The Booth Tolls for Thee

Adam Chandler
Matthew Mian
Pradeep Baliga

翻译: 周吕文[0]

COMAP Mathematical Contest in Modeling
February 7, 2005
Duke University

[0]本文的注解和排版由周吕文 (zhou.lv.wen@gmail.com) 完成, 并仅限本人课程使用. 发现翻译错误或改进方案请发邮件告知.

Abstract

In this paper, we address the problems associated with heavy demands on toll plazas such as lines, backups, and traffic jams. We consider several models in hopes of minimizing the "cost to the system", which includes the time-value of time wasted by drivers as well as the cost of daily operations of the toll plaza.

One model yields a microscopic simulation of line formation in front of the toll booths when the service rate cannot match the demand. Using hourly demand data from a major New Jersey parkway, the simulation is limited in not taking bottlenecking effects into consideration. The results, however, when subjected to threshold analysis can serve to set upper bounds on the number of booths that could potentially be suggested by any other models.

After presenting this basic model, a more general, macroscopic framework for analyzing toll plaza design is introduced. In analyzing "total cost" and allowing bottlenecking, this model is more complete than the first, and it is able to make recommendations for booth number based on data obtained from the first model. This computation melds the macro- and micro- levels, a strategy that is helpful in looking at toll booth situations.

Finally, a model for traffic flow through a plaza is formulated in the world of "cellular automata". An interesting take on microscopic ideas, the cellular automata model can serve as an independent validation of our other models.

In fact, the models mostly agree that given L lanes, a number of booths around $B = \lfloor 1.65L + 0.9 \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than x, will minimize the total human cost associated with the plaza.

本文解决有关车辆在收费站广场排队, 阻塞和堵车时大量积压的问题. 我们考虑了多种模型来最小化 "系统成本", "系统成本" 包括司机浪费的时间价值和收费站每天的运营成本.

第一个模型给出了一个微观模拟, 该模拟能够描述当服务率不能满足需求时, 收费站广场前将形成队列. 应用新泽西州某个主干道每小时的需求数据, 这个模拟的缺限在于没有考虑瓶颈效应. 然而, 通过阈值分析表明这个模型的结果可以作为收费亭需求数量的上界, 并有可能为其它模型提供参考.

在给出这样一个基本模型后, 本文介绍了一个用来分析收费站广场设计的更通用的宏观模型. 在 "总成本" 的分析和瓶颈的考虑方面, 这个模型较第一个模型更为完善, 并且该模型能够基于第一个模型的数据给出建议的收费亭数量. 这个计算融合了宏观和微观, 是一个有助于寻找收费亭的最优数量的方法.

最终, 车流经过收费站广场的过程被表述为一个 "元胞自动机" 模型. 作为一个着眼于微观的有趣的想法, 元胞自动机模型可以用来对本文其它模型的独立验证.

事实上, 三个模型几乎都给出同样的结论: 对于 L 条车道, 收费亭的数量为 $B = \lfloor 1.65L + 0.9 \rfloor$ 时, 与收费站广场相关的人力总成本将达到最小, 其中 $\lfloor x \rfloor$ 返回的是大于 x 的最小整数.

Contents

1 Introduction / 引言 **3**

2 Restatement of the Problem / 问题重述 **4**

3 Previous Work in Traffic Theory / 交通理论以前的工作 **4**

4 Properties of a Successful Model / 一个成功模型的性质 **5**

5 General Assumptions and Definitions / 全局假设和定义 **6**

6 Optimization / 优化 **7**

7 Fourier Approximation of Toll Plaza Car Entry Rate / 收费站车辆入流量的傅里叶级数近似 **9**

8 Model 1: Car-Tracking Without Bottlenecks / 模型一: 不考虑瓶颈的车辆追踪 **10**

8.1 Approach / 方法 . . . 10
8.2 Assumptions / 假设 . . . 11
8.3 Expectations of the Model / 模型的期望 . . . 12
8.4 Development of Model / 模型的建立 . . . 12
8.5 Simulation and Results / 模拟和结果 . . . 13
8.6 Discussion / 讨论 . . . 15

9 Model 2: A Macroscopic Model for Cost Minimization / 成本最小化的宏观模型 **16**

9.1 Motivation / 目的 . . . 16
9.2 Approach / 方法 . . . 16
9.3 Assumptions, Variables, and Terms / 假设, 变量和定义 . . . 17
9.4 Development / 建模 . . . 18
9.5 Simulation and Results / 模拟和结果 . . . 20
9.6 Discussion / 讨论 . . . 22

10 Model 3 – Cellular Automata: Life! (on the Road) / 模型 3 – 元胞自动机 **24**

10.1 Motivation / 目的 . . . 24
10.2 Approach / 方法 . . . 25

10.3 Assumptions / 假设 . . . 26
10.4 Development of Model / 模型建立 . . . 26
10.5 Simulation and Results / 模拟和结果 . . . 34
10.6 Discussion of Cellular Automata Traffic Model / 元胞自动机交通模型的讨论 . . . 37

11 Comparison of Results from the Models / 几种模型结果的比较 **40**

12 One Tollbooth for Each Lane / 每个车道一个收费亭 **42**

13 Conclusion / 结论 **44**

14 Potential Extension and Further Consideration **46**

Appendices **48**

A Fourier Series **48**

B Basic Car-Tracking Model Code **49**

C Cellular Automata Model Code **52**

D Model 2 Outflux [cars/min] for $L = 6$ **60**

1 Introduction / 引言

When Will Smith's name was called, announcing his receipt of the Best Male Performance award at a recent installment of the MTV Movie Awards, he bounded to the podium. Needling MTV for baiting him to attend their show, Smith snarkily quipped, "M.T.V.: My Time is Valuable". And of course, many other Americans would say the same for themselves, without Smith's irony. In an economy as driven by efficiency as America's, there certainly seems to be a predominant national mindset induced by the toil of the workweek. It's broader than just a desire to be busy or to accomplish; rather it extends to the notion of having control over one's own time, something we abhor to see wasted.

It is no stretch, either, to say that paying a toll to be able to travel to one's destination is largely viewed as an inconvenience. Americans have, for the most part, come to view having free, open roads as an inalienable right from our government. Toll roads are then aberrant and annoying.

But the vexing aspects of toll roads do not stop at the quarter or 35-cent fee, but rather include the time that drivers are forced to waste. Stopping at tolls retards the steady, quick flow of a highway, while not necessarily offering safety benefits like stopping at traffic lights (which are widely tolerated). What's worse is when heavy demand creates jams in the merging lanes exiting the booths or backs up traffic in delineated stripes of hot metal and hotter humanity entering the plaza. The time spent at a toll plaza is easily and often seen as time that could be more fruitfully spent. It is time when the drivers lose sovereignty over their personal whims and obligations.

Despite the anachronisms, imagining Sir Isaac Newton being stranded in a car at a toll plaza when the all-important apple decided to drop back at his home, or envisioning Albert Einstein sitting in a traffic jam without a pencil the moment that relativity dawned upon his own head serve to illustrate some (hyperbolic) motivation in trying to make the toll process as expedient as possible. It is certainly strange to think so, but Newton and Will Smith have something in common.

当最近一期的 MTV 电影颁奖典礼上叫到威尔史密斯的名字, 并宣布他获得最佳男演员奖时, 他来到了讲台上. 针对 MTV 官方极力地邀请他参加的行为, Smith 调侃道, "M.T.V: 我的时间是非常宝贵的". 当然, 即使没有 Smith 的调侃, 许多其它的美国人也都会说这对他们也一样. 在像美国这样一个以效率为导向的经济体中, 通过每周的辛勤劳动, 美国人似乎有一种民族自豪感. 我们讨厌看到浪费时间, 这不仅仅只是一种渴望忙碌和成功的心态, 而且更加是节约自己时间的意识.

人们通过交纳过路费才能到达目的地, 毫不夸张地说这很大程度上被看作是一种不便. 大部分的美国人都认为免费和开放的道路是政府不可剥夺的权利.

但收费公路招人厌的不止是 25 或 35 美分的过路费，而且还包括司机被迫浪费的时间. 在收费站被迫停车会妨碍高速公路稳定快速地通行, 这可不像为了安全而在交通灯前的停车 (这是被广泛接受的). 更糟的是, 当大量车辆需要经过收费站并在车道合并处和收费亭黄色线前形成阻塞时, 却还有更多车辆在进入收费站广场. 在收费站很容易浪费时间, 而这些时间本可以更用效的利用. 这个时候, 司机们就丧失了对思想和义务的主动权.

想象一下当牛顿家苹果树上的重要苹果决定落下来时, 牛顿还坐在滞留于收费站的汽车上, 或者再想象一下爱因斯坦脑子里闪现相对论时, 爱因斯坦正坐在一辆堵在路上且没有一只笔的车上, 尽管这两个例子比较过时, 仍然能说明尝试让收费站尽可能畅通的 (夸张) 动机. 虽然这么想确实比较奇怪, 但牛顿和威尔史密斯有一些共同之处.

2 Restatement of the Problem / 问题重述

Drivers have places to be and people to see, but for one reason or another, tolls must at times be collected from them. It is our goal in this paper to make the process more *optimal* for everyone involved, including the owners and operators of the booths, and of course the drivers. The only mechanism for optimization at our disposal is, presumably, adjustment of the number of booths present at a certain toll plaza, given the number of lanes entering and exiting it.

During peak hours, which occur typically when suburbanites make their way to and from work in larger cities, it is common for lines to form entering the tollbooths, as demand overtakes the fastest rates that the tolls can be collected. On the other side of the booths, too, as the (often) greater number of lanes coming out of the booths converge back down to the original number, bottlenecks and jams are wont to amass in response to the harried merging.

We seek to balance these effects, along with the cost associated to offer extra booths, in order to provide reasonable recommendations for how to minimize the waste of time and money in the toll-collecting process by adjusting the number of booths offered at a given toll plaza.

司机们要去很多地方见很多人, 但出于这样或那样的原因, 收费站有时必需向他们收费. 本文的目标就是要让这个过程对所有相关的人更优, 相关的人包括收费站的拥有者和收费员, 当然也包括司机. 对于一个给定的进出车道数量的收费站, 我们最优化的唯一机制也许就是通过调整收费亭的数量.

在大城市郊区居民上下班的高峰期, 由于通行需求超过了收费站所能处理的最大速率, 车辆进入收费站时形成队列是很常见的. 同样, 在收费站的另一端, 由于从收费站出来的多条车道又汇聚成之前的车道数, 不断的车辆合并车道引起了更大的瓶颈效应和交通堵塞.

为了给 ‘如何通过调整给定收费站的收费亭数量来最小化收费过程中时间和金钱的浪费’ 提供合理的建意, 我们寻求平衡这种交通拥堵的影响和额外增加收费亭的成本.

3 Previous Work in Traffic Theory / 交通理论以前的工作

Mark Twain famously remarked, in his disdain for arithmetic, that the answer to all mathematical problems is three. While insightful, the Twain model leaves some room for improvement in addressing the tollbooth conundrum at hand. A five-lane highway will need more than three tollbooths, but Twain wrote great novels.

There is a rather substantial literature on models for traffic flow, and most models fall into one of two categories: microscopic and macroscopic.

The microscopic models are the ones that can be said to “miss the forest for the trees”. They examine the actions and decisions made by individual cars and drivers. Often these models are called car-following models since they use the spacing and speeds of cars to characterize the overall flow of traffic. Interesting models have emerged from examining cellular automata in a traffic sense (much

对于马克吐温所不屑的数学, 马克吐温的有名言: 所有的数学问题的答案都是三. 不难发现, 吐温的结果为解决当前问题留下了改进的空间. 一条五车道的高速公路需要三个以上的收费亭, 不过吐温写了很多杰出的小说.

存在大量的有关交通流模型的文献, 并且大部分模型都划归为微观和宏观两类.

微观模型是一类可以比喻为“只见树木不见森林”的模型. 这类模型关注于每个车辆和司机的行为和决策. 由于使用车辆的间距和速度来描述整个交通的流量, 这类模型也被称为车辆跟驰模型. 一些有趣的模型来自元胞自动机对交通场景的研究和排队论 (更多的来源于元胞自动机).

more to come) and queuing theory.

Macroscopic models tend to view traffic flow in analogy to hydrodynamics and the flow of fluid streams: just as blood hurtles red blood cells through veins, vehicles pulse down streets toward their destinations. The "average" behavior is assessed, and commonly used variables include steady-state velocity, flux of cars per time, and density of traffic flow.

Some models bridge the gap, including the gas-kinetic model which allows for individual driving behaviors to enter into a macroscopic view of traffic, much like ideal gas theory can examine individual particles and collective gas [1].

The tollbooth problem is an interesting addition to the traffic literature because it involves no steady velocity, so macroscopic views may be tricky. On the other hand, specific bottlenecking events are quite complex, and microscopic ideas are certainly put to the test.

An $M/M/s$ queue (vehicles arriving with gaps determined by an exponential random variable, to s tollbooths, and service at each tollbooth taking an exponential random variable amount of time [2]) seemed appropriate at first. However, queuing assumptions did not satisfy our thirst for details about bottlenecking and about multiple lanes.

Drawing on ideas from old models, while still developing ideas more pertinent to the tollbooth problem, we were able to incorporate aspects of the situation from a small-scale into a larger-scale framework. It seems that neither micro- nor macro- will alone be adequate to capture the dynamics of a toll plaza, though our cellular automata simulation (for herky-jerky driving at lower speeds) produced some surprisingly good results (in terms of matching with other analyses).

宏观模型则倾向于将交通流类比于流体力学和流体的流动: 就像血流中的细胞通过静脉, 车辆通过街道奔向目地的. 可以得到一些行为 (变量) 的平均, 交通流的常用变量包括稳态时的速度, 单位时间的车流量, 以及车流密度.

一些模型在两类模型间建立起了桥梁, 包括气体动力学模型, 它允许单个车辆的驾驶行为转入交通的宏观视角, 就像理想气体理论既可研究单个粒子, 也可研究整个气体 [1].

收费站问题增加了交通论文的趣味性, 它涉及不稳定的速度, 所以宏观的视角可能比较困难. 另一方面, 具体的瓶颈效应相当复杂, 显然需要微观模型来测试.

一个 $M/M/s$ 的队列模型 (车辆到达的时间间隔由一个指数随机分布变量确定, 对于 s 个收费亭, 每个收费亭的服务率由指数随机变量时间给出 [2]) 初看似乎是合适的. 然而, 排队模型的假设不满足我们对瓶颈效应和多车道的细节追求.

从已有模型中借鉴想法, 同时还针对收费站问题的特点发展新的思路, 虽然本文的元胞自动机模拟 (在低速行驶的情况下) 给出了一些意想不到的好结果 (根据与其它分析的匹配程度). 但似乎无论微观还是宏观模型都不能单独的完全描述收费站的动态.

4 Properties of a Successful Model / 一个成功模型的性质

A successful toll plaza configuration should achieve the following objectives:

- Maximize efficiency of the toll plaza by reducing customer waiting time (due to bottlenecking, tollbooth lines, etc.)
- Suggest a reasonably implementable policy to toll plaza operators
- Be robust enough to efficiently handle the demands of a wide range of operating capacities

一个成功的收费站结构需要满足以下目标:

- 通过减小顾客的等待时间 (由于瓶颈, 排队等), 最大化收费站的效率.
- 为收费站广场运营者 (收费员) 提供一个合理的可行策略.
- 要有足够的鲁棒性, 能够有效地应对不同服务能力的需

- As the number of highway lanes feeding the toll plaza is increased, the optimal number of tollbooths will not decrease.

求.

- 随着进入收费广场的高速公路车道数的增加, 最优的收费亭数量不应减少.

5 General Assumptions and Definitions / 全局假设和定义

Assumptions

- **There is only one type of driver in the system**. In navigating toll plaza traffic, all drivers act according to the same set of rules. Although the individual decisions of any given driver are probabilistic, the associated probabilities are the same for all drivers.
- **Bottlenecking downstream of the tollbooths does not hinder their operation**. Vehicles which have already passed through a tollbooth may experience a slowing down due to the merging of traffic, but this effect is not extreme enough to block the tollbooth exits.
- **The number of highway lanes does not exceed the number of tollbooths**. An obvious solution to the posed problem may especially occur to those sitting still at a traffic plaza: namely, set the number of tollbooths equal to zero. The assumption above instead ensures that the number of tollbooths must be strictly positive.
- **All tollbooths offer the same service and vehicles do not distinguish between them**. We seek to improve toll plaza efficiency by optimizing the number of tollbooths – not the services they provide. While several types of tollbooth exist in practice, we have not been charged with distinguishing between them and suggesting their selective use. This is a problem of a different nature. Later, we return to this assumption and list ways in which our solution might change in response to multiple booth types.
- **The amount of traffic on the highway is dictated by the number of lanes on the highway and not the number of tollbooths**. Changing the number of tollbooths for a given number of lanes does not affect the 'demand' for the roadway.
- **The number of operating plaza booths remains constant throughout the day**.

假设

- **系统中只有一种司机**. 对于通过收费站的车流, 所有司机的行为都遵寻一套相同的规则. 虽然对于任何一个给定的司机, 其个人的决定都是随机率概事件, 但对所有司机, 这个率概都是相同的.
- **收费站下游的瓶颈并不影响收费亭的操作**. 已经通过收费亭的车辆可能会因为车道的合并而减速, 但这种效应还不至于阻碍收费亭的出口.
- **高速公路的车道数不会超过收费亭的数量**. 这是一个明显解决问题的方法, 特别是对于那些有如下设置的收费站: 即设置收费亭的数量为 0. 与以上的设想相反, 必需保证收费亭的数量严格为正.
- **所有收费亭为车辆提供相同服务, 且车辆都认为所有收费亭都是无差别的**. 我们通过优化收费亭的数量来寻求提高收费站的效率, 而不是优化收费亭的服务. 虽然实际上存在多种类型的收费亭, 但我们并没有被要求分辨不同收费亭间的差别并给出选择的建议. 这是一个完全不同的问题. 稍后, 我们将回到这个假设, 针对不同类型的收费亭, 列出改进本文解决方案的可能途径.
- **高速公路上的车流量是由高速公路的车道数量决定的, 而不是收费亭的数量**. 对于给定车道数的公路, 改变收费亭的数量并不影响公路的 '需求'(这里的需求指的是需要通过收费站的车辆数).
- **一天当中, 收费站广场运营的收费亭数量是固定的.**

Terms and Definitions

- **Take a "highway lane" to be a lane of roadway in the original highway before and after the toll plaza**. Thus, the number of 'lanes' in a given toll plaza configuration depends not on the plaza itself but on the width of the roadway before and after the toll barrier.
- **Influx** is the rate (in cars/min) of cars entering all booths of the plaza.
- **Outflux** is the rate (in cars/min) of cars exiting all booths of the plaza. It is a function of time.

术语和定义

- **定义"公路车道"为收费站广场前后的正常公路上的一条车道**. 因此, 对于给定结构的收费站广场, "车道"的数量并不由广场本身决定, 而由收费站路障前后公路的宽度决定.
- **入流量**为车辆进入收费广场所有收费亭的速率 (单位: 辆/分钟).
- **出流量**为车辆从收费广场所有收费亭出来的速率 (单位: 辆/分钟). 这是一个时间的函数.

6 Optimization / 优化

Next, we seek a method of optimization that can be used to evaluate potential solutions to the problem. How do we decide that a given toll plaza configuration is optimal? One natural way to compare potential solutions is to compute the total time drivers spend waiting in the toll plaza. It seems logical to conclude that well-designed toll plazas will require less customer waiting time than their inefficient counterparts. Although this method might offer insight, we note some serious drawbacks. Namely, this waiting time minimization disregards the standpoint of the agency operating the toll plaza. In other words, minimizing the waiting time for customers may not present convenient policy options for toll plaza operators.

Suppose a model based on waiting time minimization suggests that forty tollbooths should be used in a plaza for a six lane stretch of highway. Should operators heed this advice? Certainly, the operators of the plaza will incur significant cost in building and maintaining such a facility. In crowded areas, it may not even be possible to construct a toll plaza of this size. Furthermore, tollbooths employ personnel to serve customers without exact change. Paying additional construction, maintenance, and labor costs may not be worth the added benefit of lowering customer wait time.

We seek a more balanced method of facility optimization. This method must consider not only the customers, but also the agency operating the toll plaza. To implement this scheme, we must somehow equate customer waiting time with toll plaza operating costs.

接着, 我们寻找一种可以用来评价该问题潜在解决方案的优化目标. 我们如何确定一个给定的收费站的结构是否为最优? 一个很自然的方式就是通过计算司机在收费站广场花费的总时间来比较潜在的解决方案. 相比于效率低的收费站, 精心设计的收费站将使顾客花费更少的等待时间, 这是显而易见的. 虽然这种方法也许可以提供启发, 但我们意识到它的一些缺点. 即, 这种最小化等待时间忽视了收费站经营商的利益. 换句话说, 最小化顾客的等待时间也许不能为运营商提供实用的策略选项.

试想一个基于最小化等待时间的模型给出的建议: 对于一条六车道高速公路, 收费站中需要使用 40 个收费亭. 那运营商需要这样的建议么? 很显然, 在建造和维护这些收费亭, 收费站运营商将承担很高的成本. 在地皮紧张的区域, 甚至不太可能建造一个这样规模的收费站广场. 此外, 收费站还需要额外花钱聘用工作人员为客户服务. 可能并不值得为了减小顾客的等待时间而支付更多额外的建造, 维护和人工成本.

我们寻求一种更平衡的优化方法. 这种方法不仅必需考顾客, 还要考虑收费站运营商. 为了实施这个想法, 我们必需以某种方式把顾客的等待时间和收费站的运营成本统一起来.

We elect to use cost as a yardstick of our solutions. Cost is a convenient medium due to its ubiquity in our culture and the relative ease of its translation into time. In considering the entire plaza system, we seek the facility configuration that will generate the lowest net cost. This cost will be distributed among both parties in our system – the users and the operators.

In creating a cost optimization apparatus, we invoke the following terms and definitions:

- The general cost, C [dollars], of a toll booth is the time-value of the delays incurred at a toll plaza for each individual (driver or passenger) AND the cost associated with daily operations of the booths at the plaza. The toll fees themselves and the upstart cost of building a new plaza are NOT part of this cost.
- α is the average time-value of a minute for a car occupant.
- γ is the average car occupancy.
- N is the total number of (indistinct) tolls paid over the course of the day.
- L is the number of lanes entering and leaving a plaza. B is the number of booths in the plaza.
- Q [dollars] is the average daily operating cost of a human-staffed tollbooth.

The underlying goal of this construct is to find a reasonable number of toolbooths, B that minimizes cost C, a function of B. We formulate this function $C(B)$.

First, notice that the total waiting time per car will be WN, and so the total cost incurred by waiting time will be $W\alpha N\gamma$. General human time-value is cited as$6/hour or $\alpha = 10$ cents a minute [3]. The amount that must be expended to operate a booth for a day would then be QB. The average annual operation cost for a human-staffed tollbooth is $180,000, so we set $Q = 180000/365.25$ [4].

Reasoning that W depends on B, we now see that

$$C(B) = W\alpha N\gamma + QB$$

This is the function we'll want to minimize with respect to B (for a given L). Naturally, the knee-jerk reaction is to take its derivative and set it equal to zero, showing that the B we seek must necessarily satisfy

$$W'(B) = \frac{-Q}{\alpha N\gamma}$$

本文选择用成本来作为我们方案的评价标准. 成本的概念在我们的文化中无处不在, 并且易于转化为时间, 因此成本是一种方便的媒介. 考虑整个收费站系统, 我们寻求能够产生最小净成本的设备配置. 这个成本将来自于我们系统中的双方 – 顾客和运营商.

在建立这个优化目标函数时, 我们用到了以下的术语和定义:

- 一个收费站的总成本 C [美元] 是每个顾客 (司机和乘客) 在收费站被延误的时间价值**和**所有收费亭日常运营的相关成本. 通行费 (过路费) 本身以及建造新的收费广场的额外费用**不**是这个成本的一部分.
- α 是每个乘客一分钟的平均时间价值.
- γ 是每辆车的平均载客量.
- N 是一天收费的总次数 (非常数). 即通过的车量数.
- L 是进入和离开收费广场的车道数. B 是收费广场的收费亭数量.
- Q [美元] 是平均一天每个收费亭的人力成本 (工资).

构造这一基本的目标函数是为了寻找一个合理的收费亭数量 B, 使得总成本 C 最小化, 其中 C 是 B 的函数. 我们用 $C(B)$ 来表示这个函数.

首先, 需要注意的是所有辆车的总等待时间为 WN, 因此总的时间成本为 $W\alpha N\gamma$. 一般人的时间价值为 $6/小时或者 $\alpha = 10$ 美分/分钟 [3]. 一个收费站运营一天必需花费的成本为 QB. 一个人工运营的收费站平均年运营成本为 $180,000, 因此我们取 $Q = 180000/365.25$ [4].

显然 W 取决于 B, 我们有以下关系

这是一个我们需要最小化的关于 B 的目标函数 (对于给定的 L). 显然, 第一反应是取其导数并设为 0, 因此我们寻找的 B 必需满足

7 Fourier Approximation of Toll Plaza Car Entry Rate / 收费站车辆入流量的傅里叶级数近似

From a previous research paper's traffic flow data [3], we find the mean demand per minute (influx) of cars for a toll plaza on a given typical day. The reason the data peaks are very high at the 6am – 7am rush hour and are not as high during the 3pm – 4pm rush hour period is that the data is collected in the direction headed toward the metropolis. Thus, the main reason for traffic on a typical weekday, the workers during a business day, will be using the "toward big city" tollbooths in the morning, and these tollbooths will be far less frequented in the evening hours.

从前人研究论文中的交通流量数据 [3], 我们发现一个收费站在典型的一天中的每分钟平均需求服务的车辆数 (入流量). 数据在上午 6 - 7 点出现高峰值, 在下午 3 - 4 点出现次高峰值的原因是: 数据的收集是在通向城市方向的道路上. 因此, 一个典型工作日出现这种交通数据的主要原因是: 在工作日里, 早上工人们倾向于使用 "通向大城市的" 收费站, 而晚上的时候则较少地使用这些收费站.

Table 1: Fourier Approximation of Influx Data
入流量的傅里叶级数近似

Start Time	End Time	Hour*	Influx (cars/min)	Fourier Approx of Influx**	Start Time	End Time	Hour*	Influx (cars/min)	Fourier Approx of Influx**
12:00 AM	1:00 AM	0.5	15.44	15.16272478	12:00 PM	1:00 PM	12.5	41.72	41.63346483
1:00 AM	2:00 AM	1.5	15.32	15.42467822	1:00 PM	2:00 PM	13.5	44.54	44.44085865
2:00 AM	3:00 AM	2.5	15.16	15.18796896	2:00 PM	3:00 PM	14.5	48.88	49.29448007
3:00 AM	4:00 AM	3.5	19.90	19.81853474	3:00 PM	4:00 PM	15.5	53.20	52.55619485
4:00 AM	5:00 AM	4.5	47.09	47.22251986	4:00 PM	5:00 PM	16.5	51.61	52.21058951
5:00 AM	6:00 AM	5.5	89.95	89.61825869	5:00 PM	6:00 PM	17.5	48.38	48.16410937
6:00 AM	7:00 AM	6.5	105.9	106.4828683	6:00 PM	7:00 PM	18.6	39.72	39.50374966
7:00 AM	8:00 AM	7.5	85.52	84.72959878	7:00 PM	8:00 PM	19.5	30.51	31.11397219
8:00 AM	9:00 AM	8.5	54.68	55.57942216	8:00 PM	9:00 PM	20.5	29.48	28.86864636
9:00 AM	10:00 AM	9.5	43.11	42.42662327	9:00 PM	10:00 PM	21.5	26.82	27.19686700
10:00 AM	11:00 AM	10.5	40.16	40.49538486	10:00 PM	11:00 PM	22.5	21.21	21.26085220
11:00 AM	12:00 PM	11.5	40.85	40.83544106	11:00 PM	12:00 AM	23.5	17.22	16.91795178

* All "hour" values are averages of the start and end time, so as to accommodate the Fourier approximation.

* 所有 "时刻 (hour)" 的值都是从开始时刻 (Start Time) 到结束时刻 (End Time) 的平均值, 以用于傅里叶级数近似.

**

$$
\begin{aligned}
F(t) = 41.68 - & \ 16.38\cos(\ t\omega) - \ 18.59\cos(2t\omega) + \ 3.572\cos(3t\omega) + 7.876 \ \cos(4t\omega)\cdots \\
& - 0.5048\cos(5t\omega) - \ 2.970\cos(6t\omega) + 0.2518\cos(7t\omega) + 0.5785\cos(8t\omega)\cdots \\
& + \ 12.53\sin(\ t\omega) + 0.6307\sin(2t\omega) - \ 13.67\sin(3t\omega) + 0.4378\sin(4t\omega)\cdots \\
& + \ 6.930\sin(5t\omega) + 0.4869\sin(6t\omega) - \ 1.554\sin(7t\omega) - 0.5871\sin(8t\omega)
\end{aligned}
$$

where $\omega = \frac{2\pi}{24} = 0.2513$. This approximation fits the data points with an R^2 value of 0.9997, and a further glimpse into the coefficients can be seen in the Appendix (A).

Before settling on a Fourier Series approximation with 8 terms, we first attempt a fit with a quartic polynomial. Though we are able to find a very close fit, there is an obvious downside. The quartic polynomial will not necessarily have the same value for $t = 0$ hours and $t = 24$ hours, even though this is obligatory for a cyclic model of daily traffic influx. Therefore, we choose a Fourier Series approximation, whose main upshot is its inherent periodicity, and whose period we can define as the length of a day. Also, it is worth noting that we use an approximation, rather than the data at hand, because we need an influx rate at every minute of the day, instead of just once an hour. In order to have a value at every minute, we need an estimation with more continuity.

其中 $\omega = \frac{2\pi}{24} = 0.2513$. 数据点的这一近似拟合的 R^2 值为 0.9997, 关于拟合系数的更多细节见附录 (A).

在选用 8 阶傅里叶级数近似之前, 我们还尝试了四阶多项式拟合. 虽然我们也能找到一个非常近似的拟合, 但有一个明显的缺点. 四阶多项式拟合不能保证在 $t = 0$ 和 $t = 24$ 时刻具有相同的值, 而这对一个每日周期性的交通流量模型是必要的. 因此, 我们选择了傅里叶级数近似, 其结果自然就是周期性的, 我们可以将其周期定义为一天的长度. 再者, 值得注意的是, 我们使用了一个近似, 而不是手头的原始数据, 这是因为我们需要一天中每一分钟的入流量, 而不是每一个小时. 为了得到每一分钟的数值, 我们需要一个更连续的估计.

8 Model 1: Car-Tracking Without Bottlenecks / 模型一: 不考虑瓶颈的车辆追踪

8.1 Approach / 方法

We create the car-tracking model in order to place an upper bound on the optimal number of booths in a toll plaza configuration, given a particular number of lanes. The model looks at a typical day's influx of vehicles into a toll plaza (data from [3] and Fourier Series approximation).

As vehicles approach the tollbooths within the toll plaza, they may or may not be held up by other vehicles being served. Each vehicle is looking to get through the toll plaza as quickly as possible, and the only factor that may cause Car A,

对于给定一个特定车道数, 我们建立车辆追踪模型, 以便为收费广场收费亭的最优数量设立一个上限. 这个模型着眼于典型的某天进入收费站广场的车流量 (数据来自 [3] 及傅里叶级数近似).

随着车辆行驶并接近收费广场中的收费亭, 他们可能会或可能不会被其它正在接受服务的车辆挡道而停车. 任何一辆车都尝试尽可能快的通过收费站, 若 A 车早于 B 车到达,

which arrives earlier than Car B, to leave later than B is the random variable of service time at a tollbooth. In other words, cars do not make bad decisions concerning minimizing their wait times.

唯一能导致 A 车比 B 车更晚离开的因素是收费站的随机服务时间. 换句话说, 对于最大限度地减少自己的等待时间, 车辆不做出坏的决定.

8.2 Assumptions / 假设

- Customers are served at a tollbooth at a rate defined by an exponential random variable (a common assumption in most queuing theory [2]) with mean 12 seconds per vehicle (or 5 cars per minute).
- Traffic influx occurs on a "per lane" basis, meaning that influx per lane is constant over all configurations with varying number of lanes.
- Bottlenecking occurs more frequently when there are more tollbooths, given a particular number of lanes. This implies that omitting bottlenecking from our model will cause us to overestimate the optimal number of tollbooths for a given number of lanes, thus preserving our model as an upper bound for the optimal number of tollbooths.
- There exists a time-saving threshold such that if the waiting time saved by adding another tollbooth is under this threshold, it is not worth the trouble and expense to add the tollbooth. We assume that if an additional tollbooth does not reduce the maximum waiting time over all cars by the same amount as the average time that it takes to serve a car at a tollbooth (12 seconds = 0.2 minutes), then it is an unnecessary addition.
- An incoming car within the toll plaza will choose the tollbooth that will be soonest vacated, if all are currently occupied. If only one is vacant, the car will choose that tollbooth. If multiple tollbooths are vacant, the car will choose the one that was vacated the earliest. This last statement was created only to give the model a defined path, and it does not actually affect the waiting time in line.
- Cars make rational decisions with the goal of minimizing their wait times.

- 在一个收费亭, 顾客接受服务的速率由一个服从均值为 12 秒每辆车 (5 辆车每分钟) 的指数分布随机变量定义 (绝大多排队论模型中常用假设 [2]).
- 交通入流量来自"每条车道"上, 这意味着对于不同车道数的所有情况, 每条车道的入流量是常数.
- 对于给定的特定数量的车道, 收费亭数量越多时越容易发生瓶颈效应. 这意味着在我们的模型中忽略瓶颈效应将导致我们在给定车道数情况下过高地估计收费亭的最优数量, 这使得我们模型的结果可作为收费亭最优数量的一个上限.
- 存在一个时间节约的阈值, 如果通过增加一个额外的收费亭所节约的时间低于这个阈值, 那么就不值得去那么麻烦和费钱地增加收费亭. 我们假设如果额外增加的收费亭不能把所有车中的最大等待时间降低一辆车接收服务的平均时间 (12 秒 =0.2 分钟), 那么这个收费亭就没有必要增加.
- 如果所有收费亭都被占用的话, 在收费广场内, 一辆向收费亭驶来的汽车将会选择最快即将空闲 (结束服务) 的收费亭. 如果有一个收费亭是空闲的, 则该车直接选择这个收费亭. 如果有多个收费亭是空闲的, 则该车会选择那个最先空闲的收费亭. 最后一句声明 (指上一句话) 只是为了给模型一个确定的路径, 它实际上并不会影响在队列中的等待时间.
- 为了最小化等待时间, 汽车会做出明智的决策.

8.3 Expectations of the Model / 模型的期望

- An additional booth should not increase time spent waiting in line before the booths.
- Each additional tollbooth experiences diminishing returns in terms of time saved, because each one is used at best as often as one of the already existing tollbooths.

- 一个额外增加的收费亭不应该增加车辆在收费亭前排队等待的时间.
- 每一个额外增加的收费亭所节省的待等时间递减, 这是由于每一个新增的收费亭都和已经存在的收费亭一样以最佳状态在运营.

8.4 Development of Model / 模型的建立

As in the other models, cars arrive at the toll plaza at a rate described by the Fourier Series approximation of the data collected from [3]. Cars are considered inside the toll plaza (meaning that we begin to tabulate their waiting times) when they are either being served or waiting to be served.

Service time does not count as waiting time, so if a car enters the toll plaza and there is a vacant tollbooth, its waiting time is 0. If there are no vacant tollbooths at a particular moment, cars will form a queue to wait for tollbooths, and they will enter newly formed vacancies in the order in which they entered the toll plaza. Once a car has been served, it is considered to have exited its tollbooth and the toll plaza as a whole.

Bottlenecking has not been considered in this model, because we only wished for this model to serve as an upper bound for the number of tollbooths for a given number of lanes. Inspection of waiting time in line before the tollbooths sufficiently serves this purpose, and these waiting times will be placed under threshold analysis to determine when the marginal utility (in terms of time saved) of an additional tollbooth is negligible.

It should be noted that our car tracking model does not lend itself to cost optimization, because it is a very basic model that takes into account only how long each car waits in line for tollbooths. It does not factor in the agency controlling the tollbooths, or any costs incurred therein. The code for this model is included in Appendix (B).

与其它模型一样, 车辆到达收费广场的速率由文献 [3] 中数据的傅里叶级数近似描述. 对于正在接受服务或等待接受服务的车被认为是在收费广场内 (意思是我们开始计算他们的等待时间).

服务时长不计入等待时间, 因此如果一辆车直接进入收费广场并洽好有空闲的收费亭, 则此车等待时间为 0. 如果在某个特定时刻没有空闲的收费亭, 车辆将会形成队列以等待收费亭 (服务), 他们将按照进入广场的顺序依次进入空闲的收费亭. 一旦接受完服务, 车辆就被认为退出了收费亭和收费广场的整体系统.

瓶颈在这个模型中没有被考虑, 对于给定的车道数, 我们只希望这个模型的结果作为收费亭数量的上界. 在收费站充分服务前检查在队列中的等待时间, 对这个等待时间进行阈值分析, 以确定什么时候额外增加的收费亭的边际效用是微不足道的.

应该注意到车辆追踪模型不能给予自己成本优化, 这是因为车辆追踪模型是一个非常基本的模型, 它只考虑了每辆车在队列中等待进入收费亭的时间. 它并没有考虑控制收费亭的运营商, 或其它任何产生于其中的成本. 这个模型的程序代码见附录 (B).

8.5 Simulation and Results / 模拟和结果

We collected data from the program, for toll plazas with configurations of $L \in 1,2,3,4,5,6,7,8,16$, and each B from L up to a point where additional booths no longer had ANY noticeable effect on waiting time, where L **is the number of lanes** and B **is the number of tollbooths** in the toll plaza configuration. For example, for $L = 1$, we collect data for $1 \le B \le 5$, because each additional tollbooth beyond 5 in a one lane configuration has an extremely small, if any, effect on waiting time. NOTE: This is not the same as our cutoff for negligible marginal utility from each added tollbooth. In fact, we clarify the cutoff for negligible marginal utility below, with the help of our work for the case of 6 lanes entering the toll plaza. We choose the 6-lane configuration in order to match those examined in the models to come.

我们从模拟中收集以下结构的收费广场结果数据: $L \in 1,2,3,4,5,6,7,8,16$, 且 B 从 L 开始增长直到额外增加的收费亭对等待时间不再起任何明显的效果, 其中收费广场的结构参数 L **是车道数量**, B **是收费亭数量**. 例如, 对于 $L = 1$, 我们收集了 $1 \le B \le 5$ 的数据, 这是因为对于单车道的公路, 任何超过 5 再增加的收费亭对于等待时间如果有影响的话, 那也是非常小的. **注意**: 这和我们设定的额外增加一个收费亭带来的可忽略不计边际效用的阈值并不一样. 事实上, 我们将在下文阐述可忽略不计边际效用的阈值, 以及帮助我们研究 6 条车道进入收费广场的情况. 我们选择了 6 车道的结构, 来和后面的模型 (也选了 6 车道情况) 做对照.

Table 2: Waiting Time for Six Lane Simulation
6 条车道模拟的等待时间

Booths	Average Wait	Average Wait 2*	Maximum Wait	Marginal Utility
6	28.2683	43.0457	98.6196	N/A
7	12.1237	27.7573	55.1100	43.5096
8	5.9544	16.5089	31.6650	23.4450
9	2.4111	8.3740	15.8357	15.8293
10	0.2450	1.2188	2.7815	13.0542
11	0.0223	0.1666	0.7482	2.0333
12	0.0039	0.0653	0.3100	0.4382
13	0.0012	0.0410	0.2699	0.0401

* Average Wait 2 refers to the average wait time per car that waits for longer than 0 seconds. This is more relevant, because as more booths appear, fewer cars spend any time waiting.
** All times are in minutes.
*** These values were obtained by averaging from between 20 and 30 tests.

* 平均等待 2 指的是等待时间超过 0 秒的车辆的平均等待时间. 这是非常相关的, 因为随着收费亭的增加, 更少的车需要等待 (更多的车不需要等待, 即等待时间为 0 秒).
** 所有时间单位为分钟.
*** 这些值是从二三十次模拟的结果中求平均得到的.

To get a more detailed look, we examine some data plots of the toll plaza

为了看到更多的细节, 我们研究了一些 6 条车道 10 个收

configuration with 6 lanes and 10 tollbooths(see figure 1).

费亭的收费广场结构的数据图 (见图1).

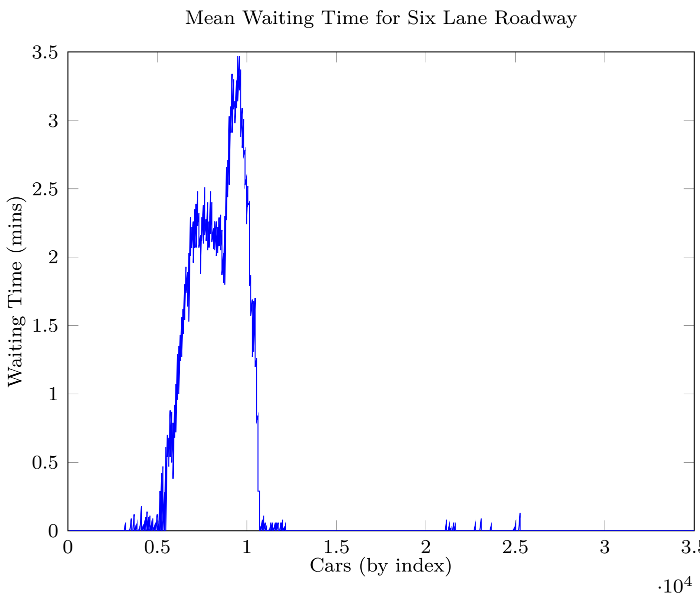

Figure 1: Waiting time per car in six lane, ten tollbooth configuration, model # 1 / 模型 # 1: 6 条车道 10 个收费亭的收费广场结构下, 每辆车的等待时间.

In choosing an optimal number of booths by threshold analysis, we seek out the first additional booth that fails to reduce the maximum waiting time for a car by at least the length of the average tollbooth service time (0.2 minutes). The column denoted "Marginal Utility" shows the amount that each additional booth reduces maximum waiting time, and we simply notice that for 13th booth, this value is 0.0401 minutes. In short, based on our assumptions, it is unnecessary to build a 13th tollbooth for a toll plaza serving 6 lanes of traffic. Thus, we optimize at $L = 6 \rightarrow B = 12$. In a similar manner, we optimize the toll plazas as table 3.

为了通过阈值分析来选出最优收费亭数量, 我们找出不能将最大等待时间至少降低一辆车接受收费亭服务的平均时长 (0.2 分钟) 的第一个额外增加的收费亭. "边际效用" 那一列显示的是每增加一个收费亭减小的最大等待时间, 我们很容易发现对于第 13 个收费亭, 这个值为 0.0401 分钟. 简而言之, 根据我们的假设, 服务于 6 车道交通的收费广场没有必要再去修建第 13 个收费亭. 因此, 我们的优化结果为 $L = 6 \rightarrow B = 12$. 以类似的方式, 我们优化收费广场的结果见表3.

Also, we wish to explore the situation in which there is one lane per booth(see table 4).

同样, 我们希望研究一条车道对应一个收费亭的情况 (见表4)

We find that with average wait times of around 30 minutes, average wait time of over 40 minutes per car that waits, and maximum wait times of around 100

我们发现平均等待时间大约为 30 分钟, 每个等待车辆的平均等待时间超过 40 分钟, 最大等待时间在 100 分钟左右,

Table 3: Optimized Number of Booths for L Lanes
L 条车道的最优收费亭数量

Lanes	1	2	3	4	5	6	7	8	16
Booths	4	5	7	8	10	12	13	16	29

Table 4: Waiting Times for L Lanes with L Booths
L 条车道 L 个收费站 $(B = L)$ 的等待时间

Lanes	Booths	Average Wait	Average Wait 2*	Maximum Wait
1	1	28.2400	37.6928	96.1541
2	2	31.6109	43.2415	103.7251
3	3	28.7137	40.8372	99.6343
4	4	30.8517	44.5677	102.4173
5	5	29.4785	44.3510	103.0286
6	6	28.2863	43.0457	98.6186
7	7	29.7364	45.7080	103.0432
8	8	27.7961	43.9531	96.2895
16	16	31.1367	48.9584	103.7946

* Average Wait 2 refers to the average wait time per car that waits for longer than 0 seconds.
** These values were obtained by averaging from between 20 and 30 tests.

* 平均等待 2 指的是等待时间超过 0 秒的辆车的平均等待时间.
** 这些值是从二三十次模拟的结果中求平均得到的.

minutes, the situation with one lane per booth is entirely undesirable.

这种一条车道对应一个收费亭的情况是完全不可取的.

8.6 Discussion / 讨论

We find that our model matches our general assumptions, and our assumptions for this model, and more importantly, that our results match our expectations. Namely, worth pointing out are the facts that the number of booths is increasing with respect to the number of lanes, that additional booths have diminishing returns on reduced waiting time (true for all L tested, and visible in the $L = 6$ chart), and that each additional booth does reduce waiting time in line.

我们发现这个模型符合我们的全局假设, 以及我们对该模型的假设, 更重要的是, 这个模型的结果与我们的期望一致. 值得指出的是, 收费亭数量随着车道数量的增加而增加, 额外增加的收费亭所节省的待等时间递减 (对所有 L 的测试结果都满足, 表中可见 $L = 6$ 的情况), 而且每个额外增加的收费亭确实减少了排队等待的时间.

The benefits of this rather simplistic model are speed and definite upper bounds. Because we know for a fact that bottlenecking will cause a configuration with fewer booths to be optimal, we know that the optimal number of booths can be no greater than those values presented in this model. Thus, this model's greatest strength is that it serves to check the values of the other two models, which provide much more detailed and accurate simulations of the situation at hand.

The obvious weakness here is that we do ignore bottlenecking, but once again, this allows us to use our results to bound the results of the other two models. In short, the results of the next two models will show that this was in fact an appropriate model in terms of setting a boundary for our soon-to-be-realized answers. With respect to the specific situation in which there is one tollbooth per lane, the toll plaza performs quite terribly. This is mainly because the influx of cars occurs much more rapidly than the service in tollbooths, and such a situation may be addressed by allowing cars to leave tollbooth lines. This would add some practicality to the model, because while in general people may wait out a tollbooth line, they would be much more likely to leave if they knew that the wait could be up to 100 minutes.

这个相当简单模型的好处就是快速, 并确定了上限. 由于我们知道这样一个事实: 瓶颈效应将导致收费广场收费亭的最优数量更少, 我们很清楚收费亭的最优数量不会比这个模型所给出的更多. 因此, 这个模型的最大优势是它可以用来检验其它两个模型的结果, (下文将建立的) 其它两个模型对于当前的问题提供了更多细节和精确的模拟.

本模型显明的缺点是我们忽略了瓶颈效应, 但是需要重申的是, 这允许我们用本模型的结果去限定其它两个模型的结果. 简而言之, 另外两个模型的结果将显示: 就为我们很快就将得到的答案设定一个边界而言, 这确实是一个合适的模型. 对于一条车道对应一个收费亭的特殊情况, 收费站广场表现相当糟糕. 这主要是因为车辆的入流量比收费亭的服务所能承受的更大, 这种情况可以通过允许车量离开等待收费亭的队列来解决. 这将为模型增加一些实用性, 因为当一般人在排队等待一个收费亭服务时, 如果得知要等上 100 分钟的话, 他们更有可能离开.

9 Model 2: A Macroscopic Model for Cost Minimization / 成本最小化的宏观模型

9.1 Motivation / 目的

Model 1 was an effective means of inputting and evaluating specific data sets. General trends could be inferred from the simulated motion of many individual cars at a toll plaza, especially regarding an upper bound on the number of tollbooths given a certain number of lanes. A theoretical framework for making those generalizations and conclusions, however, was missing. In this model we take a bird's eye view of the problem at hand to really address what we mean by the "optimization" of tollbooth processing. A decision-making apparatus is constructed that can be used with the results from Model 1 to refine our earlier observations.

模型 1 是一个输入和评估特定数据集的有效手段. 总的趋势可以从收费广场多个独立车辆运动的模拟中推断出来, 特别是关于给定车道数情况下收费亭数量的上限. 然而, 模型 1 缺少一个能给出概括与总结的理论框架. 在这个 (指当前即第二个) 模型中, 我们以一个全局的视角看待当前问题, 来真正地解决我们所谓的收费站配置的 "优化" 问题. 构造一个决策, 并且可以用我们从模型 1 中得到的结果来改进我们早期的观测.

9.2 Approach / 方法

This model operates on a "higher" level than Model 1. It is less concerned

这个模型站在了一个比模型 1 更高的层次上. 它不再关

with the details of individual vehicular motion and decision-making, but rather the general aggregate effect of the motions of the cars. The variables of this model will often be averages over a day's time.

注个体车辆的运动和决策细节, 而是所有车辆运动的整体效果. 这个模型中的变量通常是一天内的平均值.

For instance, there is no need for this model to decompose analytically the situation of two cars trying to merge into the same lane. Instead, it recognizes that beyond a certain threshold of outflux from the booths, some bottlenecking will occur. The determination of this threshold and its dependencies will be one of the problems addressed in the development of this model.

例如, 这个模型不需要细致分析两辆车试图合并到同一车道的情况. 取而代之的是, 该模型意识到从收费亭出来的车流量超过一个确定的阈值时, 就会发生一些瓶颈效应. 这个阈值的确定以及它的依赖关系将会是建立这个模型需要解决的一个问题.

Also, the concept of the "cost" of a particular plaza configuration is utilized by this model. In minimizing this cost with respect to the number B of booths given a number L of lanes, we obtain a value for B that satisfies our concept of optimality.

同样, 这个模型也用到了一个指定结构的收费广场的"成本"概念. 对于给定的 L 条车道, 通过收费亭的数量 B 来最小化成本, 我们获得了满足我们最优定义的 B 值.

Defining what this nebulous notion of "cost" should mean is the real dilemma to be addressed, but also the impetus for designing this particular model. We chose not to focus on the actual values of the tolls paid by the drivers nor the initial costs to plaza owners of renovating their plazas. Those are not the costs affecting long-term efficient design: the same people will likely be using the road despite any potential increase in tolls that might be instituted to absorb the costs of renovation. The more interesting cost to the drivers is a time-cost. As Will Smith's quote in the Introduction hinted, human time has value, and while Isaac Newton's may be worth more than others, we'll just assume a national average of \$6/hour. By calculating time wasted in waiting at toll plazas, we can get a handle on money lost from potential work unable to be completed.

"成本"这个模糊的概念究竟意味着什么, 明确其定义不仅是要解决的真正问题, 而且是设计这个特定模型的动力. 我们选择既不关注司机支付的实际通行费用, 也不关注收费广场整修的初始成本. 这些不是影响长期高效的设计成本: 尽管可能为了弥补整修费用, 会潜在地上调通进费, 同样的 (固定的) 人群仍会倾向于使用公路. 司机更关注的成本是时间成本. 正如引言中提到的威尔史密斯的话, 人的时间是有价值的, 而牛顿的时间比其它人更有价值, 我们简单地假设全美的平均时间价值为 \$6/小时. 通过计算浪费在等待收费亭的时间, 我们可以用经济损失来处理似乎无法完成的工作.

The other cost to consider is that of actual daily operations of the plazas. We combine these "costs to the system" in this model and attempt to minimize their sum with respect to the number of booths. (See Optimization Section for mathematical treatment.)

其它需要考虑的成本是该广场日常运营的成本. 我们在这个模型中兼顾了这些"系统成本", 以尝试调整收费亭数量来最小化它们的和. (见优化部分的数学处理.)

9.3 Assumptions, Variables, and Terms / 假设, 变量和定义

- We monitor traffic over the course of one day.
- W [minutes] is the average waiting time per car in the toll plaza. W accounts

- 我们监测一天过程中的交通情况.
- W [分钟] 是每辆车在收费站中的平均等待时间. W 包

for time spent in lines (W_1), service time (W_2), and bottlenecking (W_3). (The numbering reflects the progression through the toll plaza.)

含在队列中的时间 (W_1), 服务时间 (W_2) 和瓶颈时间 (W_3). (下标编号是依照通过收费站的顺序 (先排队, 再服务, 最后瓶颈).)

- F_i is a function of time giving the influx of cars per minute to the plaza from one lane. F_o gives the outflux from the booth in cars per minute (in practice, an average of at least 20 individual F_o curves since they vary probabilistically).
- r is the maximum potential service rate [cars per minute]. This is the rate each booth operates at when the influx is non-abating.
- There is an outflux barrier, K [cars per minute], above which bottlenecking takes place. It is taken to be linear in L and independent of B, and we call it the bottlenecking threshold.

- F_i 是一个时间的函数, 它给出了一条通向收费广场的车道每分钟进入广场的车辆数. F_o 给出了每分钟从收费亭出来的车辆数 (实际上, 由于出流量具有一定的随机性, 至少用了 20 个单独的 F_o 曲线来平均).
- r[车/分钟] 是最大可能的服务率. 这是收费亭在入流量不减弱 (不间断) 情况下的服务速率.
- 存在一种出流界线 K[车/分钟], 当出流量超过它时发生瓶颈效应. K 与 L 呈线性关系, 而与 B 无关, 我们称 K 为瓶颈阈值.

9.4 Development / 建模

To discover an appropriate value for B, we'll need to know more about W.

First note that W_1 and W_3 both depend on B and that $W = \sum_{i=1} W_i$. It is easily seen from the definitions that $W_2 = 1/r$.

W_1 represents the average amount of time each car spent waiting in lines before reaching the tollbooth. Thus, it begins to accumulate when the total influx of traffic exceeds the toll plaza's capacity to handle the traffic. $LF_i(t)$ gives the total influx, and Br gives the maximal rate of service. When the difference between the former and the latter is positive (represented by areas enclosed by the curves in Figure 2), we'll want to integrate over time to calculate how many cars were forced to wait in line.

Integrating again gives us the total amount of time waited by those cars (with 3600 being a scale factor for time units):

为了找到一个合适的 B 值, 我们需要知道更多关于 W 的信息.

首先, 注意到 W_1 和 W_3 都依赖于 B, 并且有 $W = \sum_{i=1} W_i$. 另外由 W_2 的定义可知 $W_2 = 1/r$.

