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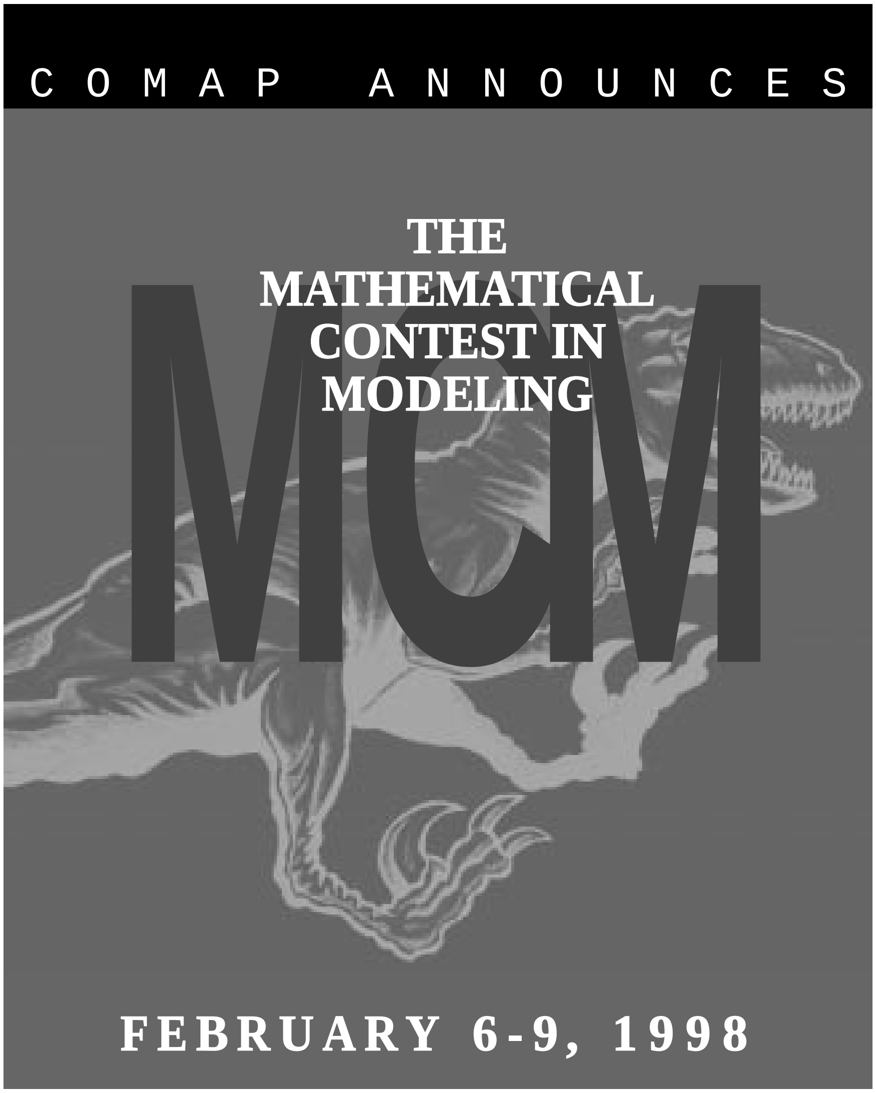

The fourteenth annual international Mathematical Contest in Modeling will be held February 6-9, 1998. The contest will offer students the opportunity to compete in a team setting, using mathematics to solve real-world problems.

For registration information, contact:
Attn: Clarice Callahan
MCM, COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA 02173
email: mcm@comap.com   voice: 781/862-7878 ext. 37

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Additional support for this project is provided by the Institute for Operations Research and the Management Sciences, the Society for Industrial and Applied Mathematics, and the Mathematical Association of America.

Publisher's Editorial

Full Plate

Solomon A. Garfunkel
Executive Director
COMAP, Inc.
57 Bedford St., Suite 210
Lexington, MA 02173
s.garfunkel@mail.comap.com

ARISE: A New High-School Curriculum

This is an incredibly exciting time for COMAP. The first in our series of ARISE Project texts has come out: *Mathematics: Modeling Our World*——Course 1, published by South-Western. Course 2 will be available in early March and Course 3 by next summer. Each course has a student text, an annotated teacher's edition, a teacher's resource guide, a solutions manual, a CD–ROM with all of our calculator and computer software, and a videotape containing introductory segments for each chapter. These texts represent the culmination of over five years of effort by project staff and by our author and field-test teams.

With the launching of this new comprehensive secondary-school curriculum, our work has just begun. Now is the time to spread the word——leadership institutes, presentations at regional and national meetings, and teacher training sessions. A great deal of our energies over the coming months and years will be devoted to putting the show on the road. For COMAP this represents new territory. We cannot rely on the slogan from *Field of Dreams*, "If you build it, they will come." We need to have a presence in the community, showing our work and explaining the goals that we are trying to attain.

ARISE began as a standards-based curriculum, part of the larger reform movement to change the content, applications, pedagogy, technology, and assessment of high-school mathematics. But what has emerged in ARISE, perhaps not surprisingly, is an *integrated curriculum built around mathematical modeling*. As with all of COMAP's materials, ARISE is a rigorous program; and also as with all of our materials, the applications and models are real, for both students and teachers.

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There has been a great deal written of late about "fuzzy math," "whole math," "politically correct math," etc. The notion that critics have put forth is that by using technology and centering on constructivist theories of learning, reformers have short-changed basic skills; that in the name of conceptual understanding for all, the curriculum is being watered down. Nonsense!

I cannot speak for all of the new curricula being written, but *Mathematics: Modeling Our World* is a great deal more rigorous and richer than the standard Algebra 1—Geometry—Algebra 2 sequence that schools now use. I honestly believe that if students entered college or the world of work having mastered these new ARISE materials, there would be reason to celebrate. One goal of ARISE has always been to increase the mathematical understanding of as large a percentage of students as possible; we believe (and have early evidence to show) that this can be achieved without diminishing the level and rigor of the curriculum.

As you can tell from the above remarks, it is difficult to work this hard and this long on a project without believing in it completely. This is an occupational hazard of working in educational reform. But all proselytizing aside, we hope that everyone will have a chance to look at these materials and give them a fair trial for what we have achieved or, at least, hoped to achieve.

HiMCM: An MCM for High Schools

There are two new COMAP projects underway that I want to describe. The first of these, called HiMCM, is under the direction of Frank Giordano, the retired chair of the U.S. Military Academy at West Point Mathematics Dept. who is now (I'm happy to say) at COMAP. HiMCM is a three-year grant from NSF to plan and test a high school version of the Mathematical Contest in Modeling. We are working closely with all of the major professional societies, with the goal of having them cooperate on running the formal contest nationwide once we have developed a model that we all feel works well. Unlike the undergraduate contest, as you might imagine, there are a number of issues of access (to technology, libraries, classrooms, etc.) and equity at the secondary level which need to be worked out. We expect to have a planning year and then run one or more versions of the contest with a set of test schools to refine our plans. We are tremendously excited by the opportunity. We feel that the undergraduate MCM, now in its 14th year, has been one of our most successful and influential programs. We look forward to having a similar influence on the secondary school community.

MathServe: Sharing Mathematical Expertise

The second project that I am pleased to announce is MathServe, under the direction of Uri Treisman (Dana Center, University of Texas at Austin), Frank Giordano, and myself. This program has been funded for three years by the Alfred P. Sloan Foundation. The purpose of MathServe is to promote discipline-based service to the community by mathematics faculty and students, primarily at the undergraduate level. Service frequently has meant helping to paint a community center or volunteering to head a Girl Scout troop. But mathematics as a discipline can and should be used in community service programs—for example, helping to plan an efficient school bus route or to devise an inventory program for a local hospital. MathServe will try to connect the service and volunteer community with mathematics departments. The centerpiece of the program will be an annual set of awards for the most effective projects, along with a special publication (perhaps an additional issue of *The UMAP Journal*) containing descriptions of those judged to be outstanding.

As you can see, COMAP has a pretty full plate. We are extremely gratified to be able to work on so many exciting ideas. And as always, we are most grateful to all of you who continue to work with us to improve the teaching and learning of mathematics at all educational levels.

About the Author

Sol Garfunkel received his Ph.D. in mathematical logic from the University of Wisconsin in 1967. He was at Cornell University and at the University of Connecticut at Storrs for eleven years and has dedicated the last 20 years to research and development efforts in mathematics education. He has been the Executive Director of COMAP since its inception in 1980.

He has directed a wide variety of projects, including UMAP (Undergraduate Mathematics and Its Applications Project), which led to the founding of this *Journal*, and HiMAP (High School Mathematics and Its Applications Project), both funded by the NSF. For Annenberg/CPB, he directed three telecourse projects: *For All Practical Purposes* (in which he appeared as the on-camera host), *Against All Odds: Inside Statistics*, and *In Simplest Terms: College Algebra*. He is currently co-director of the Applications Reform in Secondary Education (ARISE) project, a comprehensive curriculum development project for secondary school mathematics.

Reminder from the Editor

Through August 1998, manuscripts and editorial correspondence should go to:

Paul J. Campbell
c/o Lst. Prof. Pukelsheim
Institut für Mathematik der Universität Augsburg
Universitätstrasse 14
D–86135 Augsburg
Germany
voice: 011-49-821-598-2162 fax: 011-49-821-598-2280
email: `campbell@math.uni-augsburg.de`
www: `http://cs.beloit.edu/campbell/`

Modeling Forum

Results of the 1997 Mathematical Contest in Modeling

Frank Giordano, MCM Director
COMAP, Inc.
57 Bedford St., Suite 210
Lexington, MA 02173
f.giordano@mail.comap.com

Introduction

A total of 409 teams of undergraduates, from 226 schools, spent the second weekend in February working on applied mathematics problems. They were part of the twelfth Mathematical Contest in Modeling (MCM). On Friday morning, the MCM faculty advisor opened a packet and presented each team of three students with a choice of one of two problems. After a weekend of hard work, typed solution papers were mailed to COMAP on Monday. Nine of the top papers appear in this issue of *The UMAP Journal*.

Results and winning papers from the first twelve contests were published in special issues of *Mathematical Modeling* (1985–1987) and *The UMAP Journal* (1985–1996). The 1994 volume of *Tools for Teaching*, commemorating the tenth anniversary of the contest, contains all of the 20 problems used in the first ten years of the contest and a winning paper for each. Limited quantities of that volume and of the special MCM issues of the *Journal* for the last few years are available from COMAP.

Problem A: The Velociraptor Problem

The velociraptor, *Velociraptor mongoliensis*, was a predatory dinosaur that lived during the late Cretaceous period, approximately 75 million years ago. Paleontologists think that it was a very tenacious hunter and may have hunted

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in pairs or larger packs. Unfortunately, there is no way to observe its hunting behavior in the wild, as can be done with modern mammalian predators. A group of paleontologists has approached your team and asked for help in modeling the hunting behavior of the velociraptor. They hope to compare your results with field data reported by biologists studying the behaviors of lions, tigers, and similar predatory animals.

The average adult velociraptor was 3 m long with a hip height of 0.5 m and an approximate mass of 45 kg. It is estimated that the animal could run extremely fast, at speeds of 60 km / hr, for about 15 sec. After the initial burst of speed, the animal needed to stop and recover from a buildup of lactic acid in its muscles.

Suppose that velociraptor preyed on *Thescelosaurus neglectus*, a herbivorous biped approximately the same size as the velociraptor. A biomechanical analysis of a fossilized thescelosaurus indicates that it could run at a speed of about 50 km / hr for long periods of time.

Part 1

Assuming the velociraptor is a solitary hunter, design a mathematical model that describes a hunting strategy for a single velociraptor stalking and chasing a single thescelosaurus as well as the evasive strategy of the prey. Assume that the thescelosaurus can always detect the velociraptor when it comes within 15 m, but may detect the predator at even greater ranges (up to 50 m) depending upon the habitat and weather conditions. Additionally, due to its physical structure and strength, the velociraptor has a limited turning radius when running at full speed. This radius is estimated to be three times the animal's hip height. On the other hand, the thescelosaurus is extremely agile and has a turning radius of 0.5 m.

Part 2

Assuming more realistically that the velociraptor hunted in pairs, design a new model that describes a hunting strategy for two velociraptors stalking and chasing a single thescelosaurus as well as the evasive strategy of the prey. Use the other assumptions and limitations given in Part 1.

Problem B: Mix Well For Fruitful Discussions

Small group meetings for the discussion of important issues, particularly long-range planning, are gaining popularity. It is believed that large groups discourage productive discussion and that a dominant personality will usually control and direct the discussion. Thus, in corporate board meetings, the board will meet in small groups to discuss issues before meeting as a whole. These smaller groups still run the risk of control by a dominant personality. In an attempt to reduce this danger, it is common to schedule several sessions with a different mix of people in each group.

A meeting of An Tostal Corporation will be attended by 29 board members of which nine are in-house members (i.e., corporate employees). The meeting is to be an all-day affair with three sessions scheduled for the morning and four for the afternoon. Each session will take 45 minutes, beginning on the hour from 9:00 A.M. to 4:00 P.M., with lunch scheduled at noon. Each morning session will consist of six discussion groups with each discussion group led by one of the corporation's six senior officers. None of these officers is a board member. Thus, each senior officer will lead three different discussion groups. The senior officers will not be involved in the afternoon sessions, and each of these sessions will consist of only four different discussion groups.

The president of the corporation wants a list of board-member assignments to discussion groups for each of the seven sessions. The assignments should achieve as much of a mix of the members as possible. The ideal assignment would have each board member with each other board member in a discussion group the same number of times while minimizing common membership of groups for the different sessions. The assignments should also satisfy the following criteria:

1. For the morning sessions, no board member should be in the same senior officer's discussion group twice.

2. No discussion group should contain a disproportionate number of in-house members.

Give a list of assignments for members 1–9 and 10–29 and officers 1–6. Indicate how well the criteria in the previous paragraphs are met. Since it is possible that some board members will cancel at the last minute or that some not scheduled will show up, an algorithm that the secretary could use to adjust the assignments with an hour's notice would be appreciated. It would be ideal if the algorithm could also be used to make assignments for future meetings involving different levels of participation for each type of attendee.

The Results

The solution papers were coded at COMAP headquarters so that names and affiliations of the authors would be unknown to the judges. Each paper was then read preliminarily by two "triage" judges at Southern Connecticut State University (Problem A) or at Carroll College, Montana (Problem B). At the triage stage, the summary and overall organization are the basis for judging a paper. If the judges' scores diverged for a paper, the judges conferred; if they still did not agree on a score, a third judge evaluated the paper.

Final judging took place at Harvey Mudd College, Claremont, California. The judges classified the papers as follows:

	Outstanding	Meritorious	Honorable Mention	Successful Participation	Total
Velociraptor	5	37	58	134	234
Discussion Groups	4	25	43	103	175
	9	62	101	237	409

The nine papers that the judges designated as Outstanding appear in this special issue of *The UMAP Journal*, together with commentaries. We list those teams and the Meritorious teams (and advisors) below; the list of all participating schools, advisors, and results is in the **Appendix**.

Outstanding Teams

Institution and Advisor	**Team Members**
Velociraptor Papers	
"Pursuit–Evasion Games in the Late Cretaceous"	
Calvin College	Edward L. Hamilton
Grand Rapids, MI	Shawn A. Menninga
Gary W. Talsma	David Tong
"The Geometry and the Game Theory of Chases"	
Harvard University	Charlene S. Ahn
Cambridge, MA	Edward Boas
Howard Georgi	Benjamin Rahn
"Gone Huntin': Modeling Optimal Predator and Prey Strategies"	
Pomona College	Hei (Celia) Chan
Claremont, CA	Robert A. Moody
Richard Elderkin	David Young
"Lunch on the Run"	
University of Alaska Fairbanks	Gordon Bower
Fairbanks, AK	Orion Lawler
John P. Lambert	James Long
"A Three-Phase Model for Predator–Prey Analysis"	
Washington University	Lance Finney
St. Louis, MO	Jade Vinson
Hiro Mukai	Derek Zaba

Discussion Groups Papers

"An Assignment Model for
Fruitful Discussions"
East China Univ. of Science and Technology
Shanghai, China
Lu Xiwen

Han Cao
Hui Yang
Zheng Shi

"Using Simulated Annealing to Solve
the Discussion Groups Problem"
Macalester College
St. Paul, MN
Karla V. Ballman

David Castro
John Renze
Nicholas Weininger

"Meetings, Bloody Meetings!"
Rose-Hulman Institute of Technology
Terre Haute, IN
Aaron D. Klebanoff

Joshua M. Horstman
Jamie Kawabata
James C. Moore, IV

"A Greedy Algorithm for Solving
Meeting Mixing Problems"
University of Toronto
Toronto, Ontario, Canada
Nicholas A. Derzko

Adrian Corduneanu
Cyrus C. Hsia
Ryan O'Donnell

Meritorious Teams

Velociraptor Papers (37 teams)
Beijing Univ. of Aeronautics and Astronautics, Beijing, China (Li Weiguo)
Brandon University, Brandon, Manitoba, Canada (Doug Pickering)
California Polytechnic State Univ., San Luis Obispo, CA (Thomas O'Neil) (two teams)
Dalian University of Technology, Dalian, Liaoning, China (He Ming-Feng)
Duke University, Durham, NC (David P. Kraines)
East China Univ. of Science and Technology, Shanghai, China (Shao Nianci)
Experimental High School of Beijing, China (Zhang Jilin)
Goucher College, Baltimore, MD (Megan Deeney)
Harvey Mudd College, Claremont, CA (David L. Bosley)
Hebei Institute of Technology, Tangshan, Hegei, China (Wan Xinghuo)
Hope College, Holland, MI (Ronald Van Iwaarden)
Macalester College, St. Paul, MN, (Daniel A. Schwalbe)
N.C. School of Science and Mathematics, Durham, NC (Dot Doyle)
Nankai University, Tianjin, China (Ruan Jishou)
National Univ. of Defence Technology, Chang Sha, Hunan, China (Cheng LiZhi)
North Carolina State University, Raleigh, NC (Robert T. Ramsay)
Rose-Hulman Inst. of Technology, Terre Haute, IN (George Berzsenyi)
Seattle Pacific University, Seattle, WA (Steven D. Johnson)

Southeast University, Nanjing, Jiangsu, China (Xu Liang)
Swarthmore College, Swarthmore, PA (Stephen B. Maurer)
Trinity University, San Antonio, TX (Diane G. Saphire)
Tsinghua University, Beijing, China (Wang Siqun)
United States Air Force Academy, USAF Academy, CO (Scott G. Frickenstein)
Univ. of Science and Technology of China, Hefei, Anhui, China, (Yu Feng)
Univ. of Science and Technology of China, Hefei, Anhui, China (Yu Tianyue)
University of Colorado, Boulder, CO (Anne M. Dougherty)
University of Puget Sound, Tacoma, WA (Robert A. Beezer)
University of Saskatchewan, Saskatoon, Canada (Raj Srinivasan)
Wake Forest University, Winston-Salem, NC (Stephen B. Robinson)
Wake Forest University, Winston-Salem, NC (Edward Allen)
Washington University, St. Louis, MO (Hiro Mukai)
Western Washington University, Bellingham, WA (Igor Averbakh)
Western Washington University, Bellingham, WA (Saim Ural)
Wuhan Univ. of Hydraulics and Engineering, Wuhan, Hubei, China (Peng Zhuzeng)
Youngstown State University, Youngstown, OH (J. Douglas Faires)
Zhejiang University, Hangzhou, China (Fang Daoyuan)

Discussion Groups Papers (25 teams)
Colorado College, Colorado Springs, CO (Deborah P. Levinson)
David Lipscomb University, Nashville, TN (Gary C. Hall)
Eastern Mennonite University, Harrisonburg, VA (John L. Horst)
Eastern Oregon State College, LaGrande, OR (Holly S. Zullo)
Eastern Oregon State College, LaGrande, OR (Mark R. Parker)
Gettysburg College, Gettysburg, PA (James P. Fink)
Graceland College, Lamoni, IA (Ronald K. Smith)
Grinnell College, Grinnell, IA (Thomas L. Moore)
Harvey Mudd College, Claremont, CA (David L. Bosley)
Hebei Institute of Technology, Tangshan, Hegei, China (Liu Baoxiang)
Hiram College, Hiram, OH (Larry Becker)
Ithaca College, Ithaca, NY (James E. Conklin)
Kenyon College, Gambier, OH (Brian D. Jones)
National Univ. of Defence Technology, Chang Sha, Hunan, China (Wu MengDa)
Peking University, Beijing, China (Huang Sheng)
South China Univ. of Technology, Guangzhou, China (Hao Zhifeng)
Southeast University, Nanjing, Jiangsu, China (Zhu Dao-yuan)
Trinity College Dublin, Dublin, Ireland (Timothy G. Murphy)
Univ. of Northern Colorado, Greeley, CO (William W. Bosch)
University College Cork, Cork, Ireland (Martin Stynes)
University College Cork, Cork, Ireland (Gareth Thomas)
University of Colorado, Boulder, CO (Bengt Fornberg)
University of Richmond, Richmond, VA (James Davis)
University of Southern Queensland, Toowoomba, Queensland, Australia
(Christopher J. Harman)
Xidian University, Xian, Shaanxi, China (Mao Yong-cai)

Awards and Contributions

Each participating MCM advisor and team member received a certificate signed by the Contest Director and the appropriate Head Judge.

INFORMS, the Institute for Operations Research and the Management Sciences, awarded to each member of two Outstanding teams a cash award and a three-year membership. The teams were from Calvin College (Velociraptor Problem) and Rose-Hulman Institute of Technology (Discussion Groups Problem). Moreover, INFORMS gave free one-year memberships to all members of Meritorious and Honorable Mention teams.

The Society for Industrial and Applied Mathematics (SIAM) designated one Outstanding team from each problem as a SIAM Winner. Each team member received a cash prize. The teams were from Washington University (Velociraptor Problem) and from University of Toronto (Discussion Groups Problem). Both teams presented their results at a special Minisymposium at the SIAM Annual Meeting at Stanford University in July.

The Mathematical Association of America (MAA) designated one Outstanding team from each problem as an MAA Winner. The teams were from from Harvard University (Velociraptor Problem) and Macalester College (Discussion Groups Problem). The Macalester team gave a presentation at a special session of MAA Mathfest in Atlanta, GA, in August.

Judging

Director
Frank R. Giordano, COMAP, Lexington, MA

Associate Directors
Chris Arney, Dept. of Mathematical Sciences, U.S. Military Academy, West Point, NY
Robert L. Borrelli, Mathematics Dept., Harvey Mudd College, Claremont, CA
William Fox, Dept. of Mathematical Sciences, U.S. Military Academy, West Point, NY

Velociraptor Problem

Head Judge
Marvin S. Keener, Mathematics Dept., Oklahoma State University, Stillwater, OK

Associate Judges
James Case, Baltimore, Maryland
Alessandra Chiareli, Computational Science Center, 3M, St. Paul, MN

Courtney Coleman, Mathematics Dept., Harvey Mudd College,
Claremont, CA
Patrick Driscoll, Dept. of Mathematical Sciences, U.S. Military Academy,
West Point, NY
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Florida State University, Tallahassee, FL
Mario Juncosa, RAND Corporation, Santa Monica, CA
California State University Los Angeles, Los Angeles, CA
Mark Levinson, Edmonds, WA
Rick Mabry, Wolfram Research, Inc., Champaign, IL
Keith Miller, National Security Agency, Fort Meade, MD
Mike Moody, Mathematics Dept., Harvey Mudd College, Claremont, CA
Peter Olsen, National Security Agency, Fort Meade, MD
Jack Robertson, Mathematics Dept., Georgia College, Milledgeville, GA
John L. Scharf, Carroll College, Helena, MT
Lee Seitelman, Glastonbury, CT
Theodore H. Sweetser III, Jet Propulsion Lab, Pasadena, CA
Robert M. Tardiff, Dept. of Mathematical Sciences,
Salisbury State University, Salisbury, MD
Beverly West, Cornell University, Ithaca, NY
Daniel Zwillinger, Zwillinger & Associates, Arlington, MA

Discussion Groups Problem

Head Judge
Maynard Thompson, Mathematics Dept., University of Indiana,
Bloomington, IN

Associate Judges
Victor Adamchik, Wolfram Research, Inc., Champaign, IL
Karen Bolinger, Mathematics Dept., Arkansas State University,
State University, AR
Jerry Griggs, University of South Carolina, Columbia, SC
John Kobza, Virginia Polytechnic Institute and State University,
Blacksburg, VA
Daphne Liu, Dept. of Mathematics and Computer Science,
California State University Los Angeles, Los Angeles, CA
Vijay Mehrotra, Onward Inc., Mountain View, CA
Veena Mendiratta, Lucent Technologies, Naperville, IL
Don Miller, Dept. of Mathematics, St. Mary's College, Notre Dame, IN
Cathy Roberts, Northern Arizona University, Flagstaff, AZ
Kathleen M. Shannon, Salisbury State University, Salisbury, MD
Michael Tortorella, Lucent Technologies, NJ
Marie Vanisko, Carroll College, Helena, MT

Triage Session

Velociraptor Problem

Head Triage Judge
Theresa M. Sandifer, Southern Connecticut State University, New Haven, CT

Associate Judges
Therese Bennett, Southern Connecticut State University, New Haven, CT
Susanna D. Fishel, Southern Connecticut State University, New Haven, CT
Ross B. Gingrich, Southern Connecticut State University, New Haven, CT
Cynthia B. Gubitose, Western Connecticut State University, Danbury, CT
C. Edward Sandifer, Western Connecticut State University, Danbury, CT
Xiaodi Wang, Western Connecticut State University, Danbury, CT

Discussion Groups Problem
(all judges from Carroll College, Helena, MT)

Head Triage Judge
Marie Vanisko

Associate Judges
Peter Biskis
Terence Mullen
Jack Oberweiser
Philip Rose

Sources of the Problems

The Velociraptor Problem was contributed by Jack Robertson, Mathematics Dept., and William Wall, Dept. of Biological and Environmental Sciences, both of Georgia College, Milledgeville, GA. The Discussion Groups Problem was contributed by Don Miller, Dept. of Mathematics and Computer Science, St. Mary's College, Notre Dame, IN.

Acknowledgments

The MCM was funded this year by the National Security Agency, whose support we deeply appreciate. We thank Dr. Gene Berg of NSA for his coordinating efforts. The MCM is also indebted to INFORMS, SIAM, and the MAA, which provided judges and prizes.

I thank the MCM judges and MCM Board members for their valuable and unflagging efforts. Harvey Mudd College, its Mathematics Dept. staff, and Prof. Borrelli were gracious hosts to the judges.

Cautions

To the reader of research journals:

Usually a published paper has been presented to an audience, shown to colleagues, rewritten, checked by referees, revised, and edited by a journal editor. Each of the student papers here is the result of undergraduates working on a problem over a weekend; allowing substantial revision by the authors could give a false impression of accomplishment. So these papers are essentially au naturel. Light editing has taken place: minor errors have been corrected, wording has been altered for clarity or economy, and style has been adjusted to that of *The UMAP Journal*. Please peruse these student efforts in that context.

To the potential MCM Advisor:

It might be overpowering to encounter such output from a weekend of work by a small team of undergraduates, but these solution papers are highly atypical. A team that prepares and participates will have an enriching learning experience, independent of what any other team does.

Appendix: Successful Participants

KEY:

P = Successful Participation
H = Honorable Mention
M = Meritorious
O = Outstanding (published in this special issue)
A = Velociraptor Problem
B = Discussion Groups Problem

INSTITUTION	CITY	ADVISOR	A	B
ALABAMA				
University of Alabama	Huntsville	Boris Kunin		P
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University of Arizona	Tucson	Bruce J. Bayly	H	
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	Northridge	Gholam-Ali Zakeri	P	
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Loyola Marymount Univ.	Los Angeles	Thomas M. Zachariah	P	
Pomona College	Claremont	Richard Elderkin	O	P
Sonoma State University	Rohnert Park	Clement E. Falbo		P
COLORADO				
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Trinidad State Jr. College	Trinidad	George E. Leone	P	
		Robert A. Philbin	P	
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		Harry N. Newton		H
		Jonathan D. Robinson		H
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		Bengt Fornberg		M
U. of Northern Colorado	Greeley	William W. Bosch		M,P
U. of Southern Colorado	Pueblo	Bruce N. Lundberg	P	

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University of Bridgeport	Bridgeport	Natalia Romalis		P
Western Conn. State U.	Danbury	Edward Sandifer		H
DISTRICT OF COLUMBIA				
Georgetown University	Washington	Andrew Vogt	P	P
George Washington Univ.	Washington	Daniel H. Ullman	P	P
Trinity College	Washington	Suzanne E. Sands	P	P
FLORIDA				
Florida Southern College	Lakeland	William G. Albrecht	P	
Florida State University	Tallahassee	Hong Wen	H	
Jacksonville University	Jacksonville	Robert A. Hollister	H	P
		D. Neal Boehmke	P	
Stetson University	Deland	Lisa O. Coulter	P	P
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Pursuit–Evasion Games in the Late Cretaceous

Edward L. Hamilton
Shawn A. Menninga
David Tong
Calvin College
Grand Rapids, MI 49546
{ ehamil28, smenni23, dtong23 } @calvin.edu

Advisor: Gary W. Talsma

Summary

Using techniques from differential game theory, we model the velociraptor hunting problem by means of a semi-discrete computer algorithm.

By defining predator and prey behaviors in terms of simple, intuitive principles, we identify a set of strategies designed to counter one another, such that no one pure predator strategy or prey strategy defines an optimal behavior pattern. Instead, the ideal strategy switches between two or more pure strategies, in an essentially unpredictable, or protean, manner. The resulting optimum behaviors show a mixture of feints, bluffs, and true turns for the thescelosaurs, and a mixture of predictive interception and simple pursuit for the velociraptors.

Finally, using these strategies, we demonstrate a conclusive advantage for velociraptors hunting in pairs over velociraptors hunting in isolation.

Introduction

We describe hunting strategies for a velociraptor and flight strategies for its prey using a computational semi-discrete representation of a differential game of pursuit and evasion.

First, we review the formalism of traditional non-differential game theory and the extension of its principles into the analysis of differential systems, taking careful note of the unique aspects of the velociraptor problem.

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Second, we propose a minimal set of assumptions required to reduce the analytical game to a numerical time-iterated computer algorithm, and submit a small set of intuitively simple strategies for each participant to execute.

Third, we examine the implications of both the assumptions of perfect and imperfect information for the optimization of strategies for a single pursuer and single evader and then extend these conclusions to more pursuers.

Finally, we comment on the inherent limitations of this model and offer our evaluation. We conclude that for the game of imperfect information there exists no pure strategy for the prey, but that the best alternative is to behave in an unpredictable fashion with respect to the alternation of turns and feints. The predator, however, has a clearly dominant strategy of compromising between a predictive algorithm and simple pursuit.

Mathematical Formalization in Game-Theoretic Terms

Contests of interception and avoidance between a predator and its prey fall into the broad and diverse category of linear differential games. In a differential game of pursuit and evasion, two or more opposing players attempt either to maximize or to minimize their separation, subject to certain limitations on their motion.

Unlike the games of classical game theory, differential games involve the application of methodology from differential equations and make use of continuous fluctuations that define the states and objectives of the players.

In either a traditional or differential game, players seek to maximize some value, known as the *payoff function*, by selecting among a set of alternatives, with the payoff to each player determined by some function of all the players' choices. In a zero-sum game, all players are in direct competition with one another, such that the objectives expressed by their payoff functions are exactly opposite. Games of pursuit and evasion fall within this classification; the pursuer seeks to minimize the distance between itself and the evader, and the evader seeks to maximize this distance.

In the case of the velociraptor–thescelosaur pursuit–evasion game, the payoff function is a simple binary function: The only relevant outcomes are capture or escape. Such games are referred to as *games of kind*, as opposed to *games of degree*, in which the payoff can take on a larger set of possible values. In some cases, it may be helpful to consider a game of kind as being embedded in a game of degree, with the time to capture (or the separation at evasion) as the payoff function. Although this in not necessary in purely deterministic trials, the average time to capture for a particular strategy pair may provide a useful statistical measure of the efficacy of a strategy.

There are two classes of variables in a differential game: state variables, which specify the complete configuration of the entire system at any given point in time, and control variables, which participants in the game use to alter the state function in ways favorable to their attempts to maximize their own payoff function. Common state variables in traditional pursuit–evasion games are the spatial coordinates of the participants; control variables may include an angle of maximum turn, or acceleration vectors.

Games in which all players have access to a complete set of state variables at any given point in time are known as games of *perfect information*; games in which this is not true are referred to as games of *imperfect information*. Typically, games of imperfect information lack exact analytical methods of solution. Thus, one of the best methods of evaluating such games is to use a discrete model, which can be implemented in terms of a simple computational algorithm. Unfortunately, purely discrete models often run the risk of obscuring essential details of the system that may depend on continuity of values without quantization. A reasonable compromise that still permits a computational method of solution is a semi-discrete method, in which time is iterated discretely, but spatial values are allowed to extend over the entire domain of real numbers [Isaacs 1967, 42]. We chose to implement the raptor hunting game as a semi-discrete computational algorithm.

The velociraptor problem embodies one of the most interesting situations in a two-player pursuit–evasion game. If the maximum velocity of the evader is greater than the maximum velocity of the pursuer, then the evader has an optimal strategy of moving directly away from the pursuer at maximal velocity and will always successfully evade capture. Similarly, if the pursuer is superior in both speed and maneuverability, then the pursuer has an optimal strategy of moving directly toward the evader and is guaranteed of a successful capture. The case in which the pursuer is swifter while the evader is more agile, however, has no trivial solution and may require complex or nondeterministic strategies for optimal play.

Assumptions and Development of the Model

Initial Configuration

Prey animals generally ignore predators until they have moved within a well-defined flight radius; when the distance between themselves and a predator falls below this value, a flight response is triggered. We are given that the flight distance of a thescelosaurus is less than 50 m and greater than 15 m, such that the moment a velociraptor is detected within this distance, the thescelosaur will immediately flee. Similarly, we assume that the raptor has been deliberately stalking the thescelosaur with the intent of capturing it; thus, the predator will be in a set crouch position and will pursue at the first sign of flight. Although the thescelosaur may be startled or surprised, it initiates the flight and chase sequence, and thus both it and the raptor begin moving (nearly) simultaneously.

Our model allows us to specify the initial state either through selection of locations and facings for each predator and prey, or by implementing a simple probabilistic stalking model to determine the initial separation. In most cases, we set the starting separation of the predator and prey close to the minimum possible value, given to be 15 m. This is because, for most large separations, the optimal strategy of the prey is to move directly away from the predator at maximum speed until the predator is very close. Thus, differences in strategic behavior do not typically become manifest until the prey is forced to deviate from a linear path due to the proximity of the predator.

In the case of multiple predators, we assume either that they have stalked out nearly opposite sides of a circle of radius 15 m or that they start from approximately the same position.

Sensory Acuity

For the simplest level of approximation, it is sufficient to assume that all participants in a pursuit–evasion game have complete and instantaneous access to the state function for all times, such that they are involved in a game of perfect information. However, this is a relatively inaccurate assumption for most real physical systems. The ability to estimate distances and directions is always subject to random error, and vision is limited to a field of sight subtending an angle of less than 180°. Being forced to rely on other senses, usually hearing, to track a moving object is less than ideal, and can lead to sizable error in the assessment of distances in particular. These limitations are particularly problematic for predators, who have a narrowly focused forward field of vision and depend heavily on being able to estimate not only the present but the future locations of their prey.

Our model functions under assumptions of perfect information but with the introduction of both random and systematic error in sensory perception. The former is enforced by multiplying the magnitude and angle of the displacement vector between predator and prey each by a different random number between 0.95 and 1.05 before passing it to the routine governing the selection of control variables. Additionally, uncertainty in these equations due to the limited field of sight (systematic errors) are estimated using a linearly increasing statistical spread proportional to the angular displacement between the direction of sight and the observed object. The distance and angle are each multiplied by a different random number in the range $1.00 \pm s\theta/\pi$, where $s = 0.05$ for angular displacement and 0.25 for linear displacement. Finally, to emphasize the importance of visual contact, these factors are further increased for the predator by 50% to $s = 0.075$ and 0.375, respectively.

A related issue is reaction time. In the perfect information case, knowledge of the state vector is imparted instantly. A more realistic assumption is that these data become available for use only after they have been psychologically processed by the brain. Ordinarily, the process of updating awareness of the environment could be considered to be immediate; but in a contest measured in hundredths of a second, to deny the prey the ability to have an advantage over the predator in its knowledge of its own actions would obscure an essential element of the model. To prevent either dinosaur from being able to react instantaneously to changes in its environment, we delay information about the state variables of other dinosaurs by 0.05 s for the raptor and 0.037 s for the thescelosaur.

Physical Constraints on Motion

The center of mass of each of the dinosaurs is assumed to be a point particle moving in a two-dimensional plane in accordance with Newtonian mechanics. Two available options in altering the motion of a point (subject to Newtonian kinematic equations) are to impose either a linear or an angular acceleration.

The data provided by biomechanical analysis indicate that the maximum turn radius of the raptor was 1.5 m and that of the thescelosaur was only 0.5 m. (We understand the problem statement to mean that these values apply at top speed, even though this leads to the thescelosaur being capable of a 32-g turn.) This suggests, in each case, that a maximum possible angular displacement in any given time interval may be defined as $d\theta = a(dt)/v$, where v is the current velocity, dt is the length of the time interval, and a is the centripetal acceleration, where $a = v^2/r$, and v and r here are the speed and maximum radius of curvature at maximum velocity.

Without associated data on the linear acceleration of the dinosaurs, it is necessary to invoke an argument by analogy. The African cheetah is a modern predator filling an ecological role similar to that of the raptor. Cheetahs also share many of the same strategic attributes with velociraptors (high speed, limited endurance, and a turning radius inferior to that of their primary prey). One might reasonably assume that the linear acceleration capabilities of the raptor would have been similar. A cheetah can accelerate to peak velocity (over 90 km / hr) in about 2 s. However, the velociraptor has a lower maximum speed, and is also lighter than the cheetah. This suggests that the acceleration for a raptor may be somewhat greater. Operating under the assumption that the velociraptor possesses the same relative acceleration ability compared to its body size, and recognizing that the force exerted by muscle tissue is proportional to the square of the linear dimensions of the body, while body mass is equal to the cube, a factor of $(2/3)/(1.25)^{2/3} \approx .57$ is not an unreasonable correction to the acceleration of the lighter raptor (where 1.25 is an approximate ratio of the mass of a cheetah to the mass of a raptor).

Strategy

An animal might be expected to behave in accordance with straightforward heuristic principles, and each of the strategies that we tested reflects such a principle. For the sake of the simulation, we assume each raptor and each thescelosaur must adopt only one strategy at the start of a pursuit–evasion game, although we later consider the advantages that could be gained by switching strategies during the course of a chase.

Predator Strategies

- Strategy Predator-0 (default): Move directly toward the present location of the prey at the maximum possible speed without deceleration (i.e., speed cannot be decreased, even if facing away from the prey).
- Strategy Predator-1 (predictive): Estimate the future position of the prey by assuming that it will continue moving in its current direction at the present velocity, and plot an intercept course.
- Strategy Predator-2 (half-predictive): Take the average of the angles indicated by strategy Predator-0 and strategy Predator-1.

