Modeling an AR(1) process using Monte Carlo Markov Chain - Gibbs Sampling

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The R code provides an implementation of a coefficient estimate for an AR(1) process using Gibbs Sampling.

Let Y_t follow a stationary autoregressive process of order 1; i.e.

$$Y_t = c + a(y_{t-1} - c) + \epsilon \tag{1}$$

Where $\epsilon \sim N(0, \sigma^2)$.

In this case, the algorithm can be derived as follows:

We are looking for the posterior distributions of c,a and σ^2. Following the approach of Barnett, Kohn and Sheather (1995)[1], the likelihood can be expressed as:

$$p(y|c, a, \sigma^2) = p(y_1|c, a, \sigma^2) \prod_{t=1}^{n} p(y_t|y_{t-1}, c, a, \sigma^2) \tag{2}$$

Thereby,

$$p(y_1|c, a, \sigma^2) = \frac{\sqrt{1-a^2}}{\sigma\sqrt{2\pi}} e^{\frac{-(y_1-c)^2(1-a^2)}{2\sigma^2}} \tag{3}$$

All terms within the product statement can be expressed as

$$p(y_t|y_{t-1}, c, a, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-[(y_t-c)-a(y_{t-t}-c)]^2}{2\sigma^2}} \tag{4}$$

We can approximate the conditional distribution of a by including y_0 as an additional parameter, such that:

$$p(a|y, c, \sigma^2, y_0) \propto p(y|c, a, \sigma^2, y_0)p(c|a, \sigma^2, y_0)p(a|\sigma^2, y_0)p(\sigma^2|y_0)p(y_0) \tag{5}$$

[1] G. Barnett, R. Kohn and S.Sheather, Markov Chain Monte Carlo Estimation of Autoregressive Models with Application to Metal Pollutant Concentration in Sludge, Math. Comput. Modelling Vol. 22 (1995)

Thereby, the likelihood can be expressed as:

$$p(y|c,a,\sigma^2,y_0) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{\frac{-\sum\limits_{t=1}^{n}[(y_t-c)-a(y_{t-1}-c)]^2}{2\sigma^2}} \tag{6}$$

Assuming a priori independence of a,c and σ^2 we get

$$p(a|y,c,\sigma^2,y_0) \propto p(y|c,a,\sigma^2,y_0)I(-1,1) \tag{7}$$

where I(-1,1) is the indicator function that truncates (7) to (-1,1). Taking expressions that involve α gives us

$$\sum_{t=1}^{n}[(y_t-c)-a(y_{t-1}-c)]^2 = \sum_{t=1}^{n}(y_{t-1}-c)^2\left[a - \frac{\sum\limits_{t=1}^{n}(y_{t-1}-c)(y_t-c)}{\sum\limits_{t=1}^{n}(y_{t-1}-c)^2}\right]^2 \tag{8}$$

Thus,

$$a|y,c,\sigma^2,y_0 \approx N_{-1,1}\left(\frac{\sum\limits_{t=1}^{n}(y_{t-1}-c)(y_t-c)}{\sum\limits_{t=1}^{n}(y_{t-1}-c)^2}, \frac{\sigma^2}{\sum\limits_{t=1}^{n}(y_{t-1}-c)^2}\right) \tag{9}$$

Similar operations give us,

$$c|y,a,\sigma^2,y_0 \approx N\left(\frac{1}{n(1-a)}\sum_{t=1}^{n}(y_t-ay_{t-1}), \frac{\sigma^2}{n(1-a)^2}\right) \tag{10}$$

From p(σ^2) = $\frac{1}{\sigma^2}$ and

$$p(\sigma^2|y,c,a,y_0) \propto p(y|c,a,\sigma^2,y_0)p(\sigma^2) \tag{11}$$

we get

$$\sigma^2|y,c,a,y_0 \approx IG\left(\frac{n}{2}+1, 0.5\sum_{t=1}^{n}[(y_t-c)-a(y_{t-1}-c)]^2\right) \tag{12}$$

with IG being an inverse gamma distribution.

As mentioned above, we treat y_0 as another parameter:

$$y_0 = c + a(y_1 - c) + \eta \tag{13}$$

with $\eta \approx$ N(0,σ^2). Thus, it can be shown that:

$$y_0|y,c,\sigma^2,a \approx N(c+a(y_1-c),\sigma^2) \tag{14}$$