1) State space = $\{1,2,3,4\}$

Transition matrix $Q = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}$

P_{ij} = Probability of transition from state i to j.

$\Rightarrow$ a) $Q = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0 & 0.75 & 0.25 \end{bmatrix}$

b) ~~States 1 are 2~~ states 1, 2, 3, 4 are recurrent because 1 & 2 form a closed class disconnected from $\{3,4\}$ and will eventually return to 1 or 2. Similarly $\{3,4\}$ class is recurrent.

c) Stationary distributions π are st,

$\pi Q = \pi$ (they don't change on transition)

Let π $[\pi_1, \pi_2, 0, 0]$ & $\pi_1 + \pi_2 = 1$

↳ stationary ~~state~~ distribution corresponding to the class $\{1,2\}$

$\pi Q = \pi \Rightarrow \pi_1(0.5) + \pi_2 \times 0.25 + 0 = \pi_1$

$\Rightarrow \pi_2 = 2\pi_1$

& $\pi_1 + \pi_2 = 1 \Rightarrow \pi_1 = \frac{1}{3}$ & $\pi_2 = \frac{2}{3}$

$\Rightarrow \left[\frac{1}{3}, \frac{2}{3}, 0, 0\right]$ is a stationary ~~state~~ distribution corresponding to the class $\{1,2\}$

Let π' be the stationary distribution for the class $\{3,4\}$ & will eventually be in this class only

$\Rightarrow \pi' = [0, 0, \pi_3', \pi_4']$

$\Rightarrow 0.25\,\pi_3' + 0.75\,\pi_4' = \pi_3'$

$\Rightarrow \pi_4' = \pi_3' = \frac{1}{2}$

$\Rightarrow \pi' = \left[0, 0, \frac{1}{2}, \frac{1}{2}\right]$

$\exists$ two stationary distributions for the chain are $\left[\frac{1}{3}, \frac{2}{3}, 0, 0\right]$ & $\left[0, 0, \frac{1}{2}, \frac{1}{2}\right]$

Date: / /

2). Let W = team wins a game, L = team loses a game.

$P\left(\frac{W_{n+1}}{W_n}\right) = 0.8$ and $P\left(\frac{W_{n+1}}{L_n}\right) = 0.3$

$P\left(\frac{\text{Dinner}}{W}\right) = 0.7$ and $P\left(\frac{\text{Dinner}}{L}\right) = 0.2$

(a) This is a Markov chain with 2 states: Win (W) and Loss (L)
The transition matrix is $P = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$

The stationary distribution (π_W, π_L) has to be such that

$$(\pi_W, \pi_L) P = (\pi_W, \pi_L)$$

and $\pi_W + \pi_L = 1$.

$$\pi_W = 0.8\pi_W + 0.3\pi_L$$
$$\pi_L = 0.2\pi_W + 0.7\pi_L.$$
$$2\pi_W = 3\pi_L.$$
$$\pi_W = 1.5\pi_L.$$
$$2.5\pi_L = 1.$$
$$\pi_L = \frac{2}{5}.$$
$$\pi_W = \frac{3}{5} = 0.6$$

$\therefore$ The proportion of games that the team wins in the long run is 0.6.

(b). $P(\text{Dinner}) = 0.7\pi_W + 0.2\pi_L = 0.42 + 0.08 = 0.5$

$\therefore$ the proportion of games that result in a team dinner is 0.5.

(c). Since dinner is a Bernoulli process with success probability 0.5 for each game, the expected number of games for a dinner $= \frac{1}{0.5} = 2$.

3). Cat's chain has 2 states — Room1(C_1) and Room2(C_2).
It moves to the other room with probability 0.8
and stays in the current room with probability 0.2
So, the transition matrix for cat is $C = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$

Mouse's chain has 2 states — Room 1 (M_1) and Room 2(M_2)
It moves from room 1 to room 2 with probability 0.3
and from room 2 to room 1 with probability 0.6.
So, the transition matrix for mouse is $M = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$

(a) Let $\pi_C = (\pi_{C_1}, \pi_{C_2})$ denote the stationary distributions.

$$\pi_C P_C = \pi_C$$
$$\pi_{C_1} + \pi_{C_2} = 1$$
$$\pi_{C_1} = 0.2\pi_{C_1} + 0.8\pi C_2$$
$$\pi_{C_1} = \pi C_2.$$
$$\pi_{C_1} = \pi_{C_2} = 0.5$$

Let $\pi_M = (\pi_{M_1}, \pi_{M_2})$ denote the stationary distributions.

$$\pi_{M_1} = 0.7\pi_{M_1} + 0.6\pi M_2$$
$$0.3\pi M_1 = 0.6\pi M_2$$
$$\pi M_1 = 2\pi M_2.$$
$$\pi M_1 = \frac{2}{3} \text{ and } \pi M_2 = \frac{1}{3}.$$

(b). There are 4 possible states for Z_n:
(C_1, M_1); (C_2, M_1); (C_1, M_2); (C_2, M_2)
Since, the cat's and mouse's position depend only on its previous position and the cat and mouse move independently, the joint process, $Z_n = (C_n, M_n)$ is a Markov Chain because the next state depends only on the current state and not on earlier history.

