Problem 1 (Problem 1)

Solve the following differential equation:

$$\left(ye^{xy} - \frac{1}{y}\right) dx + \left(xe^{xy} + \frac{x}{y^2}\right).$$

Solution — First, we should check if the equation is exact. Using the test $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$,

$$\begin{aligned}\frac{\partial M}{\partial y} &= xye^{xy} + \frac{1}{y^2} \\ &= xye^{xy} + \frac{1}{y^2} = \frac{\partial N}{\partial x}\end{aligned}$$

Then, note that:

$$\begin{gathered}F(x,y) = \int ye^{xy} - \frac{1}{y}\,dx = \int xe^{xy} + \frac{x}{y^2}\,dy \\ e^{xy} - \frac{x}{y} + g(y) = e^{xy} - \frac{x}{y} + h(x),\end{gathered}$$

and this forces $g(y) = h(x)$. As y is not neceesarily equal to x, $g(y) = h(x) = C$, and so the solutions are of the form:

$$\begin{aligned}e^{xy} - \frac{x}{y} &= C \\ e^{xy} &= C + \frac{x}{y}.\end{aligned}$$

Problem 2 (Problem 2)
Solve the differential equation:

$$(x^4 - x + y)dx - xdy = 0.$$

Solution — First, note that there is clearly no hope of forcing linearity/seprability.

Lemma 2.1
The given differential equation is NOT exact.

Proof. Use the test for exactness:

$$\frac{\partial}{\partial y}(x^4 - x + y) = 1$$
$$\neq -1 \qquad = \frac{\partial}{\partial x}(-x),$$

□

So, given our previous observation and claim, we must add an integrating factor to make our differential equation exact.
Now, work out our needed integrating factor μ:

$$\begin{aligned}
\frac{\partial}{\partial y}((x^4 - x + y) \cdot \mu(x)) &= \frac{\partial}{\partial x}((-x) \cdot \mu(x)) \\
\mu(x) &= -\mu(x) - x\mu'(x) \\
\frac{\mu'(x)}{\mu(x)} &= -\frac{2}{x} \\
\int \frac{1}{\mu}\, d\mu &= -2 \cdot \int \frac{1}{x}\, dx \\
\log|\mu(x)| &= -2\log|x| + C \\
\mu(x) &= \frac{c}{x^2}.*
\end{aligned}$$

So now we need (we assume $c = 1$ as we can solve the differential equation with any $c \neq 0$):

$$\begin{aligned}
F(x, y) = \int \frac{x^4 - x + y}{x^2}\, dx &= \int -\frac{1}{x}\, dy \\
\frac{x^3}{3} - \log|x| - \frac{y}{x} + g(y) &= -\frac{y}{x} + h(x),
\end{aligned}$$

and noting that $g(y) = h(x) - \dfrac{x^3}{3} + \log|x|$, we have $h(x) = \dfrac{x^3}{3} - \log|x|$

and $G(y) = 0$. Thus,

$$\frac{x^3}{3} - \log|x| - \frac{y}{x} = C$$
$$\frac{x^3}{3} - \log|x| - C = \frac{y}{x}$$
$$y = \frac{x^4}{3} - x\log|x| - Cx$$

is the answer.

*Trivial solution ignored, recombined in at the end.

Problem 3 (Problem 3, version b, parts removed similar to AoPS textbook solutions)

Solve the following differential equation:

$$(5x^2y + 6x^3y + 4xy^2)\,dx + (2x^3 + 3x^4y + 3x^2y)\,dy = 0.$$

Solution — Once again we have no chance of forcing linearity or seprability, thus concentrate on exactness. Notice that:

Lemma 3.1

The given differential equation is not exact.

