CS 4804 Homework 4
Solution Sketches

1. **(10 points)**

   $\exists x : Barber(x) \wedge (\forall y : \neg Shaves(y, y) \wedge Man(y) \Rightarrow Shaves(x, y))$

2. **(10 points)** To prove that an implication $X \Rightarrow Y$ is valid is the same as showing that $X \models Y$. We proceed to do this using resolution refutation (negating Y and resolving with X to get a null clause).

   (a) 1) $(\exists x P(x)) \Rightarrow Q(A)$
   $\equiv \neg(\exists x P(x)) \vee Q(A)$
   $\equiv (\forall x \neg(P(x))) \vee Q(A)$
   $\equiv \neg(P(x)) \vee Q(A)$
   G)$\neg(\forall x P(x) \Rightarrow Q(A))$
   $\equiv \exists x \neg(P(x) \Rightarrow Q(A))$
   $\equiv \exists x \neg(\neg(P(x)) \vee Q(A))$
   $\equiv \exists x P(x) \wedge \neg(Q(A))$
   $\equiv P(B) \wedge \neg(Q(A))$

   Now calling the first goal clause $G1$ and the second $G2$, the resolution proceeds as follows

   ```
   G1
   |--1
   |--G2
   NIL
   ```

   (b) 1) $(\forall x P(x)) \Rightarrow Q(A)$
   $\equiv \neg(\forall x P(x)) \vee Q(A)$
   $\equiv (\exists x \neg(P(x))) \vee Q(A)$
   $\equiv \neg(P(B)) \vee Q(A)$
   G)$\neg(\exists x P(x) \Rightarrow Q(A))$
   $\equiv \forall x \neg(P(x) \Rightarrow Q(A))$
   $\equiv \forall x \neg(\neg(P(x)) \vee Q(A))$
   $\equiv \forall x P(x) \wedge \neg(Q(A))$
   $\equiv P(x) \wedge \neg(Q(A))$

   Now calling the first goal clause $G1$ and the second $G2$, the resolution proceeds as follows

   ```
   G1
   |--1
   |--G2
   NIL
   ```

3. **(20 points)** Predicates:

   $Blue(x)$:= Object X is blue
   $Green(x)$:= Object X is green
   $Pushable(x)$:= Object X is pushable

(a) If pushable objects are blue then non-pushable ones are green.
$(\forall x : Pushable(x) \Rightarrow Blue(x)) \Rightarrow (\forall y : \neg Pushable(y) \Rightarrow Green(y))$
$\equiv \neg(\forall x : \neg Pushable(x) \vee Blue(x)) \vee (\forall y : Pushable(y) \vee Green(y))$
$\equiv (\exists x : Pushable(x) \wedge \neg Blue(x)) \vee Pushable(y) \vee Green(y)$
$\equiv (Pushable(S) \wedge \neg Blue(S)) \vee Pushable(y) \vee Green(y)$
$\equiv (Pushable(S) \vee Pushable(y) \vee Green(y)) \wedge (\neg Blue(S) \vee Pushable(y) \vee Green(y))$

(b) All objects are either blue or green but not both.
$(\forall x : Green(x) \vee Blue(x)) \wedge (\forall x : \neg Blue(x) \vee \neg Green(x))$
$\equiv (Green(x) \vee Blue(x)) \wedge (\neg Blue(x) \vee \neg Green(x))$

(c) If there is a non-pushable object, then all pushable ones are blue
$(\exists x : \neg Pushable(x)) \Rightarrow (\forall y : Pushable(y) \Rightarrow Blue(y))$
$\equiv \neg(\exists x : \neg Pushable(x)) \vee (Pushable(y) \Rightarrow Blue(y))$
$\equiv (\forall x : (Pushable(x))) \vee (\neg Pushable(y) \vee Blue(y))$
$\equiv Pushable(x) \vee \neg Pushable(y) \vee Blue(y)$

(d) Object O1 is pushable
$Pushable(O1)$

(e) Object O2 is non-pushable
$\neg Pushable(O2)$

(f) The negated goal: There is not a green object
$\forall x : \neg Green(x) \equiv \neg Green(x)$

Now calling the first clause of a $A1$ and the second $A2$, and similarly for b, the resolution proceeds as follows

```
F
|--A2
|--B1
|--F
|--E
|
NIL
```

4. **(20 points)**

(a) $\{x/A, y/B, z/B\}$

(b) No unifier

(c) $\{y/John, x/John\}$

(d) No unifier, Occurs check

5. **(40 points)**

For simplicity, only needed statements are shown. Predicate definitions are also omitted and must be obvious from the context.