Prey Strategies

- Strategy Prey-0 (default): Move directly away from the present location of the predator at the maximum possible speed without deceleration (i.e., speed cannot be decreased, even if facing toward the predator).
- Strategy Prey-1 (constant turn): Similar to Prey-0, but every time the predator comes within 1.5 m, make a sharp 90° turn, and then go straight again.
- Strategy Prey-2 (variable turn): Similar to Prey-1, but instead of turning at 90°, turn constantly at a rate proportional to the distance to the predator.
- Strategy Prey-3 (feint and bluff): Similar to Prey-1, but instead of going on straight after the 90° turn, turn back at maximum rate by 270°.

Capture and Non-Capturability

Conventionally, in analytical studies of differential games of pursuit and evasion, the condition for capture is just the decrease of the distance between the pursuer and evader below some set value, the radius of capture. In our model, we permit capture more realistically by one of two mechanisms.

- If the raptor is within one meter of the thescelosaur, it may attempt to "lunge" and score a critical strike, thrusting the parts of its body (jaws and talons) employed to injure or subdue prey forward with a sudden burst of acceleration. This is represented by a probabilistic capture condition consisting of two factors, one dependent on distance varying linearly from 0.5 to 1.0 m, and one dependent on angle varying linearly from 0 radians to π radians. If a random number generated between 0 and 1 is less than the product of these two factors, then capture occurs.

- Alternatively, if the raptor and the thescelosaur move within 0.5 m of one another, they physically collide, with invariably catastrophic consequences for the prey. We choose not to model the possibility that the raptor may be physically injured in this case.

If capture has not been attained within 15 s, the simulation ends, and the thescelosaur is assumed to escape, as the raptor is forced to stop and rest. Although the raptor may not spend all of the 15 s at top speed, the process of accelerating and decelerating is at least as energetically demanding as running at a constant speed.

The Simulator

The simulator consists of two main portions: the outer loop in charge of updating the game status at each iteration, and the movement generators. These last, one each for the predator(s) and prey, implement the strategies. This is performed in five steps:

1. Data Acquisition Phase: The bearing and distance to each opponent is determined and recorded.
2. Data Manipulation Phase: Each of the above values is randomly permuted to simulate inaccuracies in sensory acuity.
3. Strategizing Phase: Based on the data and the chosen strategy, a "best-case" move is chosen.
4. Limitation Phase: Physical limitations are imposed on the chosen move in accordance with the associated capabilities of the animal.
5. Movement Phase: This final value is passed back to the outer loop for implementation.

If at any point the outer loop determines that capture has occurred, or that the 15-s time limit has expired, the simulation is halted and the final status is reported.

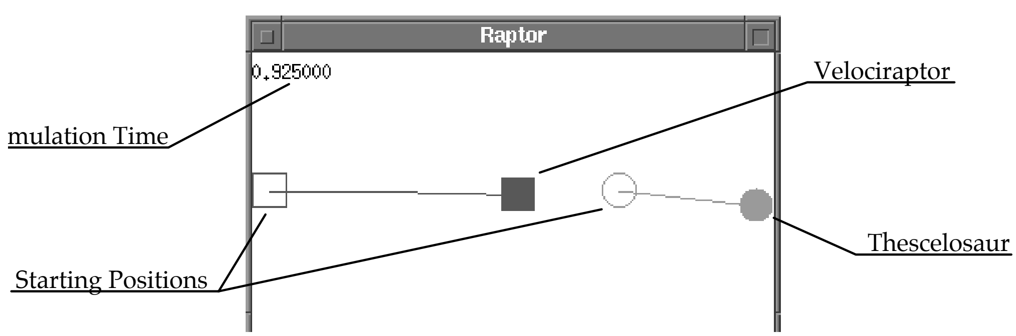

Figure 1. The simulator window.

The simulator was written using ANSI C++ (with the Hewlett-Packard standard template library). The program uses an X Windows (X11R5) display interface to graphically track the positions of the predator(s) and prey. The code was compiled and linked using the freely available C++ compiler from the Gnu Software Foundation and was executed on a Sun Microsystems SparcStation 5 workstation. Source code is available via e-mail to `smenni23@calvin.edu`.

Results

One Raptor

We studied the case of a solitary hunter in detail for both the assumptions of perfect information and for the restrictions on sensory acuity defined above. Under the first assumption, the outcome of every initial position is deterministic and repeatable. Thus every strategy by the predator is either successful, in that it results in the capture of the prey, or unsuccessful, because it does not lead to a capture within the allotted time. This leads to a natural formulation in terms of a binary matrix (**Table 1**) where 1 represents capture and 0 denotes an escape.

Table 1.
Capture in a game of perfect information.

	Prey-0	Prey-1	Prey-2	Prey-3
Predator-0	1	0	0	1
Predator-1	1	1	1	0
Predator-2	1	1	1	1

The intransitivity of the relative ranking of Predator-0 and Predator-1 (or Prey-1, Prey-2, and Prey-3) merits further comment. This corresponds to a situation in which no optimum strategy exists for the predator (or the prey) with respect to the reduced game containing only those choices. Due to the tighter turning radius of the thescelosaur, the only way the raptor can catch it as it moves through its sharp turn is by anticipating its future position. However, this leads to the potential that the thescelosaur may switch to Prey-3, a devious strategy whereby the predator is forced to overcommit as a result of a false turn by the prey, something to which the less sophisticated strategy Predator-0 was immune. The same analysis applies in reverse to the thescelosaurus; for any one strategy she commits to consistently, the raptor can switch to a new strategy capable of beating her every time. Thus, if either player insists on playing with a single deterministic strategy, the other can take advantage of it.

The most obvious option is to switch to a random selection of bluffs and real turns. This would at least guarantee that the other dinosaur could not anticipate the tactic consistently and preemptively account for it. Fortunately, in a game of perfect information, the raptor has another option: Predator-2, a deterministic strategy that combines the other two in a nonrandom way. This same alternative would not be available if the raptor suffered from a reaction time delay, as is the case with the imperfect-information variant described next.

With the introduction of effects resulting in imperfect information, the deterministic outcomes are replaced by probabilistic outcome distributions. Some strategies will still virtually always succeed in defeating others, but in many cases the outcome will be sufficiently random that it can be treated only by statistical methods. To get some idea of the relative probabilities in the game of imperfect information, we have run 10 tests at each pair of strategies, with random variation in the starting configuration. The matrix values now reported are the probability of a kill as determined by the above testing (**Table 2**). These values should be taken as rough approximations only.

Table 2.
Capture in a game of imperfect information.

	Prey-0	Prey-1	Prey-2	Prey-3
Predator-0	100%	30%	100%	70%
Predator-1	10%	70%	90%	10%
Predator-2	100%	90%	90%	30%

Many of the essential features of the game of perfect information remain: Predator-2 is still an excellent strategy, and the intransitivity between Predator-0 and Predator-1 with respect to Prey-0 and Prey-1 is also evident. However, Predator-2 is no longer a strictly dominant strategy, and Prey-2 loses much of its merit. In purely statistical terms, the optimal strategy is not to select any one pattern of behavior, but to mix them randomly. This game in intensive form has no saddle-point in pure strategies. By defining the probabilistic outcome in terms of a payoff function, however, an optimum randomized behavior defining a saddle-point in mixed strategies could probably be found.

Two Raptors

An identical analysis can be carried out for the two-raptor case (**Table 3**).

Table 3.
Capture with two predators.

	Prey-0	Prey-1	Prey-2	Prey-3
Predators 0,0	90%	50%	100%	40%
Predators 1,1	90%	100%	100%	100%
Predators 2,2	100%	100%	100%	70%
Predators 0,1	100%	100%	100%	70%

In all but two cases, the two-raptor strategy is superior to the one-raptor strategy, assuming both raptors behave the same way. If one functions in a predictive fashion (Predator-1), while the other accepts the default strategy (Predator-0), the outcome is remarkably similar to two "half-predictive" (Predator-2) raptors, suggesting that specialization of hunting roles might offer considerable advantages. (Examine, for example, the difference between the outcome of a 0,0 strategy vs. the 0,1 strategy.) These results suggest that velociraptors would have strong incentives to hunt in groups, particularly if less experienced raptors (using Predator-0) could cooperate with more experienced raptors (using Predator-1). The total number of kills by mixing two "0" raptors with two "1" raptors is not only greater than for all four hunting in isolation, it is also greater than the total kills with non-experience-mixed pairings.

Weaknesses

- Because of the lack of direct evidence for the biomechanical properties of dinosaurs, the numerical values for certain parameters, particularly the maximum linear acceleration and the reaction time delay, are matters of speculation, and subject to legitimate dispute. Arguments by analogy to modern predators are of dubious value, and if the intent is to compare the results of the simulation to the experimentally observed behavior of modern mammalian predators, then using such references as empirical parameters could be a source of positive bias.
- Because of the intrinsic limitations of a semi-discrete methodology, we cannot be assured that we have really found an optimal solution, only that a given solution is optimal with respect to the others considered.
- The strategies for the two-raptor case are unrealistic, in that they assume that the thescelosaur completely ignores the more distant of the two raptors at any time.
- The introduction of a reaction time delay substantially complicates the task of the raptor(s), and leaves then vulnerable to deception due to a fairly simple feinting turn. The model lacks the ability to "learn" even repeated behaviors without having a new strategy explicitly designed to counter them incorporated into the text of the program.
- By omitting factors such as terrain, visibility, and obstacles, the model gives considerably more advantage to predators than is the case in nature. This accounts for the unreasonably high capture percentages.

Conclusion

Our model provides conclusive support for the hypothesis that the raptors would be advantaged by hunting in pairs. Moreover, it demonstrates that there is no one optimum pursuit or evasion strategy, and that the flexibility to switch between techniques would serve to favor both the predator and the prey.

Appendix: Typical Simulator Output Illustrating Prey Strategies in Progress

While the predator strategies are fairly obvious, the prey strategies may be more difficult to visualize. The following figures illustrate each of the non-default prey strategies in the best possible situation, that is, against the correspondingly weakest predator strategy. Additionally, **Figure 4** shows the complications arising from imperfect information. Note, especially as compared to **Figure 2**, the thescelosaur's tendency to turn too far and the raptor's delayed reaction to the turn.

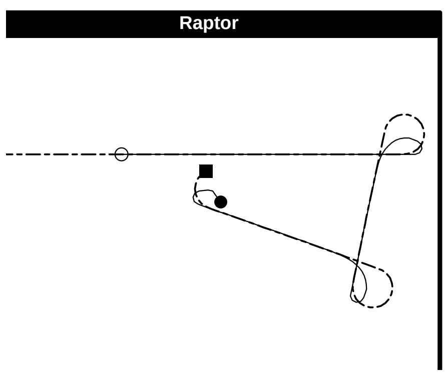

Figure 2. Prey-1 with perfect information.

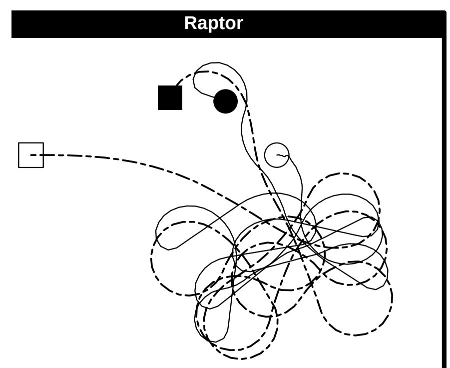

Figure 3. Prey-2 with perfect information.

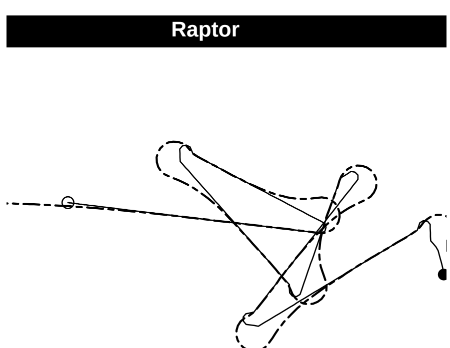

Figure 4. Prey-1 with imperfect information.

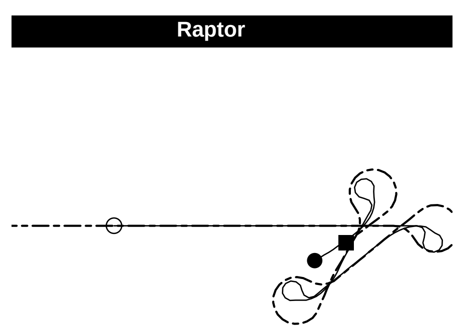

Figure 5. Prey-3 with perfect information.

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The Geometry and the Game Theory of Chases

Charlene S. Ahn
Edward Boas
Benjamin Rahn
Harvard University
Cambridge, MA 02138

Advisor: Howard Georgi

Introduction

We investigate the hunting strategies of predators and the fleeing strategies of their prey in a chase of finite time. For the velociraptor and thescelosaur, there is no overwhelming advantage for either animal; the velociraptor is faster, but the thescelosaur is more agile. We find optimal strategies for both the hunter and the hunted, for a velociraptor hunting a thescelosaur, as well as for a pair of velociraptors hunting a thescelosaur.

This original problem actually has a low probability of having occurred, as fossil remains of the velociraptor have been found only in Mongolia, while fossil remains of the thescelosaur have been found only in the Midwestern region of the United States and Canada [Weishampel et al. 1990, 270, 500]. However, this model can be useful in the study of a wide range of such problems, simply by varying the parameters. In studying these particular creatures, we may come to understand the tradeoff between speed and maneuverability.

Assumptions and Preliminary Calculations

We are given that the velociraptor moves at a speed of v_v = 60 km/h (16.7 m/s), the thescelosaur moves at a speed of of v_t = 50 km/h (13.9 m/s), and the velociraptor's hip height is 0.5 m. It is estimated that a velociraptor's turning radius is three times its hip height; thus, the velociraptor can turn in a minimum radius of r_v = 1.5 m, while the thescelosaur's minimum turning radius is r_t = 0.5 m. We assume that both always find it more advantageous

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to turn with this wide radius rather than to decelerate, stop, change direction, and re-accelerate.

Hunts are limited in time, e.g., by the maximum endurance (or patience) of the predator, or by the onset of night or day. In this case, the limit is the pitiful endurance of the otherwise fearsome velociraptor. After a burst of speed for T = 15 s, the velociraptor must stop to rest, while the thescelosaur can run for a comparatively infinite length of time. We make the additional assumption that the velociraptor must rest for more than $T(v_v - v_t)/v_t$ = 3 s, i.e., more than the time required for the thescelosaur to run as far as the maximum distance that the velociraptor could close in 15 s. Thus, the velociraptor must catch the thescelosaur in the first 15 s after the thescelosaur senses the velociraptor.

The thescelosaur first detects the velociraptor at a distance D, with 15 m < D < 50 m, while the velociraptor can detect its prey farther than 50 m away. We assume that D is not dependent upon angle; i.e., if each of a pair of velociraptors approaches a thescelosaur from a different angle, the thescelosaur detects each when its distance is less than D.

We assume that because of the position of the eyes on opposite sides of the dinosaurs' heads, their vision is virtually 360°; thus, independent of its own orientation, each is aware of the position of the opponent.

The average human reaction time is ≈ 0.1 s; the thescelosaur can turn around 180° in this amount of time. We assume that animals with the speed and agility of these dinosaurs have a considerably smaller effective reaction time, which we vary between 0.005 s and 0.05 s in our study. Furthermore, we assume that the additional burden on the senses of the thescelosaur of the presence of two velociraptors instead of one does not change its effective reaction time.

From a picture of the velociraptor [Czerkas and Olson 1987, 28, compared with Paul 1988, 363], we deduce approximate measurements: a body length of 3 m, foreclaw length 0.5 m, hip-to-foreclaw distance 0.6 m, and hip-to-head distance 1.2 m. Moreover, a running bipedal dinosaur, because of its long tail, has a center of gravity close to the hips [Alexander 1989, 69]. Based on these measurements, we assume that at top speed the velociraptor will catch anything that comes within a distance δ_v = 0.6 m of its position, which we define to be the place on its torso from which the foreclaws extend. At the widest point of its torso, the velociraptor is only 0.4 m wide; we can ignore this thickness, as it is contained well within the reach of its foreclaws. Note that the location of the center of this reach is not at the hips, which is the point from which we assume the turning radius was calculated by the scientists. However, this slight incongruity does not qualitatively change our approach to the problem.

The thescelosaur is a biped of similar size. For the velociraptor to catch it, we assume that the velociraptor must be able to grab it at the torso, as the head and tail are too thin to grab easily at 60 km/h. So, we represent the thescelosaur as a circle of radius δ_t = 0.2 m over its hips. If the grabbing region of the velociraptor intersects this circle, the thescelosaur is caught.

To facilitate calculation, we assume that both predator and prey move at full speed for the entire time of the hunt T, even when they are moving in curves, with radii of curvature no less than r_v and r_t. This assumption is not entirely reasonable, as one can calculate the centripetal accelerations to be 19 g's and 39.4 g's, respectively. Given time for further investigation, it would be appropriate to model the dinosaurs with a maximum acceleration up to a top speed.

For the second part of the problem, with two velociraptors, we assume that the velociraptors work perfectly together:

- A velociraptor has just as much incentive to let its companion catch the thescelosaur as to catch the prey itself.
- The velociraptors are perfectly coordinated and can communicate their plans.
- The velociraptors allow each other space to move; we assume that this is equivalent to preventing their grabbing regions from intersecting.

Analysis: One Velociraptor

Approach

We begin with the simple case of the velociraptor initially chasing the thescelosaur along a straight line, separated by a distance d significantly larger than the turning radii of the dinosaurs. The thescelosaur's goal is to evade the velociraptor for however much time $(T - t)$ remains before the velociraptor runs out of endurance. Thus, if $d > (v_v - v_t)(T - t)$, the thescelosaur can run directly away from the velociraptor and the velociraptor cannot close the distance in the time remaining.

But what if $d < (v_v - v_t)(T - t)$? (Certainly this will be the case if the velociraptor can approach the thescelosaur undetected to a distance closer than $(v_v - v_t)T = 42$ m.) In this scenario, the thescelosaur must make use of its superior maneuverability if it is to survive. For sufficiently large d, no matter how the thescelosaur turns, it is easy for the velociraptor to adjust its course to keep heading directly toward its prey.

Encounter Strategies

The thescelosaur must now make some decisions: When has the velociraptor come near enough for the thescelosaur to make use of its superior agility (while not getting eaten), and how should it let the velociraptor approach? We consider two representative strategies.

Encounter Strategy A

The thescelosaur initially runs directly away from the velociraptor. This costs the velociraptor time, since it can close the relative distance at a rate of only $(v_v - v_t)$. Once the velociraptor has closed to within a distance k, the thescelosaur uses its superior maneuverability to "dive" out of the way. It turns at its minimum turning radius; the velociraptor then turns at its minimum turning radius to intercept, but it is too late (see **Figure 1**). The distance k must be chosen with great care: If it is too large, the velociraptor can adjust its angle and close on the thescelosaur; if it is too small, the thescelosaur will not be able to get out of the way of the velociraptor's grabbing radius.

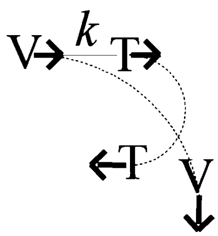

Figure 1. Encounter Strategy A.

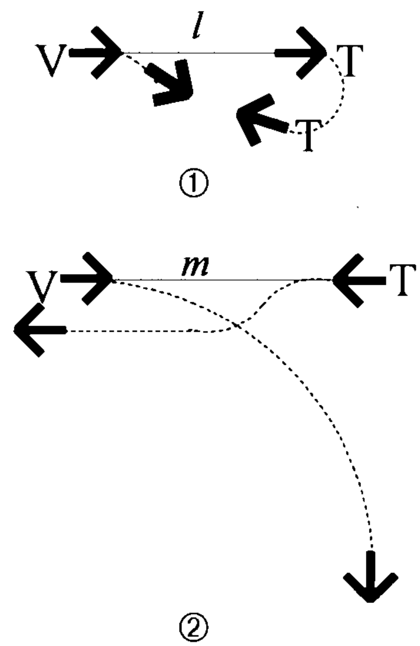

Figure 2. Encounter Strategy B.

Encounter Strategy B

The thescelosaur allows the velociraptor to close only to a distance l considerably greater than the k in Strategy A. At this point, the thescelosaur turns around and heads directly toward the velociraptor (see ① in **Figure 2**). The velociraptor, of course, continues to close; the distance between them now shrinks at a rate $(v_v + v_t)$. At an appropriate distance m, the thescelosaur again dives out of the way (see ② in **Figure 2**). Compared to Strategy A, however, the thescelosaur will be even more successful at dodging the velociraptor, as it need only change its course by a small amount to fly by the velociraptor at a relative velocity of approximately $(v_v + v_t)$. The value for m must be chosen with great care: if it is too small, the thescelosaur will not be able to stay outside the reach of the velociraptor, while if it is too large, the velociraptor will be able to compensate and intercept the thescelosaur.

Endgame

If the thescelosaur survives the encounter, the velociraptor will attempt to turn around and once again close in on its prey. The thescelosaur then has two endgame strategies.

Endgame Strategy A

Run away! If the distance between the two is greater than $(v_v - v_t)(T - t)$, the thescelosaur escapes unscathed as the velociraptor runs out of endurance.

Endgame Strategy B

This is a more daring maneuver but will take a big chunk of the velociraptor's time. Instead of running away from the velociraptor, the thescelosaurus should try to curve around it, ending up directly behind it. The velociraptor must turn around to come at the thescelosaur; because of its superior agility, the thescelosaur may be able to remain in this position relative to the velociraptor for some time. If the velociraptor starts to turn left, the thescelosaur also starts to turn left, attempting to remain 180° behind it. Because of its superior speed, however, the velociraptor will eventually outdistance the thescelosaur, and the thescelosaur will no longer be able to stay directly behind it. At this point, the thescelosaur should resort to Endgame Strategy A, as the velociraptor will soon turn around and chase it.

At the end of this post-encounter "endgame," the velociraptor will once again be chasing the thescelosaur, and we return to the "approach" phase.

Modeling the Chase: One Velociraptor

The Velociraptor Metric

How does the velociraptor get from point A to point B? More precisely: If the velociraptor is at the origin of the plane, facing in the positive y-direction, how would it get to the point (x, y)? Since we are considering the velociraptor to have constant speed, it should simply take the shortest path from the origin to the point. Unfortunately for the velociraptor, this distance is not the Euclidean metric, since the velociraptor has a limited turning radius! It cannot take a Euclidean straight-line path. So what is the appropriate path?

In **Figure 3a**, the velociraptor is at the origin facing upward. The two circles to either side represent the path of minimum turning radius. For points (such as A and B) outside these circles, the choice of minimum distance path is fairly clear: The velociraptor turns around the circle of minimum radius until it is heading directly toward the destination point; it then leaves the circle and heads straight toward the point. Representative paths are shown at right as dashed lines. Note that it is always advantageous to turn toward the side of the plane on which the destination point lies. (For the calculation of this length, refer to the **Appendix**.)

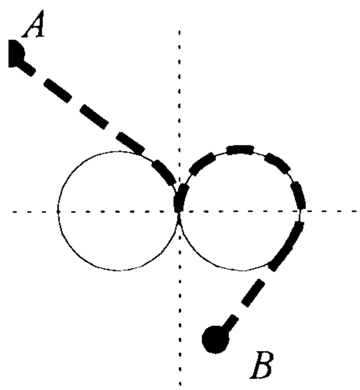

Figure 3a. How a velociraptor at the origin and facing upward gets to points A (at upper left) or B (at lower right).

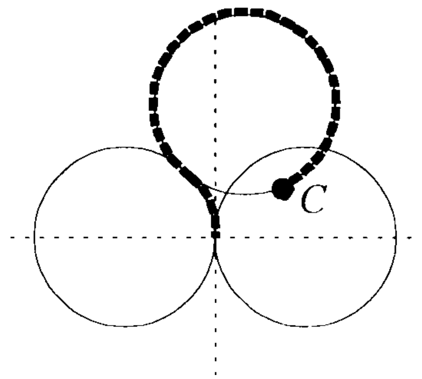

Figure 3b. How it gets to a point C inside the circle of minimum turning radius to its right.

Points inside the circles of minimum turning radius are more difficult for the velociraptor to access. It must somehow move such that destinations points inside these circles (e.g., point C in **Figure 3b**) are on or outside the circles of minimum curvature. The shortest way to do this is to turn away (in this case, to the left) from the destination point, following the other circle of minimum radius. Once the destination point lies outside the circles of minimum radius, the velociraptor follows the circles to the point. (For the calculation of this length, refer to the **Appendix**.)

We now define a new metric (velociraptor metric 1) on the plane: the distance along the curve from the origin (the location of the velociraptor, with the velociraptor facing the positive y-direction) to the point that the velociraptor follows. This metric is represented in **Figure 4** as a density plot with contour lines superimposed. Darker regions correspond to shorter distances for the velociraptor. The circles of minimum turning radius are easy to see, because of the discontinuity of the metric on the portions of the circles above the x-axis.

Thus far, we have considered the velociraptor as a point, though it has a grabbing radius of δ_v. To access a point, it need only reach any point a distance δ_v away from the desired point. We assume that the velociraptor minimizes how far it has to travel. Therefore, we replace the value of the metric at each destination point with the minimum value of the original metric on a disk of radius δ_v surrounding the destination point, yielding **Figure 5** (velociraptor metric 2). Note that the only parameters on which metrics 1 and 2 depend are the grabbing radius and minimum turning radius of the velociraptor, and that metric 1 is simply metric 2 with a grabbing radius of zero.

Now we treat a subtlety that we alluded to in the previous section. The origin of the coordinate system (the velociraptor's center of gravity) is actually

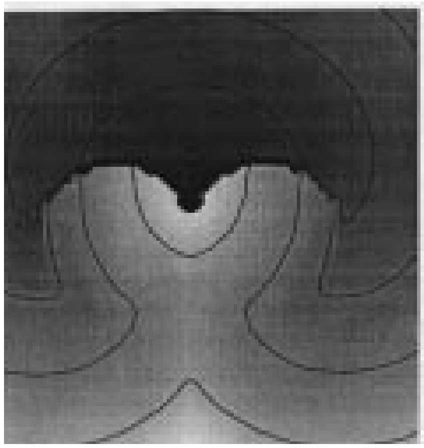

Figure 4. Velociraptor metric 1, depicted as a density plot.

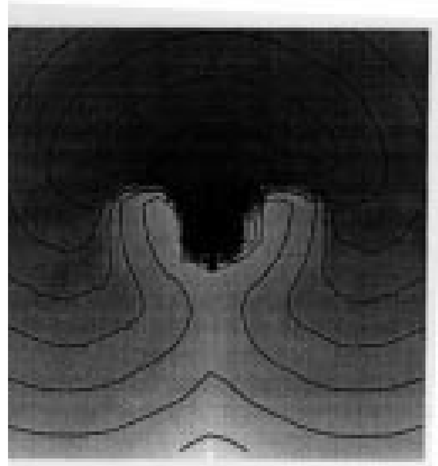

Figure 5. Velociraptor metric 2, depicted as a density plot.

0.6 m behind the center of the grabbing radius. However, one can see from **Figure 6** that given the circular and straight-line motions discussed above, the model of the situation remains exactly the same if we shift the origin to the center of the velociraptor-ball and simply change the effective minimum turning radius to $\sqrt{1.5^2 + 0.6^2}$ = 1.6 m.

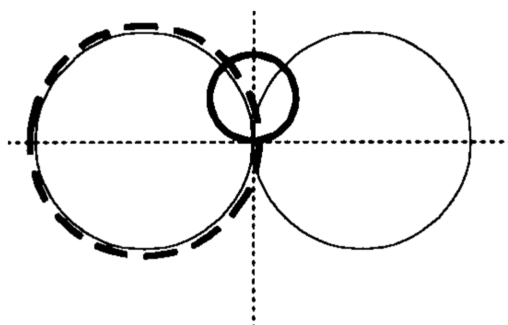

Figure 6. Result of shifting the origin to take into account the fact that the velociraptor is not a point.

We can further simplify our model in the following manner: We know that the velociraptor has caught the thescelosaur if their effective regions (circles of radius δ_v and δ_t) overlap. This is equivalent to saying that the centers of the two circles are separated by a distance less than $\delta = \delta_v + \delta_t$, or that the velociraptor has a grabbing radius of δ and the thescelosaur is a point. We thus define our metrics as described above, with effective turning radius 1.6 m and effective grabbing radius δ = 0.8 m.

Dinosaurs Have Peanut-Sized Brains

We first assume that the velociraptor acts to minimize, and the thescelosaur to maximize, the value of the velociraptor metric. We will see later that this is not sufficient to encompass all strategies (in particular, Encounter Strategy B, in which the thescelosaur heads straight for the velociraptor until the last possible moment), so we will soon make our dinosaurs a bit more sophisticated.

To evolve the system in time at a given time t, each dinosaur considers the location and heading of its opponent. It then chooses how to move during the next time step Δt. (Note that Δt is approximately equal to the effective reaction time, as it is only after the time Δt that the dinosaur can next evaluate the movement of its opponent.) As a range of options, the dinosaur chooses from a selection of arcs with length $v\Delta t$ and radii between the minimum turning radius and infinity (a line segment), shown in **Figure 7**. The dinosaur may choose the path with the most advantageous endpoint, or it may extrapolate each path several more time steps and choose from among those based on the metric evaluated at their endpoints. We vary this choice of strategy in our analysis, to optimize success rates of both predator and prey.

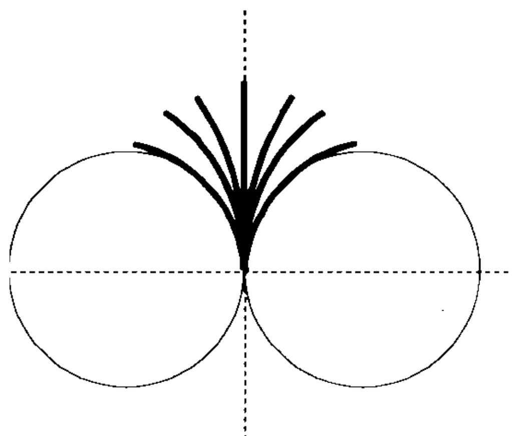

Figure 7. Possible strategies for the velociraptor.

We developed a computer simulation of the dinosaurs' behavior. Using the strategy of the velociraptor attempting to minimize the metric and the thescelosaur attempting to maximize it, we observed several phenomena discussed in the previous section. Most important, the dinosaurs chose paths similar to those used in determining the metric.

When the dinosaurs were separated by a distance larger than approximately 3 m, we observed the "approach" phase of the chase. The thescelosaur would run directly away from the velociraptor, while the velociraptor would adjust its course to trail directly behind, closing the distance. Under most circumstances, the thescelosaur would attempt to "shake" the velociraptor; but since the velociraptor was a sufficient distance behind, it was easy for it to adjust its course appropriately. Thus, we observed a rapid (on the order of a time-step of $\approx$ 0.01 s) small-amplitude oscillation of the thescelosaur's direction in the approach phase. In the simulation, once the velociraptor gets close enough to the thescelosaur, the thescelosaur adopts Encounter Strategy A. If it survives, it adopts one of the endgame strategies. In **Figures 8** and **9**, we show a hunt in which the thescelosaur successfully evades the velociraptor by using Encounter Strategy A followed by Endgame Strategy B.

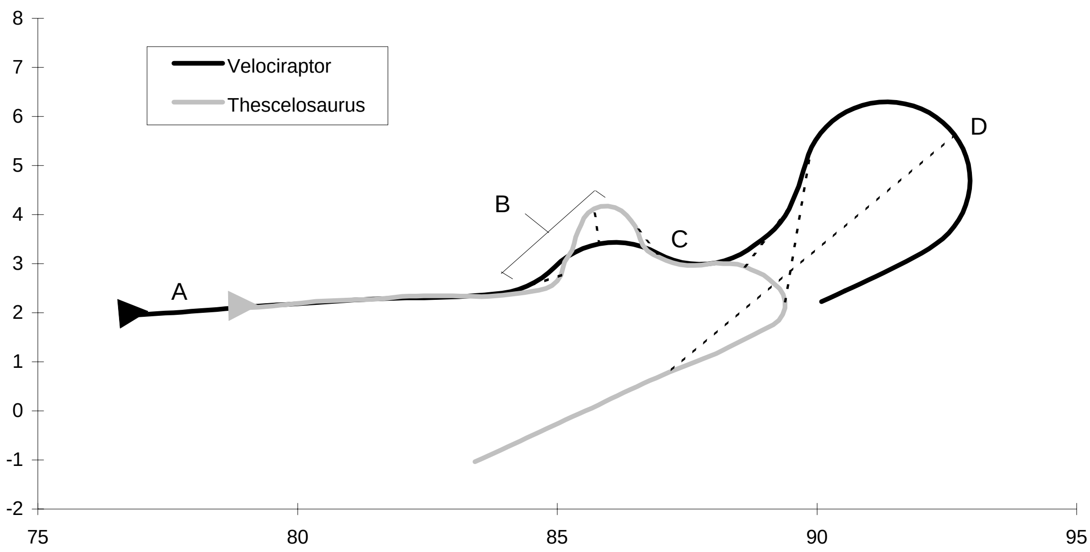

Figure 8. Encounter Strategy A, close up. Dotted lines connect points on the two curves that correspond to the same time. The velociraptor makes the big loop. This encounter strategy allows the agile thescelosaur to escape virtually every time if the velociraptor's grabbing radius is below a critical value (0.45 m). Unfortunately, the strategy fails with equal certainty if the velociraptor's grabbing radius is above that critical value. The velociraptor in this figure has a grabbing radius of 0.4 m. A few key stages are:

A. The thescelosaur runs away from the velociraptor.

B. When the velociraptor gets too close, the thescelosaur quickly turns out of the way.

C. The velociraptor cannot respond to this sudden turn fast enough, letting the thescelosaur duck behind it. Now, the velociraptor must loop around to continue chasing its meal.

D. The thescelosaur escapes before the velociraptor completes its loop.

We further found that the thescelosaur performed better using metric 2 looking only one time step ahead. Metric 2 is clearly advantageous for the prey, because this metric teaches it to stay out of the path of the predator's grabbing radius, rather than simply avoiding its center. The thescelosaur relies on its ability to maneuver quickly; thus it constantly adjusts its heading, rendering it useless for it to estimate several time steps into the future.

The velociraptor performed optimally using metric 1, looking 5 time steps ahead. We originally programmed the velociraptor to use metric 2, but it turned out to be a bit too cocky; the velociraptor was constantly disappointed as the thescelosaur barely slipped out of reach. When we changed its strategy to employ metric 1, this problem was eliminated. In a future investigation, it may be useful to make the velociraptor use metric 2 with a nonzero grabbing radius smaller than the actual value.

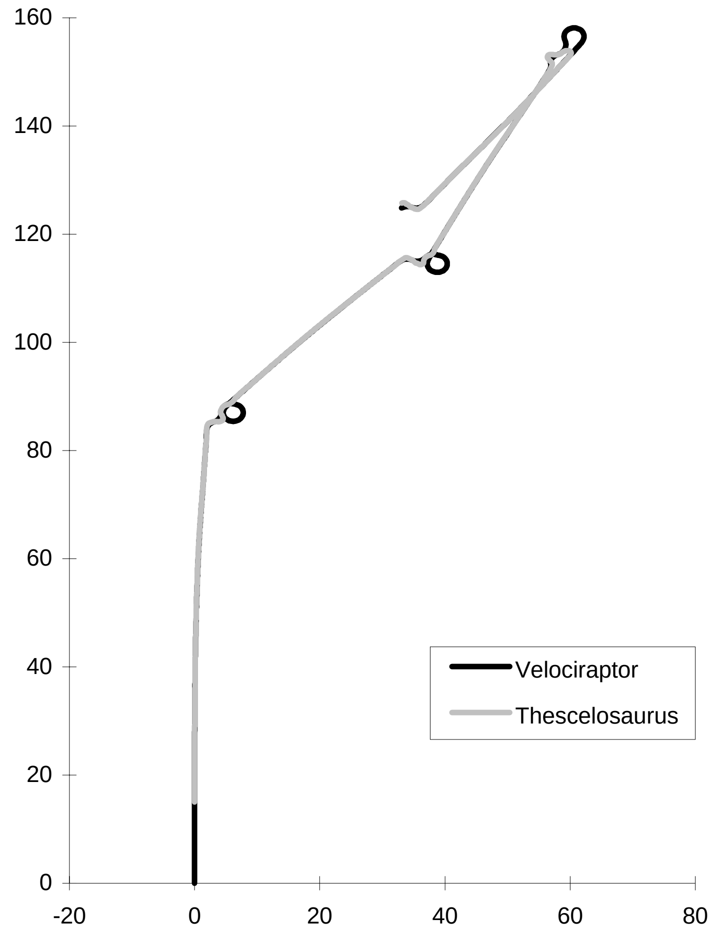

Figure 9. Encounter Strategy A, far away. After a round of Encounter Strategy A, the velociraptor soon catches up with the thescelosaur, creating a new encounter every 3.2 s. This figure shows a typical series of encounters for a 15-s chase when the thescelosaur detects the velociraptor at the minimum detection radius (15 m). Notice that the angle between the chase and escape paths is highly variable and sensitive to initial conditions. After 15 s, the velociraptor gets tired and the thescelosaur can simply run away.

We now ask what parameters allow the thescelosaur to survive. For the given speeds and minimum turning radius, the thescelosaur will always survive for values of the effective grabbing radius $\delta < 0.4$ m and is always captured for $\delta > 0.5$ m. In the region in between, the outcome is highly sensitive to initial conditions. Unfortunately for the thescelosaur, the given value of δ is actually 0.8 m. Thus, the thescelosaur should try Encounter Strategy B.

The Thescelosaur Learns to Play “Chicken”

Encounter Strategy B requires the thescelosaur to head directly toward the velociraptor, which is incompatible with the looking forward a few time steps to see which path will maximize its distance from the velociraptor (based on metric 2). Thus, we must modify our simulation to study this strategy.

We assume that the thescelosaur has sufficient time and distance to turn around and head directly toward the velociraptor when the encounter begins. We therefore consider only part 2 of Encounter Strategy B, in which the thescelosaur dives out of the path of the velociraptor just before collision.

