Utility And Risk Aversion Modeled By Differential Equations

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What People Do

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Which would you choose? It's a risky bargain doing the second option, but it's a positive expected value move. However, many people would prefer the first option over the second. Why? Is there some important hidden thershold? No! There isn't really. There's about as much difference between $10,000$ dollars and $100,000$ as between 100K and a million. There isn't a hidden thershold which mattered for this example.

A Problem With A $+EV$ Move?

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Here, the numbers I gave pointed to utility being $u(x) = C \log x$.

Utility And Decisions

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This is the expected utility, which shows how much we think we might get out of something.

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for a convave function $u(x)$.

this shows that convave functions have *some* level of risk aversion, as can be interpreted from Jensen. Thus concavity, or $u''(x)$ can be thought of as a metric of risk aversion.

Relative Risk

Notice that if your risk value changes quickly, it really doesn't matter as much if your risk is really concave. Add in some care for the real expected value to get:

Definition (Relative Risk)

The *relative risk* of a utility function is

$$\hat{R}_{\text{rel}}[u(x)] = \frac{xu''(x)}{u'(x)}$$

Absolute Risk + The Setup

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But note that people, at any instant, will likely exhibit constant risk. Thus, we have differential equations that model people's behaviour in risky situations.

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Substitution and Seperation

Note that, with $v = u'(x)$:

$$\begin{aligned}
\hat{R_{\text{rel}}}[u(x)] &= \frac{xu''(x)}{u'(x)} \\
r &= -\frac{xv'(x)}{v(x)} \\
-r \log|x| &= \log|v| + a \\
v &= Ax^r.
\end{aligned}$$

Solving The Equation

Plugging in:

$$v = Ax^r$$
$$u(x) = Ax^{r+1} + C,$$

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Substitution and Seperation For Absolute Risk

We can do a similar thing:

$$
\begin{aligned}
\hat{R}_{\text{abs}}[u(x)] &= \frac{u''(x)}{u'(x)} \\
r &= \frac{v'(x)}{v(x)} \\
rx &= \log|v| + a \\
v &= Ae^{xr}.
\end{aligned}
$$

Getting Solutions

Integrating,

$$u(x) = Ae^{xr} + C \qquad r \neq 0$$
$$u(x) = Ax + C \qquad r = 0.$$

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Remove factors of A and C and we get:

$$u(x) = Ae^{xr} \text{ or } x.$$

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A Obvious Game-Show Scam Which Works For Positive Risk

This is completely silly. Here you'd take the following risk for SOME $\epsilon > 0$:

Example (A way to get scammed)

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Do you see the problem here?

A Note

Note that we only have constant relative and absolute risk aversion for linear utility functions! Here risk is zero, and we are in a perfect land of expected values in which the second option is much better:

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Why -1 is the lowest relative risk value (That actually MAKES SENSE), and 0 for absolute

At this point, it doesn't make sense to have a relative/absolute risk this low. With the constraints imposed earlier, we have that smaller rewards are *better* than larger rewards! This is, in layman's terms: