Telco Dt.: Pg.:

Sol^n 3: $E[W_s W_t]$

if $t \geq s$ then $W_t - W_s$ is the increment in W_s after s

$\Rightarrow$ W_s & $W_t - W_s$ are independent

$\Rightarrow$ $E(W_s W_t) = E[W_s(W_t - W_s) + W_s^2]$

$= E(W_s(W_t - W_s)) + E[W_s^2]$

we know $W_s = \frac{1}{\sqrt{n}} M_{ns} \Rightarrow E[W_s] = 0$

$\Rightarrow E[W_s(W_t - W_s)] = \cancel{E[W_s]} \overset{0}{E}[W_t - W_s]$
$= 0$

$E[W_s^2] = \text{var}(W_s^2) = s$

if $s \geq t$, similarly we get $E[W_s W_t] = E[W_t^2] = t$

$\Rightarrow E[W_s W_t] = \min(s,t)$ for $s,t \geq 0$.

if $t \geq s$

W_s & $W_t - W_s$ are independent because the increments in W depend on the coin tosses after s ~~& not to~~ s which are independent of previous outcomes & W_s.

Sol^n 4: $E(W_t - W_s) = E[W_t] - E[W_s] = 0$

$\text{var}(W_t - W_s) = E[(W_t - W_s)^2]$

$= E(W_t^2 + W_s^2 - 2W_s W_t)$

$= E[W_t^2] + E[W_s^2] - 2E[W_s W_t]$

$= t + s - 2(\min(s,t))$

$= s + t - 2(s) = t - s$

Since W_t & W_s are normally distributed then W_t & W_s are jointly normal.

$$\Rightarrow \quad W_t \sim N(0,t)$$
$$W_s \sim N(0,s)$$

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then $W_t - W_s \sim N(0, t-s)$

$$\begin{pmatrix} W_s \\ W_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & s \\ s & t \end{pmatrix}\right)$$

covariance matrix

$$\begin{pmatrix} E[W_s^2] & E[W_s W_t] \\ E[W_t W_s] & E(W_t^2) \end{pmatrix}$$

$\Rightarrow$

$$(-1\ 1)\begin{pmatrix} W_s \\ W_t \end{pmatrix} \sim N\Big($$

$a = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ then $a^T \begin{pmatrix} W_s \\ W_t \end{pmatrix} \sim N\left(a^T \begin{pmatrix} 0 \\ 0 \end{pmatrix}, a^T \begin{pmatrix} s & s \\ s & t \end{pmatrix} a\right)$

$$\Rightarrow \quad W_t - W_s \sim N\left(0, (-1\ 1)\begin{pmatrix} 0 \\ t-s \end{pmatrix}\right)$$

$$\Rightarrow W_t - W_s \sim N(0, t-s)$$

the increments over ~~independe~~ non-overlapping intervals are independent by definition of Brownian motion.

Sol^n 5: $E[W_t | F_s]$ $0 \le s \le t$

$$E[W_t - W_s + W_s | F_s] = E[W_t - W_s | F_s] + E[W_s | F_s]$$

$W_t - W_s$ ~~are~~ is independent of F_s (it is independent of the information available to w upto s in the sigma algebra.

$$\Rightarrow E[W_t - W_s | F_s] = E[W_t - W_s] = 0$$
$$= E(W_s)$$

& $E[W_s | F_s] = W_s^t$ because all the information that we have in F_s resulted in W_s so we can expect it to give W_s again

$\Rightarrow \quad E[W_t \mid F_s] = E[W_s] = W_s \quad 0 \leq s \leq t$

$\Rightarrow$ Brownian motion is a martingale