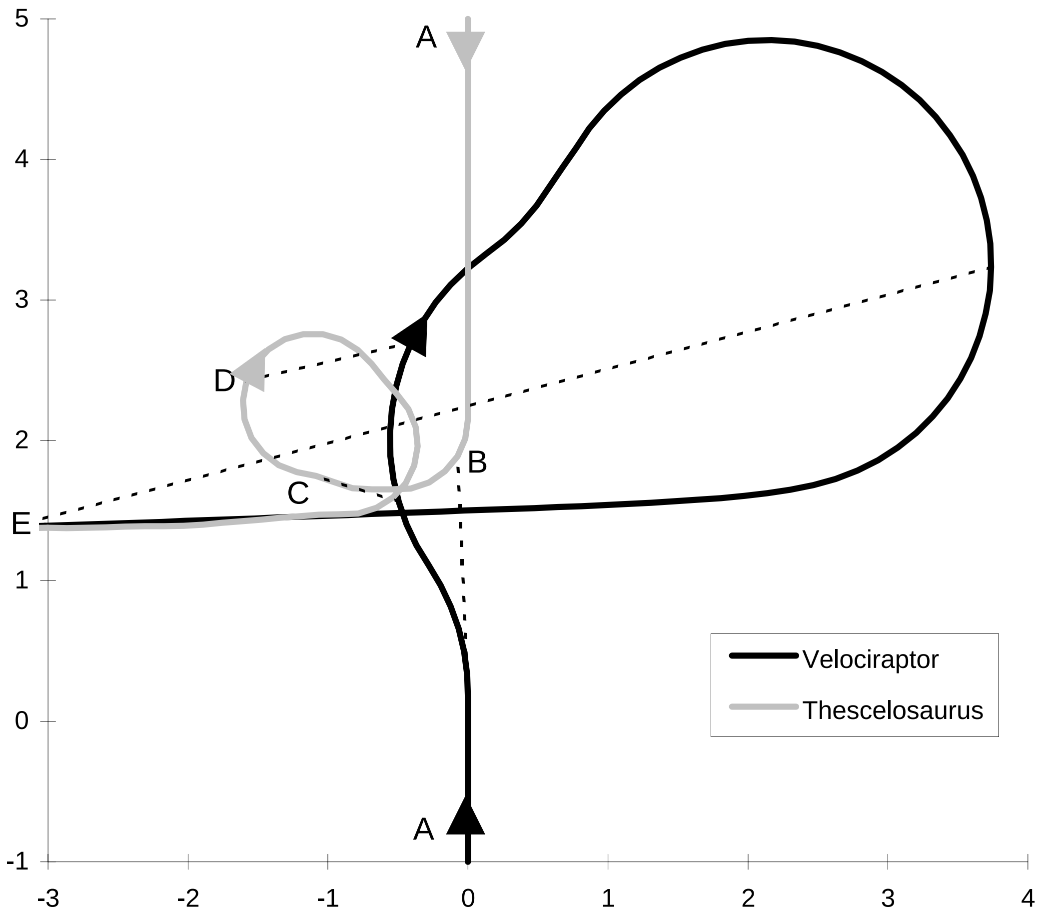

Figure 10. Encounter Strategy B. The thescelosaur must follow Encounter Strategy B if the velociraptor has a grabbing radius above 0.45 m (at which Encounter Strategy A no longer works). The velociraptor in this figure has a grabbing radius of 0.6 m. A few key stages are:

A. The thescelosaur runs towards the velociraptor.

B. When the velociraptor is too close, the thescelosaur dodges off to a side and the velociraptor follows.

C. The thescelosaur is now to the velociraptor's left, but the velociraptor curves to the right because it knows that it cannot make a sharp enough left turn to catch its meal.

D. The thescelosaur has sent the velociraptor on a huge loop while it swiftly makes a much tighter turn to come in behind the velociraptor.

E. The thescelosaur is far away even before the velociraptor completes its loop. Soon after this, the thescelosaur must turn around and run towards the velociraptor again.

We assume that once the thescelosaur begins to dodge, it is simply resuming its original strategy of maximizing distance (according to metric 2) between the two dinosaurs (see **Figure 10**). Thus, we can simulate this strategy using the original simulation, with the initial condition that the dinosaurs are heading straight toward each other separated by a small distance m, as shown in the second part of **Figure 3**.

We found that if the thescelosaur runs towards the velociraptor and starts turning when it is 2.15 m from the velociraptor, it will escape every time from a grabbing radius of 0.6 m, even if the other parameters are changed slightly. Thus Encounter Strategy B is a clear improvement over Encounter Strategy A.

A rough calculation reassures us that such a strategy indeed works for multiple passes, confirming our original assumption. After one such pass, the thescelosaur has a 7 m lead, and in the time it takes for the thescelosaur to turn around 180°, the velociraptor will gain about 1.9 m; this leaves the thescelosaur a bit of maneuvering before the critical 2.15 m turning point.

The Velociraptor Takes a Gamble, or, The Rational Dinosaurs

If the thescelosaur can escape using Encounter Strategy B, what is the velociraptor to do? We found that for a grabbing radius of 0.6 m, the 2.15 m critical distance had very little room for error—if the thescelosaur dodges too early or too late by 0.1 m, it will be caught. Thus, the velociraptor knows exactly when the thescelosaur will make its dodge.

If the velociraptor pursues its until-now optimal strategy of minimizing its metric, it will lose its dinner every time. Therefore, it should try to anticipate the movement of the thescelosaur; if the velociraptor guesses correctly which way the thescelosaur will swerve, and correct its own course accordingly, it can gain valuable time and thus catch its prey. However, if it does not guess correctly, it loses even more time than if it had merely gone straight. Moreover, the thescelosaur pursued by such a decision-oriented velociraptor, for its part, also wants to anticipate the movement of its predator.

We can model this as a game-theory problem. Consider the last possible moment before the thescelosaur must swerve. Under our original Encounter Strategy B, the thescelosaur swerves either left or right. The velociraptor, knowing this, should arbitrarily choose to swerve either left or right in this instance, giving it a 50% chance of guessing correctly and catching its prey. However, the thescelosaur knows this! So, if it is sure that the velociraptor will swerve, it should keep going straight past the critical point, and once the velociraptor swerves it can dodge the the other way an instant later. The velociraptor, knowing this, realizes it is not always to its advantage to anticipate the movement of the thescelosaur; perhaps the thescelosaur will anticipate its anticipation. In this situation, the velociraptor's optimal strategy is to keep going straight! The thescelosaur, then past the critical point, will be eaten.

Thus, it may be reasonable for the thescelosaur to move left (L), right (R), or stay straight toward the center of the velociraptor (C). The velociraptor can choose to anticipate these moves; we thus denote the velociraptor's strategy by

L, C, or R. If the velociraptor's guess is correct, we assume that it catches the thescelosaur, receiving a normalized payoff of 1, and the thescelosaur receives a payoff of 0. If the velociraptor guesses incorrectly, the thescelosaur survives the encounter, and the game will be played again at the next encounter, and so on until the velociraptor's endurance runs out.

If the thescelosaur swerves one way and the velociraptor anticipates the other (a "large miss"), then there will be a decent interval of time before the velociraptor catches up to the thescelosaur for the next encounter. If, however, one of the dinosaurs goes straight and the other swerves (a "small miss"), it will take less time for the velociraptor to catch up. Thus, there will be fewer encounters for the remainder of the hunt following a large miss than after a small miss, and thus the probability p that the thescelosaur survives the hunt after a small miss is less than the probability q that it survives after a large miss. In the small-miss case, the thescelosaur's payoff is therefore p, and that of the velociraptor is $1-p$. In the large miss case, the thescelosaur's payoff is q, and that of the velociraptor is $1-q$.

In this analysis, we have simplified the payoffs so all small misses result in the same payoffs and all large misses result in the same payoffs. This may not be entirely correct, as small misses come in two different forms: those in which the velociraptor goes straight, and those in which the thescelosaur goes straight. We have also assumed that the dinosaurs are symmetric and do not prefer one side to the other.

We would like to find any Nash equilibrium of this payoff matrix. It is clear that there is no pure equilibrium, so we look for a mixed strategy. Let a and b be the respective probabilities that the velociraptor and thescelosaur choose L. Since there is no difference between right or left, the probability of a dinosaur picking L equals the probability that it picks R. Thus, the probabilities that the dinosaurs choose strategy C are $1-2a$ and $1-2b$, respectively.

Finding the mixed Nash equilibrium is now easy. As in the elementary game-theory problem, each dinosaur wants to maximize its own expected payoff, and minimize that of the other. This occurs when the expected payoffs of the opponent are equal for any of its strategies.

Let $P_t(V|L)$ be the expected payoff for the thescelosaur if the velociraptor chooses L. Thus we have $P_t(V|L) = P_t(V|R) = (1-2b)q + bp$ and $P_t(V|C) = 2qb$. Setting these thescelosaur payoffs equal, we find $b = q/(4q-p)$.

Similarly, we have $P_v(V|L) = P_v(V|R) = a + (1-q)(1-2a) + a(1-p)$ and $P_v(V|C) = 2a(1-q) + (1-2a)$. Setting these equal, we find that a is also $q/(4q-p)$.

To determine the probabilities a and b, we must determine p and q. If there is time remaining in the chase for only one encounter, then $p = q = 1$, as the thescelosaur will not have another chance. Thus, $a = b = 1/3$, each dinosaur chooses each of its three strategies with equal probabilities, and the thescelosaur escapes 2/3 of the time! Thus, if both dinosaurs know that only one encounter remains, their expected payoffs from that encounter are 1/3 (for the velociraptor) and 2/3 (for the thescelosaur).

Now suppose (for example) that if there is a miss, there will be time for one more encounter if it is a small miss but not if it is a large miss. Thus, $p = 1$, since a large miss means that the thescelosaur survives the chase, and $q = 2/3$, since the probability of the thescelosaur surviving the next encounter is 2/3, from the previous paragraph. Therefore, $a = b = 2/5$.

Given more time for this study, time-dependent values of p and q could be determined, and we could determine the subgame-perfect Nash equilibrium of the dinosaurs for the entire chase.

Two Velociraptors: What Changes?

Approach

It is to the thescelosaur's advantage to run directly away from the velociraptors when the distance from them is large. With two velociraptors, this translates to the thescelosaur running so that the distance between it and one velociraptor remains the same as the distance between it and the other velociraptor. In this large-distance limit, the strategy for the velociraptors is also clear: They should run towards the thescelosaur using the same strategies as above.

There is one substantial difference between this case and that of one velociraptor: At the very beginning of the chase, the initial configuration is specified not only by D (which suffices for the single velociraptor and prey), but also by the angle made by the two velociraptors with the thescelosaur as the vertex. It is to the velociraptors' advantage to start out 180° apart, assuming that the thescelosaur moves in a straight line and the velociraptors continually adjust their directions to intercept it (a reasonable assumption in the far distance limit). This arrangement produces the maximum possible initial approach velocity: the velociraptor's maximum velocity v_v.

Encounter

By the time the velociraptors close in on the thescelosaur to the point that the prey will will have to start to curve, the configuration of the two velociraptors plus thescelosaur approaches one of only two cases. In the first case, both velociraptors run side by side and hence act roughly as one velociraptor with rather large turning and grabbing radii. In the second case, the velociraptors and the thescelosaur form a straight line but one velociraptor is behind another.

There are, then, two main strategies for the velociraptors: either to run side by side, or for one to run behind the other and pounce as soon as the thescelosaur starts turning. The side-by-side strategy is quite easy to model; as the two predators act as one, this case is a simple variant of the one-predator case. As one might expect, the critical grabbing radius for switching from

Encounter Strategy A to Encounter Strategy B turns out to be half that of the one-predator case.

The consecutive-velociraptor strategy, on the other hand, is a bit trickier. This strategy is meant for very small critical radii (0.3 m or less). The idea is for one velociraptor to "corral" the thescelosaur by curving toward it, even though according to the distance metric this actually makes the thescelosaur farther away. This maneuver restricts the movement of the thescelosaur by a great deal; the other velociraptor, which has been cruising behind the corraling velociraptor during this maneuver, can then circle in for the kill.

Preliminary studies using our simulation show that this strategy shows a great deal of merit. However, the simulator breaks down; the would-be corraling velociraptor curves the wrong way. Although we did not program the simulator to allow the animals to work together as the preceding paragraph implies, we are confident that this strategy will work for small radii if such experiments are done. Note also that it becomes much harder for the thescelosaur to dive between the predators, although this is probably possible for small enough critical radius.

Sensitivity of the Model

We tested the model by varying the parameters of the simulator, most notably the reaction time and the grabbing radius. Changing the curving radii and the relative velocities, while undoubtedly having a pronounced effect on the outcome, does not exhibit counterintuitive, extremely sensitive, or chaotic behavior.

On the other hand, Encounter Strategy A is rather sensitive to the reaction times of the dinosaurs. For instance, the critical grabbing radius varies from 0.4 m to 0.6 m over the range of reaction times that we tested. Changes in reaction time, along with small changes in initial position and velocity, significantly affected behavioral parameters, such as the angle between chase and escape paths. These effects have a biological interpretation: Escaping from a velociraptor using Strategy A involves exploiting a small window of opportunity during which the thescelosaur can duck out of the way before the velociraptor has time to respond. The thescelosaur must respond boldly to the slightest opportunity, resulting in a sensitivity that ensures that no two encounters are alike.

Encounter Strategy B exhibits a different kind of unpredictability. Although this strategy is less sensitive to initial conditions and reaction times, considerations of game theory require that each dinosaur choose randomly the direction in which to swerve during the encounter.

Thus, as expected, both models exhibit some unpredictable behavior. After all, the chase that we are modeling is a mortal battle of wits, not a preplanned ritual.

Strengths and Weaknesses of the Model

Our model has many strengths, perhaps the greatest of which is that the model is easy to understand: Minimization and maximization of a metric is a simple concept, and one that is not hard to implement.

Another prime strength of our model is the extreme robustness of the simulator. Not only can the simulator handle a wide range of similar scenarios simply by changing the parameters involved, but it can also handle a variety of different strategies simply by adjusting the initial conditions accordingly, as we did with Strategy B. However, we were not able to reprogram the simulator to deal with the cooperation between two predators.

Moreover, we feel that that the part of our model incorporating strategy B has many virtues to recommend it. Its robustness, as discussed in the previous section, lends it credibility as a feasible strategy. In addition, its applicability to a relatively large subset of turning radii makes it a better strategy than any other we could find. Furthermore, the game theory presented can be applied to most such finite games. However, we have a caveat in that in an actual situation involving live creatures, the prey would almost certainly not have the presence of mind to realize that running straight towards its predator would be the optimum strategy.

One of the main weaknesses of our model is the assumption that the dinosaurs go at top speed even when they are turning. More realistically, they should slow down to go around the curves at a reasonable centripetal acceleration.

Addendum: Realism Rears Its Ugly Head

The assumption that the dinosaurs are going at their top speeds even on curves is a rather poor one, due to the tremendous centripetal accelerations that would be involved. A better approximation is to model the dinosaurs' velocity on a circular arc as related to the radius of that circle. Since for centripetal acceleration, $a = v^2/4$, a first approximation is to assume that the dinosaurs can sustain an acceleration of, say, 3 g's, and can keep that maximum acceleration no matter what the radius of curvature is. Then we can model the velocity as a function of the radius as $v(r) = \min(\sqrt{a/r}, v_{\max})$.

The natural question we must ask is: Does this change the optimal strategy for the velociraptor?

A second approximation is to take into account the tangential acceleration and deceleration to the curved paths when the radius of curvature is changing.

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Appendix: The Velociraptor Metric

Outside the Minimum Turning Radius

Let the velociraptor (currently considered to be a point) be located at the origin, facing in the positive y-direction. Let the destination point be A, a distance l from the velociraptor, at an angle θ from the velociraptor's heading; thus, $A = (l\sin\theta, l\cos\theta)$. We abbreviate the minimum turning radius r_v as r. Let $B = (0, r)$ be the center of the circle of minimum turning radius, and let C be the point at which the velociraptor leaves the circle and moves along a straight line to point A. (Thus line AC is tangent to the circle.) Let $\alpha = \angle ABC$ and $\beta = \angle OBC$ (see **Figure 11**).

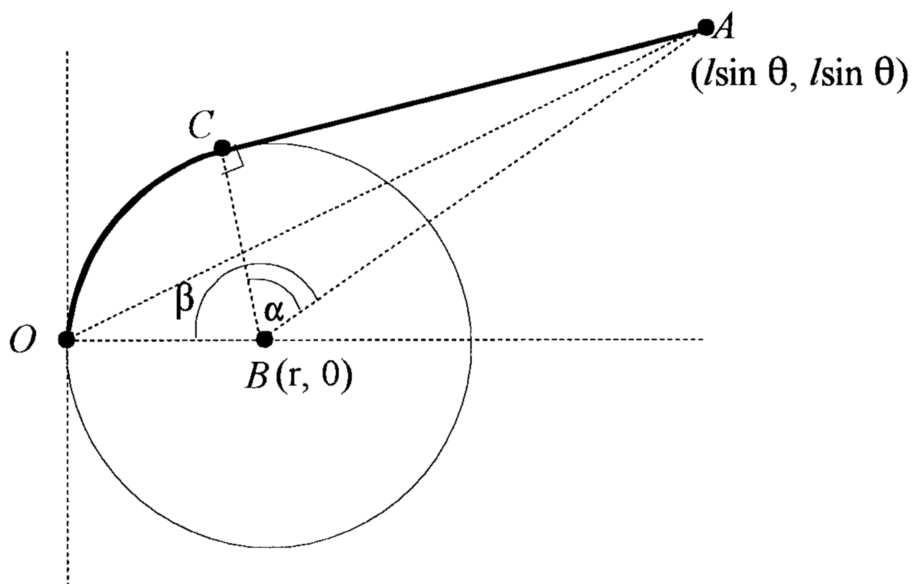

Figure 11. Diagram for the velociraptor metric outside the minimum turning radius.

The total distance that the velociraptor must travel is the length AC and the length of the arc OC, which is $r(\beta - \alpha)$. From the given coordinates, we have $AC = \sqrt{(l\sin\theta - r)^2 + (l\cos\theta)^2}$ and $BC = r$, so the length $AC = \sqrt{(l\sin\theta - r)^2 + (l\cos\theta)^2 - r^2} = \sqrt{l(l - 2r\sin\theta)}$. The law of cosines tells us that for a triangle with sides a, b, and c, the angle opposite the side with length c is $\cos^{-1}[(a^2+b^2-c^2)/2ab]$. We found AB above; we know $OB = r$ and $OA = l$; thus, we have angle β. The angle α is one of the acute angles in right triangle ABC, thus we can say that $\alpha = \tan^{-1}(AC/BC)$. Thus, the total length is

$$\sqrt{l(l-2r\sin\theta)} + r\cos^{-1}\left(\frac{r - l\sin\theta}{\sqrt{l^2 - 2lr\sin\theta + r^2}}\right) - r\tan^{-1}\frac{l(l-2r\sin\theta)}{r}.$$

Inside the Minimum Turning Radii

Let A be the destination point, P and Q the centers of the circles of minimum turning radius, and B the center of the right-hand circle of minimum turning radius after the velociraptor moves through arc OC, as shown in **Figure 12**. Let angles ϕ, α, and β be as shown. The velociraptor must move through minor arc OC and major arc AC, for a total length of $(2\pi - \phi + \alpha + \beta)$. Let A have coordinates (x, y). The Pythagorean theorem yields that $AP = \sqrt{(x+r)^2 + y^2}$ and $AQ = \sqrt{(x-r)^2 + y^2}$. The lengths PB, PQ, and AB are $2r$, $2r$, and r. Thus, we may apply the law of cosines as described in the subsection above and find the angles in question.

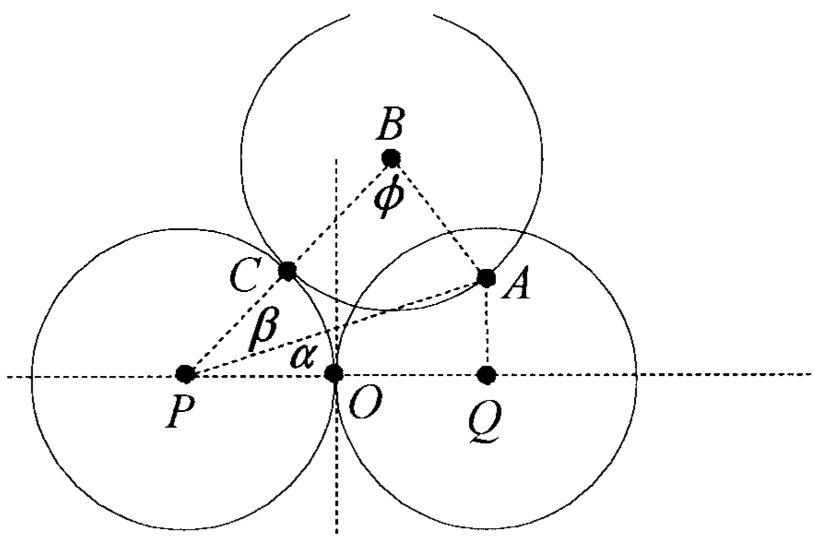

Figure 12. Diagram for the velociraptor metric inside the minimum turning radius.

Gone Huntin': Modeling Optimal Predator and Prey Strategies

Hei (Celia) Chan
Robert A. Moody
David Young
Pomona College
Claremont, CA 91711

Advisor: Richard Elderkin

Summary

We develop a model for the hunting strategy of *Velociraptor mongoliensis* pursuing *Thescelosaurus neglectus*. Regarding their characteristics, there are discrepancies between the problem statement and the literature; so we parameterize our model in terms of both physical and mechanical characteristics.

The primary locomotive differences between the animals are their relative speeds and turning radii. We show that the optimal strategies are simple, and we present equations and illustrations for the key components of the model. Since the optimal strategy for the predator includes a stochastic component, we present an equation for the probability of a successful encounter.

We also model the interaction of multiple predators and multiple prey. With a reasonable assumption regarding cooperative hunting, two or more velociraptors should have an insurmountable advantage, barring earlier detection.

Finally, we discuss an alternative approach, outlining a genetic programming solution that would evolve optimal strategies for both animals. We begin with the primitives required to evolve such a solution, and we discuss the nature of the evolution required to produce optimal solutions. We show that the evolutionary traits identified by this supposition mirror the known traits.

Background

Velociraptor, member of genus *Theropod*, lived in Central and East Asia during the Late Cretaceous period (97.5 to 66.4 million years ago). It was a fairly

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small animal, reaching a length of roughly 1.8 m and a mass of no more than 45 kg [Encyclopædia Britannica 1997]. *Velociraptor* is thought to have hunted small herbivorous dinosaurs. Current speculation is that velociraptors hunted in pairs or packs. It is estimated that a velociraptor was capable of brief bursts of speed, roughly 15–20 s in duration, of up to 60 km / hr. They possessed a sickle claw on each foot and an ossified tendon in the tail, which together enabled them to strike and slash while maintaining balance. The velociraptor is closely related to the slightly larger *Deinonychus*, indigenous to North America during the Early Cretaceous period (144 to 97.5 million years ago).

Thescelosaurus neglectus, a member of genus *Ornithopod*, lived in North America during the Late Cretaceous period. Thescelosauri were herbivores roughly 3.4 m in length. They were fast runners, capable of sustained speeds of up to 40–50 km / hr. While they were of the general type of prey that velociraptors would probably hunt, it is not apparent that the two species inhabited the same continent at the same time. It is more likely that *Deinonychus* or its descendants were predators of *Thescelosaurus*. In any event,

Mongolia during the Cretaceous period was generally arid, and vegetation was probably quite sparse. Thus, the habitat and hunting grounds of *Velociraptor* were probably large open areas with few obstacles and fewer hiding places [American Museum of Natural History 1997].

North America during the Cretaceous was quite wet and heavily vegetated, an environment probably less conducive to high-speed pursuit and better fit for stealthy hunters. Forests and steams may have provided hiding places for the prey but also served as boundaries or obstacles to escape routes.

Assumptions and Limitations

We assume that both predator and prey adopt optimal strategies. This assumption is probably not realistic, since it requires the prey to have relatively perfect information about an approaching predator. Since turning and looking would undoubtedly decrease speed, it is not clear that the prey can both maintain optimal speed and possess optimal information. However, sufficient information might be gathered from other senses to approximate an approaching predator's velocity vector and form a reasonable distance estimate.

The problem statement advises that *Velociraptor* can turn with a 1.5 m radius, and that *Thescelosaurus* can turn with a 0.5 m radius, while running at full speed. It is highly unlikely that these radii were achievable by either animal. At 60 km / hr, *Velociraptor* would experience a centripetal acceleration of 185 m / s^2, or roughly 20 g's. *Thescelosaurus* at 50 km / hr would experience 40 g's. We doubt that the animals could make these turns. A more reasonable acceleration would be 2 g's, corresponding to radii of 14.2 m and 9.8 m respectively. We are also somewhat suspicious of the assertion that *Thescelosaurus* has a shorter turning radius than *Velociraptor*. *Velociraptor*'s higher muscle-to-mass ratio, smaller size, and longer claws suggest that it should be capable of tighter turns at equivalent speeds; being more compact, it should be able tolerate higher g-forces as well.

Our model ignores the direct impact of terrain, for several reasons:

- We parameterize velocity, acceleration, and maximum duration of maximum speed for both predators and prey, so it would be easy to accommodate in our model the adverse impact of terrain on either animal.

- We are not confident that we can estimate the impact of terrain on the relative velocity of the two animals. We have not found sufficient information on their foot structures to judge their abilities to pass through non-ideal terrain.

- We are not sure what terrain is appropriate for an encounter. Mongolian terrain of the Cretaceous period is close to ideal for optimal speed, whereas North American terrain could vary significantly.

We have assumed that random movements (left or right) are captured in a probability function. This seems a reasonable guess, though the literature indicates that mammals favor a particular direction. This is a rather important issue, since the predator would presumably learn the prey's pattern, anticipate any favored turn, and improve its success rate significantly.

We parameterize acceleration as the maximum acceleration that the animal would tolerate in any direction, though we could assign different values for linear acceleration and centripetal acceleration, or separately limit positive and negative acceleration (starting up and stopping, respectively). We also assume that acceleration is constant at this maximum rate, and that changes in speed would be done prior to initiating a turn. Since optimal turning radius is achieved by decreasing speed prior to turning, this assumption is beneficial to both predator and prey. However, it is somewhat unlikely that an animal would slow down, turn tightly, then accelerate to full speed, rather than begin the turn at full speed and decrease linear velocity while turning.

We make no attempt to quantify the relative costs to each animal relative to the pursuit game. For example, since the prey dies if it loses, it would be reasonable to assume that the prey would adopt a risk-adverse strategy until capture is imminent, followed by a "try anything" strategy, including attacking the predator at the last minute. The predator, on the other hand, can make multiple attempts at the game, so it would be reasonable for it to attempt a high-risk maneuver with a reasonable probabilistic success rate, since failure implies only a delay in lunch rather than death.

For flexibility, we parameterize several variables, such as reaction time, attack success probability, and attack radius.

In the multiple predator model, we focus on the two-predator game. Two predators are virtually certain to win; adding more just doesn't seem fair.

We do not consider multiple prey models, since a single predator is not going to attempt to capture multiple prey, and multiple predators vs. multiple prey ultimately resolve down to multiple instances of "two or greater vs. one" or "one versus one."

We ignore the search costs, encounter rates, and stealth strategies, except to note that the predator benefits from minimizing the distance prior to beginning the chase and the prey benefits from maximizing the detection distance. We include a parameter for the detection distance in our model, and we compute the maximum distance at which the predator can expect success.

Finally, we make no attempt to deal with visibility, weather, presence of alternate animals, presence of other prey, obstacles, obstructions, boundaries, bodies of water, or other possibilities. Mathematically speaking, these are relatively minor omissions. Boundary conditions limit the regions of safety and danger and as such introduce variations to the probability of capture, but these calculations are fairly simple. Obstacles tend to favor the predator, since its reaction time will be longer and it will have seen the prey's response to the obstacles. This is somewhat equivalent to a shortening of the relative distance between the two animals, which can be easily accommodated in our model. Water, weather, and similar environmental considerations favor the animal with superior physical adaptations, and we are not clear as to which animal that is under each of the possible conditions. Again, the impact is most likely to be a change in relative velocities, a condition our model can accommodate.

Model 1: Single Prey, Single Predator

We consider the optimal strategy for a single predator and a single prey. We initially ignore stalking activity by the predator. At some time t_i and location d_i, the prey becomes aware of the presence of the predator and flees. The speeds of the predator and prey are v_v and v_t, the maximum duration over which the predator and prey can maintain their maximum speeds are M_v and M_t, and their minimum turning radii are r_v and r_t.

At any time during the chase, the prey has the option of changing direction, subject to the minimum turning radius. If the distance between the predator and the prey is sufficiently large—specifically, if the predator is capable of adjusting its approach trajectory to intercept all points on any circular path taken by the prey—it is never prudent for the prey to make such a direction change, since the result of such a path would decrease the net distance between the two animals without increasing the chance of escape.

If, as shown in **Figure 1**, the predator can reach the point where its minimal turning radius touches the circle representing the prey's minimum turning radius prior to the prey arriving at that point, then the prey has committed a highly disfavorable action. As long as this condition persists, the prey can maintain the maximum distance between itself and the predator by fleeing along a linear path directly away from the predator. Provided that the minimum turning radius of the prey is smaller than the minimum turning radius of the predator and the prey can execute a turn whose path crosses the intersection before the predator's does, then the prey can exploit this advantage by executing a minimum-radius turn to escape from the danger area [Howland

1974, 334–335]. Such escape is a temporary solution, since the predator will eventually adjust its approach vector and resume the chase. However, if the prey is capable of executing these maneuvers for a sufficiently long period of time, then the predator may have to abandon the chase. Therefore, the optimal strategy for the prey is to run directly away from the predator until the distance decreases to the point where the turning gambit is effective.

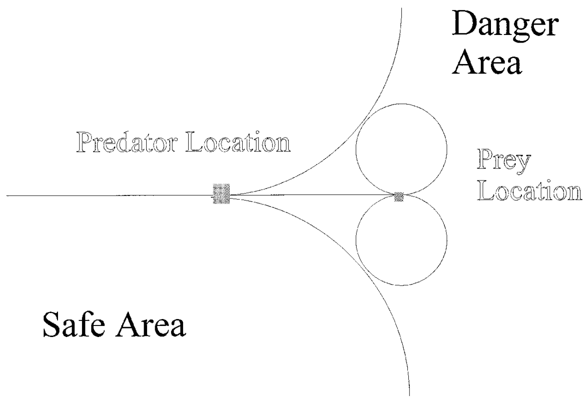

Figure 1. Ineffective turning region.

Both the predator and prey are assumed to be traveling at extremely high speeds. In the problem statement, the maximum speeds of predator and prey are 60 km / hr and 50 km / hr, with minimum turning radii 1.5 m and 0.5 m. The equation for centripetal acceleration is

$$a = \frac{v^2}{r}.$$

Therefore, the centripetal acceleration of the predator and prey, given their maximum speeds and turning radii, are

$$a_v = \frac{\left(\dfrac{60\text{ km}}{\text{hr}} \cdot \dfrac{1000\text{ m}}{1\text{ km}} \cdot \dfrac{1\text{ hr}}{3600\text{ s}}\right)^2}{1.5\text{ m}} = 185.2\text{ m}/\text{sec}^2 = 18.9\text{ g's},$$

$$a_t = \frac{\left(\dfrac{50\text{ km}}{\text{hr}} \cdot \dfrac{1000\text{ m}}{1\text{ km}} \cdot \dfrac{1\text{ hr}}{3600\text{ s}}\right)^2}{0.5\text{ m}} = 385.8\text{ m}/\text{sec}^2 = 39.4\text{ g's}.$$

These are not reasonable acceleration rates. Either the turning radii must be significantly larger, or else the animals must decelerate prior to entering the turns. We define G_v and G_t as the maximum number of g's that the animals can tolerate, either as linear or centripetal acceleration. We estimate 2.0 to be a reasonable value for both of these constants, which result in turning radii of 14.2 m for the predator and 9.8 m for the prey, or else speeds of 5.4 m / sec for the predator and 3.1 m / sec for the prey.

We include a reaction time and a deceleration period during which the animal adjusts its velocity to achieve its minimum turning radius. The ratio of the turning radii is more relevant than their actual values, since this ratio determines whether the prey will successfully reach the safe area. We therefore normalize the radii relative to the radius of the predator's minimum turn. Following the example of Howland [1974], we normalize the speeds in a similar manner. Therefore the predator's speed is arbitrarily set to 1, as is the predator's radius. The prey's speed is set to v_t/v_v, and the prey's radius is set to r_t/r_v. To create a parametric equation in dimensionless units, we normalize time, x, and y as follows:

$$t = \frac{Tv_v}{r_v}, \qquad x_v = X_v/r_v, \qquad y_v = Y_v/r_v.$$

We define the starting point of the turning gambit to be $T_0 = 0$, which implies that $t = 0$ at the beginning of the maneuver. The total time of the chase is the sum of the time spent in the linear chase, plus the time spent in the maneuver, plus the time spent following the maneuver, assuming that it is successful. In **Figure 2**, we label the four critical time events.

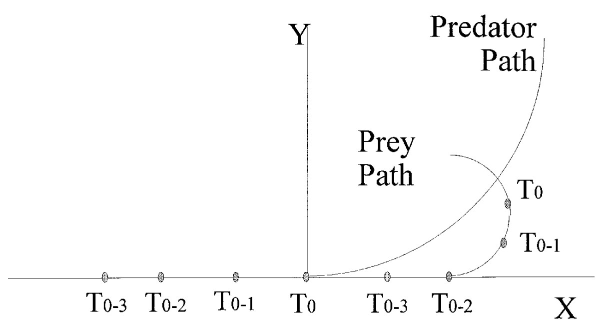

Figure 2. Turning gambit.

At T_{0-3}, the prey begins to decelerate. At T_{0-2}, the prey enters the turn. At this point the predator recognizes the turn and experiences a reaction period. At T_{0-1}, the predator begins to decelerate while the prey continues on the turn. At T_0, the predator begins his turn. The gambit is successful if the prey is able to reach the intersection prior to the predator reaching that point. Since the interval between T_{0-3} and T_0 is easily calculable, and the distance traveled by the predator is along a straight path, we calculate this time and distance separately:

$$[T_0 - T_{0-3}] \text{ s}, \qquad [(T_{0-1} - T_{0-3}V_v + (T_0 - T_{0-1})V_v + \frac{1}{2}G_v g(T_0 - T_{0-1})^2] \text{ m},$$

and begin our calculations for the turning gambit at point T_0. The equations for the two arcs are as follows:

$$x_v = \sin t, \qquad y_v = 1 - \cos t;$$

$$x_t = x_0 + r\sin\left(\frac{v[t + (t_0 - t_{0-3})]}{r}\right), \qquad y_t = r - r\cos\left(\frac{v[t + (t_0 - t_{0-3})]}{r}\right).$$

The prey reaches the safe area if it arrives at the intersection of the two arcs prior to the arrival of the predator. The intersection occurs at $x_v = x_t, y_v = y_t$. These values can be computed using the bisection method, or alternatively by Newton's method, since the predator's path is limited to the first quadrant. Using a relative speed of 0.33, a combined normalized reaction and deceleration time of 0.2 (which corresponds to a real time significantly less than 0.2 s, so in essence this is reaction time only, with no deceleration), and a normalized x_0 value of 1.06, intersection occurs at $(0.9, 0.66)$, and both animals arrived simultaneously. Thus, this starting point was a poor choice for the prey.

Once we have solved the intersection problem, it is easy to find the minimum distance required for the gambit to be successful. If detection has not occurred outside of this range, then the prey's gambit will fail and the predator will win. An effective stealth strategy is therefore beneficial to the predator.

The intersection calculated above, which reflects the assumptions in the problem statement, represents the critical point: Any x_0 less than 1.06 that does not permit the predator to catch the prey on the predator's linear path will result in escape. Assuming a 1.5 m radius for the predator, this represents a negligible elapsed time; but if we assume a more reasonable radius of 14.2 m, this corresponds to 23.9 m, or roughly 1.5 s. During this time, the predator has completed roughly one-fourth of a circle, so an additional 3.0 s will expire prior to its re-establishing a linear vector with respect to the prey. So if the prey bolted when the predator was 30 m or farther away, and M_v is 15 s or less, then the predator will have failed to capture the prey and the game is over.

In fact, with a relative radius of 0.33, the prey can repeat the winning strategy indefinitely, regardless of the actions of the predator, assuming that the predator reacts to the prey's maneuvers. Therefore, we would recommend that a single predator attempt to anticipate the optimal distance for a turning maneuver, guess the direction, and turn preemptively. If the predator notices that the prey has not turned after the beginning of the predator's preemptive move, then the predator should change to the opposite direction. This will force the prey to turn in the revised direction, decreasing the length of the arc prior to intersection with the predator's path. This results in a probability function for the predator of

$$\frac{(t_0 - t_{0-1}) + (t_{0-2} - t_{0-3})}{(t_0 - t_{0-3})} \cdot P[\text{successful attack when within range}].$$

Model 2: Multiple Predators, Single Prey

Our first objective is to define a distance function representing the prey's possible destinations, given a finite escape window. We assume that the prey can either continue in the forward direction at its maximum speed, or make a turn with a radius less than or equal to its minimum turning radius, or it can come to a full stop and resume in any direction. Although we account for acceleration and deceleration time when the animals change direction, we assume that their deceleration time going into a curve and their acceleration times coming out of curves are instantaneous. This somewhat overstates the available forward region; but it overstates it for both predator and prey, and it doesn't change the characteristics of the safety and danger regions (as we will show). So we believe that it will result in a slight understatement of the "double danger" region, and an even smaller impact upon the overall probability function.

As we discussed in the one-one model, a tighter turning radius implies a smaller speed, given a limiting acceleration rate. In **Figure 3**, the region that can be reached in a finite time period, in the forward direction, is the area in the figure marked "A".

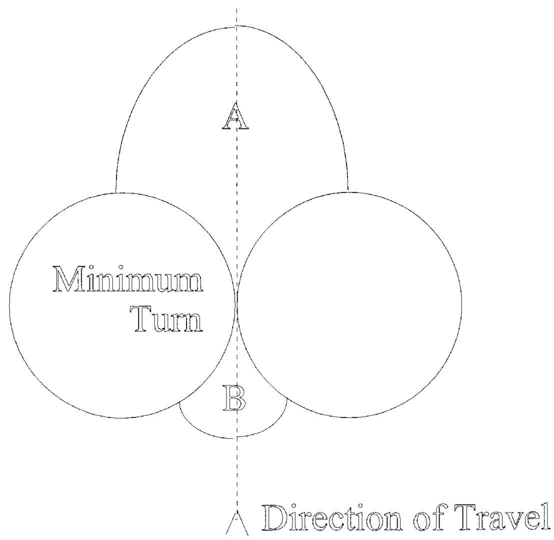

Figure 3. Accessible area.

The area marked "B" represents the reachable area if the animal comes to a complete stop, turns 180°, then resumes travel in the reverse direction and remains subject to the minimum turning radius. In fact, the animal could resume travel in any direction; however, given the reality of high-speed pursuit, it likely will resume maximum speed as quickly as possible, so the area inscribed by the figure is accurate, although the direction may not be. The area in region B and the area in region A are described by the same equation, although the time value in the B area is smaller since the deceleration time must be deducted. The equations for determining these areas are

$$r = \frac{V^2}{a}, \qquad \theta = \frac{tv}{r} = t\sqrt{\frac{a}{r}},$$

$$x = r(1 - \cos\theta), \qquad y = r\sin\theta,$$

$$\text{Area}(A) = 2\int_{V^2/M}^{\infty} r\sin\left(t\sqrt{\frac{a}{r}}\right)\, dr - 2\int_0^D \sqrt{\left(\frac{V^2}{M}\right)^2 - x^2}\, dx,$$

where

$$D = \left[1 - \cos\left(t\sqrt{\frac{a}{V^2/M}}\right)\right]\frac{V^2}{M}.$$

The calculation for Area(B) uses the same equation, except that the time value t is replaced by $(t - t_b)$, where t_b is the total time required to decelerate, stop, and accelerate to full speed in the new direction.