W_1 表示每辆车在进入收费亭之前排队等待的平均时间. 因此, 当总的车辆入流量超过了收费站所能处理的交通流量时, W_1 将不断增加. $LF_i(t)$ 表示总的入流量, Br 表示最大的服务率. 当 $LF_i(t)$ 与 Br 的差值为正时 (由图2中曲线包裹的面积表示), 我们需要对时间进行积分来计算有多少车辆被迫排队等待.

通过再次积分 (二重积分), 我们得到了这些排队车辆总的等待时间 (其中, 3600 是时间单位的比例因子):

$$3600 \int_0^{24} \int_0^t \max\left(LF_i(\tau) - Br, 0\right) d\tau dt$$

The average over the total number of cars will be this expression divided by N. In summary,

将以上表达式除以 N 得到所有车辆排队等待时间的平均值. 即

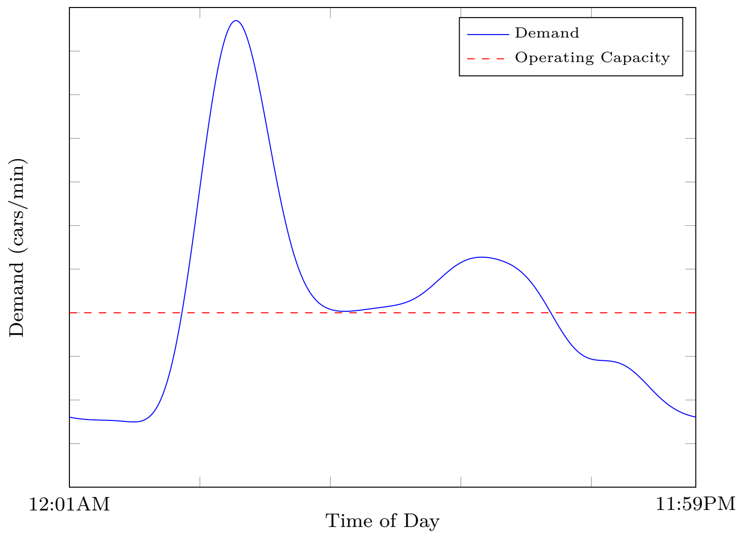

Figure 2: Enclosed area (above line, below curve) represents total number of cars in line at any time during the day. / 包裹的面积 (直线以面, 曲线以下) 表示一天中任意时间在队排的车辆总数.

$$W_1 = \frac{3600}{N}\int_0^{24}\int_0^t \max\left(LF_i(\tau) - Br, 0\right) d\tau dt$$

As it turns out, the technique to obtain W_3 is quite analogous to that we used to find W_1. When comparing the bottlenecking situation to the line-up situation just considered, analogies can be drawn from influx to outflux and from Br to K. As one might expect, we calculate W_3 as follows:

实际上，获得 W_3 方式与我们用来得到 W_1 的方式相当类似. 当比较瓶颈效应情况与刚刚考虑过的队列情况时, 可以将出流量类比于入流量, K 类比于 Br. 有人可能会想到, 我们计算 W_3 的方式如下:

$$W_3 = \frac{3600}{N}\int_0^{24}\int_0^t \max\left(F_o(\tau, B) - K, 0\right) d\tau dt$$

The problem remains of how to determine upon what variable(s) K depends. First, K is not directly dependent on B since bottlenecking should only be a result of general outflux from the booths into L lanes. K is instead indirectly dependent on the number of booths, because outflux, F_o, depends on B, and K in turn depends on outflux. K also can be considered a linear function in terms of L,

这个问题剩下如何确定 K 取决于哪些变量. 首先, 由于瓶颈效应是从收费亭中出来的总车流量进入 L 条车道的一个结果, 因此 K 不是直接取决于 B. 相反, K 间接取决于收费亭的数量, 这是因为出流量 F_o 取决于 B, 而 K 又取决于出流量. 同时, K 也以认为是 L 的线性函数, 因为 L 直接正

because L is directly proportional to influx, which, by the law of conservation of traffic, must equal outflux in the aggregate. Now, we have all the mathematical constructions in place in order to find the B that minimizes total general cost reasonably. Computationally, however, there is still a long road to hoe.

9.5 Simulation and Results / 模拟和结果

As discussed, this macroscopic model will serve as a framework in which we can analyze the data obtained from Model 1 in a cost-minimization procedure. Using the same data for traffic entering the toll plaza as was used in Model 1, this new model will now be able to address the waiting time incurred by bottlenecking and eventually minimize total general cost to the system with the correct selection of B.

As a point of illustration, this particular discussion will detail the calculation of the recommended value of B for $L = 6$. The techniques used with other values of L were completely analogous.

Clearly, with Q, α, γ, and N all independent of B, the most challenging computational procedure will be to find W as a function of B, specifically W_1 and W_3 as individual functions of B.

Given the influx function used in Model 1 (see Appendix A), we can use Mathematica to numerically integrate the expression for W_1 with a given L and $r = 5$ cars per minute (as used in the other models). (N, of course, comes from an integration of the influx expression.) A specific value of B must also be used in the evaluation, so we do the calculation for a series of B values ranging from L to $L + 7$, with $L + 7$ usually being above the upper bound from Model 1 (and step size: 0.25). A Mathematica quartic polynomial fit is then done on the resulting points $(B, W_1(B))$. This procedure will give W_1 as a function of B, as desired.

The $L = 6$ example will perhaps be illustrative. We can easily integrate to find the value $N = 92355$. We compute W_1 for values of B ranging from 6 to 13, in steps of 0.25 (for a better fitting curve). The plot of these points along with the best-fit quartic is given in Figure 3.

The recipe for $W_3(B)$ is somewhat less straight-forward since $F_o(t)$ is generated from a stochastic distribution, unlike the deterministic $F_i(t)$. $F_o(t)$ also depends on

比于入流量, 由于交通守恒, 总的入流量必然等于出流量. 到此, 我们已经建立了所需的所有数学依赖关系来寻找能合理最小化总成本的 B. 然而, 距离正真的计算, 还有一段漫长的路要走.

作为讨论, 这个宏观模型将作为一个框架, 在这个框架中, 我们可以分析从模型 1 的成本最小化中得到的数据. 使用与模型 1 相同的数据作为收费站广场的入流量, 这个新的模型现在就可以计算由瓶颈引起的等待时间了, 并最终通过选择恰当的 B 来最小化系统的总成本.

作为一个例子, 这里具体探讨一下 $L = 6$ 时 B 的推荐值的详细计算过程. 对于 L 取其它值的情况, 所使用的计算方法完全类似.

显然, Q, α, γ, 以及 N 都与 B 无关, 计算过程的最大挑战在于得到函数 W, 其中 W 是 B 的函数, 特别的, W_1 和 W_3 是 B 的两个独立函数.

给定模型 1 中使用的入流量函数 (见附录A), 对于给定的 L 及 $r = 5$ 辆车每分钟 (其它模型中也用这个参数), 我们可以使用 Mathematica 来对表达式进行数值积分求解 W_1. (当然, N 来自入流量表达式的积分.) 在评价中必需使用一个具体的 B 值, 因此我们计算了从 L 到 $L + 7$ 的一系列 B 值 (并且步长为 0.25), 其中 $L + 7$ 通常高于模型 1 得到的上界. 然后通过 Mathematica 的四次多项式拟合得到了点 $(B, W_1(B))$. 和预料的一样, 这个过程将得到 B 的函数 W_1.

$L = 6$ 的例子或许将被说明. 我们可以简单地积分得到 $N = 92355$. 对于 B 从 6 取到 13, 步长为 0.25(为了更好的曲线拟合), 我们计算了 W_1 的值. 图3给出了这些点以及相应的最佳四次拟合曲线.

不同于确定性的 $F_i(t)$, $F_o(t)$ 是从一个随机分布产生的, 因此 $W_3(B)$ 的计算就没有那么直接了当了. $F_o(t)$ 也同样取

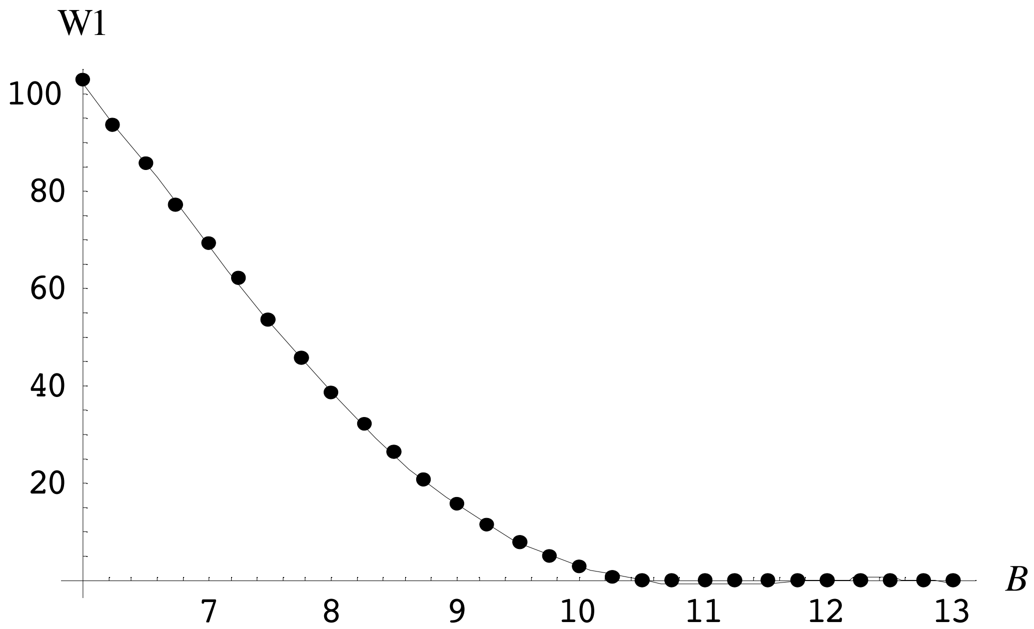

Figure 3: Plot of W_1 for B ranging from 6 to 13 (actual points with quartic fit) / W_1 和 B(从 6 取到 13) 的图 (实际点和四次拟合曲线)

B, a significant complication. At least 20 trials of each case (L, B) were run under the first model, and the averaged outcomes of their outflux functions become the function we used for outflux in this model's simulation, henceforth referred to just as $F_o(t, B)$.

决于 B, 但关系非常复杂. 对于每一组值 (L, B), 至少用模型 1 测试了 20 次, 这些测试的出流量函数的平均值作为我们在本模型模拟中使用的出流量函数, 下文将称之为 $F_o(t, B)$.

Once $F_o(t, B)$ was determined for each salient value of L, we used surface-fitting software (Systat's TableCurve3D) to generate an expression for outflux as a function of time AND the number of booths. We use this expression in the compound integral for W_3, and used Mathematica to integrate numerically for the same values of B selected earlier (to determine the plot for $W_1(B)$). As before, we generate a scatter plot of points $(B, W_3(B))$ and do a Mathematica quartic polynomial fit to the data. This is the function of B we seek.