Solⁿ 4: There are 3 types of squares

a) 4 corner squares → have 3 squares where they can go

b) edge ones ($6 \times 4 = 24$) have 5 options

c) center squares → $6^2 = 36$ have 8 options.

the detailed balance condition

$$\pi(i) P(i \to j) = \pi(j) P(j \to i)$$

if we assume that $\pi(i) \propto K \cdot$ No. of options (i) ~~No of options (i)~~

$\Rightarrow$ $\left(\frac{K \cdot \text{No. of options}(i)}{\cancel{\text{No. of option}}}\right) \times \frac{1}{\text{No. of options}(i)} = K$.

$\pi(i) \propto K \cdot$ No. of options (i) satisfies the detailed balance conditions

~~⇒ it is reversible~~

⇒ it is a stationary distribution. (Reversibility ⇒ stationary distribution)

$\Rightarrow$ $\pi(i)$ for corner squares is $\frac{K}{3}$

$\pi(i)$ for edge squares is $\frac{K}{5}$

$\pi(i)$ for center squares is $\frac{K}{8}$.

Since the markov chain is irreducible (it can reach to all 64 states eventually) & it is aperiodic

⇒ this stationary distribution is unique.

$4\frac{K}{3} + 24\frac{K}{5} + 36\frac{9 K}{82} = 1 \Rightarrow \frac{323K}{30} = 1$ $k = \frac{3}{32}$

Soln 5: a) there is a non-zero probability that stock price will not return to the initial price eventually
$\Rightarrow$ it is transient

b) Since stock price is not recurrent $\Rightarrow$ No stationary distribution of stock price exist.

c) Probability very close to 0 (1000 simulation)

Soln 6: a) from a permutation g, i can get $^{26}C_2$ different permutations h by swapping 2 alphabets

all are equally likely $\Rightarrow$ $p(g \to h) = \frac{1}{325}$

If we can get h by 1 swap.
$p(g \to h) = 0$ in all other cases

Since this markov chain of 26! permutations irreducible (can eventually reach all states) & aperiodic
$\Rightarrow$ unique stationary distribution exists

stationary distribution $\pi(i) = \frac{1}{26!}$ $\forall$ $1 \le i \le 26!$

(because the states are symmetric

$\hookrightarrow$ satisfies
$\hookrightarrow$ consider the transition matrix $Q_{26! \times 26!}$

g_1 g_2

Q_{ij} = Pq transition from i to j.

g_1
g_2

Then $\pi Q = \frac{1}{261}\left[\text{sum } q \text{ rows}, \text{ sum } q \text{ rows}, \ldots\right]$
$\left[\frac{1}{261}, \frac{1}{261}, \ldots \frac{1}{261}\right]$ $\quad$ of Q

& sum of rows of Q for a particular column is the total probability of reaching that state from all other states which is essentially 1

$$\Rightarrow \pi Q = \left[\frac{1}{261}, \frac{1}{261} \ldots \frac{1}{261}\right] = \pi$$

$= \left[\frac{1}{261}, \frac{1}{261} \ldots \frac{1}{261}\right]$ is the unique stationary distribution

b) Detailed balance condition /

$$\pi(g)\, q(g\to h) = \pi(h)\, q(h\to g)$$

$q(g\to h)$ can be written as probability of proposal found in 1st part $\times$ Probability of acceptance of proposal,

$\downarrow z(g\to h)$ $\quad$ $\to w(g\to h)$

Consider $\pi(g) \propto s(g) = \dfrac{s(g)}{\sum_{g'} s(g')}$

Probability of proposal $(g\to h)$ = Probability of proposal $(h\to g)$

but probability of acceptance are different in the 2 cases

$$s(g)\, \cancel{z(g\to h)} \times w(g\to h) = s(h)\, \cancel{z(h\to g)}\, w(h\to g)$$

if $s(g) \leq s(h)$

telco Dt.: Pg.:

then $w(g \to h) = 1$ & $w(h \to g) = \frac{s(g)}{s(h)}$

$\Rightarrow s(g)\, w(g \to h) = s(g) = s(h)\, w(h \to g)$

& vice versa

$\Rightarrow s(g)\, w(g \to h) = s(h)\, w(h \to g)$

$\Rightarrow \pi(g) \propto s(g)$ obeys the detailed balance condition
markov chain is reversible & has
$\Rightarrow$ stationary distribution proportional to the score of the permutation $s(g)$.

Contributions: Garvit $\to$ 1, 4, 5, 6
Jayaraman $\to$ 2, 3, 5