Given our area equation, we now have the necessary tools to develop an optimal chase strategy. The probability of a successful hunt is

$$\begin{aligned} P[\text{successful hunt}] = \{ &\text{Overlap}(\text{Prey}, \text{Predator1}) + \text{Overlap}(\text{Prey}, \text{Predator2}) \\ &- 2 \times \text{Overlap}(\text{Predator1}, \text{Predator2}) \} \\ &\times P[\text{successful hunt with single predator}] \\ &+ \text{Overlap}(\text{Predator1}, \text{Predator2}) \times P[\text{success of two predators}], \end{aligned}$$

where Overlap(A, B) is a function computing the area of the overlapping region between areas A and B. This function can be constructed as a combination of two iterations of the same area integral used previously, with adjustment for the relative positions and orientations of the two areas. We have not constructed this function during this project.

Given our probability function, and the existence of an Overlap function, we can numerically solve for the optimal displacement and orientation vectors that maximize the value of P[successful hunt]. With reasonable values of the two success functions, we would expect a strategy of converging attack, with one predator remaining sufficiently behind, such that the probability of capturing the prey throughout its entire region is at least P[successful hunt with a single predator], as shown in **Figure 4**.

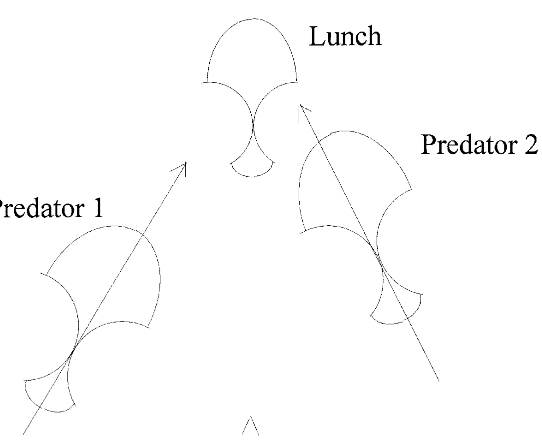

Figure 4. Optimal attack vectors.

Note that we have drawn the accessible regions quite smaller than their actual size, to illustrate the relationships without the clutter of overlapping lines and curves. Any strategy where the predators are parallel to the x-axis is inferior, since at some point in the chase the B region of the prey will be uncovered by either probability function. This in turn gives rise to an effective prey strategy. It is in the prey's interest to have such an uncovered region exist. Therefore, the prey's best strategy is to alter course in the direction of the trailing predator, in an attempt to achieve a normal escape vector with respect to the lines between the two predators. If the prey is able to get directly between the two predators, with a direction vector directly towards one of the two, it can employ the strategy outlined in the first model, effectively reducing the problem to a single-predator problem. These simple assertions address all of the possible cases of the two-one problem.

Adding additional predators increases the successful hunt probability and effectively eliminates the prey's linearization strategy, since getting three or more predators collinear is more difficult than two. Whether this total probability is superior to the case where two or more separate games (of one-one or two-one) are conducted concurrently depends on the probability values assigned to each function. With the exception of assuring that all predators avoid collinearity, the strategy employed by each predator in a group, except for the lead predator, is identical. This results in the prey being coerced into a spiral path with decreasing predator-to-prey distances. In any case of two or more predators, the prey is in a very poor situation.

Limitations of the Model

As with most mathematical models, our model has a number of limitations.

- Our model deals with a one-time chase and gives no indication of the result of a second encounter between the same prey and the same predator. We have not explored the learning that each animal undergoes between chases.

- Our model deals solely with the optimum distance at which the prey should initiate a turning strategy and does not take into account the scenario of detection of the predator by the prey. If the distance at which the prey first detects the predator is smaller than the optimal distance at which the prey should employ its turning maneuver, then the prey will not escape successfully.

- Although the probability functions that assign probabilities of a successful hunt to various areas of overlap exist, these functions may be difficult to derive explicitly.

- Our models do not rely extensively on the actual biological or mechanical aspects of the dinosaurs themselves, and this limitation is probably the most difficult one to overcome, since the only data regarding these dinosaurs are found in fossils.

- We have limited knowledge of the *Velociraptor*'s hunting habits. We are uncertain whether they are search-oriented creatures (as we have assumed from the outset) or whether they wait and hide for their prey.

- We have treated the dinosaurs as point masses. In reality, even if their trajectories do not cross at exactly the same time, it is still possible that the two dinosaurs come sufficiently close that the predator can reach the prey, or the predator may otherwise jump out to capture the prey.

Discussion of Alternatives

Our model addresses the most obvious variable characteristics of a predator-prey relationship but it does so in an idealized manner. We assume that the prey and the predator are making relatively informed decisions and that they are making rational choices based on perceived information. While we are confident that we in fact do model the behavior of the animals, we are uncomfortable with the premise that the animals are solving optimization problems while running for their lives. In this section, we discuss a method by which the predator and prey can essentially choose the expected actions but that does not require the assumption of advanced cognition. Specifically, we are searching for general strategies that optimize success in all of the various $n \times m$ models while minimizing both the information-gathering and cognitive-processing requirements.

Our method uses genetic programming to develop both the prey and predator models. The essence of this method is the generation of "chromosomes" that contain "genes" in a manner that selects for fitness. In this context, the chromosomes represent entire programs and the genes represent individual program steps. Given their differing objectives, the predators and prey have different genes and chromosomes. The process of genetic programming is to generate randomly several hundred "individuals," to test each individual's fitness, then to select the most fit individuals for reproduction.

Reproduction involves copying the individuals in the culled pool and randomly applying certain mutations. At the end of the reproduction stage, the new pool of individuals is again tested; and the process is repeated until the pool is comprised exclusively of fit individuals.

It is fairly easy to develop a predator model that can always converge on a stationary prey or set of prey. Koza [1992] describes two genetic programs relating to ant behavior. The first involves a trail of food with an objective of passing over each item in a limited amount of time [1992, 251–257]. Using an extremely limited function set consisting of

- terminal actions `Move`, `Right`, and `Left`;

- decision functions `If-food-present-do-next-else-do-subsequent`, `Do-two-actions-sequentially`, and `Do-three-actions-sequentially`;

- 20 genes,

it took only 21 generations to evolve a program that successfully located 89 of 89 food objects. Expanding the function set to a total of nine operations, including `Drop-pheromone` and `Move-to-adjacent-pheromone`, with 47 genes, resulted in a single program produced by generation 38 that could be executed by each member of an ant colony, with the result that the colony was able to locate and transport 144 food objects from two locations within a limited time period [1992, 310–317].

We developed a set of primitives for both the prey and predator programs that could be used to evolve effective strategies genetically. The requirement of adapting to the presence of one or more additional predators makes both the prey and predator programs significantly more complex than Koza's ant functions, but they are entirely reasonable as possibilities for higher animals. Thus, we would expect to need a chromosome with roughly 1,000 genes, and we would expect a minimum of several hundred generations before an adequate program evolved. If dinosaur cognition is primarily instinctive, as is the case for ants, one might argue that the predators are unlikely to survive a learning process of such a duration. On the other hand, if the dinosaurs learn their hunting strategies, then an individual need only participate in a few hundred hunts to master or develop a successful technique. The prey do not get the opportunity to learn from their mistakes. Thus, we would suspect that their responses need to be more instinctual. We would therefore argue that necessary conditions for this mechanism to be adopted by both prey and predator would be r-favored reproduction by the prey and relatively high intelligence in the case of the predators. These traits are consistent with current theory regarding both animals.

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Lunch on the Run

Gordon Bower
Orion Lawler
James Long
University of Alaska Fairbanks
Fairbanks, AK 99775

Advisor: John P. Lambert

Introduction

We devised two different models for this pursuit: a purely mathematical model, and a computer model. Both models use the following assumptions:

- All animals are represented as points at their respective centers of mass.
- The simulations and chase both begin when the predator, slowly sneaking up on the prey, is spotted by the prey.
- The chase lasts 15 s or until the prey is killed, whichever occurs first.
- No animal may travel at more than its specified maximum speed.
- Each time that the distance from the prey to the predator is at a local minimum, the prey takes a chance of being killed.

Table 1 summarizes notation used in the paper.

Introduction to the Mathematical Model

The mathematical model also assumes that it takes negligible time to start and stop moving but that the turning radius imposes a maximum angular velocity. This assumption makes the analysis possible without a computer but neglects the finite acceleration capability of real animals. This model can analyze only the one predator/one prey situation; it works quickly and well.

We analyze two predator strategies and one prey strategy. The prey strategy is to run directly away from the predator until the predator gets closer than

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some critical distance, then make a sharp turn to the left or right. The prey can just squeak by the predator, whose larger turning radius makes it lose a few meters. The predator strategies considered were the hungry predator, who always heads straight for the current location of the prey if possible, and the maximal-turning predator, who uses knowledge of the prey's strategy to turn more sharply than the hungry predator in an effort to cut off the prey.

We calculate the probability of the prey's survival, given parameters about the predator and prey paths and the distribution of initial separations.

Introduction to the Computer Model

We use an iterative algorithm to figure out where the predators and prey go and which prey survive. By running several thousand 15-s scenarios, we determined how survival rates change depending on predator strategy, prey strategy, initial separation, the elevation and terrain, animal reaction times, and so on.

The computer model supports n predators chasing m prey across any terrain. It takes into account acceleration/deceleration, reaction times, two predator strategies, and three prey strategies. Written in C++ and running on a PowerPC 604–based 132 mHz workstation, the program can calculate 600 15-s scenarios using a 1-ms timestep in 1 min.

Non-Chasing Phases of the Hunting

Locating Prey

Thescelosaurus is similar in size to *Velociraptor*, so a single kill would provide the predator with sufficient meat for several meals; *Velociraptor* need bring down only one victim every few days. Assuming that each predator-prey interaction is independent, the number of chases required to kill one animal has a geometric distribution with parameter $1 - P$, the probability of a kill on one chase. Thus the expected number of chases per kill is $1/(1 - P)$, with variance $P/(1-P)^2$. Assuming a typical value for $P = 0.3$, the predator usually succeeds on the first or second attempt. More than four chases will be required only 1% of the time.

Stalking

A predator, having discovered prey as yet unaware of the predator's presence, normally attempts to approach the prey stealthily, in an effort to reduce the length of the inevitable high-speed chase. Likewise, the prey normally attempts to be vigilant, making it difficult for the predator to approach unnoticed.

Table 1.
Notation.

(x_1, y_1)	location of the predator, *Velociraptor mongoliensis*
(x_2, y_2)	location of the prey, *Thescelosaurus neglectus*
v_1	speed of the predator; usually 50/3 m/s, or 60 km/h
v_2	speed of the prey; usually 125/9 m/s, or 50 km/h
r_1	full-speed turning radius of the predator; usually 3/2 m
r_2	full-speed turning radius of the prey; usually 1/2 m
$r_v = v_1/(v_1 - v_2)$	ratio of predator speed to closing speed, i.e., how far the predator must run in a straight-line chase to get 1 m closer to the prey; usually 6
θ_1	orientation of the predator, in radians
θ_2	orientation of the prey, in radians
t	time
$t_{\max}$	maximum chase duration (running time for the predator); usually 15 s
h_I	separation of predator and prey at the beginning of the chase
h_B	separation of predator and prey when the prey attempts to break away by turning
$d_I = r_v(h_I - h_B)$	distance traveled by predator during initial straight-line pursuit
d_B	distance traveled by predator during the prey's breakaway maneuver
d_G	distance traveled by predator to regain ground lost during a successful breakaway maneuver by the prey
$S_i(d)$	closest approach of predator and prey during the first d meters traveled by the predator during the ith phase of the chase ($i = 1$: initial straight pursuit; $i = 2$: breakaway maneuver; $i = 3$: catch-up pursuit following successful breakaway)
$p_S(S)$	probability that the prey survives one close approach to the predator, during which its closest approach to the predator is S
$p_H(h_I)$	probability that the prey survives a complete attack, given initial separation h_I
$f(h_I)$	probability density function for the initial separation of predator and prey
a, b	shape and scale parameters for a gamma distribution
ν, ω	shape parameters for a beta distribution
P	probability of prey surviving a typical attack

Neither of these behaviors appears explicitly in our models. Instead, our models calculate $p_H(h_I)$, the conditional probability of the prey's survival, given the separation at the start of the chase, and combine p_H with $f(h_I)$, the probability density function for the initial separation. We account for the stealth of the predator and the vigilance of the prey by our choice of f. The more attentive the prey, the higher the mean of f; the more stealthy the predator, the lower the mean. We model f using gamma and beta distributions, whose parameters can be altered easily to reflect conditions.

Analytic Model for Trajectory Selection

Model Development

The problem statement does not specify the acceleration capabilities of predator or prey, but it does specify turning radii (which, considering the speeds involved, are perhaps too small). We choose to permit infinite acceleration, i.e., at any given time, each animal may select any speed between zero and its maximum speed. However, to preserve the turning-radius limitations, we also introduce a maximum angular velocity.

The maximum angular velocity is calculated from the maximum speed and minimum turning radius. Given a speed v and radius r, turning $180^\circ = \pi$ radians requires traveling πr m, which can be done in $\pi r/v$ s. Thus, we obtain

$$\left|\frac{d\theta}{dt}\right|_{\max} = \frac{v}{r},$$

which, using the constraints in the problem, results in maximum angular velocities of 100/9 rad/s for *Velociraptor* and 250/9 rad/s for *Thescelosaurus*.

Further, we assume that the predator begins running directly toward the prey and the prey begins running directly away from the predator.

The pursuit and evasion strategies consist of recipes for choosing new angular and linear velocities, given current positions and velocities.

Strategy for Predator and Prey Far Apart

If the predator is sufficiently far away, since the predator tires quickly, the prey can escape by simply outrunning the predator. In general, this is possible when $v_1 t < v_2 t_{\max} + h_I$, or, rearranging terms, when $h_I > (v_1 - v_2)t_{\max}$. In the situation of the problem statement, $(v_1 - v_2)t_{\max} = 125/3$ m.

If the predator and prey are closer than $(v_1 - v_2)t_{\max}$ meters, but the separation is still large in comparison to the turning radii, then neither predator nor prey gains any benefit from being the first to deviate from a straight path. If the predator turns, it increases the minimum path length required to catch up to prey moving straight ahead. In addition, when the prey sees the predator turn one way, the prey can respond by turning in the opposite direction, further increasing the separation between them. Similarly, if the prey deviates from its straight course, it is inviting the predator to turn in the same direction; the predator's arc will lie inside the prey's, meaning that the prey has unnecessarily helped the predator to catch up.

Strategy When Predator and Prey Are Near

If the prey is close to the predator, a sharp turn by the prey may enable it to reach a "safe zone" that the predator—faster, but handicapped with a wider turning radius—cannot enter without stopping and turning around.

Since the prey's goal is to reach the "safe zone" while giving the predator a minimum amount of time to respond, the prey should always make a minimum-radius turn. But when is the ideal moment to make the turn? If the prey turns while the separation is still wide, it will run into the waiting jaws of the predator; if it waits too long, it may be within the predator's grasp before the turnaround maneuver is complete. The answer depends on what strategy the predator uses to respond to the prey's sharp turn, which depends on how much the predator knows about the prey's intentions.

If the predator turns itself around and catches up to the prey again, a second breakaway maneuver may be necessary, then a third, and so on. Even if the idealized "point animals" never touch other, the breakaway maneuver is not risk-free. We assign the prey a probability of surviving each breakaway, which depends on the closest approach distance.

By computing $d\left[(x_1-x_2)^2+(y_1-y_2)^2\right]/dt$ and seeing how it depends on dx_1/dt and dy_1/dt, we learn that the predator should continue moving forward only if the prey is in front of, rather than behind or directly alongside, the predator. Computationally, this amounts to finding the angle between the predator's course and the azimuth from predator to prey:

$$\text{Move forward if}\quad \left|\theta_1-\tan^{-1}\left(\frac{y_2-y_1}{x_2-x_1}\right)\right|<\frac{\pi}{2},$$

with the minor caveat that it may be necessary to add 2π to or subtract 2π from θ_1 to ensure that θ_1 and the calculated (normally the principal) inverse tangent differ by π or less. This decision procedure tells the predator only *if* it should move forward, not *how fast*; it is often not always best for the predator to travel at top speed.

For a predator always running at top speed, it is not particularly difficult to compute the locus of all paths that it can follow in a given time. For a predator with speed v and turning radius r at the origin, initially moving along the y-axis, the region is bounded by three curves: the two circles of radius r centered at $(\pm r, 0)$ and the arc described by the parametric equations

$$x=\pm r\left[1-\cos u+\left(\frac{vt_{\max}}{r}-u\right)\sin u\right],\qquad y=\pm r\left[\sin u+\left(\frac{vt_{\max}}{r}-u\right)\cos u\right]$$

for $u\in[0, vt/r]$. The resulting mushroom-shaped region is shown in **Figure 1**. The problem is isomorphic to the familiar geometric problem of a swinging rope hanging between two tangent circles and closely related to the swinging rope hanging from the cusp of a cycloid (i.e., a pendulum of uniform period). These problems are considered in various common references, such as Wells [1991].

For the parameter values of the problem statement, the prey can always get outside of this locus by executing a sharp turn, provided the prey is less than about 1.57 m away. (We conjecture that the condition is $h_B<\pi/2$, but we did not have the time to prove it.)

The Hungry Strategy

"Run directly toward the prey, if possible; if not, turn to face the prey, moving in its general direction if possible." This type of strategy is traditionally called

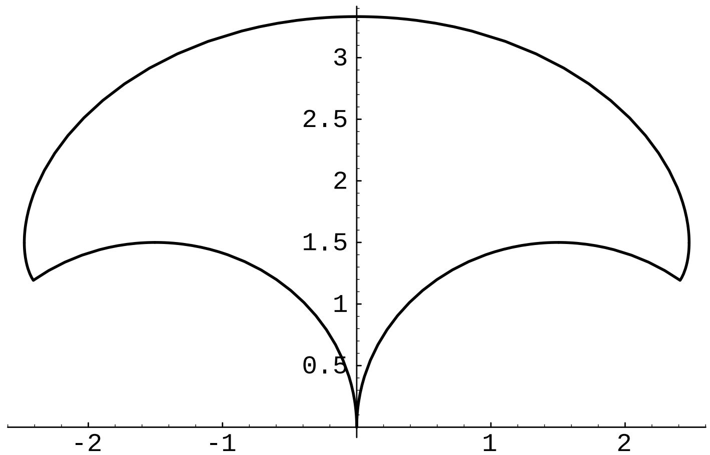

Figure 1. Locus of predator's paths 0.2 s after breakaway.

a "greedy algorithm"; in the context of this problem, "hungry" seems a more evocative name.

For the predator to select an angular velocity, it needs to know how the angle between his course and the prey's position is changing. This amounts to calculating the derivative of $\tan^{-1}[(y_1 - y_2)/(x_1 - x_2)]$, which is

$$\frac{y_2 - y_1}{(x_2 - x_1)^2 + (y_2 - y_1)^2}\left(\frac{\partial x_2}{\partial t} - \frac{\partial x_1}{\partial t}\right) - \frac{x_2 - x_1}{(x_2 - x_1)^2 + (y_2 - y_1)^2}\left(\frac{\partial y_2}{\partial t} - \frac{\partial y_1}{\partial t}\right),$$

where

$$\frac{\partial x_i}{\partial t} = v_i \cos\theta_i, \qquad \frac{\partial y_i}{\partial t} = v_i \sin\theta_i.$$

Note that all quantities in this formula—x_1, x_2, y_1, y_2, v_1, v_2, θ_1, and θ_2—are assumed known to both animals at all times. In our basic hungry algorithm, the predator charges ahead at full speed all the time (which turns out to be slightly wasteful in certain situations.)

A computer simulation plotted paths resulting from a hungry predator pursuing prey that executed a breakaway maneuver at arbitrary distance h_B from the predator. The ideal moment for the prey to begin turning is when predator and prey are 1.62 m apart; this guarantees that the predator and prey never come nearer to one another than 0.88 m. The predator runs 1.94 m before the hungry algorithm demands that it stop. (Note the inefficiency of the predator's course; it was shown above that for $h_B > 1.57$ m, there is a predator path that prevents the prey's escape.)

Following the breakaway maneuver, the prey is standing very close to, but slightly behind, the predator. At this point, the prey's greater maneuverability is sufficient to prevent the predator from turning around to face the prey. Unable to move toward the prey until facing toward it, the predator will come to a halt and rotate in place. We assumed that time spent rotating but not moving forward does not count in the predator's 15-s allotment.

Given the predator's angular velocity limit of 100/9 rad/s, the prey can retreat to a distance of 1.25 m from the predator (the greatest radius at which a speed of 125/9 m/s and an angular velocity of at least 100/9 rad/s are compatible) and still prevent the predator from turning around.

The prey neither gains nor loses by remaining at the 1.25 m distance; since the prey can't run in a circle forever, eventually it has to make a break for it again. We could not determine analytically the precise distance that the prey could travel before the predator catches up and forces another breakaway attempt.

If the prey begins to run directly away from the predator instead of circling, the predator has to run about 14 m under the strict greedy algorithm to force another breakaway maneuver. By turning in place a while longer, a smarter predator could cut this down to just under 12 m.

Alternatively, the prey could simply begin to run in a straight line. The hungry predator then requires about 15 m to catch up. By turning in place longer, the predator can reduce this to 13 m. Experimentation indicates that an optimal run-at-full-speed strategy is unlikely to change these numbers significantly.

The distances ($h_B = 1.62$ m, $d_B = 1.94$ m, $d_G \approx 14$ m, $S(d_B) = 0.88$ m) are used in a statistical procedure, described later, that uses the probability density of h_I to calculate the prey's chance of surviving for 15 s. The procedure returned survival probabilities of $P = 0.20$ for a reference beta density, $P = 0.28$ for a reference gamma density, $P = 0.44$ for a modified beta density, and $P = 0.43$ for a modified gamma density.

A numerical model discussed later implements a similar greedy algorithm. The results of that algorithm were qualitatively similar, though the exact distances, turning points, and so on were slightly different.

The Maximal Turn Strategy

The hungry strategy is a good all-around strategy for a predator that does not know what its prey is likely to do next. However, an intelligent predator might notice that, once the prey has begun its tight circle, it plans to continue circling tightly. This more intelligent predator, rather than aiming at the prey, might respond by turning in the same direction, as sharply as possible, in an effort to cut off the prey's anticipated escape.

Suppose the predator is at the origin, running along the y-axis, and the prey is at $(0, h_B)$, also running along the y-axis, when the prey initiates a breakaway maneuver. If the predator responds by simultaneously making a minimum-radius turn in the same direction, then the paths of predator and prey are given by the following parametric equations:

$$x_1 = r_1\left[1 - \cos\left(\frac{v_1 t}{r_1}\right)\right], \qquad y_1 = r_1 \sin\left(\frac{v_1 t}{r_1}\right)$$

$$x_2 = r_2\left[1 - \cos\left(\frac{v_2 t}{r_2}\right)\right], \qquad y_2 = h_B + r_2 \sin\left(\frac{v_2 t}{r_2}\right).$$

With the aid of a Mathematica-type application or a simple program, it is quite simple to calculate $\min\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}$. This figure represents the closest approach of predator to prey during the breakaway. The prey's survival depends on choosing an optimal breakaway time, i.e., the optimal h_B is given by $\arg\max\{\min\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}, h_B\}$, which comes to h_B = 0.66 m, d_B = 1.52 m, $S(d_B)$ = 0.25 m. This indicates that the prey stands a much smaller chance of surviving a single encounter with this more intelligent predator. Should it survive one breakaway maneuver, the earlier remarks about circling at 1.25 m and attempting to flee apply almost unchanged; we have $d_G \approx 14$ m for this case also. The calculated survival probabilities were $P = 0.03$ for the reference beta distribution, $P = 0.15$ for the reference gamma distribution, $P = 0.21$ for the modified beta distribution, and $P = 0.29$ for the modified gamma distribution.

A Refined Trajectory Model

It is sometimes right for the predator not to run at full speed. Neither the hungry nor the maximal-turn strategies are optimal—the predator does better by turning in the same direction as the prey and simultaneously slowing down. This prevents the prey from exploiting the time during which the prey and predator are running in opposite directions on a non-collision course.

Can such a strategy be determined? Doing so requires an algorithm for minimizing travel time given initial position, initial velocity, and final position, subject to the constraints that $v(t) \leq v_1$ and $|d\theta_1/dt| \leq v_1/r_1$. Then we must determine what point along the prey's circular path is the best destination, with the prey trying to maximize, and the predator trying to minimize, the closest-approach distance. We could not solve this problem analytically nor implement a good numeric approximation, so it is probably unrealistic to expect a Cretaceous dinosaur to calculate the solution in its head instantaneously. So this ideal strategy, though of theoretical interest, likely would not be usable.

Flaws of the Analytic Approach

The ability of both dinosaurs to select any speed they wish, that is, to be capable of brief bursts of essentially infinite acceleration, is unrealistic. Neither animal can leap into the chase at full speed or stop on a dime. The predator has to waste some time running in the wrong direction if the prey survives a breakaway. The effect is that d_G, in a practical sense, will always be in the 20–25 m range, not 14 m as proposed above. The prey's survival probability is correspondingly higher. **Table 2** gives P for both the hungry and maximal-turn strategies, recalculated using d_G = 22 m. The results agree more closely with the computer simulations than do the earlier numbers for 14 m.

Table 2.
Survival rates with revised d_G.

Distribution	P for hungry strategy	P for maximal-turn strategy
Reference beta	.38	.03
Reference gamma	.35	.15
Modified beta	.51	.21
Modified gamma	.49	.29

Effect of Changing Minimum Turning Radii

Extreme forces are involved in the sharp high-speed turns that we have discussed. Doubling the turning radius of each species would halve the force that each animal would need, increasing the realism of the scenario. The shape of all trajectories would remain unchanged but the scale would be doubled. This increase in realism would greatly improve the prey's chances: The closest approaches would be 1.76 m and 0.50 m, resulting in much lower death rates than the current 0.88 m and 0.25 m. Similarly, d_I would remain unchanged, but d_B and d_G would be doubled, which would substantially reduce the number of breakaway cycles that prey would have to endure to survive a 15-s chase. **Table 3** gives P, recalculated using the doubled values, including $d_G = 44$ m.

Table 3.
Survival rates with doubled turning radii.

Distribution	P for hungry strategy	P for maximal-turn strategy
Reference beta	.84	.08
Reference gamma	.94	.19
Modified beta	.95	.27
Modified gamma	.95	.33

Implications

When the prey is forced to pass very close to the predator to break away, its chance of surviving a single breakaway maneuver is very low, and the chance of surviving two or more such turns is virtually nil. In this case, the probability of survival is essentially equal to the probability of spotting the predator at a distance of 125/3 or more meters. Thus, increased vigilance of the prey would be expected to carry a strong evolutionary reward. On the other hand, if the prey can escape easily during a breakaway cycle (e.g., hungry predator and doubled radii), the predator is exerting minimal selection pressure against inattentive prey.

Comparing our calculated probabilities with a subjective assessment of the survival rates of modern prey, it seems that the hungry strategy with tight turning radii, and the maximal-turn strategy with doubled radii, yield the most plausible results. This seems reasonable: The strategies involved are simple and "obvious" enough that moderately intelligent animals could probably implement them.

Converting Trajectory Descriptions to Survival Probabilities

Overview of Procedure

The trajectories produced by all of our models have some common structural features. In particular, the trajectories can be divided into three phases:

- The first phase is a straight-run segment, during which the predator gains on, but is virtually never able to kill, the prey.
- The second phase is the breakaway maneuver, during which the prey makes its closest approach to the predator and runs a significant risk of being killed.
- The third phase is a brief period of almost-straight chasing as the predator attempts to reclaim the distance the prey gained via a successful breakaway.

Each trajectory, regardless of the model that produced it, begins in phase one, then alternates between phases two and three for the remainder of its length. Within a given trajectory, all complete phase-two episodes will be the same length, as will all complete phase-three episodes. The final segment, during which the predator's allotted time expires, will of course be incomplete.

We assume that the prey has a nonzero chance of being killed each time the distance between predator and prey reaches a minimum. We model the chance of survival of each encounter as a function of minimum separation, p_S. The probability of the prey surviving the entire 15-s encounter depends primarily on the number of close encounters, i.e., the number of times the prey must attempt the breakaway maneuver. This number depends on two things. The first is the length of the phase-two and phase-three episodes. This depends only on the model design; we calculated these lengths for each model we subjected to the statistical procedure. The second factor is the length of the initial sprint. This depends on the optimal breakaway separation h_B, which depends on model design, and on the separation at the beginning of the chase, h_I. It is thus possible to plot p_H as a function of h_I.

The initial separation will be different every time the predator goes hunting. Initial separation is influenced by terrain and vegetation, visibility conditions, the alertness of the prey, the stealth of the predator, and countless other factors. Rather than trying to account explicitly for the effect of each factor, we opt instead to treat h_I as a random variable, the shape of the probability density function being chosen based on these factors. Generally speaking, bare ground, good lighting, and attentive prey cause the mean of $f(h_I)$ to be high, while obstacles, fog, darkness, or particularly stealthy stalking on the predator's part cause the mean of $f(h_I)$ to be low. In addition, the problem statement proposed 15 m and 50 m as minimum and maximum values for h_I.

Once a distribution of h_I has been selected, we can use $p_H(h_I)$ and $f(h_I)$ together to determine P, the probability that prey survive a 15-s attack, given the conditions specified by model design, the choice of p_S, and the choice of f.

Method of Computing P

Three numbers and one function, based on the model design, are required for the computation of P. The three numbers are

- h_B, the optimal separation for attempting the breakaway maneuver;
- d_B, the distance traveled by the predator during a breakaway attempt; and
- d_G, the distance traveled by the predator while regaining ground lost during a successful breakaway.

The function $S(d)$ is the minimum separation during the first d meters traveled by a predator during a phase. During phase one, the function $S_1(d)$ is linear; during phase two, $S_2(d)$ decreases rapidly in a complicated way to a minimum value; that minimum value is characteristic of the model design. The definition of $S_3(d)$ is slightly different, since phase three begins with the prey close to but behind the predator, out of harm's way. The separation rapidly increases to a maximum, then drops approximately linearly as the predator catches back up to the prey, following an ever-straighter path. We define $S_3(d)$ as the minimum separation after maximum separation has already been reached, but infinite before that time. (Some of the path-determining models assume that, immediately after a successful breakaway, the prey can briefly move about in perfect safely.) For the purposes of computing p_H and P, S_2 is the only function of real interest, since the prey has virtually no chance of dying unless S is small.

It is necessary to choose a function p_S that relates the minimum separation to the prey's probability of surviving a close encounter with the predator. The choice of this function is arbitrary, subject to some obvious constraints:

- $p_S(0) = 0$: If the predator and prey actually contact each other, the prey will surely be killed. (Actually, one could realistically let $p_S(0)$ be a small positive number; it is plausible that the prey still has a slight chance of surviving a direct assault by the predator.)
- $p_S(x) \to 1$ as $x \to \infty$: If the predator never comes close to the prey, the prey clearly will survive.
- $\dfrac{dp_S(x)}{dx} \geq 0$: It is always safer to be farther away from the predator.

Velociraptor's most potent weapon was the large claw on each foot, and its legs were approximately 0.5 m long. Its mouth and forearms posed a significant but much smaller danger to the prey [Sattler 1983]. On the basis of these facts, it seems reasonable that the prey is in great danger if it is within one leg length (0.5 m). A distance of two leg lengths (1 m) ought to bring safety from the claws but not from the jaw. At distances significantly greater than 1 m, the danger should be negligible. We decided that $p_S(1) = 0.8$ seemed like a reasonable figure. For general distance x, we use

$$p_S(x) = \left(1 - \frac{1}{1+4x^4}\right) = \frac{4x^4}{1+4x^4},$$

though any of several other S-shaped functions would serve as well.

Calculating p_H as a function of h_I is the most complex portion of the computation of P. The function is piecewise defined. Letting

$$d_I = \left(\frac{v_1}{v_1 - v_2}\right) h_I,$$

we treat the three phases of the trajectory separately, saving phase two for last because it is computationally most difficult.

Phase One: If $d_I \geq v_I t_{\max}$, then $p_S(h_I) = p_S(h_I - (v_1 - v_2)t_{\max})$.

If the prey is able to outrun the predator, then the closest approach of predator and prey occurs at time $t_{\max}$, when the predator is forced to abandon the chase. Unless the prey was about to make its first breakaway attempt, probability of survival is essentially 1. On **Figure 2**, this phase produces a long plateau at the right.

Phase Three: If $d_I + k(d_B + d_G) - d_G \leq v_1 t_{\max} \leq d_I + k(d_B + d_G)$ for some $k \in \{1, 2, 3, \dots\}$, then let

$$d^* = v_1 t_{\max} - d_I - k(d_B + d_G) + d_G, \qquad p_H(h_I) = p_S[S_2(d_B)]^k p_S[S_3(d^*)].$$

This looks like an uglier computation than it is. Unless h_B is very small, $S_3(d^*)$, the final term in the product, representing the probability of surviving the beginning of a brief chase, is very close to 1. The first term in the product is simply the probability of surviving a single breakaway maneuver, raised to the kth power; each breakaway attempt is viewed as an independent event for the purposes of this calculation. In **Figure 2**, this phase produces the equally spaced plateaus that occupy most of the left portion of the plot.

Phase Two: If neither of the above is true, then

$$d_I + k\left(d_B + d_G\right) < v_1 t_{\max} < d_I + k\left(d_B + d_G\right) + d_B,$$

for some $k \in \{0, 1, 2, 3, \dots\}$; and letting $d^* = v_1 d - k\left(d_b - d_G\right)$, we have $p_H(h_I) = p_S\left[S_2(d_B)\right]^k p_S\left[S_2\left(d^*\right)\right].$

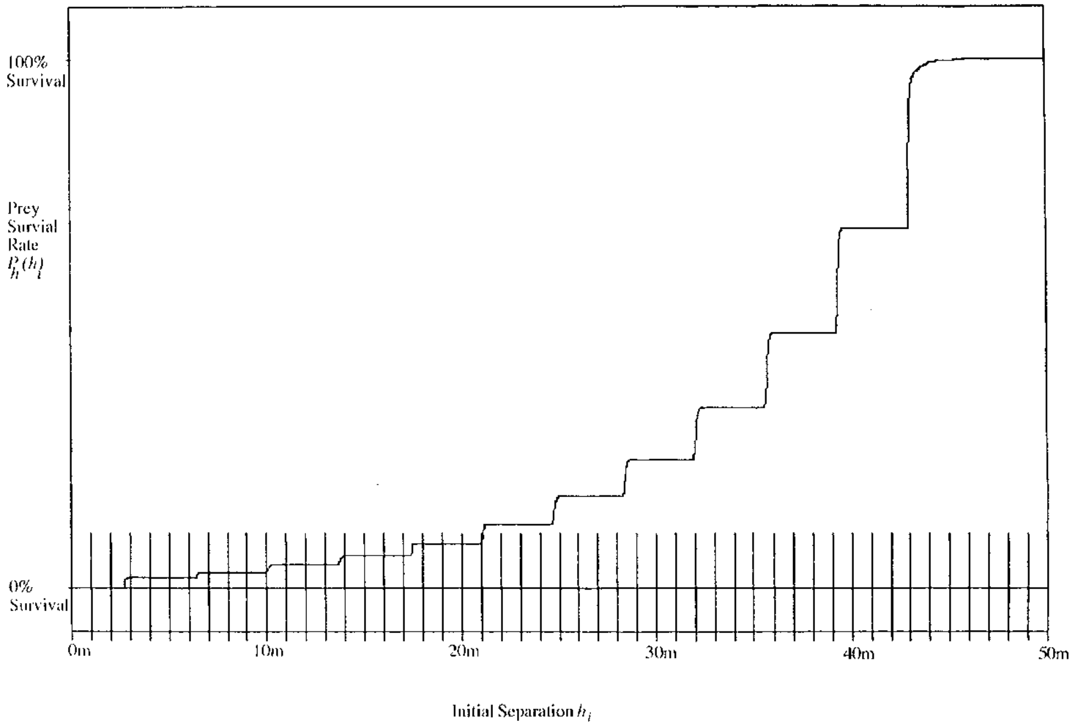

Figure 2. Prey survival rate vs. initial separation (mathematical model).

Phase two comes into play if the 15 s expires while the prey is in the act of attempting to break away. There is no way around computing $S_2\left(d^*\right)$ this time; S_2 has to be calculated numerically from a simulation of the phase-two trajectory. It makes a great deal of difference to the prey exactly how close the closest approach is: The prey is almost twice as likely to survive at 36 cm as at 30 cm! The term raised to the kth power represents the previously completed breakaways, while the last term is the probability of surviving the final, incomplete, breakaway. In **Figure 2**, this phase produces the S-shaped pieces connecting the plateaus on the left side of the graph.

Figure 2 is typical of most graphs of p_H. If S is never ignored, then $p_H(h_I)$ is a continuous monotonically increasing function with domain $[0, \infty)$ and range $[0, 1)$. As $p_S(S_2(d_B))$ becomes small, $p_H \to 0$ for all $h_I < (v_1 - v_2)t$.

The conditional probability of survival given initial separation is given by $p_H(h_I)$. Hence, by the law of total probability [Freund 1992], the unconditional probability of survival is given by $\int_0^\infty p_H(h_I) f(h_I)\, dh_I$, which, once p_H has been calculated, is extremely easy to calculate numerically.

For our tests, we use the gamma and beta distributions, two flexible and widely known families. We adapt the descriptions by Evans et al. [1993].

Gamma-distributed variates can take on any positive value. The density function has two parameters, with α controlling the shape and β controlling the scale:

$$f_{\text{gamma}}(x) = \left(\frac{1}{\beta^\alpha \Gamma(\alpha)}\right) x^{\alpha-1} e^{-x/\beta}.$$

The gamma distribution has mean $\alpha\beta$, mode $(\alpha-1)\beta$, and variance $\alpha\beta^2$. To reflect the conditions of the problem, we selected Gamma(6,5) as our reference gamma distribution; this particular density is less than 15 about 8% of the time and greater than 50 about 7% of the time. To determine how much it benefits the prey to be alert, thereby increasing the chance of seeing the predator at considerable distance, we repeated our calculations using the Gamma(6,6) distribution.

Beta variates, in their original form, have a domain of $[0,1]$. Multiplying by 35 and then adding 15 produces a beta-distribution on $[15,50]$, reflecting the stipulations of the problem statement. The transformed beta density function has two parameters, ν and ω, which together determine the shape of the density:

$$f_{\text{beta}}(x) = \left(\frac{\Gamma(\nu+\omega)}{35\Gamma(\nu)\Gamma(\omega)}\right) \left(\frac{x-15}{35}\right)^{\nu-1} \left(\frac{50-x}{35}\right)^{\omega-1} \qquad \text{for } 15 \le x \le 50,$$

with mean $15 + 35\nu/(\nu+\omega)$, mode $15 + 35(\nu-1)/(\nu+\omega-2)$, and variance $1225\nu\omega/[(\nu+\omega)^2(\nu+\omega+1)]$.

