$\underline{\underline{\text{Sol}^n 1}}$: required P = 1 − P(no letter is in correct envelope)

$= 1 - \frac{D(N)}{N!} \rightarrow$ Derangement.

$$D(N) = N!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} \cdots \frac{-(1)^n}{n!}\right)$$

$$\frac{D(N)}{N!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} \cdots \frac{(-1)^n}{n!})$$

$$\Rightarrow 1 - \frac{1}{2!} + \frac{1}{3!} \cdots + \frac{(-1)^{n+1}}{n!} \rightarrow \text{required probability}$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots$$

$$1 - e^{-1} = \lim_{n \to \infty} 1 - \frac{1}{2!} + \frac{1}{3!} \cdots \frac{(-1)^{n+1}}{n!}$$

however terms get smaller as n increases, hence for n = 50, we can approximate it as $1 - e^{-1}$.

$\underline{\underline{\text{Sol}^n 2}}$: This problem is similar to Monty hall problem.

if i initially choose the ~~door~~ gift with 1000 dollars & the host opens either of the ~~other~~ gifts then it is not beneficial to switch $\Rightarrow$ NO.

if I choose either of the bad gifts then the host opens the other bad one so if I switch i get 1000. $\Rightarrow$ Yes & Yes for both cases

$\Rightarrow \frac{2}{3}$ probability of getting the 1000 gift if I switch

$$\Rightarrow \text{Expected winnings} = \frac{2}{3} \times 1000 + 0 = \frac{2000}{3}$$

Sq 3: a) $P(A\cap B|C) = \frac{P(A\cap B\cap C)}{P(C)} = \frac{P(B\cap C)\,P(A|B\cap C)}{P(C)}$

$= P(B|C)\,P(A|B\cap C)$

$\Rightarrow$ True

b)

$\cancel{P(A\cap B|C) \times P(C) = P(A\cap B) \times P(C|A\cap B)}$

Assuming its true

$\Rightarrow P(A\cap B|C) = P(A|C)\,P(B|C)$

multiplying by $P(C)$ on both side

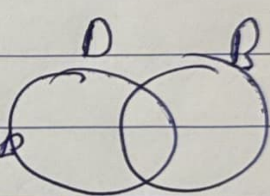
$P(C)\,P(A\cap B|C) = P(A\cap B) = P(A)\,P(B) = \cancel{P(A)\,P(B|C)}$ $P(B)\,P(A|C)$

since $\cancel{P(B) \neq P(B|C)}$ $P(A) \neq P(A|C)$

$\Rightarrow$ contradiction $\Rightarrow$ False

c) $P(A|D\cap B^c) > P(A|D\cap B)$ & $P(A|D^c\cap B^c) > P(A|D^c\cap B)$

$\cancel{P(A\cap B)}$ $\cancel{P(A) = P_A(A\cap D) + P_A(A\cap D^c)}$



$P(A|B)$ intersection of A with what is common to D & B

$= P$ of $A\cap(D\cap B)$

$+ P_A(A\cap(D^c\cap B))$ ↳ area of B in D^c,

$P(A|B) \Rightarrow P(D|B)\,P(A|D\cap B)$

$+ P(D^c|B)\,P(A|D^c\cap B)$

$P(A|B^c) = P(D|B^c)(P(A|D\cap B^c) + P(D^c|B^c)\,P(A|D^c\cap B^c)$

$> P(A|D\cap B)$ $\qquad P(A|D^c\cap B)$

can't comment about $P(D|B)$, $P(D|B^c)$

$P(D^c|B)$ & $P(D^c|B^c)$

$\Rightarrow$ False.

Sol^n 4: a) $P(X=n) = \frac{\alpha}{n^3}$

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then $\sum P(X=n) = \sum \frac{\alpha}{n^3} = 1$ ~~given~~ $\frac{6}{\pi^2}$

converges.

$E(X) = \sum n \frac{\alpha}{n^3} = \sum \frac{\alpha}{n^2}$

$\hookrightarrow \frac{\pi^2}{6}\alpha$ finite.

$E(X^2) = \sum n^2 \frac{\alpha}{n^3} = \sum \frac{\alpha}{n} \rightarrow$ diverges

b) probability distribution function $f(u) = \frac{\alpha}{u^p}$.
$u > 0$.

$\int_0^\infty \frac{\alpha}{u^p} du$ converges for $p > 1$

$E(X) = \int_0^\infty \frac{\alpha}{u^{p-1}} du$ converges for $p > 2$

$E(X^2) = \int_0^\infty \frac{\alpha}{u^{p-2}} du$ diverges for $p \leq 3$

$\Rightarrow$ on taking p in $(2,3]$ we get desired result.