(a) A professor is either a Assistant, Associate, or Full.
$\forall x : \text{Professor}(x) \Rightarrow \text{Rank}(x, Assistant) \vee \text{Rank}(x, Associate) \vee \text{Rank}(x, Full)$
$\equiv \forall x : \neg\text{Professor}(x) \vee \text{Rank}(x, Assistant) \vee \text{Rank}(x, Associate) \vee \text{Rank}(x, Full)$
$\equiv \neg\text{Professor}(x) \vee \text{Rank}(x, Assistant) \vee \text{Rank}(x, Associate) \vee \text{Rank}(x, Full)$

(b) If someone has a rank, then they are a professor.
$\forall x : (\exists y : \text{Rank}(x, y)) \Rightarrow \text{Professor}(x)$
$\equiv \forall x : \neg(\exists y : \text{Rank}(x, y)) \vee \text{Professor}(x)$
$\equiv \forall x : (\forall y : \neg\text{Rank}(x, y)) \vee \text{Professor}(x)$
$\equiv \neg\text{Rank}(x, y) \vee \text{Professor}(x)$

(c) No one can evaluate a Full professor.
$\forall xy : \text{Rank}(x, Full) \Rightarrow \neg Evaluate(y, x)$
$\equiv \forall xy : \neg\text{Rank}(x, Full) \vee \neg Evaluate(y, x)$
$\equiv \neg\text{Rank}(x, Full) \vee \neg Evaluate(y, x)$

(d) If someone is an Assistant professor, then an Associate professor can evaluate them.
$\forall xy : \text{Rank}(x, Assistant) \wedge \text{Rank}(y, Associate) \Rightarrow \text{Evaluate}(y, x)$
$\equiv \forall xy : \neg\text{Rank}(x, Assistant) \vee \neg\text{Rank}(y, Associate) \vee \text{Evaluate}(y, x)$
$\equiv \neg\text{Rank}(x, Assistant) \vee \neg\text{Rank}(y, Associate) \vee \text{Evaluate}(y, x)$

(e) If someone is an Assistant professor, then a Full professor can evaluate them.
$\forall xy : \text{Rank}(x, Assistant) \wedge \text{Rank}(y, Full) \Rightarrow \text{Evaluate}(y, x)$
$\equiv \forall xy : \neg\text{Rank}(x, Assistant) \vee \neg\text{Rank}(y, Full) \vee \text{Evaluate}(y, x)$
$\equiv \neg\text{Rank}(x, Assistant) \vee \neg\text{Rank}(y, Full) \vee \text{Evaluate}(y, x)$

(f) There are at least two people who can evaluate Dr. Strangelove.
$\exists xy : \text{Evaluate}(x, Strangelove) \wedge \text{Evaluate}(y, Strangelove) \wedge \neg(x = y)$
$\equiv \text{Evaluate}(S1, Strangelove) \wedge \text{Evaluate}(S2, Strangelove) \wedge \neg(S1 = S2)$

i. $\text{Evaluate}(S1, Strangelove)$
ii. $\text{Evaluate}(S2, Strangelove)$
iii. $\neg(S1 = S2)$

(g) There are at most two people who can evaluate Dr. Strangelove.
$\forall xyz : \text{Evaluate}(x, Strangelove) \wedge \text{Evaluate}(y, Strangelove) \wedge \text{Evaluate}(z, Strangelove) \Rightarrow$
$x = y \vee x = z \vee y = z$
$\equiv \forall xyz : \neg(\text{Evaluate}(x, Strangelove) \wedge \text{Evaluate}(y, Strangelove) \wedge \text{Evaluate}(z, Strangelove)) \vee$
$x = y \vee x = z \vee y = z$
$\equiv \forall xyz : \neg\text{Evaluate}(x, Strangelove) \vee \neg\text{Evaluate}(y, Strangelove) \vee \neg\text{Evaluate}(z, Strangelove) \vee$
$x = y \vee x = z \vee y = z$
$\equiv \neg\text{Evaluate}(x, Strangelove) \vee \neg\text{Evaluate}(y, Strangelove) \vee \neg\text{Evaluate}(z, Strangelove) \vee$
$x = y \vee x = z \vee y = z$