The ratio of ν to ω controls the mean of the distribution, while the variance is inversely proportional to $(\nu+\omega)$. We selected the Beta(2,3) distribution as our reference beta distribution. It has a mean, mode, and central "hump" shape very similar to the reference gamma distribution, but no tails sticking out below 15 and above 50. We also examined the Beta(3,2) distribution, with the same shape but a higher mean and mode, to check the effect of increased prey alertness. **Figures 3** and **4** show the four chosen densities. Note, however, that nothing in the calculation procedure limits us to the use of these densities.

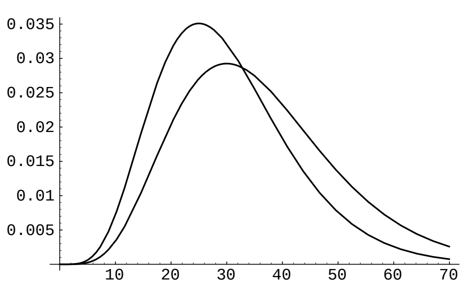

Figure 3. Reference and modified gamma densities.

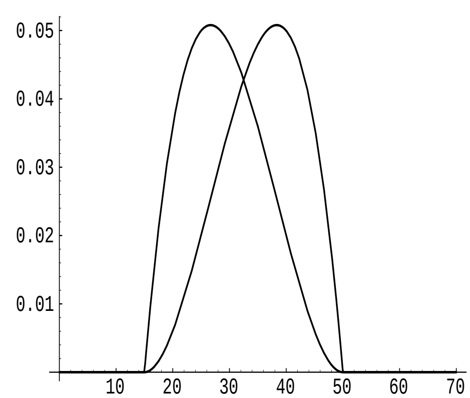

Figure 4. Reference and modified beta densities.

The Computer Simulation Model

Model Overview and Assumptions

The computer model uses physics-based motion, where accelerations are finite, instead of the more-theoretical motion of the mathematical model (in which acceleration is not considered). While there is no explicit limit on the turning radius (it isn't even calculated in the simulation), when we restrict our creatures' acceleration to the assumed value, then at top speed the animals can turn in no less than their minimum turning radius. Hence, although the mathematical and computation models' methods differ, the two models give the same answer.

Computer Model Assumptions

- The predator(s) slowly sneak up on the prey as long as the prey doesn't start to run away. At some distance h_I (where $15\text{ m} < h_I < 50\text{ m}$), the prey notices the approach and starts running.
- Each animal has a maximum speed.
- Each animal has a maximum acceleration, derived from $a = \dfrac{v^2}{r}$, where v is the animal's maximum speed and r is its minimum turn radius at top speed. We further assume that this acceleration can be applied in any direction—centripetal, tangential, or a combination thereof.
- At each instant, each animal figures out in which direction it would like to apply its acceleration, based on its present position and the position of every other animal. However, an animal knows only the position of the other animals 20 ms ago (its reaction time).
- Whenever a predator stops getting closer to the prey (i.e., the predator has made a close pass by the prey), a probability-of-death function, of the smallest distance between the predator and prey, is evaluated. The prey's probability of death depends only on the closest approach distance.
- Simulation continues until the prey dies or 15 s elapses.

Model Design

Our simulation models each animal as a point at its center of mass, which can accelerate in any direction. The simulation outputs smooth curves of the paths of the centers of motion of the animals, by iteratively solving these vector differential equations using Euler's method (outlined below):

$$\frac{d\vec{P}}{dt} = \vec{V}, \qquad \frac{d\vec{V}}{dt} = \vec{A},$$

where $\vec{P}$, $\vec{V}$, and $\vec{A}$ are the position, velocity, and acceleration as functions of time.

The simulation cycle begins by determining the optimal direction for the animal to accelerate, based on the animal's strategy. The animal accelerates at its maximum acceleration in that direction. Values for upper bounds on acceleration are parameters to the simulation, derived from the minimum turn radius at top speed via the central acceleration formula, $a = v^2/r$.

The acceleration vector is then added to the local elevation gradient multiplied by the acceleration of gravity, making it easier to go down a hill than up. The elevation gradient is determined from a bilinearly interpolated elevation grid, which is read in from an elevation file.

Once the animal has decided on a direction, the simulation applies one step of Euler's method to update its position and velocity vectors. This method of solving differential equations works by noting that the first two terms of the Taylor series expansion of a function of time depend only on the present condition of the system:

$$f(t+h) = f(t) + h\,\frac{df}{dt}.$$

Hence, if we know an animal's position, velocity, and acceleration at some time t, we can figure out a first-order approximation for where the animal will be at time $t+\Delta t$ by substituting the vector equations into the Taylor series expansion:

$$\vec{P}(t+\Delta t) = \vec{P}(t) + \Delta t \vec{V}(t), \qquad \vec{V}(t+\Delta t) = \vec{V}(t) + \Delta t \vec{A}(t).$$

Taking $\Delta t = 0.001$ s, and given the initial conditions and an acceleration vector, our simulation computes the velocity and position of each animal at each time step.

Life and Death in the Computer

In the computer model, each time the prey passes near the predator, a probability-of-death function is evaluated. The formula that we chose for the probability of death, given minimum separation, is

$$P_{\text{death}}(x) = \frac{1}{1+4x^4},$$

whose graph is shown in **Figure 5**. In the notation of the section **Converting Trajectory Descriptions** . . . , this function is $(1 - p_S)$.

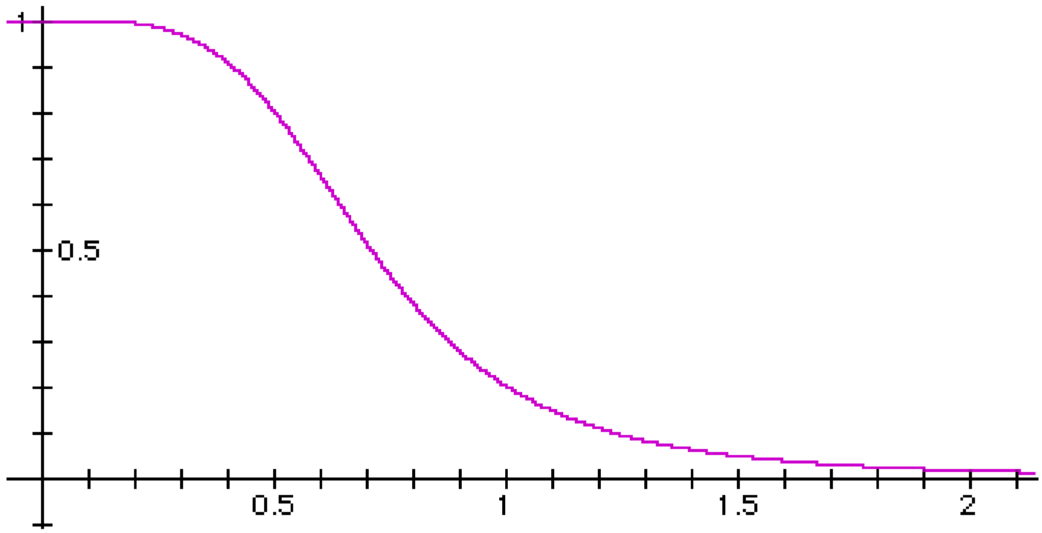

Figure 5. Probability of death vs. closest approach to predator (in meters).

Modeled Hunting Strategies

Every orientation of two animals can be simulated by starting the predator at the origin and the prey a distance h_I along the x-axis, where h_I varies in this simulation from 15 m to 50 m.

The Hungry Hunter

A hungry hunter heads straight for the current position of the closest prey. This is the only strategy that we model both analytically and numerically.

The Smart Hunter

A smart hunter determines the point where it can intercept the closest prey and heads straight for that point. A derivation of the quadratic equation that the smart hunter for the intercept point is given in **Appendix A**. [EDITOR'S NOTE: Omitted for space reasons.]

Modeled Evasion Strategies (One Predator)

The Frightened Prey

Frightened prey flee straight away from the nearest predator. These prey always die if the predator can close the distance between them before its 15 s are up. At a closing rate of 2.78 m/s, the prey will always be overtaken and die if h_I, the initial separation of the predator and prey, is less than 41.7 m.

The Smart Prey

When the nearest predator is far away, smart prey act like frightened prey and flee straight away. But when the predator closes to a critical distance h_B (1.619 m—see **Appendix B** for derivation [EDITOR'S NOTE: Omitted for space reasons]), the prey darts either to the left or to the right. By using its much smaller turn radius, the prey buys some distance, which is again closed by the predator, whereupon the prey can dart again.

For the two-animal case, there are two possibilities for what this looks like: smart prey versus hungry predator, and smart prey versus smart predator. As **Figures 6–7** show, the prey succeeds in outrunning the predator for 15 s.

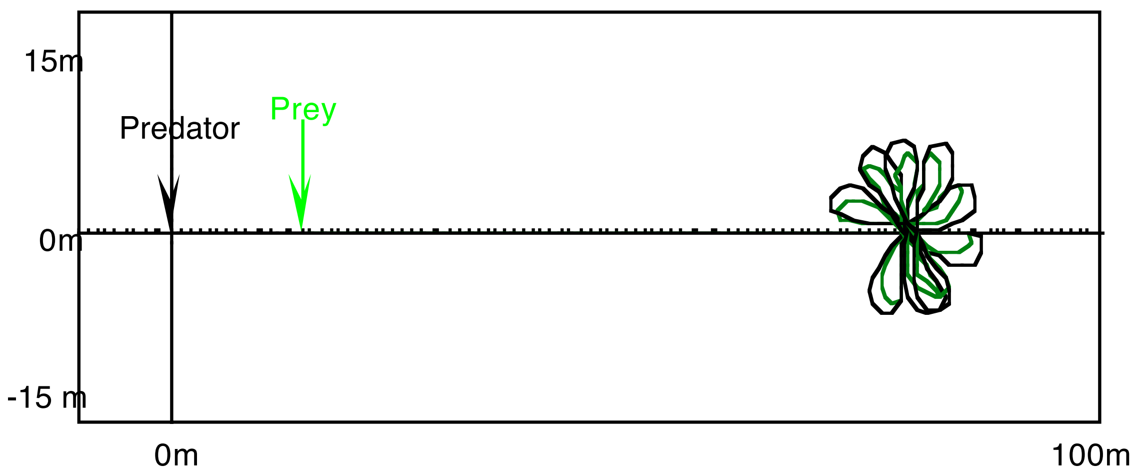

Figure 6. Smart prey vs. hungry predator, $h_I = 15$ m.

The Gradient Prey

Gradient prey are the same as smart prey when the nearest predator is very close (less than h_B) or when there is only one predator. When there are two or more predators, the gradient prey runs in the direction of least danger, i.e., along the gradient of the danger function. If the danger from each predator decreases with the inverse square of its distance, and the danger from each predator is added to produce the danger function, then the danger gradient can be computed by adding the gradient of the danger from each predator:

$$\bigtriangledown \left(\sum \frac{1}{d_i^2(x,y)} \right) = \sum \bigtriangledown \left(\frac{1}{d_i^2(x,y)} \right).$$

This can be done quickly and easily in the simulation. The gradient prey is different from the smart prey only when there are two predators; their graphs and analysis are presented in the next section.

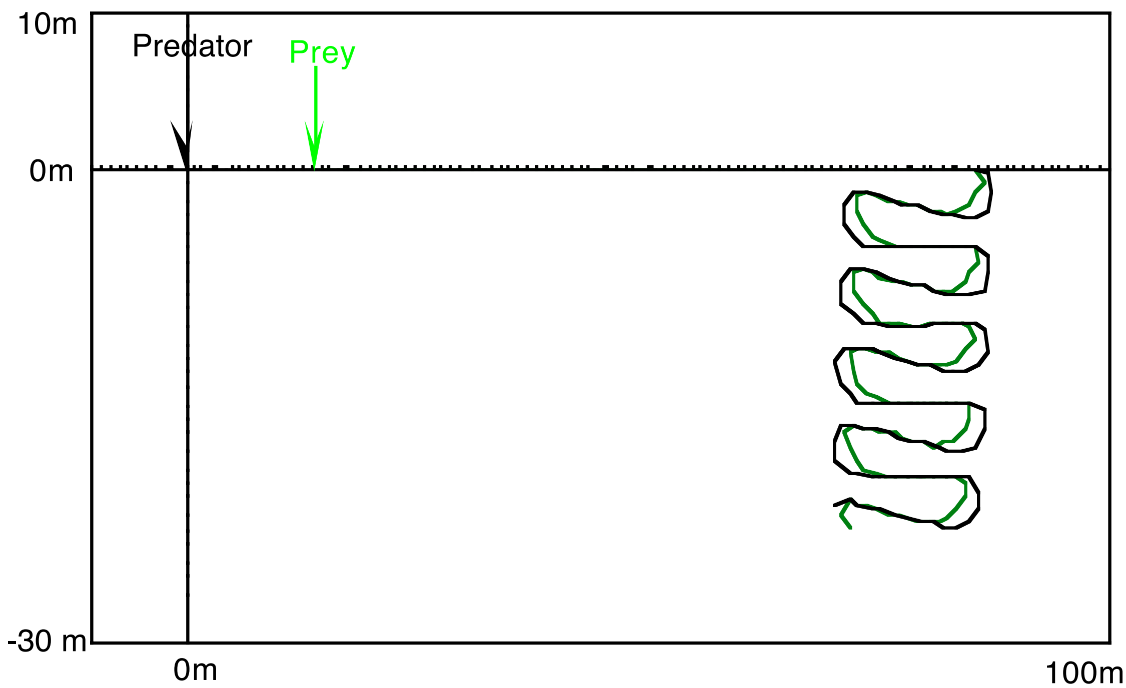

Figure 7. Smart prey vs. smart predator, $h_I = 15$ m.

The Two-Predator Situation

We determined that a good strategy for the predators is for the second predator to follow the first one (by about 8 m), since our prey's strategies hinge around getting behind the first predator.

Frightened Prey

Because frightened prey always die unless they detect the predator(s) more than 41.7 m away, there is no advantage in chasing them with two predators.

Smart Prey and Gradient Prey

Figures 8–11 illustrate typical trajectories taken by smart and by gradient prey when pursued by smart and hungry predators. Each plot shows $x = 60$ to 90 m, $y = -30$ to 5 m. Initial positions are $(-8, 0)$ for predator 1, $(0, 0)$ for predator 2, and $(15, 0)$ for the prey.

Computational Results

If the animals started running with separation determined by the reference beta distribution, the estimates of overall survival rates and their standard errors (2,000 runs) are shown in **Table 4**. Note that the error bars for some strate-

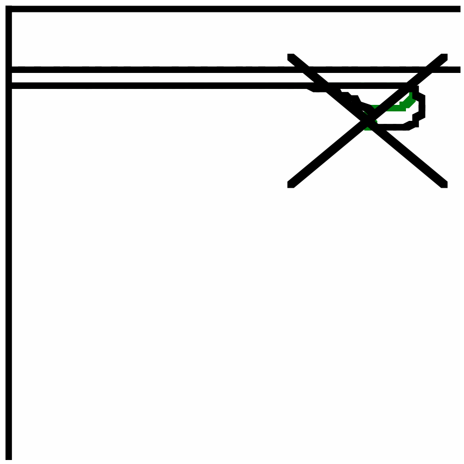

Figure 8. Smart prey, smart predators.

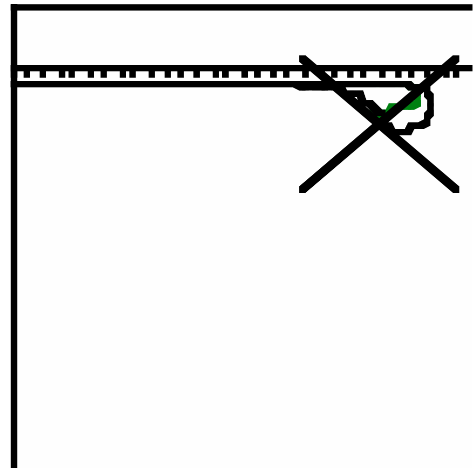

Figure 9. Smart prey, hungry predators.

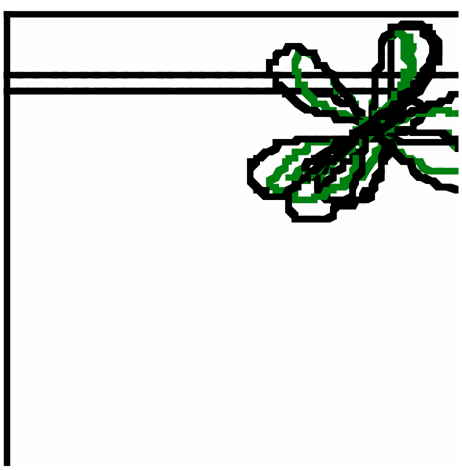

Figure 10. Gradient prey, smart predators.

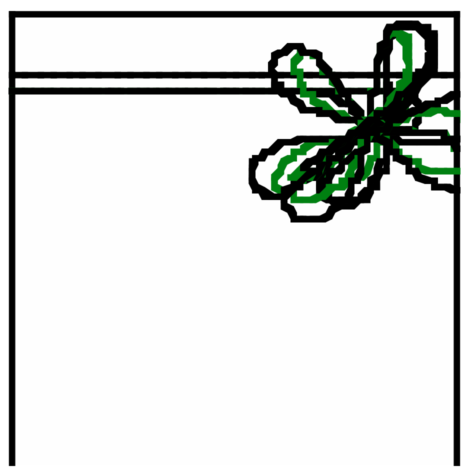

Figure 11. Gradient prey, hungry predators.

gies overlap; for instance, we cannot be certain that smart prey do significantly better than gradient prey against two hungry predators.

Prey Strategies

Frightened Prey: Always fleeing the predator unconditionally leads to the lowest survival rates. This is a bad choice.

Smart Prey: The smart prey, which darts to the side when the predator closes to some critical distance behind it, did quite well in the single-predator runs. This is the best overall evasion strategy for *Thescelosaurus*.

Gradient Prey: On two-predator runs, the gradient prey did quite well against the smart predators but more poorly than the smart prey against the hungry ones!

Table 4.

Estimated survival rates.

Prey strategy	One hungry predator	One smart predator
Frightened prey	$3.2 \pm 0.4\%$	$3.9 \pm 0.4\%$
Smart prey	$35.7 \pm 1.1\%$	$35.5 \pm 1.1\%$
Gradient prey	$35.7 \pm 1.1\%$	$35.5 \pm 1.1\%$
	Two hungry predators	Two smart predators
Frightened prey	$3.2 \pm 0.4\%$	$3.9 \pm 0.4\%$
Smart prey	$3.7 \pm 0.4\%$	$4.0 \pm 0.4\%$
Gradient prey	$3.5 \pm 0.4\%$	$9.4 \pm 0.7\%$

Predator Strategies

Hungry Predator: This predator killed the most prey in both the single-predator and hungry-predator scenarios; it is never misled by the prey's movements. Our data indicate that this is the best overall hunting strategy for *Velociraptor*: Always head straight for the prey.

Smart Predator: This predator is too easily misled by the prey's movements, and, except alone against smart prey, is always worse than the hungry predator. It is especially poor in the two-predator scenario against the gradient prey.

Conclusion

The single most important factor in survival is the distance h_I between the prey and predator at the moment of detection. This is because the most successful prey strategy is to flee the predator directly until the predator closes to another critical distance, h_B (1.6 m), then make a sharp turn. By utilizing its smaller turn radius in this fashion, the prey can get slightly ahead of the predator. The predator rapidly closes this distance, whereupon the prey can pull the same trick again. Each time it does so, however, it takes another chance that the predator will win. Hence the number of breakaway turns is the single most important factor in determining chances of surviving one predator. The chances of survival are very low ($< 4\%$) in the two-predator case when one predator trails the first by 8 m.

Figure 12 reproduces **Figure 2**, a predicted-probability-of-survival graph produced by the mathematical model, for a smart prey pursued by a hungry hunter. **Figure 13** graphs the estimated survival probability from the computer model (50,000 runs, 100 at each h_I). The graphs, aside from the static produced by the random variation of our small sample size, are nearly identical.

Using the mathematical model, we found that changing the turn radii affects the sharpness of curvature of the probability graphs but does not diminish the importance of h_I. With the computational model, and with many different hunting and evasion strategies, h_I was the most important factor in every case.

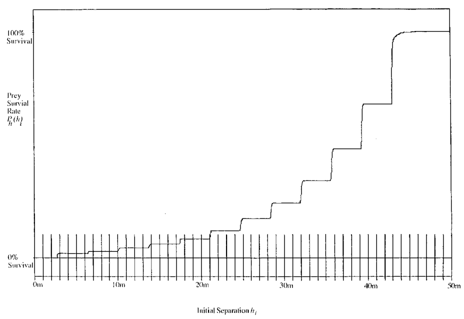

Figure 12. Prey survival rate vs. initial separation (mathematical model).

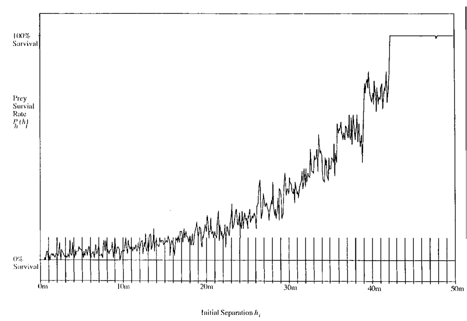

Figure 13. Prey survival rate vs. initial separation (computer model).

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A Three-Phase Model for Predator–Prey Analysis

Lance Finney
Jade Vinson
Derek Zaba
Washington University
1 Brookings Drive, CB 1040
St. Louis, MO 63130

Advisor: Hiro Mukai

Abstract

We model the hunt as a game of three explicit stages: the stalk, the attack, and the subdual. We implemented this model in Matlab to simulate a velociraptor hunting a thescelosaurus and an African lion hunting a gazelle.

During the attack phase, also known as the *macroscopic game*, the speed constraints are binding but the curvature and acceleration constraints are not. The predator's goal for this game is to minimize the time that it takes to intercept the prey. We introduce a two-predator strategy that is space-optimal for both the predator and prey and conjecture that it is also time-optimal.

In the subdual phase, also known as the *microscopic game*, the time constraints are insignificant. In this game, the predator's goal is to minimize its closest approach to the prey. The prey's strategy is to use its smaller turning radius to outmaneuver the predator until time runs out.

Based on our model and simulations, we conclude that the hunting strategies of an African lion and of a velociraptor differ. The lion has a slower maximum speed but has greater acceleration and is more maneuverable than a gazelle. Conversely, the velociraptor has greater speed but less maneuverability than a thescelosaurus. An ambush is the most effective strategy for a pair of lions, and chasing from behind is the most effective strategy for a pair of velociraptors.

The three-phase model may apply to other situations, such as a guided missile chasing an aircraft.

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Introduction

We have structured our model in accordance with a modified version of Elliott et al.'s framework that divides the hunt into three distinct sections: stalk, attack, subdue [1977].

Stalking refers to the predator's attempt to reduce the predator-prey distance while the prey is unaware of the predator's actions. The predator has the choice of whether to use a sneaking approach or a running approach. In the *sneaking* approach, the chance of detection by the prey is given by a probability distribution based on the predator-prey distance. In the *running* approach, the predator rushes the prey, attempting to benefit from the element of surprise while sacrificing the chance that the prey remains unaware of its position.

The attack phase refers to an active approach taken by the predator seeking to maximize the chance of predator-prey contact. The attack phase effectively begins when the prey detects the predator and ends when the predator-prey separation is reduced to a specified level, beginning the subdue stage.

The subdue stage can be viewed as microscopic, as opposed to the macroscopic attack stage. Curvature constraints become increasingly important when the predator-prey distance is reduced to this level, and the chance of physical contact approaches certainty. When the predator achieves a specified level of separation (the *capture radius*), the prey is deemed captured.

In the stalking phase, the predator seeks to minimize the effective separation. As the predator sneaks toward the prey, there is a point at which the risk of a detection balances the benefit of a closer approach. We find (depending on the choice of reflex times and a probability distribution for the detection range) that the optimal target distance for a stalk is 24.7 m.

The two-player phase begins when either the prey or predator starts running and the other reacts. We model the positions $x_P(t)$ ("P" for pursuer) of the predator and $x_E(t)$ ("E" for evader, as in Hajek [1975]) of the prey as moving subject to maximum speed, minimum curvature radius, and eventually acceleration constraints. Let $R_{\min}$ be the closest approach of the predator to the prey within 15 s; it is the predator's goal to minimize $R_{\min}$ and the prey's goal to maximize it. We show that for the given parameters of the velociraptor and thescelosaurus, the two-player game may be decomposed into phases: the macroscopic and microscopic games.

In the macroscopic game, distances are large enough that the curvature constraints are nonbinding. Since velociraptors have a turning radius of 1.5 m, the macroscopic game begins at about 3 to 5 m of separation. The predator's goal is to minimize the time needed to approach the prey and enter the microscopic game; it is the prey's goal to maximize this time. We show analytically that the prey's best macroscopic strategy when facing a single predator is to run directly away, and (ignoring the 15-s time limit) we explain the optimal macroscopic strategy for two predators and one prey.

The microscopic game is more difficult to analyze directly; we instead simulated it by computer. Compared with the macroscopic game, the microscopic game takes an insignificant amount of time. Since the microscopic games starts after an extended macroscopic game, we assume that predator and prey enter the microscopic game at maximum velocity. Thus, the only consideration of the microscopic game is the closest approach R. We summarize the decomposition in **Table 1**.

Table 1.

Summary of model stages.

	Lion hunt	Our model	Predator's goal	Prey's goal
Phase 1	Stalk	One-player game	Minimize effective distance	
Phase 2	Attack	Macroscopic game	Minimize pursuit time	Maximize pursuit time
Phase 3	Subdue	Microscopic game	Minimize closest approach	Maximize closest approach

We prove upper bounds for the closest approach of the predator to the prey given that a microscopic game commences. Based on the bounds, smaller bounds found through simulation, and the sizes of a velociraptor and a thescelosaurus, we conclude that in a microscopic game, a lone velociraptor can achieve physical contact. Since lions are successful in killing 71% of the large prey that they touch [Elliott et al. 1977] and only 17% of all prey hunted (which includes what they do not touch) [Stander 1992], we argue by analogy that the velociraptors always win the microscopic game.

In our model, the only random factor is the distance the predator can approach undetected. Similarly, the stalking phase of a lion hunt is the most important factor in determining success [Elliott et al. 1977].

In our model, the prey live if and only if the effective distance at the end of the stalking phase is greater than $(S_P - S_E)t_{\max}$. If the value of $t_{\max}$ is exactly 15 s, then during the stalking phase the velociraptor has a trivial strategy: Approach until some critical point is reached, then jump.

Basic Model: The Two-Car Problem

The logic behind the two-car problem proposed by R. Isaacs [Hajek 1975; Isaacs 1965] is the basis of our model. Car P (the pursuer) chases car E (the evader). If car P ever gets closer to car E than a specified distance δ (the capture radius), then car P wins the contest. Both cars have minimum turning radii ρ_P and ρ_E and move at constant speeds S_P and S_E. In the special case of a perfect capture, $\delta = 0$ signifies that P captures E only if their positions coincide exactly. The two-car problem with perfect capture was solved exactly by E. Cockayne [Cockayne 1967; Cockayne and Hall 1975].

Theorem 1 (Cockayne). *P can capture E from any initial state if and only if $S_P > S_E$ and $S_P^2/\rho_P \geq S_E^2/\rho_E$.*

Much research in differential games was conducted during the Cold War for military purposes. For example, the two-car problem might model a dogfight between two airplanes or one boat chasing another. Tellingly, roughly half the articles with direct military application or support were written in Russian, half in English.

Additions to the Model

We modify the basic model to incorporate the hunting strategies of the African lion. We assume that the probability of capture depends on the closest approach to the prey:

$$A = \min_t |x_P(t) - x_E(t)|.$$

The paths $x_P(t)$ and $x_E(t)$ of the centers of gravity of the pursuer and evader satisfy the maximum speed and minimum curvature radius constraints. The predator tries to minimize, and the prey tries to maximize, A. We introduce delay times γ_P and γ_E, typically 0.05 s, which may be thought of as either reflex times or imperfect information [Schreuer 1976]. For instance, P can only react to the actions that E took γ_E ago.

The performance data for lions and their prey [Elliott et al. 1977, Stander 1991] do not include the turning radius; those for velociraptors and thescelosauri do not include forward acceleration. Hence, we assume for all species a constant ratio f of maximum forward acceleration a to maximum lateral acceleration S^2/ρ. A value of $f = 0.5$ gives reasonable values for the inferred constants.

Table 2.

Model parameters for different species.

Values for S, ρ, and K are from the problem statement or from Hajek [1975]; the others are inferred.

	S (m/s)	ρ (m)	K (1/s)	S^2/ρ (m/s^2)	$S^2 f/\rho = Ks$ (m/s^2)	force in turn (g's)	baton distances (m)
Velociraptor	16.6	1.5	5.6	18.5	93	1.89	3.0
Thescelosaurus	13.9	0.5	13.9	386	193	39.4	1.0
African lion	14.3	10.5	0.68	19.4	9.7	1.98	21
Thomson's gazelle	27.1	80	0.17	9.2	4.6	0.94	159
Zebra	16.4	26.4	0.31	10.2	5.1	1.04	53
Wildebeest	14.7	18.9	0.39	11.4	5.7	1.16	38

The Stalk: A One-Person Game

The stalking phase is the most important factor affecting the success of a lion's hunt [Elliott et al. 1977]. During the stalk, the predator tries to minimize its effective separation from the prey. The effective separation accounts for the actual separation, the acceleration capabilities of the two species, and which player jumps first.

For the predator, the advantage of sneaking closer is a decrease in the actual separation. The disadvantage is the risk of being noticed and losing the element of surprise.

We define the effective separation to have the following property. If the prey runs directly away from the predator, then starting from a standstill and assuming that one species surprises the other is "equivalent" to starting from the effective separation with both species running at full speed. By "equivalent," we mean that the time that it takes the predator to catch the prey is the same. Explicitly,

$$R_{\text{effective}} \approx R_{\text{actual}} - b_E + b_P - \begin{cases} -\gamma_E S_E, & \text{if predator jumps first;} \\ +\gamma_P S_P, & \text{if prey jumps first.} \end{cases}$$

The prey and predator approach their maximum speed asymptotically but never reach it. However, our model parameters imply that with a 10-m head start, the thescelosaurus will be traveling at nearly full speed. Since the velociraptor is only slightly faster, it too must be running at near full speed when it captures the thescelosaurus. The approximation above would be equality if full speed was attained by both species before contact.

The constants b are "baton distances." For example, the baton distance b_P is the initial separation that a stationary velociraptor on a relay team needs from its teammate in order to accelerate to full speed and receive the baton as the teammate catches up. By the acceleration assumptions of our model, $b = S/K = \rho/f$.

The reflex term can be either positive or negative depending on which species reacts first. If P jumps first, we assume that E immediately notices, takes γ_E s to react, and loses an effective distance of $\gamma_E S_E$ m. If E notices the predator and jumps first, the predator needs γ_P s to react and loses an effective distance of $\gamma_P S_P$. Since the prey is not anticipating flight, but the predator is anticipating a detection, we assume that $0.05 = \gamma_P < \gamma_E = 0.20$.

We now devise a cumulative probability distribution function $P(x)$ for the distance at which the prey first notices the stalking predator. We wish to fit a twice-differentiable and easily analyzable function P to the constraints $P(15) = 0$ and $P(50) = 1$. We choose

$$P(x) = \frac{x - 15}{35} - \frac{\sin\left(\dfrac{2\pi(x-15)}{35}\right)}{2\pi}.$$

The predator will try to stalk until reaching its target separation $x = R$ and then jump; of course, the predator may have to jump sooner if detected. The expected effective distance is 24.7 m, as discussed below.

Figure 1 shows the expected effective distance as a function of the target distance. For our choice of parameters and probability distribution, there is a unique minimum, the optimum target separation for the stalking predator. We locate the minimum at 24.7 m by taking the derivative of the expected effective distance with respect to target distance. The derivative shown in **Figure 2** does not adequately represent the predator's disadvantage in advancing from 20 m to 15 m, because the strategies with targets 20 m and 15 m vary only in the rare case that the predator reaches 20 m undetected. We account for this effect by considering the expected benefit for a predator at x m to advance infinitesimally and obtain the conditional derivative by dividing by $P(D)$. The conditional derivative is shown in **Figure 3**.

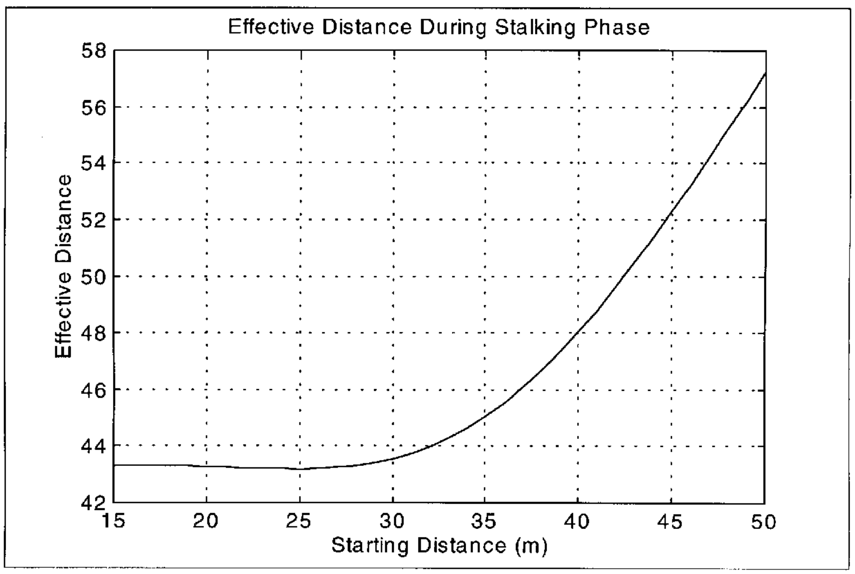

Figure 1. Effective distance.

The Macroscopic Game

This phase of the game is the simplest strategically for both the predator and the prey. The predator's goal is to use the least amount of time to reach the prey. The prey's goal is to use as much time as possible before being reached by the predator. Using simple trigonometry, it is easily shown that the prey will then run directly away from the predator to extend the time spent in the chase. Obviously, the best strategy for the predator is to run directly toward the prey. Because of differences in top velocity, the predator will reach the prey and the microscopic game will begin in $R_{\text{effective}}/(S_P - S_E)$ s, where $R_{\text{effective}}$ is the initial effective distance between the prey and the predator.

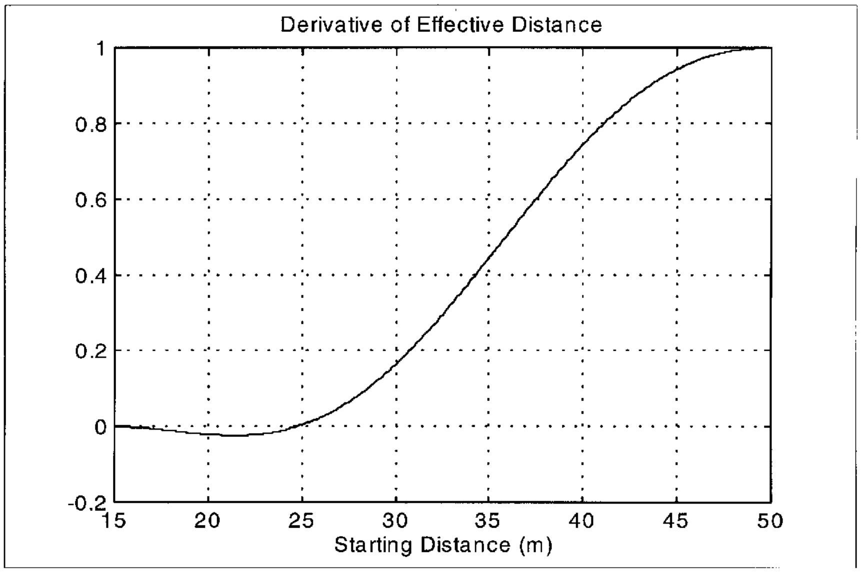

Figure 2. Derivative of effective distance.

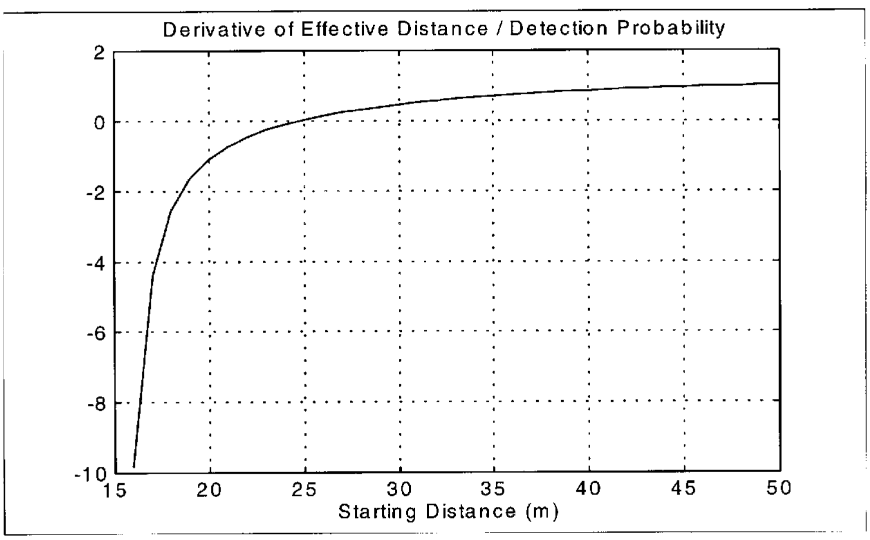

Figure 3. Conditional derivative of effective distance.

The Microscopic Game

We prove in **Theorem 2** that the predator, simply by running directly toward the prey, can approach very close to the prey no matter what the prey does. The following lemma is instrumental in this proof. When the predator is as close as **Theorem 2** allows, then the microscopic game has begun.

Lemma. *Suppose that the predator always tries to run directly at the prey. Then the predator can run directly at the prey until the separation distance is* $\rho_E(S_E/S_P)$ *meters.*

Proof: We may assume that for large distances the predator can line up with the prey. The predator can maintain this orientation as long as the predator's ability to change direction is greater than the prey's ability to change the direction of of the vector from predator to prey:

$$\frac{|x'_E|}{d} \leq \frac{S_P}{\rho_P}.$$

Since $|x'_E| < S_E$, the condition $d \geq \rho_P S_E/S_P$ suffices. $\square$

A similar result holds for the prey: If the prey's strategy is to run directly away from the predator, then the prey can keep the predator directly behind it at least until the separation distance is $\rho_E(S_P/S_E)$ m.