一旦所有 L 的关键值对应的 $F_o(t, B)$ 被确定后, 我们就可以使用曲面拟合软件 (Systat's TableCurve3D) 来生成出流量作为时间和收费亭数量的函数表达式. 我们在重积分中使用这个表达式来求解 W_3, 并对早已选定 (用来确定绘制 $W_1(B)$ 图) 相同的那些 B 值用 Mathematica 进行数值积分. 像之前一样, 我们绘制了 $(B, W_3(B))$ 的散点图, 以及相应数据的 Mathematica 四次多项式拟合曲线. 这就是我们寻求的关于 B 的函数.

It is important to note that correlation coefficient values (R^2) for the surface

必须要注意的是: 曲面拟合的相关系数 (R^2) 都落在 0.84

fits all fall between 0.84 and 0.95, which are certainly acceptable values. All of the quartic fits have correlation coefficients very near 1.

Again, this process is best illustrated with the concrete $L = 6$ example. The data used for the surface fit (shown below in Figure 4) is included in Appendix D. As in determining $W_1(B)$, the values of W_3 are shown for values of B from 6 to 13 (step size: 0.25), along with the fitted quartic (see below). Now we have $W_3(B)$, and are ready to do the cost minimization. In general, dW/dB will be a cubic (as a result of the quartic fits), and so three solutions emerge. In all cases examined, only one is an appropriate value to minimize C (when rounded). These values are summarized below in Table 5.

到 0.95 之间, 这当然是可以接受的值. 所有四次拟合的相关系数都非常接近于 1.

再次, $L = 6$ 的具体例子是这个过程的最好说明. 用于曲面拟合的数据包含在了附录D中. 与确定 $W_1(B)$ 一样, 对于 B 从 6 取到 13(步长: 0.25), W_3 的值及相应的四次拟合也被得到 (见下文). 自此我们得到了 $W_3(B)$, 并准备去做成本最小化. 通常, dW/dB 将会是三次函数 (由于拟合是四次的), 因此将出现三个解. 验证所有情况发现, 只有一个解 (四舍五入) 是最小化 C 的合适值. 这些值汇总在了表5中.

Table 5: Model 2 Final Recommendations
模型 2 的最终建议

Lanes	1	2	3	4	5	6	7	8	16
Booths	3	5	6	7	9	11	12	14	27

9.6 Discussion / 讨论

This model really amounts to a theoretical way to calculate total waiting time for drivers based on general ideas of traffic flow. The values given above in the summary table are actually quite reasonable and satisfy many of the expectations we laid out characterizing a successful model of the tollbooth problem. The recommended B- values increase monotonically with L, as expected, and they also fit in under the "upper bounds" produced in Model 1, as hoped. Clear from the table above, too, is that a **1-booth per lane setup was never chosen as optimal**. In other words, $B = L$ never seems to be a minimizing value for B in $C(B)$ (for given L). This is because, as we can see from the graphs above of W_1 and W_3, while bottlenecking is zero, waiting time in line before the tollbooths is much higher, thus diminishing the effect of bottlenecking.

Given the model's success, it may be disheartening to acknowledge its lack of robustness. The formulation of this model glosses over the fine details of traffic behavior, so any adjustments to fine-scale aspects of traffic, such as the addition of

基于交通流的一般思路, 这个模型真的相当于一个理论的方法来计算司机们总的等待时间. 上述汇总表中的数据值是非常合理的, 并满足了我们对一个成功收费站问题的模型的许多期望. 这些推荐的 B 值随 L 单调递增, 正如所料, 像期望的那样, 它们也在模型 1 给出的 "上界" 之下. 同样, 从上面的表中还可以清楚的看到, **每条车道一个收费亭的设置从来都没被选作最佳**. 换句话说, $B = L$ 似乎从来没有给出一个 $C(B)$ 的最小值 (对于给定的 L). 我们可以从关于 W_1 和 W_3 的图中看出, 当瓶颈效应为 0 时, 在队列中等待收费亭的时间变得非常高, 从而消除了瓶颈效应.

(上文) 给出了模型的成功, 我们必需承认其缺乏鲁棒性, 这可能令人沮丧. 这个模型的公式掩盖了交通行为的具体细节, 因此, 以任何特定的手段对交通方面的任何精细调整都

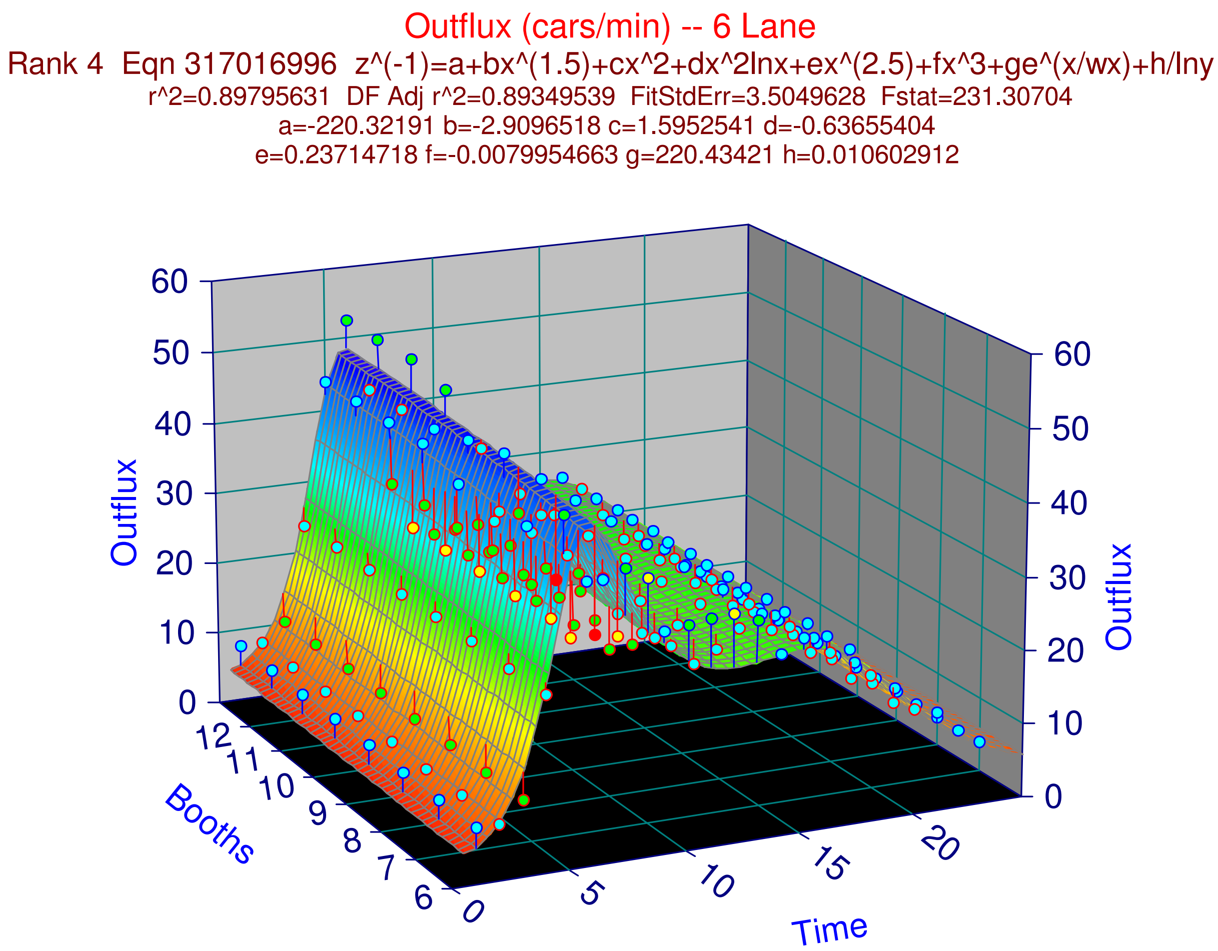

Figure 4: Surface fit for outflux function with respect to time (hrs) and number of booths for $L = 6$ / 对于 $L = 6$, 出流量关于时间 (hrs) 和收费亭数量的函数曲面拟合.

a potential E-ZPass lane (to be discussed later), would be impossible to implement in any specific manner. Perhaps the rate r of service could be adjusted higher for the given scenario, but changing lanes before the tollbooths in anticipation of such

将不可能实现, 比如可能存在额外增加的电子收费车道 (将在后文讨论). 在这种情况下, 也许可以通过调高服务率 r, 但是司机为了进入这样 (电子收费) 的车道而在收费亭前更换

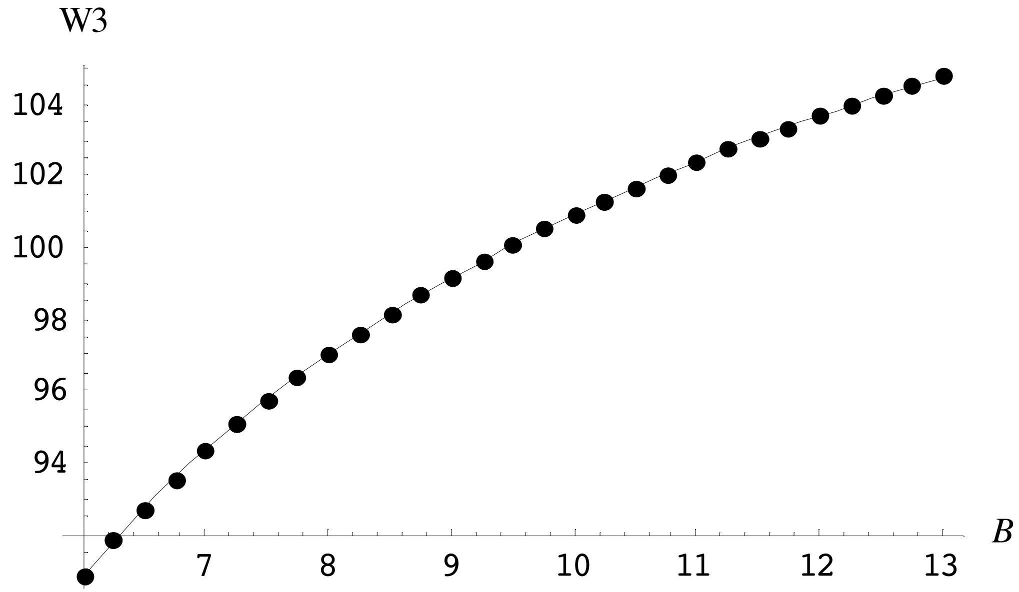

Figure 5: Plot of W_3 for B ranging from 6 to 13 (actual points with quartic fit) / W_3 和 B(从 6 取到 13) 的图 (实际点和四次拟合曲线)

a lane would be difficult to capture with this model. Nevertheless, the model works quite well, it seems, for the given formulation of the problem.

车道的行为很难在这个模型中得到描述. 然而, 对于能够给出问题的公式描述, 这个模型似乎干得不错.

10 Model 3 – Cellular Automata: Life! (on the Road) / 模型 3 – 元胞自动机

10.1 Motivation / 目的

In approaching the problem of toll plaza optimization, we have heretofore considered traffic flow in a macroscopic sense. To avoid the complexity of accounting for individual cars, we have used continuous flow parameters averaged over the course of a day. Although this approach offers mathematical tractability, it may be overlooking important behavior. What effect does the discreteness of traffic have on the nature and solution of the problem?