(h) Dr. Strangelove is a professor.
$\text{Professor}(Strangelove)$

(i) Dr. Knowitall is an Associate professor.
$\text{Rank}(Knowitall, Associate)$

(j) Dr. Gray is an Full professor.
$\text{Rank}(Gray, Full)$

(k) Dr. White is an Full professor.
Rank($White, Full$)

(l) The goal is the rank that Dr. Strangelove holds.
$\neg(\exists x : \text{Rank}(Strangelove, x))$
$\equiv \neg\text{Rank}(Strangelove, x)$

The refutation then proceeds as follows (the keyword AND is used to denote the resolution at each step):

$\neg\text{Rank}(Strangelove, x)$
AND $\neg\text{Professor}(y) \vee \text{Rank}(y, Assistant) \vee \text{Rank}(y, Associate) \vee \text{Rank}(y, Full)$
$\equiv \neg\text{Professor}(Strangelove) \vee \text{Rank}(Strangelove, Assistant) \vee \text{Rank}(Strangelove, Full)$
AND Professor($Strangelove$)
$\equiv \text{Rank}(Strangelove, Assistant) \vee \text{Rank}(Strangelove, Full)$
AND $\neg\text{Rank}(x, Full) \vee \neg Evaluate(y, x)$
$\equiv \text{Rank}(Strangelove, Assistant) \vee \neg Evaluate(y, Strangelove)$
AND Evaluate($S1, Strangelove$)
$\equiv \text{Rank}(Strangelove, Assistant)$
AND $\neg\text{Rank}(x, Assistant) \vee \neg\text{Rank}(y, Associate) \vee \text{Evaluate}(y, x)$
$\equiv \neg\text{Rank}(y, Associate) \vee \text{Evaluate}(y, Strangelove)$
AND Rank($Knowitall, Associate$)
$\equiv \text{Evaluate}(Knowitall, Strangelove)$.

Save this fact for later and lets continue on a different track. Lets go back to the statement (proven above) that Dr. Strangelove must be an Assistant professor and proceed from there.

Rank($Strangelove, Assistant$)
AND $\neg\text{Rank}(x, Assistant) \vee \neg\text{Rank}(y, Full) \vee \text{Evaluate}(y, x)$
$\equiv \neg\text{Rank}(y, Full) \vee \text{Evaluate}(y, Strangelove)$
AND Rank($Gray, Full$)
$\equiv \text{Evaluate}(Gray, Strangelove)$.

Agan, save this fact for later. Continue:

Rank($Strangelove, Assistant$)
AND $\neg\text{Rank}(x, Assistant) \vee \neg\text{Rank}(y, Full) \vee \text{Evaluate}(y, x)$
$\equiv \neg\text{Rank}(y, Full) \vee \text{Evaluate}(y, Strangelove)$
AND Rank($White, Full$)
$\equiv \text{Evaluate}(White, Strangelove)$.

At this point, we have identified three people who can evaluate Dr. Strangelove. This clearly violates one of the statements given in the problem, so we can aim to arrive at a contradiction.

Evaluate($White, Strangelove$)
$AND \neg\text{Evaluate}(x, Strangelove) \vee \neg\text{Evaluate}(y, Strangelove) \vee \neg\text{Evaluate}(z, Strangelove) \vee x = y \vee x = z \vee y = z$
$\equiv \neg\text{Evaluate}(y, Strangelove) \vee \neg\text{Evaluate}(z, Strangelove) \vee White = y \vee White = z \vee y = z$
AND Evaluate($Gray, Strangelove$)
$\equiv \neg\text{Evaluate}(z, Strangelove) \vee White = Gray \vee White = z \vee Gray = z$
AND Evaluate($Knowitall, Strangelove$)

$\equiv White = Gray \vee White = Knowitall \vee Gray = Knowitall$
AND $\neg(White = Gray)$
$\equiv White = Knowitall \vee Gray = Knowitall$
AND $\neg(White = Knowitall)$
$\equiv Gray = Knowitall$
AND $\neg(Gray = Knowitall)$
$\equiv false$

Since the unification $\{x/Associate\}$ was made for the negated goal statement, Dr. Strangelove is an Associate professor. This proof uses set-of-support resolution, with the initial set consisting of the negated goal. Linear resolution could also be used as it is complete.