Theorem 2. *A predator with maximum speed* S_P*, turning radius* ρ_P*, and reflex time* γ_P *is guaranteed to approach within* $\rho_P(S_E^2/S_P^2) + \gamma_P S_E$ *of a prey with maximum speed* S_E*.*

Proof: First, we consider the zero-reflex case when $\gamma_P = 0$. The predator can run directly at the prey until the separation is $\rho_P(S_E/S_P)$. Then the predator continues in a straight line to the point where the prey is at this instant. In the time that it takes to reach this point, the prey travels at most $\rho_P(S_E^2/SD_P^2)$ m.

For nonzero reflex time, the predator should always choose the point where the prey was γ_P s ago. When the predator comes within $\rho_P(S_E^2/S_P^2)$ of the pursued point, it is within $\rho_P(S_E^2/S_P^2) + \gamma_P S_E$ of the prey. $\square$

The only physical advantage of the prey over the predator is the prey's much smaller turning radius. However, to use this advantage, the prey must allow the predator to approach close by. Assuming perfect capture and instantaneous reaction, and more than 1.4 m separation, if the prey turns as hard as it can, then the predator can capture it. In our model, with nonzero reaction time, the predator instead aims directly for the prey. **Figure 4** shows a typical result of such a turn: The prey can maneuver more quickly and the predator must make a larger turn, using more time. **Figure 5** shows the minimum distance between the centers of gravity for the predator and the prey at different turning distances, assuming instantaneous reactions but using the parameters for velociraptor and

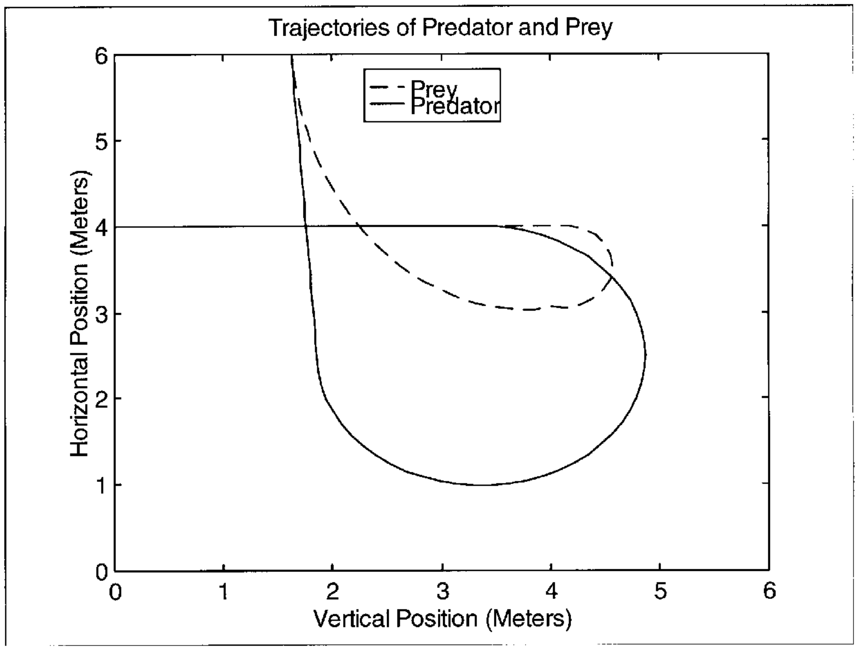

Figure 4. Trajectories of hard turn maneuver.

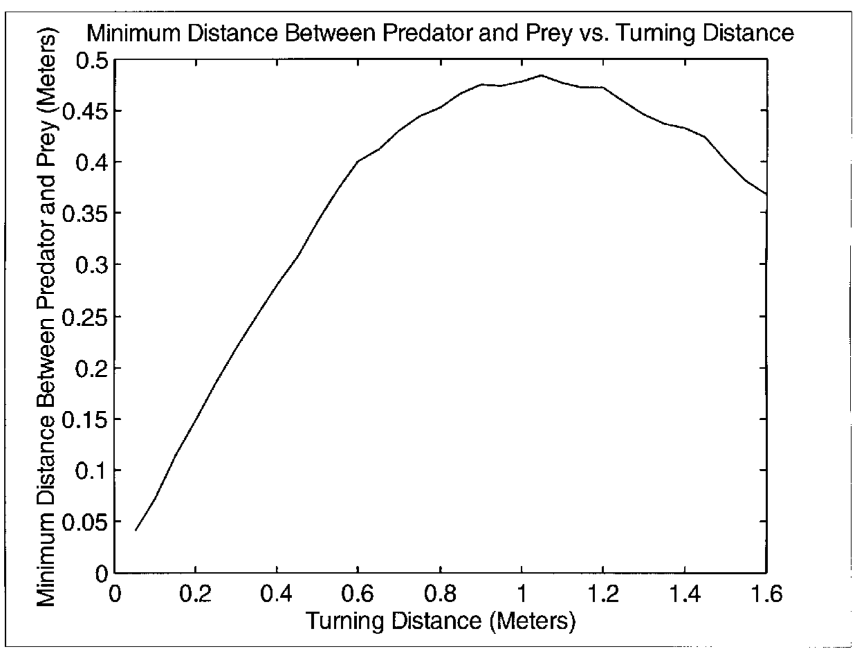

Figure 5. Minimum approach vs. turning distance.

thescelosaurus. The predator is kept the farthest from the prey if the prey turns when the predator is about 1 m behind. **Figures 6–7** show the effect of reaction time on the optimal time to turn (optimality is keeping the predator as far away as possible). **Figure 6** shows what the minimum approaches are with various reaction times; and **Figure 7** shows at what separation the prey should turn to obtain that minimum approach, given the reaction times.

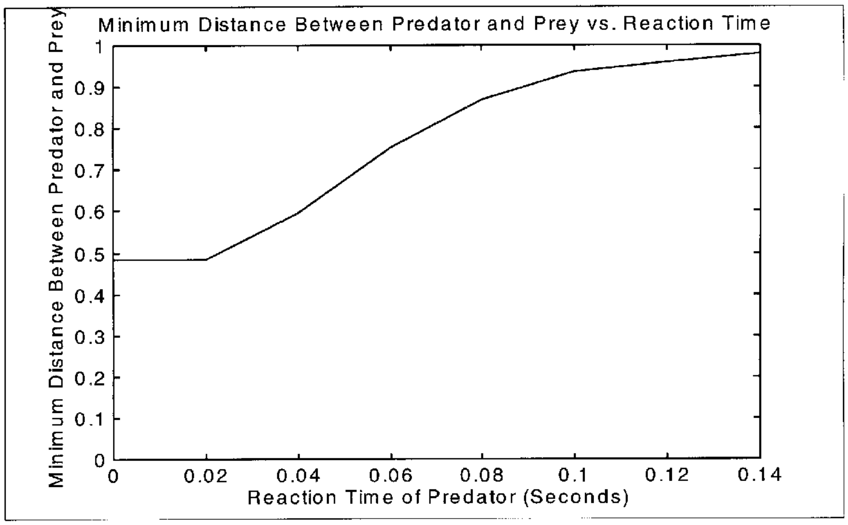

Figure 6. Minimum approach vs. predator reaction time.

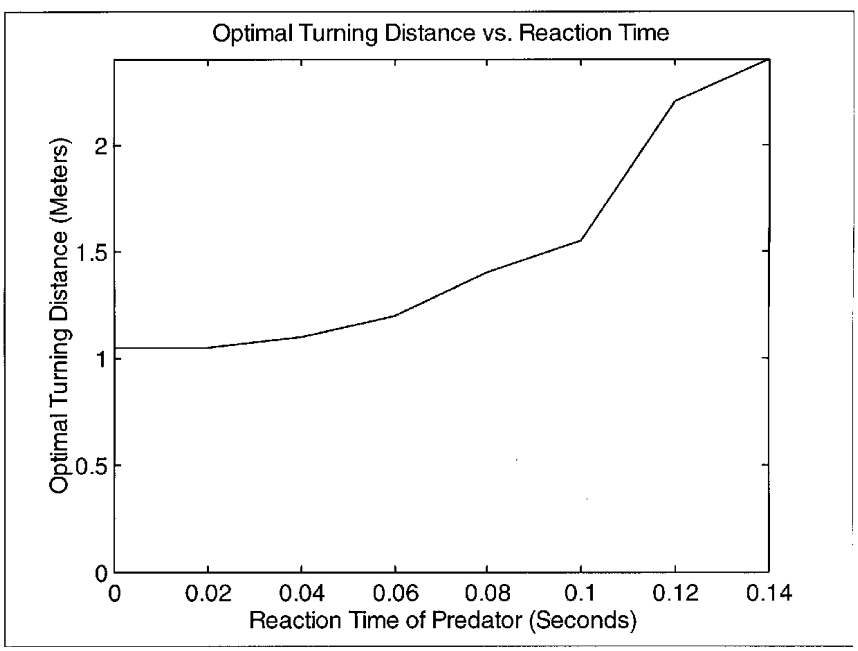

Figure 7. Optimal turning distance vs. predator's reaction time.

Hunting Strategy of Multiple Lions

For a multiple lion hunt, Stander [1992] proposes a three-part framework to model the predators' strategies. The most effective strategy for the lion is a coordinated ambush, with a probability of success of 26% for large prey. A single lion initiates the attack, driving the prey towards the other lions lying in wait. Multiple lions have been observed to use this strategy in 52% of their large-prey hunts.

Another strategy is convergence, in which two lions jointly initiate the attack phase, pursuing the prey from the same direction. For large-prey hunts with multiple lions, this strategy is pursued 14% of the time with a probability of success of 14%.

The least effective strategy for group hunting is an uncoordinated ambush, which usually occurs when one lion startles the herd before the others are in position to receive the prey. While Stander observed a 34% occurrence rate for large prey hunts, not a single animal was killed under this approach out of 68 attempts [1992].

The Two-on-One Macroscopic Game

The two-predator game is similar to the one-predator game in several respects. In both, the goal of the predator is to reach the prey as soon as possible. With two predators, one goal for the prey would be to avoid being contacted by any predator as long as possible; another goal would be to avoid being contacted by both predators as long as possible. The second goal implies that a situation in which one predator is quickly encountered might be preferable if that meant that the time until the second predator joins the fight is delayed.

For the sake of simplicity, assume that the pursuers are faster than the evader and have no constraints on turning, acceleration, or time. The optimal strategies for each goal are provided by circles of Apollonius.

For two points P and E in the plane and a constant k, the locus of all points Q that satisfy $PQ = k \cdot QE$ is a circle (for $k \neq 1$), known as a *circle of Apollonius*. The points P and E are the positions of the predator and prey, and k is the ratio S_P/S_E of their speeds. For $k > 1$, the circle contains E. As the pursuer and evader proceed, their positions and the circle change; the area of the circle decreases continuously as the pursuer and evader approach each other.

We omit proofs of the following results.

Theorem. *Regardless of the pursuer's strategy, the evader can reach any point in or on the boundary of the original circle of Apollonius. If the pursuer follows an optimal strategy, the evader can reach the boundary only by traveling in a straight line.*

Theorem. *If the evader runs in a straight line, the fastest way for the pursuer to capture the evader is by following a straight line.*

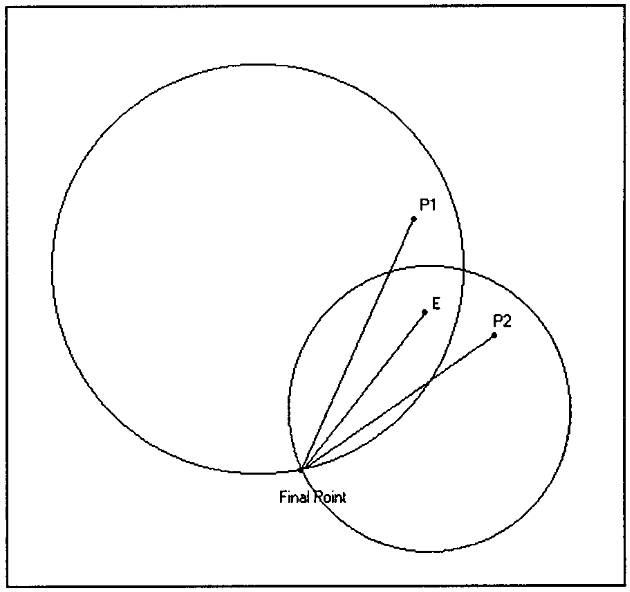

Figure 8. Example of circles of Apollonius.

Assuming that the prey can survive a microscopic game with one predator but not with two, the prey's goal in the microscopic game is to delay the entrance of the second predator into the microscopic game. Assuming further that the microscopic game takes up very little space compared to the macroscopic game, the prey's optimal macroscopic strategy is to engage the first predator as far from the second as possible. Thus, the prey aims for the point on the Apollonius circle with P_1 (the closer predator) that is farthest from P_2.

Based upon the different assumption that the prey's macroscopic goal is to delay the first contact with either predator, we conjecture that the optimal strategy for the prey is to run towards an intersection of the circles of Apollonius with the two predators, if such a point exists. If none exists, the prey's optimal strategy is to run away from the closest predator.

The Ambush

The ambush is an effective strategy for lions because they have greater maneuverability and acceleration than their prey. Indeed, Stander empirically confirms that the most effective hunting strategy for multiple lions is a coordinated ambush. He witnessed a 27% success rate for the coordinated ambush strategy vs. an average success rate of 15%. In addition, Stander notes that while the lions pursued a coordinated ambushed strategy in 68% of the total hunts, the figure increases to 87% if only large prey are considered [1992].

Figure 9 shows the results of our model simulating an ambush of a gazelle by a lion. We assume, based on Stander's observations, that the lion's ambush is 15 m from the gazelle. At this distance, the probability of a single lion capturing a gazelle from behind is essentially zero. The gazelle, after being scared by a second lion, runs in a direction that, if unaltered, would pass within 10 m of the hidden lion. Just after the gazelle starts running, the hidden lion leaps out. The gazelle accelerates through a turn in order to escape the lion but is unsuccessful. We experimented with different angles for the lion and different degrees of aiming for a point in front of the gazelle. In this simulation, we assumed zero reaction time for the gazelle.

We also considered a velociraptor attempting to ambush a thescelosaurus. We assume, since the agile thescelosaurus has 15 m to accelerate, that it is traveling at full speed when near the ambush. The constant speed assumption makes an analytic solution possible. Let D be distance at which the thescelosaurus would pass from the ambush if it continues in a straight path. We assume a reaction time of $\gamma = 0.2$ for the thescelosaurus, whose strategy is to try and cut away from the ambush as soon as the predator reveals itself. We assume also that the predator's strategy is to run in a straight line to intercept the thescelosaurus after the thescelosaurus has turned by an angle of θ. Based on these assumptions, the velociraptor's leap must be perpendicular to the thescelosaurus's original path. For a zero capture radius, the maximum distance D from which the ambush will be successful is

$$D(\theta) = S_P\left(\frac{\rho_E\theta}{S_E}+\gamma_E\right)-\frac{S_P}{K_P}\left\{1-\exp\left[-K_P\left(\frac{\rho_E\theta}{S_E}+\gamma_E\right)\right]\right\}-\rho_E(1-\cos\theta).$$

The optimum value of θ is 1.0681 radians, which allows a range of 1.51 m for the ambush, as seen in **Figure 10**. Note that in our simulation of the velociraptor's ambush, all of the assumptions except zero capture radius favor the velociraptor, so we are confident that 1.5 m is an upper bound for the maximum distance of effective ambush.

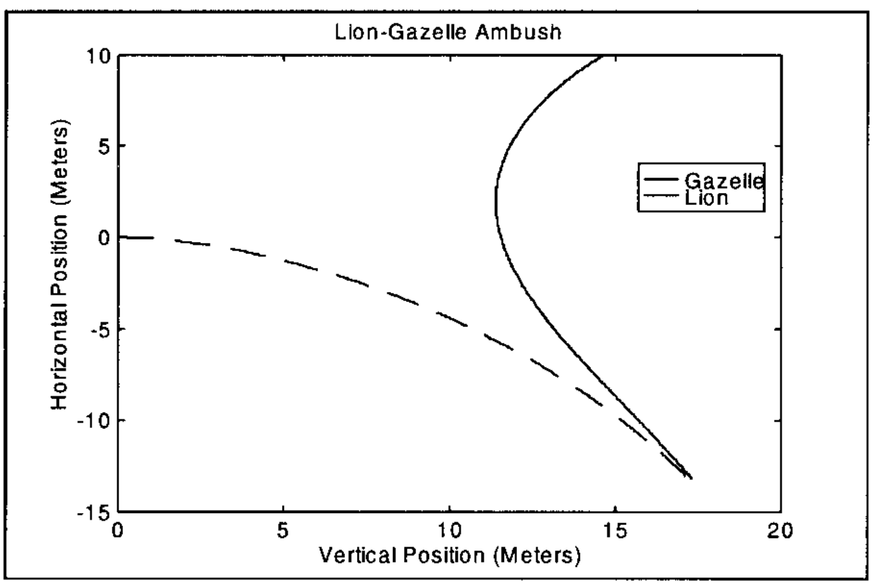

Figure 9. Ambush of a gazelle by a lion.

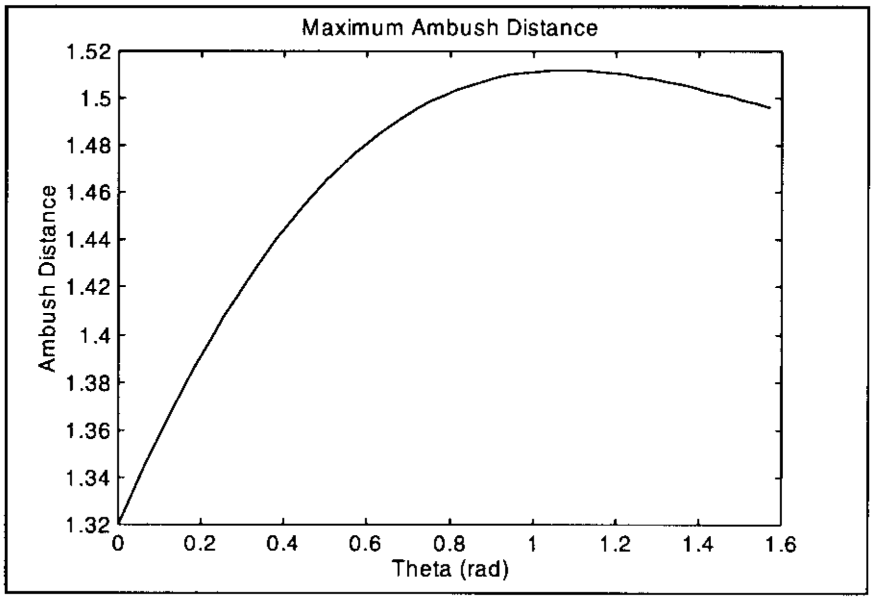

Figure 10. Maximum ambush distance by a velociraptor.

Our model accurately predicts that the lion should be able to capture a gazelle from 10 m and that a velociraptor should be able to capture a thescelosaurus from 1.51 m. This difference explains why the ambush is a useful strategy for the lion but not for the velociraptor. The lion has a slower maximum speed but has greater acceleration and is more maneuverable than a gazelle, while the velociraptor has greater speed but less maneuverability than a thescelosaurus. An ambush is the most effective strategy for a pair of lions, and chasing from behind is the most effective strategy for a pair of velociraptors.

Testing the Model

We determined reasonable parameter values for the velociraptor hunt by comparing the predictions of our model with the success rates of different strategies for the African lion. Our model could be validated by testing it on another known species, such as the cheetah, for which there are data. One particularly significant assumption that should be tested is the ratio f of maximum forward acceleration to maximum lateral acceleration (f), whose value of 0.5 was chosen because it produced realistic results in the lion–gazelle hunt.

Extensions

We could make the capture a stochastic process. One alternative scoring function [Friedman 1970, Marec and Van Nhan 1977] is to integrate some function of time and the positions with respect to time:

$$B = \int \mu(t, x_P(t), x_E(t))\, dt.$$

The probability of a capture is e^{-B}. We briefly considered both

$$\mu = \frac{1}{|x_P - x_E|} - \frac{1}{3}$$

and

$$\mu = 1 + \cos\left(\frac{\pi|x_P - x_E|}{3}\right)$$

for $0 \leq |x_P - x_E| \leq 3$ but disagreed with their predictions. The "chicken maneuver" from Cockayne's proof of **Theorem 1**, in which the prey charges at the predator but swerves at the last moment, gives the prey a good score. Although the predator and prey come close together, they are close for only a short time, hence the integral with respect to time is small.

For the velociraptor–thescelosaurus hunt, the closest approach in the chicken maneuver is less than the size of the prey and would probably result in a kill. We conclude that the probabilistic approach is not appropriate for the given model parameter values.

Since the closest approach scales linearly with turning radius, the probabilistic approach may be appropriate for situations in which the turning radius is much larger than the predator and prey, such as a missile chasing an airplane.

Strengths and Weaknesses

We took advantage of the fact the turning radii and the distances needed to accelerate to full speed are negligible compared with the overall distances. This fact allowed us to decompose the chase into a macroscopic phase and a microscopic phase. The players have different restrictions and goals in each phase. We feel that the decomposition of a difficult problem into more manageable problems is the major strength of our approach.

The assumptions that $t_{\max} = 15$, $\rho_P = 1.5$, $\rho_E = 0.5$, $S_P = 16.67$, and $S_E = 13.89$ are unrealistic for a biomechanical analysis of dinosaurs. In particular, a thescelosaurus turning at full speed in a radius of 0.5 m has a centrifugal acceleration of 39 times the force of gravity. We might fix the g-force by linearly scaling the given constants ρ and $t_{\max}$ and scaling the assumed reflex time. Since our model is invariant under this change of scale, the strategic analysis will be unchanged, with the possible exception that the capture radius stays constant and thus the microscopic game does not always favor the velociraptor.

A disadvantage of our model is that the transition between the macroscopic and microscopic games is not well defined. The exact transition would be unimportant if the initial distances are much larger than the curving radii, but we were not able to quantify their importance.

A second disadvantage is that the decomposition does not apply in all situations, since the macroscopic phase explicitly assumes that the predator's maximum speed is greater than the prey's. For instance, the macroscopic phase does not apply to the lion–gazelle hunt, because the lion is slower than its prey.

We could not find general strategies for the microscopic game by mathematical analysis. Instead, we used simulations to conjecture and test strategies.

An unforeseen advantage of our model is its potential application to other situations in which the initial distances are large compared to the turning radii, such as air combat or naval warfare.

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Judge's Commentary: The Outstanding Velociraptor Papers

John S. Robertson
Dept. of Mathematics and Computer Science
Georgia College and State University
Milledgeville, GA 31061–0490
jroberts@mail.gac.peachnet.edu

Introduction

The life sciences provide a particularly fertile area in which a mathematical modeler can apply the craft. Physics and engineering are the traditional playgrounds of applied mathematicians, while biology and its subdisciplines have often been thought of as areas in which little or no mathematics is necessary—soft subjects, if you will. That stereotype is just not true. (Just look at any volume of *The UMAP Journal*!)

Paleontology is a particularly rich area for the mathematical modeler because there are simply no data available from observations. It is not possible, for example, to watch a tyrannosaur hunt down its dinner or a pterodactyl soar through the air. What we do know about ancient animals is based on a lot of inferential detective work, something that often requires a substantial amount of applied mathematics.

The success of Michael Crichton's novels *Jurassic Park* and *The Lost World* (and their extraordinary popularity as movies under the masterful direction of Steven Spielberg) only served to raise the interest in many people—young ones, in particular—in paleontology. In that sense, the timing for this problem could not have been better.

The Importance of Assumptions

The modeling process hinges critically on the assumptions made. In the cases of the velociraptor and the thescelosaurus, paleontologists provided data about certain physical characteristics of these dinosaurs, based on analysis of fossil remains. The paleontologists needed help in analyzing hunting strategies used by the raptors.

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The data provided suggested that unrealistically high forces were present as either animal made a sharp turn, a maneuver almost certainly necessary for survival. Most teams realized this; and since they were unable to obtain revised data from the paleontologists, they made reasonable moderating assumptions. This type of difficulty is not unknown in the real world, and it was important to the success of the teams that they identified the data provided as containing an evident weakness and reacted in some reasonable and appropriate way. For example, quite a few teams drew on information readily available in the literature concerning mammalian species similar in size and behavior traits to the dinosaurs in the problem. This enabled the teams to adjust the data in a realistic way, raising the likelihood that their subsequent results would be of real utility to their clients.

Another major issue that had to be treated with assumptions dealt with the geometry and mechanisms of the stalk, the chase, and the capture. The best teams provided clear, detailed thinking about their choices. Good work here inevitably eased the problem of interpretation that the teams eventually had to face.

The Choice of Models

Once teams clarified their assumptions, they set to the task using a surprising diversity of approaches. Some teams were able to formulate models that used just algebra and geometry and no calculus. Others used differential equations, and one of the best papers submitted used differential game theory.

In almost every case, teams turned to a computer to perform model calculations. The judges saw all manner of approaches to this. Computer algebra systems were very popular, as was Matlab. And many teams used a good old-fashioned programming language (C++ in most cases) for this purpose.

This problem was especially well suited to graphical interpretation of one sort or another, and most teams provided graphs and charts that depicted the conduct and outcome of chases with one or two predators. Graphical analysis is particularly important when working with clients who may not grasp all the technical mathematics (such as paleontologists). The old saying—that a picture is worth a thousand words—is decidedly true in this case, as the illustrations produced by teams were absolutely vital to their analysis or the model's predictions.

Analysis of Results

The very best papers gave a thorough treatment to an interpretation of their results and predictions. In most case, this involved a consideration of model weaknesses uncovered by those results. These weaknesses often were traced back to assumptions made at the beginning of the process. This step is also important to the modeler. The client is not always in a position to assess the existence, never mind the seriousness, of weaknesses. Such weaknesses are not necessarily fatal, but they do serve to point out where better data, more accurate assumptions, or different methods are called for. This leads to an important interaction between the modeler and the client. Without this, the whole process loses its focus.

Conclusion

The Outstanding papers showed remarkable depth of insight into both the biology and the mathematics used, as well as into the process of melding the two together. The judges had a great time reading them. It was evident that the topic stimulated a intense interest on the part of the contest participants. No doubt the dinosaur problem has prompted many students to contemplate investigating biology as an exciting and fruitful area of work for a mathematician.

About the Author

Dr. John S. Robertson is Chair and Professor of Mathematics and Computer Science at Georgia College and State University, the alma mater of the great southern writer Flannery O'Connor. He describes himself as a "dirt-under-the-fingernails" applied mathematician, and is as fascinated by the applications of mathematics to other disciplines as he is by the mathematics itself. He and his family are happily ensconced in the Deep South, from where he enjoys watching weather reports about all the winter snow that falls in the northeastern United States. He doesn't miss it a bit.

An Assignment Model for Fruitful Discussions

Han Cao
Hui Yang
Zheng Shi
East China University of Science and Technology
Shanghai, China 200237

Advisor: Lu Xiwen

Introduction

We present a boolean programming model to solve a practical problem of giving assignments for a meeting.

Because the objective function of the model is nonlinear and because there are too many variables in the model, the problem is quite difficult to solve by means of general methods from integer programming. We use a greedy algorithm to get an initial feasible solution, then we optimize locally and iterate to approach the optimal solution.

We believe that our algorithm solves the given problem quite well. For the possibility that some board members will cancel at the last minute or that some not scheduled will show up, we give an adjustment method that makes the fewest necessary changes in assignments.

Our ideas admit of generalization. The parameters, such as the number of members, the number of types of attendees, and the number of different levels of participation can be varied, and the model and the algorithm always give a good solution. The model has the following advantages:

- It solves the given problem successfully, and it can generate a group of quite optimized solutions quickly.
- The model is general; it can give quite good solutions for different parameter values.
- It has lots of applications.

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Assumptions

- Three kinds of members attend the meeting:
  - senior officers (6),
  - in-house members (9),
  - other members (20).
- The whole meeting is divided into two stages:
  - A.M.: The morning meeting includes 3 sessions, each session consists of 6 discussion groups, and each group is led by a senior officers.
  - P.M.: The afternoon meeting includes 4 sessions, each session consists of 4 discussion groups, and no senior officer attends.
- The assignments should satisfy the following two criteria:
  - For the morning sessions, no board member should be in the same senior officer's discussion group more than once.
  - No discussion group should contain a disproportionate number of in-house members.
- No member changes groups during the meeting.

Table 1.

Description of the variables.

X	strategy vector; x_{ijk} means member i is or is not in group k in session j
P	dividing matrix
P_j	the dividing matrix of session j
Q	acquaintance matrix
Q_j	the acquaintance matrix of session j
Q_{sum}	the summary acquaintance matrix, $Q_{\text{sum}} = \sum_{j=1}^{7} Q_j$
$f(X)$	objective function, the number of 0s in matrix Q_{sum}
$g(X)$	objective function, the square of the norm of the matrix $T = Q_{\text{sum}} - k \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix}$, where k is a constant.

Analysis and Model Design

Preparation Knowledge

Divide the set $S = \{s_1, \ldots, s_M\}$ into n groups $G_1, \ldots, G_n$ and represent the division by the $m \times n$ matrix $P = (p_{ij})$, where $p_{ij} = 1$ if $s_i \in G_j$ and 0 otherwise. We call P the *dividing matrix*. We also consider the $m \times m$ matrix $Q = (q_{ij})$, where $q_{ij} = 1$ if s_i and s_j are in the same group ($i \neq j$, $1 \leq i, j \leq m$) and 0 otherwise (in particular, 0 on the diagonal). We call this the *acquaintance matrix*. We have a basic theorem that relates the dividing matrix and acquaintance matrix.

Theorem. *Let P be a dividing matrix. Then the corresponding acquaintance matrix is $Q = PP^T - E$, where E is a matrix of all 1s.*

Proof: Because each s_i can be in only one group, only one element of each row of P is 1 and the others are 0. We can easily calculate the elements of $Q = (PP^T - E)$:

$$q_{ij} = \begin{cases} \sum_{k=1}^{m} p_{ik} p_{jk}, & i \neq j; \\ \sum_{k=1}^{m} p_{ik}^2 - 1, & i = j. \end{cases}$$

If P_{ik} and P_{jk} are not both 1, then s_i and s_j and are not in the same group, so $q_{ij} = 0$.

If P_{ik} and P_{jk} are both 1, then s_i and s_j are in the same group, so $q_{ij} = 1$. □

Set

$$x_{ijk} := \begin{cases} 1, & \text{if member } i \text{ is assigned to group } k \text{ in session } j; \\ 0, & \text{otherwise.} \end{cases}$$

For our problem, i ranges from 1 to 29 (let $i = 1, \ldots, 9$ be the in-house members), j from 1 to 7, and k from 1 to 6. For each assignment (session) j, there is a dividing matrix P_j and a corresponding acquaintance matrix Q_j.

Constraints

- Each member is assigned to only one group in each session:

$$\sum_{k=1}^{6} x_{ijk} = 1, \qquad i = 1, \ldots, 29; j = 1, \ldots, 3;$$
$$\sum_{k=1}^{4} x_{ijk} = 1, \qquad i = 1, \ldots, 29; j = 4, \ldots, 7.$$

- Each discussion group should contain a proportionate number of in-house members in each session:

$$1 \le \sum_{i=1}^{9} x_{ijk} \le 2, \qquad k = 1,\ldots,6; j = 1,\ldots,3;$$
$$2 \le \sum_{i=1}^{9} x_{ijk} \le 3, \qquad k = 1,\ldots,4; j = 4,\ldots,7.$$

- In the morning session, each of six discussion groups is led by a senior officer and no board member should be in the same senior officer's discussion group more than once:

$$0 \le x_{i1k} + x_{i2k} + x_{i3k} \le 1, \qquad i = 1,\ldots,29; k = 1,\ldots,6.$$

We seek a 0–1 three-dimensional matrix X that satisfies all these constraints. How can we judge whether it is good for our purposes?

Objective Function

The problem requires that the assignments should

- mix all the board members well,
- have each board member with each other board member in a discussion group the same number of times while minimizing common membership of groups for the different sessions.

Consider an extreme situation: If the 29 members are divided into just one group, the goal of mixing well is satisfied but the number of repetitions (session after session) is greatest. That is, when the number of the dividing groups is small, more sessions will increase the repetitions. At the same time, we must avoid too many people (thereby discouraging productive discussion) and a group being controlled or directed by a dominant personality. *Therefore, during the course of model design and solution, we consider only the situation of the 29 board members divided as equally as possible into groups.* We think that such a plan should minimize the number of repetitions, though we can't prove this claim. So we divide each morning session into 6 groups of 5, 5, 5, 5, 5, and 4, and each afternoon session into 4 groups of 7, 7, 7, and 8. And how can we describe or judge whether the results of the seven divisions are good mathematically? The number of times that each member meets may be calculated by following formula:

$$Q_{\text{sum}} = \sum_{j=1}^{7} Q_j = (q_{ij}^{\text{sum}})_{29\times 29} \qquad s,t = 1,\ldots,29,$$

where

$$q_{ij}^{\text{sum}} = \sum_{l=1}^{3}\sum_{k=1}^{6} x_{ilk}x_{jlk} + \sum_{l=4}^{7}\sum_{k=1}^{4} x_{ilk}x_{jlk}, \qquad i,j = 1,\ldots,29.$$

The matrix Q_{sum} is called the *summary acquaintance matrix* of the division. Considering the goals, we think that the final summary acquaintance matrix should have as few zero elements as possible. It would be ideal for each nondiagonal element to be 1, with each main diagonal element 0.

Altogether, the assignments provide

$$3 \times \left(4 \times \binom{5}{2} + \binom{4}{2}\right) + 4 \times \left(3 \times \binom{7}{2} + \binom{8}{2}\right) = 532$$

chances for two individuals to meet each other. On the other hand, there are only $\binom{29}{22} = 406$ pairs of members. So every pair meets $K = 532/406 \approx 1.31$ times on average. Let

$$T = (t_{ij})_{29\times 29} = Q_{\text{sum}} - K \begin{pmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & \cdots & 0 \end{pmatrix}$$

and let

$$f(X) = \text{the number of } 0 \text{ elements in } Q_{\text{sum}}, \quad \text{and} \quad g(X) \ = \|T\|^2 = \sum_{i=1}^{29}\sum_{j=1}^{29} t_{ij}^2,$$

where

$$t_{ij} = \begin{cases} \sum_{l=1}^{3}\sum_{k=1}^{6} x_{ilk}x_{jlk} + \sum_{l=4}^{7}\sum_{k=1}^{4} x_{ilk}x_{jlk} - K, & i,j = 1,\ldots,29,\ i \neq j; \\ \sum_{l=1}^{3}\sum_{k=1}^{6} x_{ilk}^2 + \sum_{l=4}^{7}\sum_{k=1}^{4} x_{ilk}^2, & i = j = 1,\ldots,29. \end{cases}$$

When the function $f(X)$ is minimized, the goal of mixing well will be satisfied best; and we think that an assignment would have each board member in a discussion with each other board member the same number of times when the function $g(X)$ attains a minimum.

Model

We have two objective functions to minimize, $f(X)$ and $g(X)$, and constraints as indicated earlier. This is a standard 0–1 integer programming and multiobjective programming problem. The task now is to solve this model.

Model Solution

Analysis

Although the constraints are linear, the objective functions are nonlinear. For this kind of integer programming problem, there is no general method to get the optimal solution efficiently; in fact, it is an NP-complete problem. For our problem instance, with 986 variables, the solution space has size at least least 6^{986}, so the method of exhaustion is infeasible. We must devise an efficient algorithm.

We find a good feasible solution and adjust it iteratively to approach the optimal solution, arriving at an acceptable approximately optimal solution.

Initial Feasible Solution

We use the greedy heuristic to get an initial feasible solution. The heuristic assigns the 29 members into each group one by one in each session. Before each member is assigned into one of groups, we examine which would be the the best group for the member.

Iteration

Because there are two objective functions, we use the strategy of multiobjective programming. We program the problem first using the function $f(X)$ until its value cannot be reduced. We can always get assignments that minimize $g(X)$ when $f(X)$ is minimized.

In the first step, we first adjust the vector for the seventh division to reduce the value of $f(X)$, then we perform exchanges (see below) to reduce $g(X)$ without affecting $f(X)$.

Then we similarly adjust the vector for the sixth division, then the fifth, and so on. Repeating the procedure until the values of the objective functions cannot be reduced further in this way, we obtain an approximate optimal solution.

Permutation

We still have more than 10^{80} different combinations, regardless of whether the members are divided into 6 or 4 groups in one session. It is still impossible for a computer to investigate each possibility. So we give a simple strategy. We exchange the seats of two members who are not assigned to the same group. If the exchange reduces the value of the objective function, it is acceptable; otherwise, we refuse to accept it. We repeat the procedure over all such pairs, thereby minimizing the value of the objective function.

Steps of the Algorithm

[EDITOR'S NOTE: For space reasons, we omit the detailed pseudocode of the algorithm.]

Solution

Using the greedy heuristic gives the plan of **Table 2** as the initial feasible solution, which has $f(X) = 209$ and $g(X) = 28.25$. Applying our iteration yields the result in **Table 3**, with $f(X) = 81$ and $g(X) = 19.44$.

Table 2.

Assignment table for initial feasible solution from greedy heuristic.

Morning.

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
Session 1	1 7 13 19 25	2 8 14 20 26	3 9 15 21 27	4 10 16 22 28	5 11 17 23 29	6 12 18 24
Session 2	4 11 14 21 26	5 10 15 23 28	6 12 16 20 29	1 8 17 24 27	2 7 18 22	3 9 13 19 25
Session 3	2 9 15 22 27	1 11 16 21 29	4 7 17 24 28	3 12 18 23	6 8 13 19 25	5 10 14 20 26

Afternoon.

	Group 1	Group 2	Group 3	Group 4
Session 4	3 7 11 14 20 22 27	1 5 9 16 18 24 26	4 8 12 15 19 23 28	2 6 10 13 17 21 25 29
Session 5	2 7 10 16 19 23 27	3 6 11 15 17 22 26	4 8 12 13 18 21 28 29	1 5 9 14 20 24 25
Session 6	1 6 10 14 18 21 27	2 5 11 13 19 22 26 29	3 7 12 16 20 24 28	4 8 9 15 17 23 25
Session 7	1 6 11 13 18 22 25	2 7 9 15 20 23 28	3 5 12 16 19 24 27	4 8 10 14 17 21 26 29

Table 3.

Assignment table after application of iteration.

Morning.

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
Session 1	1 6 13 19 24	2 3 14 25 26	5 8 10 15 27	7 16 20 22 28	9 11 21 23 29	4 12 17 18
Session 2	3 10 11 26 28	5 12 19 22 23	6 16 20 21 29	1 2 17 24 27	4 7 15 18	8 9 13 14 25
Session 3	5 7 14 17 29	4 13 15 16 21	1 9 19 26 28	3 11 18 23	6 8 12 25 27	2 10 20 22 24

Afternoon.