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Sol^n 5: $E(M) = \sum_{M=1}^{N} M \sum_{r=1}^{n} {}^nC_r \cdot \frac{1}{N^r} \cdot \frac{(M-1)^{n-r}}{N^{n-r}}$

$$= \frac{1}{N^n} \sum_{M=1}^{N} M \sum_{r=1}^{n} {}^nC_r (m-1)^{n-r} 1^r$$

$$= \frac{1}{N^n} \sum_{m=1}^{N} m \left(m^n - (m-1)^n\right)$$

$$\frac{1}{N^n} \sum_{m=1}^{N} \left[m^{n+1} - (m-1)^{n+1}\right] - (m-1)^n$$

→ telescopic series:

$$\frac{1}{N^n} \quad \cancel{1^{n+1}} - 0^{n+1}, \quad 2^{n+1} - \cancel{1^{n+1}}, \quad \ldots, \quad N^{n+1} - (N-1)^{n+1}$$

$$= \frac{1}{N^n} \left(N^{n+1} - \sum_{M=1}^{N} (m-1)^n \right)$$

$$= \frac{1}{N^n} \left(N^{n+1} - (1^n + 2^n \ldots + (N-1)^n) \right)$$

$$\boxed{E(M) = N - \sum_{t=1}^{N-1} \left(\frac{t}{N}\right)^n}$$

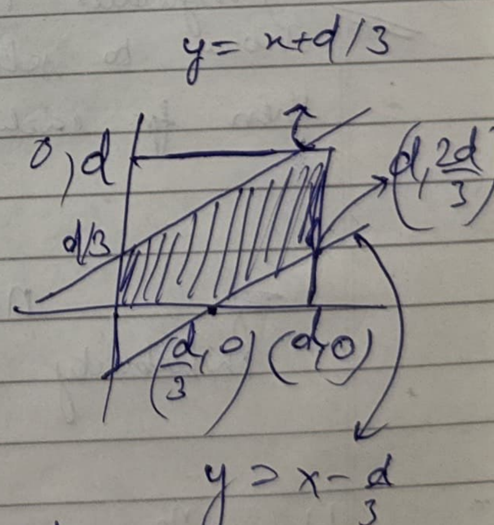
Sol^n 6: $0 \le x, y \le d$

we need $P(|x-y| \le \frac{d}{3})$

$y > x - d/3$

&

$y < x + d/3$



$$P = \frac{d^2 - 2 \times \frac{1}{2} \times \frac{2d}{3} \times \frac{2d}{3}}{d^2}$$

$$= 1 - \frac{4}{9} = \frac{5}{9}$$

7). Let's take the situation where:

- there are $n+1$ inhabitants in the town $(0,1,2,\cdots,n)$ with the rumor starting from person 0.
- at each step, the current person tells the rumor to a randomly chosen person from the remaining n persons excluding himself.

(a). At each step, the favourable cases to whom the rumor can be passed $= n-1$ (excluding himself and the originator)
Total number of cases $= n$ (excluding only himself).
probability at each step $= \frac{n-1}{n}$.
Total number of steps $= r$.
$\therefore$ the probability that the rumor will be told r times without returning to the originator $= \left(\frac{n-1}{n}\right)^r$.

(b). In this case, no person should hear the rumor twice.
Let the rumor start with the person 0.
At each step k (k ranging from 1 to r), k people have heard the rumor (0^{th} person $+$ $k-1$ others).
Total number of people to whom the rumor should be passed
$= n+1-k$
(Total number of people $(n+1)-k$ (the number of people) who have already heard it
k includes the teller himself as well).
Total number of people available to hear the rumor $= n$.
$\therefore$ the probability at each step $= \left(\frac{n-k+1}{n}\right)$.
The probability after r steps $= \prod_{k=1}^{r} \frac{n-k+1}{n} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{n^r}$

$$= \frac{n!}{(n-r)!\, n!}$$

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8). As $A_1, A_2, \ldots, A_n$ are independent,

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} P(A_i^c) = \prod_{i=1}^{n} (1 - P(A_i))$$

For all $x \in [0,1]$, $e^{-x} \geq 1 - x$.

$$\prod_{i=1}^{n} (1 - P(A_i)) \leq \prod_{i=1}^{n} e^{-P(A_i)}$$

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) \leq e^{-P(A_1) - P(A_2) - P(A_3) - \cdots - P(A_n)}$$

Hence, proved.

9). Let F and G be distribution functions on $\mathbb{R}$.

- $F(x) = P(X \leq x)$ and • $G(x) = P(Y \leq x)$ for some random variables x and y.

