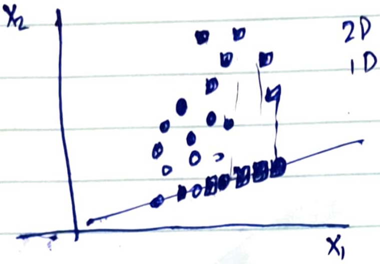
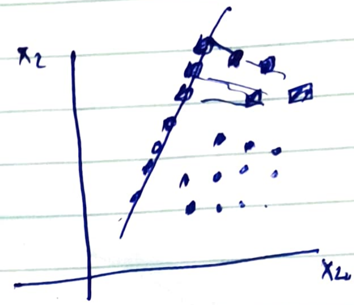
Linear Discriminant Analysis

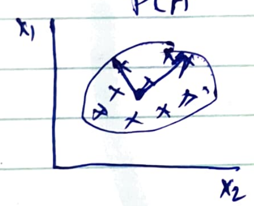
LDA is supervised, consider the class label and finds components which separates the classes most.



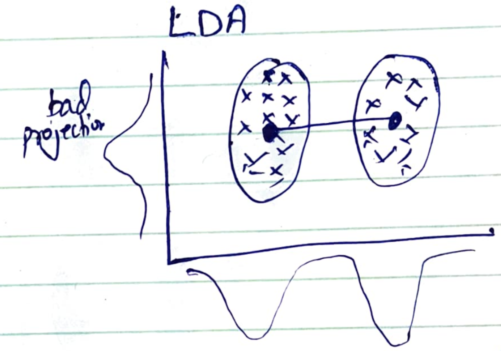


This line succeeded in separating the two classes & in the meantime reducing the dimensionality of our problem from two features (X) to only scalar value y

PCA



LDA



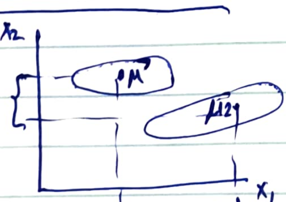
good projection, separates classes well

* PCA component axes that maximize the variance

* LDA; maximizing the component axes for class-separation

How to choose a line

This axis yields better class separability



This axis has larger distance means

Fisher Discriminant Ratio

$$J(w) = \frac{|\tilde{\mu}_1 - \tilde{\mu}_2|^2}{\tilde{s}_1^2 + \tilde{s}_2^2}$$

← maximize

← minimize

Maximize the $J(w)$

Objectives → to minimise variability within a class (Inter class scatter)

→ to increase the b/w class variability (Between Class Scatter)

Within-class scatter matrix S_W

$$S_W = \sum_{i=1}^{c} S_i$$

$$S_i = \sum_{x \in D_i}^{n} (x - m_i)(x - m_i)^T$$

Between class scatter matrix S_B

$$S_B = \sum_{i=1}^{c} N_i (m_i - m)(m_i - m)^T$$

⇒ LDA does not change the location but tries to provide more class separability & draw a decision region b/w the given classes.

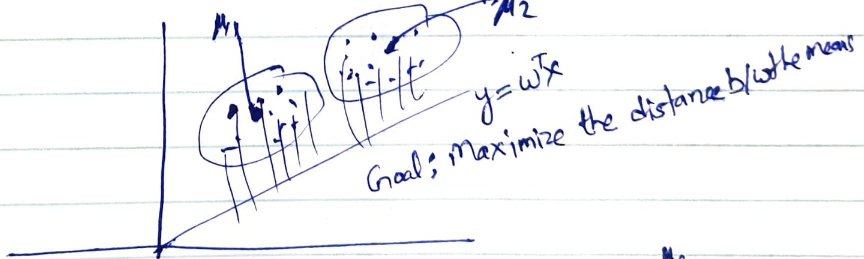
⇒ LDA is a supervised ML method that is used to separate two groups/classes.

⇒ LDA objective is to maximize the separability b/w the two groups so that we can make the best decision to classify them

⇒ LDA is like PCA which helps in dimensionality reduction but it focuses on maximizing the separability among known categories by creating a new linear axis & projecting the data points on that axis.