	Group 1	Group 2	Group 3	Group 4
Session 4	4 6 11 14 20 23 27	7 8 9 16 18 24 26	2 5 17 19 21 25 28	1 3 10 12 13 15 22 29
Session 5	1 4 10 14 16 23 25	5 6 11 15 17 22 26	2 8 13 18 19 27 28 29	3 7 9 12 20 21 24
Session 6	6 9 10 18 21 22 27	1 2 11 12 14 15 16 19	7 8 13 17 20 23 26	3 4 5 24 25 28 29
Session 7	1 5 11 13 18 20 25	2 6 9 15 23 24 28	3 7 10 16 17 19 27	4 8 12 14 21 22 26 29

We compare these two plans in **Table 4**. The initial feasible solution satisfies the criterion of balanced assignment. The final solution as optimized by the iterative method not only keeps the balance of the initial solution but also reduces the objective functions $f(X)$ and $g(X)$. We believe that the final solution satisfies the criteria of the problem. It attains the two goals of the problem, and it reduces not only the number of pairs that fail to meet (to 26) but also the number of multiple meetings.

Table 4.
Comparison of initial feasible solution with result after iteration.

Number of times a pair meets	0	1	2	3	4	5
Initial feasible solution from greedy heuristic	90	154	119	33	9	1
Solution after iteration	26	253	102	25	0	0

Adjusting Assignments

Let's consider how to adjust the assignments when some board members cancel at the last minute or some not scheduled show up. We could just renumber the attendees and solve the problem again. But in real life, we would not like to adjust the assignments too much. So we give another strategy for adjustment.

Case 1

When some members not scheduled show up at the last minute, we consider only how to assign without changing the given assignments. We use an adjustment method that is similar to the greedy heuristic. The additional members are assigned into the groups one by one. We always try to find the best group that the member should be in, according to the given constraint of keeping the assignments balanced and making the attendees mixed well.

Case 2

When some board members cancel, we delete one absentee at a time. We select one of the board members of the same type from the original assignments whose absence will lead to best mixing, that is, whose absence would reduce the value of the objective functions most. Then we let that member replace one of the absentees and delete the absentee in the list of assignments.

Case 3

When some board members cancel and some not scheduled show up at the same time, we classify them according to their types. Let a stand for the number of absentees and b stand for the number of additional members. For the members of the same type, we do the following operation:

- If $a = b$, then use the additional members to replace the absentees.
- If $a < b$, then first replace all of the absentees by some of the additional members. Then assign the remaining additional members using the method of Case 1.
- If $a > b$, then replace all of the additional members by some of the absentees, keeping the balance of the assignments. Then delete the remaining absentees using the method of Case 2.

Table 5 compares the results of such adjustment with the greedy heuristic and with the iterative algorithm, for several cases of absentees and additional members.

Table 5.

Comparison of results for the greedy heuristic (G), the iterative algorithm (I), and the adjustment algorithm (A) for various cases of absentees and additional members.

	Situation		Method	Number of times that a pair meets					
Senior	In-house	Other		0	1	2	3	4	5
6	9	20	G	90	154	119	33	9	1
			I	26	253	102	25		
			A	no adjustment needed					
6	9	21	G	88	171	132	39	5	
			I	33	253	128	21		
			A	33	264	108	28	2	
6	9	19	G	76	145	126	29	2	
			I	24	236	100	16	2	
			A	26	235	93	24		
6	10	20	G	87	164	146	36	2	
			I	35	247	134	19		
			A	33	263	104	28	2	
6	8	20	G	74	140	140	24		
			I	26	231	103	17	1	
			A	23	237	95	23		

The adjustment strategy and the iterative algorithm are both satisfactory. It seems that the solution from the adjustment strategy should be as good as that from the iterative method; but in several cases (e.g., for 8 in-house members), the solution from the adjustment strategy is better.

Extension of the Model

Our model and solution method are completely general and apply to number of members, kinds of members, and levels of participation.

Assume that

- there are d types of attendees,
- the whole meeting is divided into w stages,
- there are S_i sessions for the ith stage, and
- each session consists of G_i discussion groups for the ith stage.

The assignments should also satisfy the following requirements:

- Each member can be assigned to only one group in each session.
- The attendees of type α are to be divided equally in a session.
- The whole assignment must always be balanced.
- An attendee of type L is not allowed to meet a particular senior officer more than c times in stage r.

Assume that there are b_i members for the ith type of attendee. Then there are $m = \sum_{i=1}^{d} b_i$ attendees, and we number them from 1 to m. The whole meeting involves w stages with S_i sessions in the ith stage, for a total of $S = \sum_{i=1}^{w} S_i$ sessions. We assume that each session in the ith stage is divided into discussion groups. We define all variables analogously to the simpler setting of the original problem, and we arrive at a generalized boolean programming problem.

Strengths and Weakness of the Model

The model has quite good practicality, and the given algorithm has little time complexity. For the given problem size, our C program for the greedy algorithm and iterative method runs in less than 5 min on a Pentium–100 computer. That means the model can give a list of assignments quickly when the number of attendees is not too large. The assignments produced are close to the optimal solution.

The model is quite easy to extend to different numbers of attendees, numbers of groups, types of attendees, and levels of participation.

We can adjust to last-minute changes either by re-running our program or (to minimize effect on assignments already made) by using our adjustment procedure.

The weakness of the model is that there is some difference between the real optimal solution and the solution obtained from the model.

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Using Simulated Annealing to Solve the Discussion Groups Problem

David Castro
John Renze
Nicholas Weininger
Macalester College
St. Paul, MN 55105

Advisor: Karla V. Ballman

Introduction

Our task is to assign 29 corporate board members to a sequence of discussion groups organized into seven sessions, of which three are morning sessions led by senior officers and four are afternoon sessions not led by senior officers. We wish to find a combination of group assignments that best satisfies the following objectives:

- In the morning sessions, no board member should be in the same senior officer's discussion group twice.
- No group should have too many in-house members (there are nine in-house corporate employees among the 29 members).
- The number of times any two board members are in the same group should vary as little as possible.
- No two groups should have a large number of common members.

The specific case given assumes that all members will participate in all sessions and that there will be six discussion groups for each morning session and four for each afternoon session. Our model ought to be capable of producing good answers quickly under these assumptions and of adjusting to more general configurations of board members. In particular, it should be able to adjust to small changes, such as individual additions or subtractions of board members, without recalculating all assignments from scratch.

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Model Design Considerations

This problem is essentially one of optimization. The different potential assignments of board members form a solution space, and we must optimize the "fitness" of our assignments (how well they satisfy our objectives) over that solution space. Thus we have to consider the issues that come up when designing any optimizer: time and space efficiency, flexibility, and optimality of the solution. We also have concerns particular to this problem.

Multiple Objective Satisfaction

Most optimization problems involve maximizing a single objective function; here we have four objectives. The problem statement doesn't tell us whether, for example, to consider minimizing common group membership more important than minimizing the number of times any two board members meet. Presumably, we want to use some sort of weighted combination of the objectives as the function to optimize over the solution space—but determining how to combine and weight them is nontrivial.

Large Solution Space

The number of ways of assigning the board members to the discussion groups is enormous. Since each board member goes into seven sessions, we have a total of $29 \times 7 = 203$ variables; since there are six possible assignments for each member in the morning sessions and four for the afternoon sessions, the total number of possible solutions is on the order of $6^{87} \times 4^{116} \approx 3 \times 10^{137}$. Furthermore, no matter how we define our objective function, the solution space will probably contain a large number of local optima. This has two implications. First, it's probably impossible to find a global optimum (and it isn't absolutely necessary here). Second, we need a solution model that doesn't require searching over a significant fraction of the solution space.

Fast Readjustment

Since board members may drop out or in at the last minute, we want a way of taking a precomputed solution and adjusting it for small changes in the member configuration. This adjustment should be significantly faster than a complete recomputation and should also involve fewer changes of assignment.

Simulated Annealing

We chose simulated annealing as the basis for our model, as it offers the best chance of constructing an effective solution-finding algorithm.

The Simulated Annealing Process

The simulated annealing algorithm can be described as follows:

1. Start with an objective function that is evaluable for every point in the solution space, a randomly chosen point in that space, and an initial "temperature" value.
2. Evaluate the initial point's objective function value.
3. Perturb the point by a small amount in a random direction.
4. Calculate the objective function value of the perturbed point.
5. If the new function value is better than the old one, accept it automatically, staying at the perturbed point.
6. If the new function value is worse, decide probabilistically whether to stay at the perturbed point or go back to the old one. The probability of not staying at the new point should depend on how low the temperature is and how much worse the new point is than the old one.
7. Lower the temperature slightly and go to step 3. Iterate until the temperature is close to 0 or the solution stops changing.

The algorithm essentially does a hillclimb through the solution space, with occasional random steps in the wrong direction. The temperature controls how likely the algorithm is to take a step the wrong way; at the beginning, the algorithm jumps almost completely randomly around the solution space, but by the end it almost never takes a wrong step. The idea is that the random steps allow the hillclimber to avoid getting stuck at local minima or maxima. In most implementations of simulated annealing, the probability of acceptance of a wrong-way step is $e^{-d/T}$, where d is the difference between the old and new objective functions and T is the temperature. The temperature typically starts at a value sufficient to give almost any degree of wrong-way movement a significant probability of acceptance, and decreases exponentially from there.

The original idea for simulated annealing comes from statistical mechanics. The molecules of a liquid move randomly and freely at high temperatures; when the liquid is cooled and then frozen, the molecules essentially "search" for the lowest energy state possible. The minimum energy state occurs when the molecules are arranged in a specific crystalline structure. If you cool the liquid slowly, the molecules will have time to redistribute themselves and find this structure; if you cool it quickly, they will typically get "stuck" at a higher-energy, noncrystalline state.

Reasons to Use Simulated Annealing

A number of factors influenced our decision to apply simulated annealing to our problem:

- It's fast. Simulated annealing requires evaluation of the objective function over a relatively small number of points to achieve its result.
- It's simple. Everything except the fitness function evaluation can be done in well under 100 lines of C code. It isn't difficult to understand or debug.
- It lends itself well to discrete solution spaces and nondifferentiable objective functions. As long as you can provide a random jump between solution points and a way of evaluating the function at any point, it can work; many other methods require a continuous solution space, or demand that you evaluate the function's derivative.
- It has a track record of success. Simulated annealing has been used for applications as diverse as stellar spectrum analysis and chromosome research.
- We were all well acquainted with simulated annealing, and one of us had previously used it to solve a similar partitioning problem.

Alternatives

Heuristic Algorithms

One could try to get a solution by using some sort of intuitive rules about how to rearrange things, much as a human secretary would. This would likely be too slow for a large number of members, though, and coming up with good intuitive rules that a computer can implement is difficult.

Gradient Methods

Traditional gradient descent methods are another possibility. These, however, generally require that the objective function's derivative be evaluated or at least estimated, and our objective function is extremely difficult to express mathematically. Furthermore, they run the risk of getting stuck at local minima.

Integer Programming

Integer programming is commonly used for discrete optimization problems like this one. But none of us had experience implementing it, and we feared that the large size of the solution space might make it too slow.

Genetic Algorithms

We could establish a pool of potential solutions that would "evolve" toward an optimal solution through a process analogous to natural selection. But this, too, would likely be too slow and complex for our problem.

Modeling the Solution

We designed and coded in C a program that uses simulated annealing to solve a general set of discussion-group problems, including the one given.

The Data Structure

Our approach to the problem began with data structure design: We needed a way to encode the whole list of assignments that would allow us to code the annealing process straightforwardly, and preserve important testing and organization data for analysis of the annealing results.

The data structure `partition` has three levels of organization:

1. The top level, `partition`. This contains a lot of data pertaining to the whole solution structure: the number of morning and afternoon sessions, the number of groups in each type of session, and the total number of members and of in-house members participating in each session. It also contains some data relevant to the objectives concerning pairs of members and common membership (more on that below).

2. The list of groups contained in the `partition`. Each `group` corresponds to one of the discussion groups in one of the sessions and has a size variable.

3. The list of people contained in each `group`. Each `person` has a name, a number and a flag indicating whether they're in-house. Thus each instance of `person` corresponds to one member's participation in one session. Note that this allows members to participate in only some of the sessions.

The Objective Function

We chose as the objective function a weighted sum of four subfunctions. Each subfunction takes a partition and calculates how close it comes to satisfying one of the objectives given in the problem. The results are all expressed in terms of "badness," or the degree to which the partition fails to satisfy an objective; thus our problem becomes the minimization of the "badness function."

Senior officer nonrepetition

The first objective we consider is that no board member should be in the same senior officer's morning group twice. Our first subfunction simply takes the morning groups headed by each senior officer and checks for repetitions in their member lists. The total number of repetitions—that is, the total number of times a board member is in the same senior officer's morning group more than once—is multiplied by one weight to form the first part of the badness function. Since we want zero repetitions, and want the annealer to stay with solutions with zero repetitions once it finds them, we also have a "zero bonus" that subtracts a given amount from the badness if there are zero repetitions.

In-house members

The second objective states that no group should have a disproportionate number of in-house members. Our second subfunction calculates the amount of disproportionality in the distribution of in-house members among groups. For each session, it uses the number of total members and in-house members participating in the session to compute ideal floor and ceiling values for the number of in-house members in each group. (If the number of in-house members is a multiple of the number of groups in the session, then the floor and ceiling are the same; otherwise they differ by one.) Then it goes through each group in the session and tests to see if the number of in-house members is more than the floor or less than the ceiling; if so, it adds to the disproportionality sum the difference between the floor or ceiling and the actual number. So, if the ideal ceiling were two in-house members per group, a group with four in-house members in it would add two to the disproportionality sum.

We want the disproportionality sum, like the repetition sum, to be zero, and so implement a zero bonus; we set this equal to the disproportionality weight rather than making it separately adjustable, for reasons discussed below.

Pairings of board members

Our third objective is to try to ensure that each board member meets each other board member approximately the same number of times. Our third subfunction does this by computing ideal floor and ceiling values for that meeting number, just as the second does for the proportion of in-house members. In the `partition` structure we keep a matrix whose elements correspond to the number of times each pair of board members are in the same group. The third subfunction recalculates this matrix by looking at each pairing in each discussion group; derives the ideal floor and ceiling values from the matrix; and then goes through the matrix again, adding up the instances in which the number of times a pair of members meets is less than the floor or more than the ceiling. This sum becomes the *pairwise anomaly count*.

We also calculate the maximum number of times any two board members meet, and use that as well as the pairwise anomaly count in the badness calculation. The reasoning behind this is not only because it's desirable for most pairs of members to meet the same number of times, but also to ensure that no pair of members is in a hugely disproportionate number of groups together. We don't want a solution with one pair that is in every group together but with no other anomalous pairs to be preferable to a solution with many slightly anomalous pairs but with no hugely anomalous ones.

Here, we have no zero bonus for the anomaly sum, because the meeting criterion ought to be satisfied "as much as possible," rather than absolutely, and because it's impossible in many cases (including ours) to drive it to zero.

Common membership

Our final objective is to ensure that no two groups have a disproportionate number of common members. The fourth subfunction is almost precisely analogous to the third. Again, we have a matrix in the `partition` structure, listing for each pair of groups how many members they have in common; again, we calculate this matrix, use it to derive an ideal mean, and find the sum of deviations from that mean and the maximal deviation. Here, however, we don't count as deviant groups those that have fewer than the mean number of common members. It doesn't matter if two groups have no members in common, only if they have too many.

The Annealing Iteration Process

Once we have the "badness function" described above, we perform simulated annealing. The relevant implementation details are:

- The perturbation: A single swap of two members is the unit of random perturbation. Our program randomly chooses a session, two discussion groups within that session, and a member in each group, and then swaps them.

- The initial configuration: The file-reader dumps the members from the file into the `partition` in order, giving an extremely bad configuration. Our program does random swaps on that configuration for a random number of times (between 1 and 32,767), producing a "shuffled" initial configuration.

- The starting temperature setting and the rate of exponential decay: These are two key variables that determine how long the annealing takes and what the temperature profile is.

  For the starting temperature, we just use the badness value of the starting configuration; this means that at the starting temperature, a solution as bad as our starting one would have a 10% chance of being accepted as a step away from a perfect (zero-badness) configuration.

  For the decay, we used a variety of different rates. Multiplying the temperature by 0.998 at the end of each iteration tends to give the best time/accuracy tradeoff. Slower decay makes for long annealing runs that don't get much better; faster decay doesn't give the process enough room to "jump around" randomly at the beginning, resulting in much worse solutions.

Setting the Weights

We anneal in an attempt to minimize the following badness function:

$$\begin{aligned}\texttt{badness(partition)} = w_0 * \texttt{reptotal} - w_1 * \texttt{repzero} + w_2 * \texttt{disprop} \\ - w_2 * \texttt{diszero} + w_3 * \texttt{pairanom} + w_4 * \texttt{maxpair} \\ + w_5 * \texttt{commanom} + w_6 * \texttt{maxcomm}\end{aligned}$$

where

- `reptotal` is the number of times a member is in the same senior officer's group twice;
- `disprop` is the disproportionality sum for in-house members;
- both `repzero` and `diszero` are 1 if `reptotal` and `disprop` both are 0, and both are 0 otherwise;
- `pairanom` and `maxpair` are the anomaly score for pairwise meetings and the maximum number of times a pair meets;
- `commanom` and `maxcomm` are the anomaly score for group common membership and the maximum number of members a group has in common; and
- $w_0, \ldots, w_6$ are integer weights.

How should we set the weights? We did so by trial and error. The set of weights $w_0 = w_1 = 1200$, $w_2 = 1000$, $w_3 = 400$, $w_4 = 4000$, $w_5 = 100$, and $w_6 = 500$ produces extremely good results for a 9,000-iteration annealing run on the standard problem configuration. Such a run takes about 5 min on an HP 712/60 workstation. Using these weights, we could repeatedly produce solutions that had no senior officer repetitions, no instances of in-house member disproportion, no pair of members that met more than three times, and no pair of groups with more than three members in common.

We used somewhat different weights for a longer annealing run to produce our very best solution, and also adjusted the weights to produce solutions for different session configurations, as we will describe below. But the set of defaults above appeared to work remarkably well for a variety of configurations.

Some things to note about the default weights:

- The zero bonus for senior officer nonrepetition equals the minimization weight. That works so well that we hard-coded in the same equality for in-house disproportionality, rather than making another adjustable zero bonus.
- The ratio of pairwise minimization weight to pairwise maximum weight is 1 to 10. That means that the algorithm considers reducing the pairwise anomaly score by 10 equivalent to reducing the pairwise maximum by 1. Lower ratios tend not to force the pairwise maximum low enough; higher ratios tend to prevent the annealer from occasionally making the pairwise maximum higher by 1 on its way to a much lower pairwise anomaly score.

- The common membership weights are quite small. We found that good common membership configurations were strongly correlated with good pairwise meeting configurations; that is, configurations in which no pair of members met an inordinate number of times also tended to be configurations in which no two groups had too many common members.

Our Solution

Using extra-long runs and adjusting the weights by trial and error, we eventually produced the solution in **Tables 1–2**. The in-house members are 1–9 and the non-in-house members are 10–29.

Table 1.

Assignments by discussion group.

Session	Group	Members
Morning 1	1	27 22 20 17 1
	2	4 28 21 3 13
	3	10 11 8 29 18
	4	14 19 24 7 23
	5	15 9 5 16 12
	6	6 25 2 26
Morning 2	1	13 29 2 6 28
	2	10 16 9 19 25
	3	7 23 20 22 12
	4	5 11 26 17 3
	5	8 14 20 27 4
	6	24 21 1 15
Morning 3	1	16 8 19 21 5
	2	2 18 22 1 23
	3	15 26 17 9 14
	4	12 25 4 29 27
	5	2 13 20 11 6
	6	28 7 3 10
Afternoon 1	1	27 26 24 2 8 10
	2	6 3 22 15 19 4 18
	3	11 16 23 5 25 14 1 28
	4	21 9 13 17 29 7 12
Afternoon 2	1	23 6 5 21 10 27 12
	2	3 15 29 14 2 25 20
	3	26 9 1 8 28 19 22 13
	4	7 24 4 17 16 11 18
Afternoon 3	1	15 4 1 13 2 10 16
	2	5 24 12 25 22 17 8
	3	21 28 14 20 7 26 18 6
	4	27 23 11 19 3 9 29
Afternoon 4	1	7 25 27 13 19 5 18
	2	21 14 22 11 10 2 9
	3	12 29 3 6 16 24 26 1
	4	4 23 17 15 28 20 8

Table 2.
Assignments by board member.

	Morning 1	Morning 2	Morning 3	Afternoon 1	Afternoon 2	Afternoon 3	Afternoon 4
In-house members							
1	1	6	2	3	3	1	3
2	6	1	2	1	2	1	2
3	2	4	6	2	2	4	3
4	2	5	4	2	4	1	4
5	5	4	1	3	1	2	1
6	6	1	5	2	1	3	3
7	4	3	6	4	4	3	1
8	3	5	1	1	3	2	4
9	5	2	3	4	3	4	2
Other members							
10	3	2	6	1	1	1	2
11	3	4	5	3	4	4	2
12	5	3	4	4	1	2	3
13	2	1	5	4	3	1	1
14	4	5	3	3	2	3	2
15	5	6	3	2	2	1	4
16	5	2	1	3	4	1	3
17	1	4	3	4	4	2	4
18	3	5	2	2	4	3	1
19	4	2	1	2	3	4	1
20	1	3	5	1	2	3	4
21	2	6	1	4	1	3	2
22	1	3	2	2	3	2	2
23	4	3	2	3	1	4	4
24	4	6	5	1	4	2	3
25	6	2	4	3	2	2	1
26	6	4	3	1	3	3	3
27	1	5	4	1	1	4	1
28	2	1	6	3	3	3	4
29	3	1	4	4	2	4	3

How Good Is This?

In this configuration, no member is ever in the same senior officer's morning discussion group twice. No group contains a disproportionate number of in-house members (the morning groups contain 1 or 2, the afternoon groups 2 or 3).

No pairs of members are in the same group together more than 3 times. Of the possible pairs of members, 40 never meet; 214 meet once; 138 meet twice; and 14 meet three times. Thus the pairwise anomaly score is 54, and the vast majority of members meet one another a "reasonable" number of times (the mean number of meetings is about 1.3). It is possible to achieve a configuration in which no two members meet more than twice, but not while preserving the other objectives.

Also, no two discussion groups have more than two members in common.

This is as low as possible.

We conclude that this configuration satisfies all the objectives well and three of the four perfectly; it is likely extremely close to the global optimum.

Typical Results on the Standard Configuration

A typical annealing run, with default weights, will zero out the senior officer repetition and in-house disproportionality. It will also reduce the maxima of member-pair meetings and group common membership to 3. It will not usually reduce maximum group common membership to 2, and it typically gives a pairwise anomaly score of 50 to 60.

Thus, the standard annealing run's result isn't quite as good as our best; but it's nearly as good and much easier to achieve. It considers only about 9,000 different sets of assignments in reaching its solution, and takes about 6 min.

Generalizing the Model

More and Different Board Members

We tested the annealing process on data sets with a variety of different numbers of board members, ranging from 20 to 100. We also tried keeping the number of members at 29 and adjusting the proportion of in-house members. Finally, we tried keeping the existing set of 9 in-house and 20 other members but changed the number of senior officers, the number of morning and afternoon sessions, and the number of groups in each afternoon session. In all cases, we annealed with the same default weights used for the standard configuration.

The model responded extremely well in each case. **Table 3** shows the times required for a full annealing run on the standard session configuration with various numbers of members.

Table 3.

Run-time results.

Members			Run-time
In-house	Others	Total	
5	10	15	2:56
9	20	29	6:38
12	27	39	8:25
15	33	48	9:53
18	41	59	22:06
30	70	100	58:01

We also tried changing the profile to 4 in-house and 25 other members, and to 14 in-house and 15 other members. In both cases, the annealing still drove the in-house disproportionality to zero and reduced the pairwise and commonality maxima to 3.

We tried a total of five different changes in the session / group profile; these ranged from decreasing the number of groups in each morning session to two to having only one morning session and six afternoon sessions. In almost all cases, the annealing produced solutions that were as good as could be expected, given the limitations of the sessions and groups.

The only exception occurred when we tried having three officer-led groups for each of the three sessions. Then the default weights failed to minimize the senior officer nonrepetition criterion, which is much harder to achieve with three groups than with six. Increasing w_0 to 2,000 and rerunning gave much better results.

Different Levels of Session Participation

The data structure and annealing code are designed to deal transparently with members who attend some but not all of the sessions, by considering each member's participation in each session as a separate variable. We tested our code on several different session-participation variations, including:

- making the in-house members not attend the afternoon sessions,
- making some of the non-in-house members not attend the morning sessions, and
- introducing new members who go only to one morning session.

In all cases, the annealing produced good results—zero senior officer repetitions, no more than two instances of in-house disproportionality, maximal pairwise meetings, and group common memberships at one more than the ideal mean.

Adjusting Quickly to Last-Minute Changes

Another important consideration is how well our model can deal with small changes in the configuration of board members—one or two new board members added or deleted, say. We investigated two approaches to adjusting an already-annealed configuration, reannealing and flip-path search. Both are motivated by the idea that the new configuration's best solution should be quite close to the old one's. We tested these approaches on several small modifications: single and double additions and deletions to all sessions, plus additions to only one or two sessions.

Reannealing

Since we want to stay in the neighborhood of the old configuration and to take less time than the original annealing, we use a much lower starting temperature. Experimentation shows that dividing the regular starting temperature by 1,000 works best. The new configuration often is very different from the old one. This would be undesirable in real-world applications, where you might want to make as few changes to group assignments as possible.

Greedy path search

The alternative is to try to reduce the badness function with as few changes as possible. One way is to try all possible single flips in the configuration, then try all possible combinations of two possible single flips, and so on, always looking for the flip sequence that produces the lowest badness function value.

This approach quickly becomes too slow. There are 2,310 possible flips in the standard configuration, and evaluating all of them takes more than 1 min on an HP 712/60; thus, evaluating all two-flip sequences would take at least 20 h. An alternative, much faster approach is to try all single flips, take the one that produces the lowest badness function, perform that flip, try all single flips from the resulting flipped configuration, and so on. We call this a *greedy path strategy*, and in our tests it brought the new configuration's badness function down acceptably close to the old one's in seven to ten flips.

Furthermore, we observed during our tests of the greedy path search that it tended to do best after doing one flip involving groups in each of the seven sessions. This was especially true when we added a new member to all of the sessions. This makes sense, because it ought to take one flip to put a new member in the "right" place in each session.

So we tried the following variation: Perform the best of the possible flips involving groups in the first session, then perform the best of the flips involving the second session, and so on, to the last session. This requires considering all of the 2,310 possible flips only once. This does not work nearly as well as the original greedy search, probably because this approach fixes the order in which the best flips in the sessions are taken.

We finally found a workable hybrid approach. This approach starts out by finding and taking the best flip from all the sessions, then takes the best flip in each session in order, and finally again finds and takes the overall best flip. It runs in roughly 5 min for the standard configuration and gives test results about as good as for the original greedy path search. It doesn't always work as well as reannealing, but it never requires more than nine (in the standard configuration) single-flip changes.

Improving the Model

Complexity

Our algorithm runs in time approximately proportional to the square of the number n of board members. The pairwise part of the badness calculator goes through a matrix including all $n(n-1)/2$ pairs of members. The run-time is also quadratic in the number of groups, because of the pairs-of-groups common membership matrix.

It would be nice to develop a badness function that runs in linear time and produces a good approximation to our original function value. We could also make smaller efficiency improvements, such as developing a way to update efficiently the pairwise and commonality matrices with each single flip instead of recalculating them on each iteration. Time considerations prevented us from doing this (we tried doing it with the pairwise matrix but never got it to run significantly faster than straight recalculation). We believe that our algorithm runs remarkably well as it is; finding a near-minimum over a solution space of 10^{137} points in 6 min is no small task.

Flexibility

The present model allows only two kinds of group sessions—morning and afternoon—and assumes that all sessions have the same number of groups of the same size (or differing by only one). It wouldn't be too difficult to extend the `partition` structure to allow for a more complex session structure, such as one that would allow for different numbers of senior officers at each morning session.

We could also add different types of board members and specify new objectives based on them. Perhaps, for example, one might want to stipulate that some board members are new, and that new board members should all be in the same discussion groups so that they can get to know each other (or at least that every new member should meet every other member once).

Finally, we could try to devise a general method for setting annealing weights, given a configuration. The default standard weights appear to work well for a large range of configurations, but they are almost certainly not optimal for all configurations.

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Meetings, Bloody Meetings!

Joshua M. Horstman
Jamie Kawabata
James C. Moore, IV
Rose-Hulman Institute of Technology
Terre Haute, IN 47803

Advisor: Aaron D. Klebanoff

Introduction

We present a model and three algorithms for solving the problem of scheduling An Tostal Corporation's upcoming board meeting. To determine how well a schedule will fit An Tostal Corporation's needs, we establish a badness function based on assumptions as to what kinds of schedules are desirable. These assumptions include that a good mix of people in meetings is pivotal for effective idea-sharing.

The algorithms that we compare are based on random selection, greedy assignment, and greedy assignment followed by hill-climbing. The random algorithm places board members at random. The greedy algorithm assigns one member at a time by examining which possibility is locally optimal, hoping for a solution that is globally optimal. The greedy algorithm with hill-climbing enhances the greedy algorithm's effectiveness by tweaking the schedule in a manner that will cause the schedule to edge closer and closer to optimality.

We also provide a simple algorithm that will allow a secretary to handle any unforeseen additions or cancellations. This algorithm is designed to alter as few board members' schedules as possible while still obtaining a good mix of people.

Finally, we summarize the strengths and weaknesses of the algorithms presented, present a sample solution for use by the An Tostal Corporation, and conclude that the "greedy-twiddle" algorithm is most effective and provides a very good solution nearly every time.

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Assumptions and Justifications

- All discussion groups in a given session should be of approximately the same size. A discussion group that is too large will not facilitate good discussion. If any of the discussion groups are too small, then some of the discussion groups will be too large (since there are a fixed number of discussion groups and board members), and we will have the same problem.
- Since concurrent discussion groups will be of roughly equal size and we are to put a proportionate number of in-house board members in each discussion group, each concurrent discussion group should have approximately the same number of in-house members.
- All board members are treated equally, with the exception of in-house employees as outlined in the problem statement. There are no special needs considered in the assignment of schedules, such as a board member desiring to avoid a particular individual.
- All of the senior officers are present for the three morning meetings.
- The schedules will not be adjusted for board members who are late or have to leave during the middle of the day.
- When the secretary is performing a last-minute change to the schedule, it is desirable to minimize the number of members who experience a schedule adjustment. That is, a change that completely rearranges one member's schedule is better than a change that slightly alters many members' schedules.

Defining the Model

A natural model is a hypergraph in which each vertex corresponds to a board member and each edge corresponds to a discussion group. However, the added constraints, such as the in-house board members' even distribution and the limitation that each board member see each officer at most once, disrupt the structure of the hypergraph enough to render many of the well-known theorems inapplicable. On the other hand, some of the matrices derived from the hypergraph model, such as the adjacency matrix and the incidence matrix, are quite useful for our computations.

With this as the underlying model, each board member corresponds to a row and a column of the adjacency matrix, and to a row of the incidence matrix. A column of the incidence matrix corresponds to a discussion group. Each board member is represented by a number from 1 to 29, with 1 through 9 representing the in-house board members (corporate employees).

A good schedule is one that does a good job of meeting the following criteria:

- The total number of board members in any two concurrent discussion groups does not differ by much.
- The number of in-house board members in any two concurrent discussion groups does not differ by much.
- In the morning sessions, no board member is in multiple discussion groups led by the same senior officer.

We design our algorithm so that every schedule produced satisfies the third condition; so we attempt to devise a schedule that does a good job of meeting the first two criteria. We measure how good a schedule is by defining a function that assigns a "badness" to each schedule. First, we define an *encounter* as one instance of two board members being in the same discussion group. We want our badness function to penalize, by increasing the badness, a schedule that contains a pair of board members who have too many encounters. Making sure that no board members have too many encounters automatically ensures that no board members have too few encounters. Since there is a fixed number of encounters (as long as the discussion group sizes stay constant), then if one pair of board members has too few encounters, it is only because another pair has too many. We define e_i to be the number of pairs of board members who encountered each other i times. A reasonable function for the badness b of a particular schedule S is

$$b(S) = \sum_{i=2}^{\infty} e_i \cdot 4^{i-2}.$$

By making the badness penalty increasingly higher for each additional encounter between a pair of board members, we create a function that is minimized when every pair of board members has the same number of encounters. This badness function is not good enough, though, since such a function also must take into account the distribution of the in-house board members. We will say that all schedules for which the numbers of in-house members in any two concurrent discussion groups differ by no more than one are equally good. However, a very heavy penalty will be imposed for each discussion group that contains a number of in-house board members that is more than one away from the average number of in-house board members in a discussion group. Define d to be the number of discussion groups in the entire schedule that fall into this category; then we can revise our badness function to

$$b(S) = 1000d + \sum_{i=2}^{\infty} e_i \cdot 4^{i-2}.$$

This penalty prevents deviation from an even distribution of in-house board members. Minimizing the badness of a schedule will take care of the first two criteria of an optimal schedule referred to above. The minimum theoretical value for $b(S)$ is 129, calculated using the Pigeonhole Principle.

Random Selection Algorithm

There is a huge search space, containing more than a googol (10^{100}) of schedules, so a brute-force attack is out of the question. To provide a basis for comparison, we see how effective a random selection algorithm is. This algorithm assigns each board member to a random discussion group in each session, making sure that no board member is in multiple discussion groups led by the same senior officer. Although we impose no limit on the size of a discussion group, we hope that a random algorithm will distribute the board members evenly. We implemented this algorithm on a computer and ran it 1,024 times. **Figure 1** displays the badness of these executions. The results are not good.

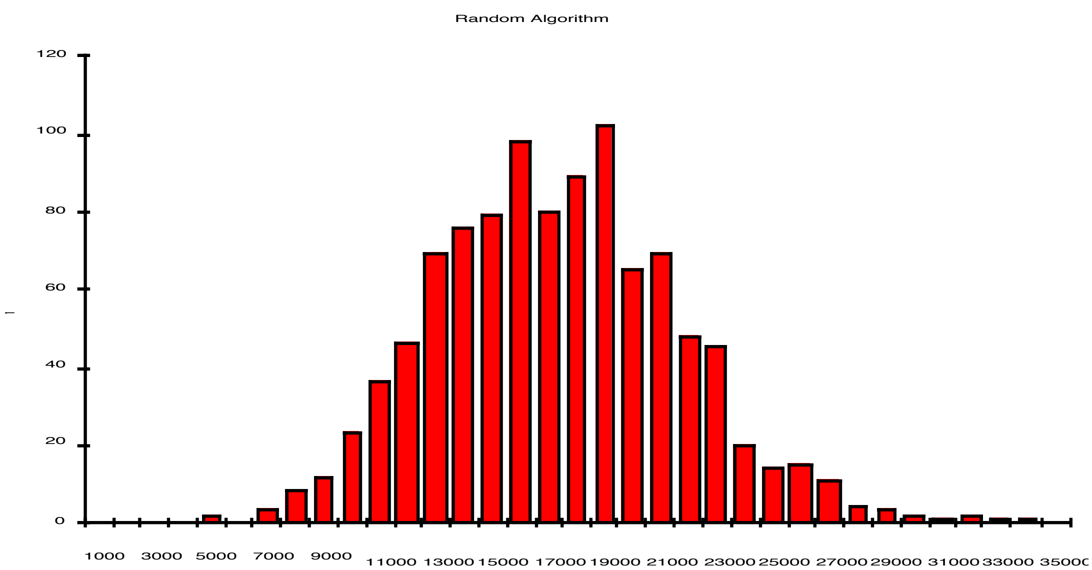

Figure 1. Badness results for random assignments.

Greedy Algorithm

We sought a heuristic requiring a minimum amount of backtracking, and a greedy algorithm seemed logical. We assign the in-house board members separately before assigning the rest of the board members. Each board member is placed in the first available position that minimizes the number of encounters that board member has had with any other board member. The algorithm, shown in **Figure 2**, works by assigning all of the in-house board members to each session, then going back and filling each session with the remaining board members. The order in which board members are placed and the order in which the various discussion groups are tried is random. This feature facilitates distributing the board members evenly, by eliminating regular patterns that may occur when placing the board members in the same way every time.

```
for member from 1 to 9 (in random order)
   for session from 1 to 3
     select the groups in which member has never been
     of them, choose the groups containing the fewest other members
     of those, choose the group containing members with whom member has been
       the fewest times
     place member in this group
   for session from 4 to 7
     select the groups containing the fewest other members
     of those, choose the group containing members with whom member has been
       the fewest times
     place member in this group
for session from 1 to 7
   for member from 10 to 29 (in random order)
     set greedlevel to 0
     select a group not led by an officer already encountered by member
     repeat
       does this group have a member with more than greedlevel encounters?
         If no, place member in this group
         If yes, select a group that hasn't been tried yet
           If every group has been tried, increment greedlevel and consider all
            groups untried
     until member is placed
```

Figure 2. Greedy algorithm.

Figure 3 shows the badness from 1,024 runs of the greedy algorithm. The greedy algorithm is a *dramatic* improvement over random assignment—all badnesses are below 300, while the best that random assignment could do was more than 5,000. However, there are many final schedules that the greedy algorithm can never reach. For example, in first several sessions, no board member will see any other more than once, since the greedy algorithm won't place two board members who have already encountered each other together in a meeting until it is unavoidable. The best schedule, on the other hand, may have a pair of board members who encounter each other in both the first and second sessions; but the greedy algorithm will never find it.

Greedy-Twiddle Algorithm

We can do better by using hill-climbing together with the greedy algorithm. This means that we make small changes to our schedule, determine whether the slightly modified schedule is better or worse, using our badness function, and proceed making small adjustments until we can do no better. The greedy algorithm helps a lot, in that we start our hill-climbing from a schedule that is much better than a random assignment. We call the small changes *twiddles*. Each twiddle consists of swapping two board members between meetings in a

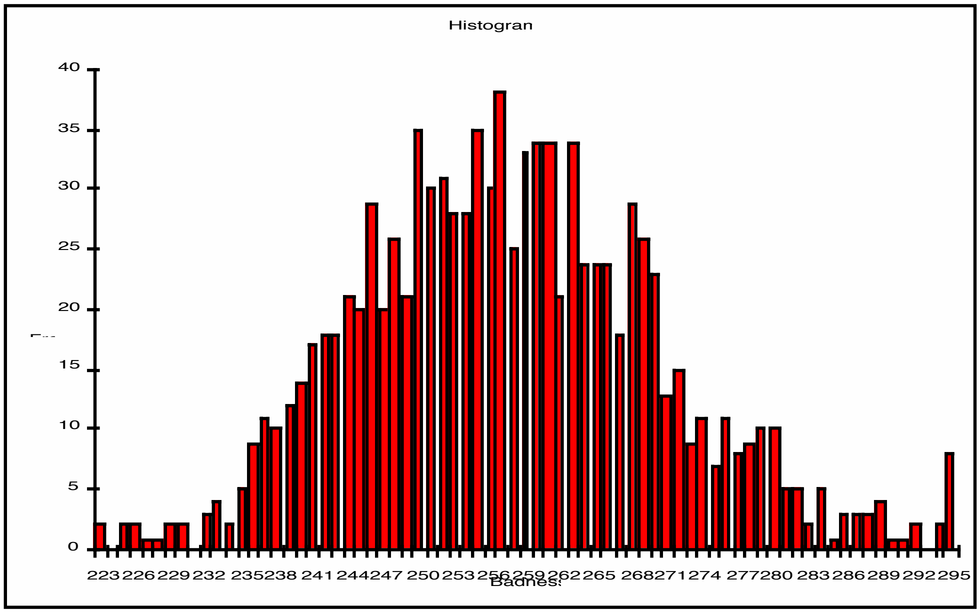

Figure 3. Badness results for assignments made by the greedy algorithm.

session or simply moving a board member from one meeting to another. After twiddling, we compute the badness function and determine if the twiddle made our schedule better or worse. If it made it worse, we undo it. We continue until no swap or move improves our schedule.