Proof. Use the test for exactness:

$$\begin{aligned}&\frac{\partial}{\partial y}(5x^2y + 6x^3y^2 + 4xy^2)\\ &= 5x^2 + 12x^3y + 8xy\\ &\neq 6x^2 + 12x^3y + 6xy\\ &= \frac{\partial}{\partial x}(2x^3 + 3x^4y + 3x^2y),\end{aligned}$$

□

Notice that we need for an integrating factor:

$$\begin{aligned}&(5x^2 + 12x^3y + 8xy)\mu(x,y) + (5x^2y + 6x^3y^2 + 4xy^2)\mu_y(x,y)\\ &\quad = (6x^2 + 12x^3y + 6xy)\mu(x,y) + (2x^3 + 3x^4y + 3x^2y)\mu_x(x,y)\\ &(-x^2 + 2xy)\mu(x,y)\\ &\quad = (2x^3 + 3x^4y + 3x^2y)\mu_x(x,y) - (5x^2y + 6x^3y^2 + 4xy^2)\mu_y(x,y).\end{aligned}$$

As the left and right side could both be polynomials, try $\mu(x,y) = x^a y^b$:

$$\begin{aligned}&(-x^2 + 2xy)\mu(x,y)\\ &\quad = (2x^3 + 3x^4y + 3x^2y)\mu_x(x,y) - (5x^2y + 6x^3y^2 + 4xy^2)\mu_y(x,y)\\ &\quad - x^{2+a}y^b + 2x^{a+1}y^{b+1}\\ &\quad = 2ax^{2+a}y^b + 3ax^{3+a}y^{b+1} + 3ax^{1+a}y^{b+1}\\ &\quad - 5bx^{2+a}y^b - 6bx^{3+a}y^{b+1} - 4bx^{a+1}y^{b+1}\\ -x^2 + 2xy &= 2ax^2 + 3ax^3y + 3axy - 5bx^2 - 6bx^3y - 4bxy\\ -x + 2y &= 2ax + 3ax^2y + 3ay - 5bx - 6bx^2y - 4by\\ 0 &= (2a + 1 - 5b)x + (3a - 6b)x^2y + (3a - 4b - 2)y,\end{aligned}$$

thus $a = 2b$, $-b + 1 = 0$, $b = 1$, $a = 2$. So we need to solve the *exact* differential equation:

$$(5x^4y^2 + 6x^5y^2 + 4x^3y^3)\,dx + (2x^5y + 3x^5y^2 + 3x^4y^2)\,dy = 0.$$

Notice that both sides will be of the form after integration $xyG(x,y)+H_x(x)$ (or same for y) thus $F(x,y) = xyG(x,y)$. So we only integrate one equation and get:

$$\int 5x^4y^2 + 6x^5y^2 + 4x^3y^3\, dx$$
$$= x^5y^2 + x^6y^2 + x^4y^3,$$

thus we need:e

$$x^5y^2 + x^6y^2 + x^4y^3 = C^*.$$

* Here we ignore $y(x) \equiv 0$, added back later.

Problem 4 (Problem 4, generalized)
Solve a Ricatti Equation ("First Order Quadratic Differential Equation"):

$$y' = A(x)y^2 + B(x)y + C(x).$$

with $y' = 1 + y^2 - x^2$.

Solution — First let's try $y(x) = \frac{1}{v(x)} + y_1(x)$ where y_1 is a particular solution. Then note that:

$$-\frac{v'(x)}{v(x)^2} + y_1'(x) = A(x)\left(\frac{1}{v(x)} + y_1(x)\right)^2 + B(x)\left(\frac{1}{v(x)} + y_1(x)\right) + C(x) \tag{4.1}$$
$$-v'(x) = A(x) + 2A(x)v(x)y_1(x) + B(x)v(x) \tag{4.2}$$
$$v'(x) + v(x)(2A(x)y_1(x) + B(x)) = -A(x). \tag{4.3}$$

Solving for our y,

$$v'(x) + v(x)(2A(x)y_1(x) + B(x)) = -A(x) \tag{4.4}$$
$$v'(x) + 2xv(x) = -1 \tag{4.5}$$
$$e^{x^2}v'(x) + 2xe^{x^2}v(x) = -e^{x^2} \tag{4.6}$$
$$v(x) = -\frac{1}{e^{x^2}}\int e^{x^2}\, dx \tag{4.7}$$
$$y(x) = \frac{1}{-\frac{1}{e^{x^2}}\left(\int e^{x^2}\, dx + C\right)} + y_1(x). \tag{4.8}$$

Project/Problem 5: in seperate beamer presentation