⇒ LDA doesn't work on finding the PC, it basically looks at what type of point/features/subspace gives more discrimination to separate the data.

⇒ LDA objectives to find a line that maximizes the class separation.



Goal :- to find the best set of w which gives the maximum separation i.e the distance b/w the two means is maximum.

However, the distance b/w the projected means is not a very good measures since it does not take into account the standard deviation within the classes.

How to define which class is better?

1. The data where the invariance within the class is minimum & the variability among the other classes are maximum is considered to be good.
2. The solution proposed by Fisher is to maximize a function that represents the diff b/w the means, normalized by a measures of the within-class (intra-class) variability, or the so-called scatter.
3. Note : Scatter = Variance.
4. For each class, we defines the scatter, an equivalent of the variance as: (sum of squared diff. b/w the projected samples & their class mean).

$$\tilde{S}_i^2 = \sum_{y \in w_i} (y - \tilde{\mu}_i)^2$$

$$\boxed{y_i = w_i^T x}$$

$\Rightarrow$ The scatter of the projection can be expressed as a function of the scatter matrix in the x feature space

$$\tilde{S}_i^2 = \sum_{y \in w_i} (y - \tilde{\mu}_i)^2 = \sum_{x \in w_i} (w^T x - w^T \mu_i)^2 = \sum_{x \in w_i} w^T (x - \mu_i)(x - \mu_i)^T w = w^T S_i w$$

$$\boxed{\tilde{S}_1^2 + \tilde{S}_2^2 = w^T S_w w}$$

Similarly $(\tilde{\mu}_1 - \tilde{\mu}_2)^2 = (w^T \mu_1 - w^T \mu_2)^2 = w^T \underbrace{(\mu_1 - \mu_2)(\mu_1 - \mu_2)^T}_{S_B} w = w^T S_B w$

$\therefore$ Finally the Fisher criterion in terms of S_w and S_B as

$$J(w) = \frac{w^T S_B w}{w^T S_w w}$$

For maximum

$$\frac{d}{dw}[J(w)] = 0$$

$C_1 \Rightarrow X_1 \Rightarrow (X_1, X_2) = \{(4,1),(2,4),(2,3);(3,6),(4,4)\}$
$C_2 \Rightarrow X_2 \Rightarrow (X_1, X_2) = \{(9,10),(6,8),(9,5),(8,7),(10,8)\}$

$$\mu_1 = \left[\frac{4+2+2+3+4}{5}, \frac{1+4+3+6+4}{5}\right] = [3 \quad 3.6]$$

$$\mu_2 = \left[\frac{9+6+9+8+10}{5} \quad \frac{10+8+5+7+8}{5}\right] = [8.4 \quad 7.6]$$

$$S_1 = \sum_{x \in w_1} (x-\mu_1)(x-\mu_1)^T$$

[S_1 is the covariance matrix for class C_1
S_2 " " " " " " C_2]

Covariance

$$[x-\mu_1] = \begin{bmatrix} (4-3) & (2-3) & (2-3) & (3-3) & (4-3) \\ (1-3.6) & (4-3.6) & (3-3.6) & (6-3.6) & (4-3.6) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 0 & 1 \\ -2.6 & 0.4 & -0.6 & 2.4 & 0.4 \end{bmatrix}$$