Figure 4 shows the badness of 679 runs of the greedy-twiddle algorithm, which shows significant improvement on the simple greedy algorithm—all badnesses are below the best that the greedy algorithm could do.

Table 1 shows our final recommendation to the An Tostal Corporation on scheduling their meeting; it is the best schedule from the 679 runs mentioned above.

Secretary's Algorithm

To maintain the near-optimality of the schedule while changing as few members' schedules as possible, the secretary will do the following when performing last-minute changes:

- Determine the net change in the number of in-house members who will be attending and the net change in the number of out-of-house members who will be attending. If you can pair up an in-house member who canceled and an in-house member who added, then just give the added member the schedule originally assigned to the canceled member, and similarly for out-of-house members. Do this for as many pairs as possible.

- Treat the situation for in-house members as follows:

Table 1.
Recommendation on scheduling.

Session	Members
Morning 1	1 3 10 12 26
	2 8 17 21 25
	6 13 22 28
	5 11 18 23 27
	4 14 15 19 24
	7 9 16 20 29
Morning 2	7 11 13 17 24
	1 4 15 22 27
	8 9 16 18 26
	6 10 14 20 25
	3 5 21 28 29
	2 12 19 23
Morning 3	4 5 25 29
	6 7 11 19 26
	3 15 17 20 23
	9 12 13 21 24
	2 10 16 18 22
	1 8 14 27 28
Afternoon 1	2 3 6 18 24 26 27 29
	8 9 11 12 14 15 22 25
	1 5 13 16 19 20 21
	4 7 10 17 23 28
Afternoon 2	5 6 7 12 15 18 21
	2 4 11 14 20 26 28
	1 9 17 19 22 24 29
	3 8 10 13 16 23 25 27
Afternoon 3	5 8 20 22 23 24 26
	3 7 9 18 19 25 28
	1 2 10 11 13 15 29
	4 6 12 14 16 17 21 27
Afternoon 4	5 9 10 15 17 26 27
	4 8 12 13 18 19 20 29
	1 6 11 16 23 24 25 28
	2 3 7 14 21 22

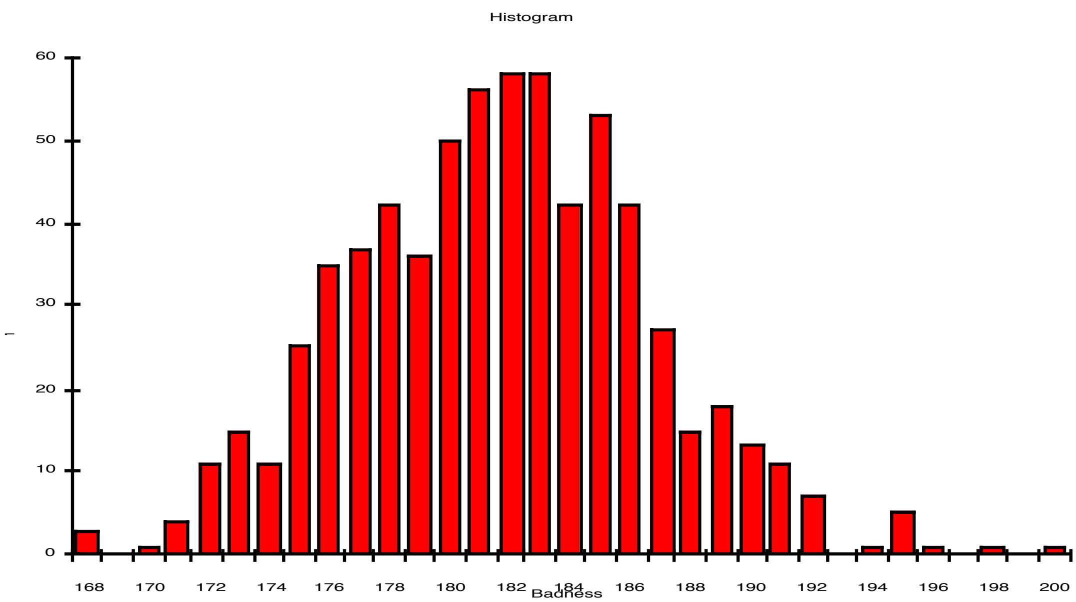

Figure 4. Badness results for assignments made by the greedy-twiddle algorithm.

- If there are still in-house cancellations that could not be paired with in-house additions:
  - ∗ Remove from each discussion group the member(s) who canceled.
  - ∗ If this makes the number of in-house members in a discussion group disproportionately low, move an in-house member from a group with more than the optimal number of in-house members to this group. Move as few members as possible, choosing first from members whose schedules have already been changed. Repeat for each discussion group.
- If there are still in-house additions that could not be paired with in-house member cancellations: For each session, place each added member in a discussion group by following these steps:
  - ∗ Eliminate all discussion groups led by a senior officer whom the member has already encountered.
  - ∗ From the discussion groups remaining, eliminate any group that has more in-house members attending than any other remaining discussion group.
  - ∗ For each remaining group, determine which member has had the most encounters with the member to be added. Choose the group for which this number of encounters is the lowest.

• Treat the situation for out-of-house members in analogous fashion.

Strengths and Weaknesses

We detail the the strengths and weaknesses of the various algorithms in **Table 2**.

Table 2.
Strengths and weaknesses of the algorithms.

Algorithm	Strengths	Weaknesses
Random	very simple, fast, requires no computer	bad schedules uneven group sizes
Greedy	relatively simple, fast on a computer	mediocre schedules, need computer for large problems
Greedy-twiddle	very good schedules	slow, difficult to program
Secretary's	secretary could do it by hand, many schedules stay unchanged	not as good as completely recomputing the schedule

Other Applications

Although the problem that motivated this model has many details unique to its situation, the model that we have developed has many other applications, such as a similar meeting with different numbers of board members or discussion groups, and also for other situations.

Consider, for example, the case of Mardi Gras Junior High School. The 120 eighth-grade students there are required to take four core classes: English, mathematics, history, and science. All four classes are offered during each of four class hours. The students also take one of six concurrent home rooms and one of six concurrent study halls. They must attend each of the core classes exactly once, but a student may have the same teacher for home room and study hall.

The school board believes that the students' educational experience can be enhanced by ensuring that the students are mixed as well as possible. That is, every student should have roughly the same number of classes with every other student. Furthermore, the board feels that the 30 honors students should be distributed so that each class (including home room and study hall) has a proportionate number. The model and algorithms developed for use at the An Tostal Corporation could be applied with minimal changes at Mardi Gras Junior High School.

A Greedy Algorithm for Solving Meeting Mixing Problems

Adrian Corduneanu
Cyrus C. Hsia
Ryan O'Donnell
hsia@math.toronto.edu
{ a.corduneanu, ryan.odonnell } @utoronto.ca
University of Toronto
Toronto, Ontario M5S 3G3
Canada

Advisor: Nicholas A. Derzko

Introduction

We consider the problem of how to best arrange a large number of people into small discussion groups so that the groups are well mixed. It is important to have well-mixed groups because any meeting runs the risk of being controlled or directed by a dominant personality. Thus, we wish to ensure schedules give different mixes of people for each group.

This problem relates directly to the case of An Tostal Corporation. The company wants to place its board members in small groups within each session so that the board members are well mixed throughout the day. The company's schedule must also satisfy other constraints:

- At the morning session, there is a senior officer assigned to each group, and no board member is to be placed with any senior officer more than once.
- A percentage of the board members are in-house members, and the company wishes that no group should have a disproportionate number of in-house members.

To solve the problem, we first develop a scoring system for schedules. There must be a schedule or schedules that achieve the best possible score; but the total number of schedules is astronomical, so we cannot check all of them. Consequently, the problem reduces to finding as good a schedule as possible in

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a short period of time. To find good schedules, we wrote a computer program in C that uses a greedy algorithm. The algorithm places the board members into the schedule one by one, at each step making the schedule placement that gives the best possible score. Our algorithm also uses a switching procedure to improve on the greedy placement at each step.

Assumptions

- The day is split into a number of sections, and each section has a fixed number of sessions and a fixed number of groups per session. We assume that the senior officer constraint applies in a fixed subset of the sections. For these sections, we assume that the number of senior officers is the same as the number of discussion groups, and that there are at least as many discussion groups in an officer-led section as there are meeting sessions (not assuming this makes the problem unsolvable).
- The secretary in charge of scheduling can change the parameters of our computer program to reflect the discussion day at hand.
- It is not vitally important to get the perfect mix of group members; close-to-optimal solutions are acceptable.

Choosing an Incidence Scoring System

We need to develop some scoring system to provide a total ordering on the set of all possible configurations; this way, there will be some best configurations, and our goal is a configuration with a score that is as good as possible. Although there are a number of side constraints imposed by the An Tostal Corporation (relating to the in-house members and the senior officers), we assume that our computer algorithm will take care of these and that the main criterion—mixing well—is what our score should measure.

The obvious first choice for an incidence score is just the *incidence sum*. However, this scoring system turns out to have a major flaw: The minimal incidence sum is easy to achieve, but in many cases the incidence elements vary widely. An incidence matrix composed mostly of 1s and 2s is better than one with lots of 0s but also lots of 5s, 6s, and 7s; we don't want to minimize just overall incidence, we want to try to minimize everyone's incidence, which presumably involves keeping everyone's incidence about the same. At the other extreme, the variance or standard deviation of the incidence elements in the lower incidence matrix could be used as the scoring system. However, using just variance does nothing to keep the overall incidence low; it merely keeps all the incidence elements close. As a compromise, the scoring system

Table 1.

Definitions and notations (in parentheses, values for the An Tostal problem instance).

Day	The time block that encompasses all the meetings for a given problem.
Session	A block of time within a day, to which meetings are devoted. Each person goes to only one meeting during a session.
Groups	Actual meetings attended by sets of board members.
Section	A set of sessions with the same number of groups per session.
Configuration	An assignment of people into groups so that each person is in one and exactly one group per session (sometimes referred to as a *schedule*).
S	The number of sections. (2)
$N_1, \ldots, N_S$	The number of sessions per section. (3, 4)
$G_1, \ldots, G_S$	The number of groups per session. (6, 4)
B	The number of people to be scheduled (board members). (29)
I	The number of board members who are in-house members. We number the board members $1, \ldots, B$, and $1, \ldots, I$ are the in-house members. (9)
O	The number O such that the senior officer constraint applies to sections $1, \ldots, O$. For simplicity, we number the groups in any officer-led section $1, \ldots, G$ and regard each officer as leading the same group number for the entire section. Thus, a configuration satisfies the officer constraint if and only if every board member is in a different group number (within each officer-led section). (1)
Incidence	The incidence of person X with person Y is the number of times that X is in the same group as Y within a given configuration.
Incidence matrix	The incidence matrix for a given configuration is the $B \times B$ matrix $IM = (a_{ij})$, where a_{ij} is the incidence of person i and person j. Elements of this matrix are *incidence elements*.
Lower incidence triangle	Since any incidence matrix has zeros down its main diagonal and is symmetric, we often consider just the lower triangle minus the diagonal, i.e., a_{ij} for $j < i$.
Incidence score	A rating given to the configuration that reflects how well mixed it is, a function of the configuration's lower incidence triangle. Lower scores are better.
Incidence sum	The sum of the incidence elements in the configuration's lower incidence matrix, which gives the total number of common memberships over all groups.
Optimal configuration	A configuration with the lowest possible incidence score.

that we use is a sum of squares:

$$\text{the incidence score of } IM = \sum_{1 \leq i \leq j \leq B} a_{ij}^2.$$

This scoring system requires the incidence elements to be close to one other. At the same time, if we can bound this score, we can also automatically bound the incidence sum using the Cauchy-Schwartz inequality (see **Theorem 3**).

Theoretical Lower Bounds for Scores and Sums

We wish to find a theoretical lower bound for all possible incidence sums for a given day. We first find a closed form for this total.

Theorem 1. *For any configuration, the incidence sum is*

$$\sum_{\text{all groups } G} \binom{n_G}{2},$$

where n_G is the number of people in the group G.

Proof: The sum of the incidence elements in the lower incidence matrix of a configuration is the same as the sum over all possible pairs in all the groups. For each group with n_G people, there are $\binom{n_G}{2}$ pairs. The total number of pairs for all groups is the indicated sum. □

This closed form allows us to calculate a lower bound in terms of the problem parameters.

Theorem 2. *For a given day, the incidence sum of configurations is bounded below by*

$$\frac{1}{2} B^2 \sum_{i=1}^{S} \frac{N_i}{G_i} - \frac{1}{2} B \sum_{i=1}^{S} N_i.$$

(For An Tostal, this value is 529.25.)

Proof: From the previous theorem, we have that a configuration's incidence sum is

$$\sum_{\text{all groups } G} \binom{n_G}{2} = \sum_{\text{all groups } G} \frac{n_G^2 - n_G}{2} = \frac{1}{2} \sum_{\text{all groups } G} n_G^2 - \frac{1}{2} B \sum_{i=1}^{S} N_i.$$

To get a lower bound for this value, we apply the Cauchy-Schwartz inequality [Plank and Williams 1992, 46] to each section, since the number of groups may differ across sections. We have

$$\left(\sum 1^2\right) \left(\sum n_G^2\right) \geq \left(\sum n_G\right)^2 = B^2 N_i^2,$$

where the sums are each over all groups G of section i. This yields

$$(G_i N_i)\left(\sum n_G^2\right) \geq B^2 N_i^2,$$
$$\sum n_G^2 \geq \frac{N_i}{G_i} B^2,$$

with equality holding iff all the n_G are equal (this may not be achievable in our discrete case, as these numbers must be integers). So taking the sum of all these inequalities over all the possible sections, we have

$$\sum_{\text{all groups } G} n_G^2 \geq B^2 \sum_{i=1}^{S} \frac{N_i}{G_i}.$$

Thus, we reach the conclusion claimed, with equality holding when in each section the groups have the same number of people. □

This minimal value cannot be achieved when the number of people cannot be divided evenly among the groups in the sessions. However, by distributing the people as evenly as possible among the groups, the sum will be as small as possible.

Fact. *The minimal sum for An Tostal Corporation is 532.*

Proof: Distributing the 29 people as evenly as possible among the 6 groups in each morning session we get groups of 4, 5, 5, 5, 5, and 5 (in some order); for the 4 groups in the afternoon sessions, we get groups of 8, 7, 7, and 7 (in some order). The resulting incidence sum is then

$$3\left[\binom{4}{2} + 5\binom{5}{2}\right] + 4\left[\binom{8}{2} + 3\binom{7}{2}\right] = 532. \qquad \square$$

Theorem 3. *For a given day, a configuration's incidence score is bounded below by*

$$\frac{B}{2(B-1)}\left(B\sum_{i=1}^{S}\frac{N_i}{G_i} - \sum_{i=1}^{S} N_i\right)^2.$$

For An Tostal, this value is about 689.9.

Proof: By using the Cauchy-Schwartz inequality and **Theorem 2**, we have that

$$\left(\sum_{1\leq i\leq j\leq B} a_{ij}^2\right)(1+\cdots+1) \geq \left(\sum_{1\leq i\leq j\leq B} a_{ij}\right)^2$$
$$\geq \left(\frac{1}{2}B^2\sum_{i=1}^{S}\frac{N_i}{G_i} - \frac{1}{2}B\sum_{i=1}^{S} N_i\right)^2,$$

so

$$\left(\sum_{1\le i\le j\le B} a_{ij}^2\right)\binom{B}{2} \ge \frac{1}{4}B^2\left(B\sum_{i=1}^{S}\frac{N_i}{G_i} - \sum_{i=1}^{S} N_i\right)^2$$

and

$$\left(\sum_{1\le i\le j\le B} a_{ij}^2\right) \ge \frac{B}{2(B-1)}\left(B\sum_{i=1}^{S}\frac{N_i}{G_i} - \sum_{i=1}^{S} N_i\right)^2. \qquad \square$$

The first line demonstrates that the incidence score can be used to bound the incidens sum from above. Hence, incidence scores are also bounded below and incidence sums are squeezed in between. So, if we achieve a small incidence score, we also achieve a small incidence sum.

In our case, the minimum sum of squares cannot be achieved, for similar reasons that the minimum sum could not be achieved. We can still calculate the minimum possible sum of squares by distributing the B people as evenly as possible among the groups of a session. By doing so, we can find a closed form that may be achieved.

Theorem 4. *The minimum incidence score possible given a fixed minimum incidence sum MS and a fixed number of people B is*

$$(2d+1)MS - d(d+1)\binom{B}{2},$$

where

$$d = \left\lfloor \frac{MS}{\binom{B}{2}} \right\rfloor.$$

For An Tostal, this minimum incidence score is 784.

Proof: We wish to make the incidence elements as close as possible, which means we must have a number of them, say a, which have value d, and the rest, say b, which have value $(d+1)$. The total $(a+b)$ must be the total number of pairs of people, $\binom{B}{2}$. The minimum sum of the incidence elements, MS, is then $da+(d+1)b$. Solving for a and b in the two equations, we get $b = MS-\binom{B}{2}$ and $a = (d+1)\binom{B}{2} - MS$. This gives as our incidence score the value

$$ad^2 + b(d+1)^2 = (2d+1)MS - d(d+1)\binom{B}{2},$$

which upon substitution for a and b, and some manipulation, yields the desired result. $\square$

Fact. *Any incidence matrix for the An Tostal case must contain at least some values greater than 1 in the optimal distribution.*

Proof: By the previous **Fact**, the optimal distribution has sum of incidence elements 532. The greatest number of distinct incidences is $\binom{29}{2} = 406$. Thus, at least one person will have an incidence of at least 2. $\square$

This fact means that in our results for the An Tostal problem, we expect to see at least 2s and 1s in any incidence matrix.

Theoretical Bounds on Computer Run Time

We show that the total number of possibilities is exponential in B.

Theorem 5. *The total number of possible configurations is*

$$\left(\prod_{i=1}^{S} G_i^{N_i}\right)^B .$$

For An Tostal, this is $(6^3 4^4)^2 9 > 3 \times 10^{137}$.

Proof: For each of the B people, this person may be in any of the G_i groups for any of the N_i sessions for all possible sections i ranging from 1 to S. $\square$

We can do better than this by considering only cases in which every group in a session has at least one member. However, even if we have an even distribution (e.g., 4, 5, 5, 5, 5, 5 / 8, 7, 7, 7 for An Tostal), we have a corresponding bound.

Theorem 6. *The total number of possible configurations with even distribution is bounded below by*

$$\prod_{i=1}^{S}\left[\frac{B!}{\left(\left\lceil \frac{B}{G_i}\right\rceil !\right)^{G_i}}\right]^{N_i} .$$

For An Tostal, this is

$$\left[\frac{29!}{(5!)^6}\right]^3 \left[\frac{29!}{(8!)^4}\right]^4 > 3 \times 10^{105}.$$

Proof: In the even distribution, each group in a session of section i has at most $\lceil B/G_i \rceil$ people. Given a group of $n_1 + \cdots + n_k$ numbers with n_1 1s, ... , and n_k

ks, the total number of ways of rearranging them is given by the multinomial coefficient

$$\binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}.$$

Therefore, in each session, the total number of possible group placements is bounded below by

$$\frac{B!}{\left(\left\lceil \dfrac{B}{G_i} \right\rceil !\right)^{G_i}}.$$

Thus, the total number of possible configurations with even distribution is bounded below by the claimed quantity. $\square$

How We Implement a Greedy Algorithm

The algorithm that we use to generate configurations has two main ingredients, which we call *Greedy Placement* and *Switching*. The principal ingredient is Greedy Placement; Switching is merely a tweak to give slightly better scores.

Greedy Placement proceeds through the B board members one by one; on the ith iteration, it finalizes the placement of person i in all the groups. The placements are “greedy,” that is, we “[make] the choice that looks best at the moment” [Cormen et al. 1996, 329]. At each iteration, the algorithm looks at every possible way to place the person subject to the senior officer restriction and chooses the one that leads to the best possible incidence score. The first I board members are the in-house members, so Greedy Placement distributes the in-house members as evenly as it distributes all the board members. Thus, if the algorithm is successful overall, it also satisfies the criterion that no group should have a disproportionate number of in-house members.

Finally, we describe the Switching add-on. After every iteration of Greedy Placement, we do Switching. Switching looks at our current configuration and tries to find a case where, within a given session, it is possible to switch the placement of two board members and consequently get a better score. If there is such a case, we make the switch and reevaluate the configuration to see if there is another useful switch. We continue making switches until we are in a state from which any switch would be detrimental; then we move on to the next iteration of Greedy Placement. In this way, we get a score that is a local minimum after every iteration. Although many switches could take place at each iteration, causing an unpredictable increase in running time, we found that the average number of switches per iteration was about 1.

Switching introduces two complexities.

- We must make sure not to make any switches that would cause the senior officer restriction to be violated.

- If we allow all other switches, we run the risk of switching in-house members with non-in-house members, which destroys our argument that the number of in-house members per group will not be disproportionately high. To counter this, we restrict Switching to switch in-house members only with in-house members and non-in-house members with non-in-house members.

Justification for Our Algorithm

- Trying every possible configuration is impossible, as **Theorems 4–5** show.
- Our algorithm is fast. When the An Tostal day was run on an SGI Challenge, our algorithm took less than 45 sec. Even when it was run on a Pentium 90 MHz with 32 MB RAM, it took only about 7 min. Thus, if any board members don't show up, or extra board members do, the secretary can surely calculate a new assignment in under an hour.
- Our algorithm is flexible. All parameters to the problem (i.e., every variable defined near the beginning of this paper) can be altered simply in a text file.
- Our algorithm always gives a configuration that satisfies the senior officer criterion. Since the configuration satisfies the main criterion of mixing well (see below), it also does not have a disproportionate number of in-house members in any group.
- Our algorithm produces configurations that satisfy the main criterion (mixing well). To demonstrate this, we created some different days (parameter setups). We ran our algorithm for these cases and compared the scores produced by the algorithm with calculated lower bounds from **Theorems 2** and **3** (see the **Results** section below).

Results

To determine the effectiveness of our algorithm, we ran a number of test days. In all cases, the incidence sum found by the algorithm is very close to the minimal theoretical sum; this means that the total number of common memberships is essentially minimized. Also, in each case the incidence score is quite small. In cases similar to the An Tostal problem, the scores found never exceeded the theoretical bound by more than 14%. Even in a huge test case, the score exceeded the theoretical bound by only 29%. This shows that another important constraint is achieved by our algorithm: Each board member meets others a similar number of times.

We present in **Table 2** the results of our greedy algorithm as run on the An Tostal day. **Table 3** shows the distribution of incidence elements and gives other data.

Table 2.

Recommendation to An Tostal.

The in-house members are numbers 1 through 9.

	Morning Section			
	Group 1 Officer 1	Group 2 Officer 2	Group 3 Officer 3	
Session 1	1 4 14 22 25	2 9 12 21 28	3 10 15 23 27	
Session 2	2 7 18 19 27	1 8 13 23 25	4 9 17 20 29	
Session 3	3 9 17 23 26	4 10 19 20 24	1 11 12 18	
	Group 4 Officer 4	Group 5 Officer 5	Group 6 Officer 6	
Session 1	5 6 18 20 26	7 11 17 24 29	8 13 16 19	
Session 2	3 11 16 22 28	5 10 14 21 26	6 12 15 24	
Session 3	2 13 14 15 29	6 8 16 22 27	5 7 21 25 28	
	Afternoon Section			
	Group 1	Group 2	Group 3	Group 4
Session 4	1 5 9 16 23 24 27	2 6 10 13 17 22 26 28	3 3 7 12 14 19 20 25	4 8 11 15 18 21 29
Session 5	1 3 6 17 19 21 27 29	2 5 11 15 20 22	8 9 10 14 18 24 25 28	4 7 12 13 16 23 26
Session 6	1 2 10 15 16 17 25	7 8 9 20 21 26 27	3 5 12 13 18 22 24 29	4 6 11 14 19 23 28
Session 7	1 7 15 19 22 26 28 29	2 3 4 16 18 21 24	6 9 10 11 13 25 27	5 8 12 14 17 20 23

Table 3.

Distribution of incidence elements.

Incidence	0	1	2	3	4+
Number of incidence elements	33	226	134	13	0

Mean: 1.39	Standard Deviation: 0.67
Incidence Score: 879	Incidence Sum: 533
Incidence Score Lower Bound: 784	Incidence Sum Lower Bound: 532

Note that the officer constraint is indeed satisfied, the in-house members are in as even a distribution as possible, and that the resultant score was very close to its theoretical lower bound.

Limitations of Our Model

- Our algorithm does not guarantee an optimal incidence score. We have found cases where an optimal solution is known but is not found by our algorithm.
- Our greedy algorithm with switching may take too much time for very large parameter sizes.
- Our algorithm also does not guarantee a minimal incidence sum. On the other hand, the incidence sums we got in our test were very close and should really be good enough.
- We were able to provide only theoretical lower bounds on incidence scores; these lower bounds may not be achievable.
- Our algorithm is not easy to change. A secretary could not easily change it to use a different incidence scoring system or to consider additional constraints.

Conclusion

We generalized the problem to allow for any number of board members and in-house members; any values for sections, sessions, and groups; and any number of officer-led sessions. We defined a scoring system to evaluate possible configurations that successfully encapsulated the well-mixing criterion. Following this, we determined theoretical lower bounds on scores in terms of the problem parameters. We created a computer program that uses a modified greedy algorithm to come up with good schedules according to our scoring system; the program also took care of the other An Tostal constraints. We used this algorithm to provide the An Tostal Corporation with a well-mixed schedule for their original problem. Finally, we ran tests to verify that our algorithm came up with schedule scores close to the theoretical lower bounds. We believe that the results more than adequately achieved the criteria specified by the original problem, and that our algorithm is a valuable tool for use in scheduling similar planning days.

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Judge's Commentary: The Outstanding Discussion Groups Papers

Donald E. Miller
Department of Mathematics
Saint Mary's College
Notre Dame, IN 46556
dmiller@saintmarys.edu

Making the An Tostal situation particularly open-ended is the fact that a "good mix" of board members is not clearly defined. This also makes it a particularly realistic problem. In practice, it is not uncommon that those requesting the solution don't know exactly what they want or what is possible. They look to the modeler for these answers and related suggestions. While the problem statement provides some guidance of the mixing desired, it allows for a lot of interpretation by the modeler. Thus, in order to evaluate any set of board member assignments to the seven sessions it is imperative that some sort of measure of the "goodness" of a particular solution be identified.

To establish this measure, or objective function, it is necessary to make several assumptions. These assumptions can be made by answering questions such as:

- Is the third meeting of two board members worse than the second?
- Is the second meeting of two in-house members worse than the second meeting of two regular members?
- How should the second meeting of an in-house member with a regular member be evaluated?
- Does increasing the time between sessions in which two members are in the same group reduce the "cost?"
- How does the "cost" of having two board members fail to meet compare to that of having them meet more than once?

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The name "An Tostal" has no real meaning as far as the problem is concerned. It was the name of the spring-quarter weekend of celebration just before final exams at Kent State University, where I studied as an undergraduate.

- Is common membership in the form of A-B-C worse than common membership of the form A-B and C-D?

It should not come as a surprise that wide variations in these assumptions and thus in how to measure success were employed by the modeling teams. The quality and justification of such assumptions were weighted heavily by the judges in their evaluation of papers. While varied assumptions were reasonable, others considered unreasonable include:

- It is better to have a member skip a session than to be in a session with the same member again.

- To minimize common membership, the number of groups for the afternoon sessions may be increased from four to five.

- To ensure that everyone meets at least once, there will be only one group for the seventh session.

A common weakness of papers in past competitions has been their failure to provide both a functional model and a solution. Frequently, they have provided either a "brute force" solution or a simulated solution without a model. At the other extreme are those providing creative models without demonstrating their functionality in solving the problem at hand.

In the statement of this year's problem, we attempted to avoid these pitfalls by calling for a solution to the current problem as well as a simple algorithm that can be used in the event that the problem parameters are changed. Thus, papers providing only a "brute force" solution for the existing problem were screened out early in the process. The judges were unanimous in their opinion that the general quality of the entries was improved over those of a year ago. There seemed to be an increased understanding and a willingness to discuss what form a realistic solution might take as well as the related mathematics and bounds on solutions.

While methods of solution varied from the brute-force listing, with creative matrix methods of accounting, to orthogonal Latin squares, the most common methods were simulated annealing and the greedy algorithm. However, few teams addressed the generalized problem for future meetings with any or all parameters changed. One team that used the greedy algorithm noted that local optimization—that is, optimization at the session level—does not guarantee global optimization, and thus it may be desirable to allow a second encounter of two members early in the day.

Several teams used a different algorithm to make last-minute changes than to make the initial assignments. One team doing this made a conscious effort to alter drastically the schedule of a few people rather than modify slightly several schedules. One judge commented that the team must have contained a social scientist, while another retorted, "Or someone who had worked with college faculties." Another team noted that in measuring success it is desirable to consider balance (balance of membership of groups) and mixture (pairs that meet a disproportionate number of times). To accomplish this, the geometric mean of these two measures was minimized. Another team decided that if a pair of members must meet twice during the day, this would be less "costly" if the time between these two meetings were maximized.

The four papers judged outstanding had many similarities. All decided it would be desirable to keep the size of the groups for each session as equal as possible. With this agreement, most observed that 532 pairings would need to be made in order to complete the day's schedule, a value achieved in their final solution only by the team from East China University of Science and Technology. Some argued effectively that uneven group sizes increase the number of pairings needed. All four teams recognized that 406 pairings are necessary if each board member were to meet each other board member. That is, for a group of 29 people it would take 406 handshakes for each to shake hands with every other person. In fact, all these teams reported the number of times each of these pairings (handshakes) occurred in their final or "best" solution, as shown in **Table 1**. For example, the University of Toronto team reported that 33 of the pairs never met, 226 of the pairs met exactly once, 134 of the pairs met twice, and 14 of the pairs met three times.

Table 1.

Occurrences of numbers of pairs in final solutions.

Team	Number of pairs			
	0	1	2	3
East China Univ. of Science and Tech.	26	253	102	25
Macalester College	40	214	138	14
Rose Hulman Inst. of Tech.	32	218	152	4
University of Toronto	33	226	134	14

In spite of the fact that the teams used quite different objective functions, these results are quite similar. Some of the differences might be predicted from the difference in objective functions. The team from Rose-Hulman Institute of Technology got only four groups meeting three times, since their penalty for this situation was modeled as powers of four. The team from Macalester College had the largest number of pairs that never met, 40. This is because they were the only one of the four teams that looked beyond individual pairs in developing their objective function. They even reported that for their solution, "No two discussion groups have more than two members in common."

Three of the four Outstanding papers used a form of the greedy algorithm to obtain a solution. The other, from Macalester College, used simulated annealing. This paper stood out for its explanation of how simulated annealing is used and for its strong objective function. This function was the sum of four objectives and included a penalty for more than one repeat pairing in a group. In addition, the paper contained a nice proof that if no two pairs of board members are together more than two times, then some pair is never together.

The paper from the Rose-Hulman team stood out for its comparison of three solution methods and the fact that it was well written for the intended audience. Schedules made randomly, with the greedy algorithm and a modified greedy algorithm, were compared statistically. The random method could then be used as a baseline for comparison of the other methods.

Finally, the paper from the University of Toronto team provided some excellent proofs on bounds for solutions. These results were proven in general and then demonstrated for the An Tostal situation.

About the Author

Donald Miller is Associate Professor and of Chair of Mathematics at Saint Mary's College. He has served as an associate judge of the MCM for five years and prior to that mentored two Meritorious teams. He has done considerable consulting and research in the areas of modeling and applied statistics. His current research, with a colleague in political science, involves the statistical analysis of the politics related to the adoption of state lotteries and state approval of other forms of gambling. He is currently a member of SIAM's Education Committee and Past President of the Indiana Section of the Mathematical Association of America.

Practitioner's Commentary: The Outstanding Discussion Groups Papers

Vijay Mehrotra
CEO
Onward Inc.
888 Villa St., Suite 210
Mountain View, CA 94041
vijay@onward-net.com
http://www.onward-net.com

As an operations management consultant, I am used to dealing with difficult problems, incomplete information, and unclear objectives. My profession requires

- a willingness to wrestle with such assignments by understanding the key business goals and issues;
- a desire to solve the problems by finding the right roles for the right people, models, processes, and information systems;
- and an ability to "sell" our solutions by presenting both our methods and our results clearly to diverse and demanding audiences.

That's why I loved this problem.

This problem is a decidedly nontrivial combinatorial optimization problem with lots of different dimensions. It features 7 different time slots, with multiple concurrent meetings per time slot, 3 different classes of people, and a whole lot of restrictions on how these people are to be scheduled. Moreover, the students were also asked to create a solution method that

- could be run by an individual with no technical knowledge,
- could be re-run in less than one hour if inputs were changed slightly, and
- was sufficiently general enough to tackle slightly different versions of the same problem, with different parameters or more general constraints.

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In summary, it is a tough, practical problem requiring creative thinking, thorough analysis, and diverse skills.

That's why I was so impressed with the winning papers (and with several other high-quality submissions).

Because this is not a cookie-cutter problem, there is no "right" way to solve it. The problem's complexity—different types of board members, different sessions with different constraints, different parameter values—prevents teams from using a standard modeling framework and turning the crank. Accordingly, the best papers, including the winners published here, tackled the problem with many different methods, including simulated annealing, greedy algorithms, graph theory, and integer programming.

From the practitioner perspective, this is a key aspect of what the MCM should be teaching: Though the vast majority of academic curriculum in mathematics is organized around specific methods (e.g., multivariable calculus, probability theory, linear programming, etc.), the actual problems that we face as practitioners are rarely well defined and often require us to stretch beyond what we has been presented in the classroom. The top submissions in this competition reflect an appreciation for this fact, revealing creativity in the way in which different models and methods are leveraged to tackle a difficult and very real type of problem.

Young practitioners often become discouraged at the lack of direct application (and direct appreciation) of their freshly minted skills. Buried in large organizations or saddled with narrow responsibilities, many leave the mathematical sciences for other pursuits. This is a distressing outcome, because of the following paradox: When models do not fit real problem / decisions, it is often merely evidence that the problem is difficult, which is precisely why we, and our customers, need structured frameworks and models to help solve them! Yet too few of our graduates understand this notion.

The skills that mathematics students possess, even at what most faculty members consider to be an elementary level, are powerful problem-solving tools, when applied creatively and thoughtfully. In addition to modeling skills and mathematical insights, these tools are brought to life by a strong understanding of the business context, by the ability to use the computer to actually solve the problem, and also by the ability to effectively communicate the nature of the problem, the description of the solution methodology, and the results of the analysis.

However, the same mathematical tools are of limited value when detached from actual problems and viewed in isolation. In this year's competition, papers which looked for a simple "turnkey" solution, or conversely solved the mixing problem narrowly and without regard to extensions or modifications, were not evaluated favorably. The nonstandard and dynamic nature of the problem is an important aspect that had to be addressed in order for a submission to impress the judges in the competition, just as it would in practice in industry and business.

Conversely, in almost all of the highly rated entries, there were a number of characteristics that appeared over and over again. These common themes are evident in all of the papers published here, and are discussed below.

First of all, the best papers all demonstrate a clear understanding of the problem and its competing objectives. All of the desired elements (minimal common membership, maximal interaction between different board members, in-house representation, senior officer group restriction) are explicitly included in the analysis, whether represented as model constraints, as part of an objective function, or as part of the post-solution verification. These award-winning submissions all had a good sense of how these different elements of the problem were in conflict with one another, while also developing solution techniques to reconcile these competing goals.

Once again, in practice, this is something that we face on a regular basis. In countless presentations throughout my career, I have faced questions like "How did you account for X?" and "Where does Y fit into what you did?" In today's world, where the volume of data grows ever faster but valuable information and knowledge are increasingly hard to find, it is a major challenge for analysts to determine what to include, what not to include, and why. As Einstein said, "Things should be made as simple as possible, but not any simpler." The process of identifying the key aspects of a problem, decidedly a black art, is a huge part of what this competition offers to its participants.

Another common theme in this year's winning entries was an understanding of the power of good abstraction. As you read each of the papers published here, you will see a precise and well-presented mathematical formulation of the problem that they propose to solve, along with a clear description of how this abstract problem formulation relates to the "real" problem that is being addressed. In turn, the quality of the abstraction that was selected is directly related to

- the adaptability of the solution to different problems or slightly different conditions, and
- the computational feasibility of the selected solution method.

Note that there were several elegant formulations that couldn't be solved and many clever solutions that couldn't be extended. None of those papers appears here.

Finally, each of the winning entries took the time to examine critically the quality of the scheduling solution generated by their modeling methods. It is challenging to define a standard for what a "best" solution is, yet this type of yardstick is essential for assessing how well a specific method works. Once more, this is something that we as practitioners struggle with, both in trying to determine how successful we have been and in identifying areas where we can improve.

While my primary purpose is to celebrate and illuminate the best of the best of this year's papers, I think it is important to step back and examine the gauntlet that is the MCM. Competitions can be frightening and / or overwhelming, especially when we don't quite know what we're doing, we've got very little time to do it, and we've no choice but to work with other people to get it done. Contests can bring out the best in us, especially when we are desperately trying to do our best, for we can discover a deeper-than-imagined capacity for hypothesizing, learning, assessing, analyzing, and cooperating. In some sense, the MCM is just an extreme case of what we struggle with today in our projects and in our careers.

Today, in business and in life, we typically don't have all the information about the problems, don't always know who the judges are and what they are thinking, aren't sure what the absolute best approach is, and are perpetually time-constrained. Our choices are clear: either not to participate (or go through the motions half-heartedly), because of all of the uncertainties; or to dive in, think hard, work with our teammates, struggle and fall down a few times, take our best shot, clean up our mess as best we can, and explain clearly what we did and why we did it. The MCM gives students a chance to go through this experience at a relatively young age, to make use of things that they have learned already, and to learn a good deal more simply by going through the process.

About the Author

Dr. Vijay Mehrotra is the co-founder and CEO of Onward, an operations management consulting firm based in Mountain View, CA. He has been a management consultant since 1987, specializing in the application of appropriate mathematical models to key business problems. He has worked with clients in many industries, including semiconductor manufacturing, call center operations management, container shipping, electric power, and sales and marketing management. Vijay holds a Ph.D. in Operations Research from Stanford University and a B.A. in Mathematics and Economics from St. Olaf College.