We define the convolution of F and G to be

$$H(x) = \int_{-\infty}^{\infty} F(x-y)\, dG(y) \quad \text{or} \quad H(x) = \int_{-\infty}^{\infty} G(x-y)\, dF(y)$$

To show $H(x)$ is a distribution function, we need to show:

(i) H is non-decreasing.

(ii) $\lim_{x \to -\infty} H(x) = 0$

(iii) $\lim_{x \to \infty} H(x) = 1$

(iv) H is right-continuous.

(i). Let $x_1 < x_2$;

$$H(x_1) = \int_{-\infty}^{\infty} F(x_1 - y)\, dG(y) \text{ and } H(x_2) = \int_{-\infty}^{\infty} F(x_2 - y)\, dG(y).$$

$\because$ F is non-decreasing and $x_1 < x_2$, $H(x_1) \leq H(x_2)$

(ii). $\lim_{x \to -\infty} H(x) = \int_{-\infty}^{\infty} \lim_{x \to -\infty} F(x-y)\, dG(y)$

$\because$ F is a distribution function $\lim_{x \to -\infty} F(x-y) = 0$.

$\therefore \lim_{x \to -\infty} H(x) = 0$.

(iii). $\lim_{x \to \infty} H(x) = \int_{-\infty}^{\infty} \lim_{x \to \infty} F(x-y)\, dG(y) = \int_{-\infty}^{\infty} (1)\, dG(y) = 1$ as G is a distribution function.

(iv). $\lim_{h \to 0} H(x+h) = \int_{-\infty}^{\infty} \lim_{h \to 0} F(x+h-y)\, dG(y) = \int_{-\infty}^{\infty} F(x-y)\, dG(y) = H(x)$.

Since, $H(x)$ satisfies all the four properties, $H(x)$ is also a distribution function.

Interestingly, $H(x)$ is the distribution function of the sum $X+Y$.

Sol^n 10 : $I_{[0, X(\omega)]}(x) = 1$ if $x < X(\omega)$

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$$\Rightarrow \int_{\Omega} \int_0^{\infty} I_{[0, X(\omega)]}(x)\, dx\, dP(\omega)$$

$$= \int_{\Omega} \int_0^{X(\omega)} dx\, dP(\omega)$$

$$\Rightarrow \int_{\Omega} X(\omega)\, dP(\omega) = E(X).$$

$$\int_{\Omega} \int_0^{\infty} I_{[0, X(\omega)]}(X)\, dx\, dP(\omega) = \int_0^{\infty} \left(\int_{\Omega} I_{[0, X(\omega)]}(x)\, dP(\omega) \right) dx.$$

1 if $X(\omega) > x$.

$$= \int_0^{\infty} P(X > x)\, dx$$

$$\int_0^{\infty} 1 - F(x)\, dx$$

Sol^n 11 : $E(e^{uX}) = \int_{-\infty}^{\infty} e^{ux} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

$$\frac{1}{\sigma\sqrt{2\pi}} \quad \int_{-\infty}^{\infty} e^{ux - \frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$ux - \frac{(x-\mu)^2}{2\sigma^2} = \frac{-1}{2\sigma^2}\left(x^2 + \mu^2 - 2\mu x - 2\sigma^2 ux \right)$$

$$= \frac{-1}{2\sigma^2}\left(x - \mu - u\sigma^2 \right)^2 + \frac{1}{2}u^2\sigma^2 + u\mu$$

$$\Rightarrow e^{\frac{1}{2}u^2\sigma^2 + \mu u} \cdot \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu-u\sigma^2)^2}{2\sigma^2}} dx$$

$$x - u\sigma^2 = t$$

$\Rightarrow I = 1$ (Total probability)

$$\Rightarrow E(e^{uX}) = e^{u\mu + \frac{1}{2}u^2\sigma^2}$$

b) $X = N(\mu, \sigma^2)$

$E(X) = \mu$

$$E(e^{uX}) = e^{u\mu + \frac{1}{2}\sigma^2 u^2} \geq e^{u\mu} \rightarrow e^{uX}$$

$$\rightarrow \boxed{E(e^{uX}) \geq e^{uE(X)}}$$

Practical section

Total paths = ${}^{2n}C_n$

lets say the person crosses the diagonal at (t, t) to $(t, t+1)$ then for each corresponding path from $(t, t+1)$ to (n, n) there is one right move than up & taking the reflection of the path about the line joining $(t, t+1)$ & (n, n) gives a corresponding path having one more U to $(n-1, n+1)$

Total no. of ways to reach $(n-1, n+1)$ ${}^{2n}C_{n+1}$

& each path has a one - one correspondence with paths that cross the diagonal

$\Rightarrow$ Total valid path = ${}^{2n}C_n - {}^{2n}C_{n+1} = \dfrac{{}^{2n}C_n}{n+1}$