$(x-\mu_1)(x-\mu_1)^T$

$$\underset{2\times1}{\begin{bmatrix} 1 \\ -2.6 \end{bmatrix}} \underset{1\times2}{[1 \quad -2.6]} = \begin{bmatrix} 1 & -2.6 \\ -2.6 & 6.76 \end{bmatrix} \text{ - (i)}$$

$$\begin{bmatrix} -1 \\ 0.4 \end{bmatrix} [-1 \quad 0.4] = \begin{bmatrix} 1 & -0.4 \\ -0.4 & 0.16 \end{bmatrix} \text{ - (ii)}$$

$$\begin{bmatrix} -1 \\ -0.6 \end{bmatrix} [-1 \quad -0.6] = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.36 \end{bmatrix} \text{ - (iii)}$$

$$\begin{bmatrix} 0 \\ 2.4 \end{bmatrix} [0 \quad 2.4] = \begin{bmatrix} 0 & 0 \\ 0 & 5.76 \end{bmatrix} \text{ - (iv)}$$

$$\begin{bmatrix} 1 \\ 0.4 \end{bmatrix} [1 \quad 0.4] = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 0.16 \end{bmatrix} \text{ - (iv)}$$

$$S_1 = \sum (x-\mu_1)(x-\mu_1)^T$$
= i.e adding (i),(ii),(iii),(iv) & (v) & take avg

$$S_1 = \begin{bmatrix} 4 & -2 \\ -2 & 13.2 \end{bmatrix} / 5$$

$$S_1 = \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 2.64 \end{bmatrix}$$

<u>Now For S_2</u>

$$[x_2-\mu_2] = \begin{bmatrix} (9-8.4) & (6-8.4) & (9-8.4) & (8-8.4) & (10-8.4) \\ (10-7.6) & (8-7.6) & (5-7.6) & (7-7.6) & (8-7.6) \end{bmatrix}$$
$$= \begin{bmatrix} 0.6 & -2.4 & 0.6 & -0.4 & 1.6 \\ 2.4 & 0.4 & -2.6 & -0.6 & 0.4 \end{bmatrix}$$

$$\begin{bmatrix} 0.6 \\ 2.4 \end{bmatrix} [0.6 \quad 2.4] = \begin{bmatrix} 0.36 & 1.44 \\ 1.44 & 5.76 \end{bmatrix}$$

$$\begin{bmatrix} -0.4 \\ -0.6 \end{bmatrix} [-0.4 \quad -0.6] = \begin{bmatrix} 0.16 & 0.24 \\ 0.24 & 0.36 \end{bmatrix}$$

$$\begin{bmatrix} -2.4 \\ 0.4 \end{bmatrix} [-2.4 \quad 0.4] = \begin{bmatrix} 5.76 & -0.96 \\ -0.96 & 0.16 \end{bmatrix}$$

$$\begin{bmatrix} 1.6 \\ 0.4 \end{bmatrix} [1.6 \quad 0.4] = \begin{bmatrix} 2.56 & 0.64 \\ 0.64 & 0.16 \end{bmatrix}$$

$$\begin{bmatrix} 0.6 \\ -2.6 \end{bmatrix} [0.6 \quad -2.6] = \begin{bmatrix} 0.36 & -1.56 \\ 1.56 & 6.76 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 9.2 & -0.2 \\ -0.2 & 13.2 \end{bmatrix} / 5$$

$$S_2 = \begin{bmatrix} 1.84 & -0.04 \\ -0.04 & 2.64 \end{bmatrix}$$

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$$S_W = S_1 + S_2 = \begin{bmatrix} 0.8 & -0.4 \\ -0.4 & 2.64 \end{bmatrix} + \begin{bmatrix} 1.84 & -0.04 \\ -0.04 & 2.64 \end{bmatrix}$$

$$S_W = \begin{bmatrix} 2.64 & -0.44 \\ -0.44 & 5.28 \end{bmatrix}$$

Step Now Calculate Between class Scatter Matrix (S_B)

$$S_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T$$

$$= \begin{bmatrix} (3-8.4) \\ (3.6-7.6) \end{bmatrix} \begin{bmatrix} (3-8.4) & (3.6-7.6) \end{bmatrix} = \underset{2\times1}{\begin{bmatrix} -5.4 \\ -4 \end{bmatrix}} \underset{1\times2}{\begin{bmatrix} -5.4 & -4 \end{bmatrix}} = \begin{bmatrix} 29.16 & 21.6 \\ 21.6 & 16 \end{bmatrix}$$

$$\boxed{S_B = \begin{bmatrix} 29.16 & 21.6 \\ 21.6 & 16 \end{bmatrix}}$$

Step Find the best LDA Projection Vector.

Eigen Value $S_W^{-1} S_B w = \lambda w$ $\quad\therefore S_W^{-1} S_B w - \lambda w = 0$

$$|S_W^{-1} S_B - \lambda I| = 0$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$S_W^{-1} = \begin{bmatrix} 2.64 & -0.44 \\ -0.44 & 5.28 \end{bmatrix}^{-1} = \frac{1 \times \begin{bmatrix} 5.28 & 0.44 \\ 0.44 & 2.64 \end{bmatrix}}{(2.64)(5.28) - (-0.44)(-0.44)}$$

$$= \frac{1}{13.7456} \begin{bmatrix} 5.28 & 0.44 \\ 0.44 & 2.64 \end{bmatrix}$$

$$= \begin{bmatrix} 0.384 & 0.032 \\ 0.082 & 0.192 \end{bmatrix}$$

$$\therefore S_W^{-1} S_B - \lambda I = 0$$

$$\begin{bmatrix} 0.384 & 0.032 \\ 0.082 & 0.192 \end{bmatrix} \begin{bmatrix} 29.16 & 21.6 \\ 21.6 & 16 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{vmatrix} 11.89-\lambda & 8.81 \\ 5.08 & 3.76-\lambda \end{vmatrix} = 0 \Rightarrow (11.89-\lambda)(3.76-\lambda) - 8.81 \times 5.08 = 0$$

$$\Rightarrow 44.7064 - 11.89\lambda - 3.76\lambda + \lambda^2 - 44.7548$$

$$\lambda^2 - 15.65\lambda - 0.04 = 0$$

$$\lambda = 15.65$$

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Eigen Vector $= S_w^{-1}[\mu_1 - \mu_2]$

$$= \begin{bmatrix} 0.384 & 0.032 \\ 0.032 & 0.192 \end{bmatrix} \begin{bmatrix} -5.4 \\ -4 \end{bmatrix} = \begin{bmatrix} -2.0736 - 0.032 \\ -0.1728 - 0.768 \end{bmatrix}$$

$$= \begin{bmatrix} -2.2016 \\ -0.9408 \end{bmatrix}$$

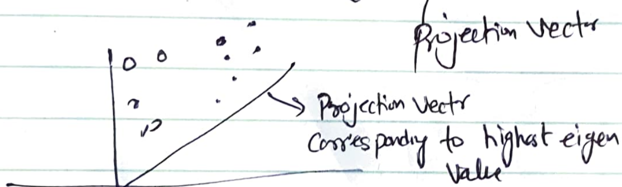
Ste

Step · Dimension Reduction

$$y = w^T x$$

x — Input data Sample

w — Projection Vector



"Children must be taught how to think, not what to think."

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Eigen Vector for $\lambda = \lambda_1 (15.65)$

$$\begin{bmatrix} 11.89-\lambda & 8.81 \\ 5.08 & 3.76-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$(11.89-\lambda)v_1 + 8.81 v_2 = 0$$

$$5.08 v_1 + (3.16-\lambda) v_2 = 0$$

$$\frac{v_1}{-8.81} = \frac{v_2}{11.89-\lambda} = t \qquad U_1 = \begin{bmatrix} -8.81 \\ 11.89-\lambda \end{bmatrix}$$

$$\|U_1\| = \sqrt{(8.81)^2 + \underbrace{(11.89-\lambda)^2}_{(-3.76)^2}} = \sqrt{77.6161 + 14.1376}$$

$$= \sqrt{91.7537} = 9.5788$$

Unit eigen vector $e_i = \frac{U_1}{\|U_1\|} = \begin{bmatrix} -8.81/9.5788 \\ -3.76/9.5788 \end{bmatrix} = \begin{bmatrix} -0.9197 \\ -0.3925 \end{bmatrix} = \begin{bmatrix} 0.91 \\ 0.39 \end{bmatrix}$