在处理收费广场的优化问题上, 目前为止我们站在了宏观角度考虑了交通流. 为了避免逐个车辆计算的复杂性, 我们使用了经过一天过程平均的连续流量函数. 虽然这种方式易于数学处理, 但它可能忽略了重要的行为. 交通的离散性对问题的性质和解答有什么样的影响?

At its heart, traffic movement is a discrete phenomenon. Any caffeinated

问题的核心是交通运动是一个离散的现象. 任何工作日

weekday morning commuter would attest to this. Traffic lights, bottlenecks, and construction zones can turn a daily commute into a stop-and-go nightmare. Although steady-state traffic flow may be well-characterized by continuous equations, vehicles and tollbooths may not always lend themselves to such convenient description. Each vehicle is operated by a driver whose behavior and decisions may not necessarily be predictable. Drivers introduce the variable elements of reaction time, attentiveness, and following distance to traffic analysis models. Vehicles and tollbooths are rigid objects and occupy specific positions. While “collisions” of continuous fluid elements may be of little consequence, they can be devastating in a traffic analog.

早晨的繁忙通勤都证明了这一点. 交通灯, 瓶颈, 以及施工区都会把一个日常通勤转化为停停走走的噩梦. 虽然稳态的交通流可以通过连续方程描述, 车辆和收费站不可能总是让自己能够被这样简单地描述. 每辆车都是由一个司机来驾驶, 车辆的行为和决策不一定是可预测的. 司机的存在向交通分析模型中引入了反应时间, 注意力和跟驰距离等变量. 车辆和收费站是实际存在的事物并占据一定的位置. 虽然连续流体微元的“碰撞”可能没有什么严重的后果, 但它们会给交通的模拟带来障碍.

Simply put, a continuous model of traffic may be neglecting the very factors that give rise to traffic congestion and jamming. The smoothness assumed in the continuous macroscopic approach could be inappropriate for a toll plaza. To address this possibility, we turn to cellular automata theory to develop a discrete, microscopic model. The results of this analysis have the potential either to confirm the results of our continuous approach or perhaps cause us to question the soundness of our initial assumptions.

简单来说, 一个连续的交通模型可能忽略了引起交通拥堵和干扰的重要因素. 对于收费站广场问题, 连续宏观方法中的平滑假设可能是不恰当. 为了消除这种可能的不恰当, 我们改用元胞自动机理论来建立一个离散的微观模型. 这一分析的结果也许可以确认我们连续方法的结果, 或者可能导致我们质疑我们最初假设的合理性.

10.2 Approach / 方法

In the cellular automata model of toll plaza dynamics, each cell is designated as a vehicle, a vacancy, or a barrier to traffic flow. The model follows individual vehicles through the plaza and computes the time that each one spends waiting. By summing the time spent waiting by all vehicles in the plaza, we arrive at an indication of the plaza’s operating efficiency. Using measurements of waiting times, we may optimize the number of booths for any given number of lanes using either of the strategies discussed earlier.

在收费广场的动力学元胞自动机模型中, 每个元胞被指定为一个车辆, 一个空位, 或一个交通流的障碍. 该模型追踪着每辆车通过广场的过程, 并计算每辆车的等待时间. 通过对广场上所有车辆的等待时间求和, 我们得出了收费广场的服务效率指标. 对于任何给定的车道数或者正在使用之前讨论过的策略, 基于等待时间的计算, 我们可以优化收费亭的数量.

In the cellular automata model, time, position, and vehicle identity are each discrete quantities. Position is specified as a cell or location in a matrix, while time is incremented using a convenient time step. Individual vehicles are created and followed through the plaza. This approach contrasts with that of the macroscopic models, which consider only the overall flow of traffic. By observing individual components of the traffic, the cellular automata model may help elucidate the

在元胞自动机模型中, 时间, 空间位置和车辆标识都是离散的量. 空间位置用元胞或者矩阵中的某个位置表示, 使用适当的时间步长来递增时间. 车辆被一个个创建出来, 然后依次通过收费广场. 这种方法与仅考虑整体交通流量的宏观模型形成了鲜明的对比. 通过观测交通流的单个组件, 元胞自动机模型可能有助于阐明由收费站引起交通堵塞的相关机

mechanisms that give rise to congestion associated with tollbooths.

制.

At its essence, the cellular automata model provides a means of creating vehicles, manipulating them through a virtual plaza, and quantifying the efficiency of the plaza configuration. Vehicles progress through the plaza as a function of time and the surrounding geometry. In any given time step, a vehicle may advance forward, change lanes, or sit still. Vehicles enter the plaza from a stretch of road containing a specific number of lanes. As a vehicle approaches the string of tollbooths, the road widens to accommodate the booths (given that there are more tollbooths than lanes). There is a specific delay associated with using a tollbooth. Once a vehicle leaves a booth, it must merge into a roadway containing the original number of lanes. During the course of a vehicle's trip, it may encounter obstacles that temporarily prevent it from moving.

从本质上说, 元胞自动机模型提供了一种手段来创造车辆, 控制它们通过一个虚拟的收费广场, 并量化收费广场配置的效率. 车辆通过广场的过程随时间和周围几何特征的变化而推进. 在任何特定的时间步中, 车辆可以前进, 改变车道, 或停顿. 车辆从一段包含特定数量车道的公路上进入收费广场. 随着车辆接近收费站的边界, 道路变得更宽以容纳收费亭 (因为收费亭的数量比车道的数量更多). 存在一个与接受收费服务相关的特定时间延迟. 一旦车辆离开收费亭, 它必须合流到一条包含原始车道数的公路上. 在车辆行驶的过程中, 它可能会遇到障碍, 并暂时地停止移动.

10.3 Assumptions / 假设

In addition to the general assumptions about toll plazas and traffic flow listed at the beginning of this paper, the cellular automata model requires an additional set of specific assumptions (to be distinguished from governing dynamics rules):

- The plaza consists of three types of cells: occupied cells, vacant cells, and 'forbidden' cells (which represent cells that vehicles are not permitted to occupy – i.e. the boundary of the plaza).
- Cells represent a physical space that may accommodate a standard vehicle and a comfortable buffer region on either side.
- All vehicles are the same size.

We will revisit and reevaluate each of these assumptions in the subsequent discussion section.

除了本文开头列出的对收费广场和交通流的总体假设外, 元胞自动机模型还需要一些额外的特殊假设 (注意与动力学控制规则区分):

- 收费广场由三种类型的元胞组成: 被占据的元胞, 空置的元胞, 和' 禁止' 的元胞 ('禁止' 元胞表示车辆不允许占用的元胞 –即广场的边界).
- 元胞代表了一个可容纳标准车辆的物理空间和车辆前后适当的缓冲区域.
- 所有车辆具有相同的尺寸.

在随后的讨论中, 我们将重新审视和评估这些假设.

10.4 Development of Model / 模型建立

In formulating a two-dimensional cellular automata model of toll plaza traffic dynamics, we give considerable attention to the feasibility of computational implementation. As analytical treatment of a two-dimensional automata model is prohibitively difficult, we seek only numerical solution to the task at hand. The following section discusses our formulation of a cellular automata model and presents

对于创建的这个收费广场交通动态二维元胞自动机模型, 其计算实现的可行性, 我们给予相当的重视. 求解一个二维的元胞自动机模型的解析解比登天还难, 我们只能求解当前问题的数值解. 接下来的部分将讨论我们建立的元胞自动机模型, 并给出了模型实施的一般性描述. 对于这个模型, 编程

a general description of the model's implementation. For this model, coding and numerical analysis are done using MatLab software.

和数值分析都是使用的 MATLAB 软件.

A Virtual Toll Plaza

In devising a toll plaza, we accommodate the discrete nature of position in our model. To represent the merging or diverging of lanes, we use a simple stair-step approach like that shown in Figure 6.

虚拟收费广场

在构造一个虚拟收费广场时, 我们允许我们的模型中包含空间的离散性. 如图6所示, 为了表示合并或发散的车道, 我们使用一种简单的阶梯近似.

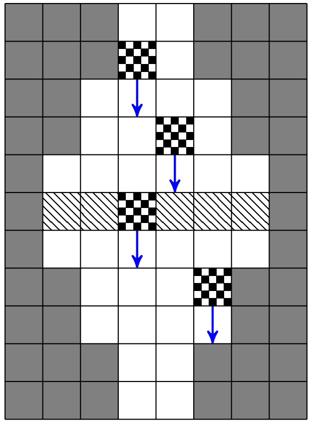

Figure 6: Schematic representation of simple plaza construction. For this case, the plaza contains two lanes and six booths. Solid (dark) cells represent positions that vehicles are not permitted to occupy, while checkered cells represent vehicles, and the striped cells represent tollbooths. The direction of motion is fixed (but arbitrary if the configuration is symmetric). White cells are vacant. Arrows indicate the direction of motion. / 简单收费站广场结构示意图. 对于这种情况, 广场包含两条车道和六个收费亭. 实心 (灰暗) 元胞代表禁止车辆占用的位置, 而黑白相间格子元胞代表车辆, 条纹元胞代表的收费亭. 运动的方向是固定的 (如果收费亭结构是对称的, 则是任意). 白色元胞是空置的. 箭头指示运动方向.

To implement this concept in MatLab, we create a large matrix representing the physical system. Occupied cells are labeled with ones while vacant cells are numbered with zeros. For easy visualization, 'forbidden cells' are marked with '-888'. In some cases, occupied cells are flagged and assigned numbers other than one. This will be discussed in more detail later.

The plazas shown above are very simple and do not represent the actual cases simulated. Typically, a long corridor is added to the top and bottom of the system. In this way, vehicles can establish a regular, steady rate before arriving at

为了在 MATLAB 中实现这个构想, 我们创建了一个大矩阵来代表实际的系统. 被占据的元胞被标为 1, 而空置的元胞则被标为 0. 为便于可视化, ' 禁止元胞' 被标记为 "-888". 在某些情况下, 被占据的元胞由指定不为 1 的其它数字标记. 这将在后面更详细地讨论.

上面显示的广场是很简单的, 并不代表实际的模拟实例. 通常, 一个较长的通道被添加到系统的顶部和底部. 这样, 车辆就可以在到达收费站之前和离开收费站之后达到正常和稳

-888	-888	-888	0	0	-888	-888	-888
-888	-888	-888	1	0	-888	-888	-888
-888	-888	0	0	0	0	-888	-888
-888	-888	0	0	1	0	-888	-888
-888	0	1	0	0	0	0	-888
-888	0	0	1	0	0	0	-888
-888	0	0	0	0	0	0	-888
-888	-888	0	0	0	0	-888	-888
-888	-888	0	0	0	0	-888	-888
-888	-888	-888	0	1	-888	-888	-888
-888	-888	-888	1	0	-888	-888	-888

Figure 7: Visual representation of simple plaza from Figure 6 built in MatLab. Ones represent occupied cells, zeros represent vacant cells, and '-888' denotes a forbidden (inaccessible) cell. As a convention, cars in the model traveled from the top of the matrix to the bottom. 图6所示的简单收费广场在 Matlab 环境中的直观表示. 1 代表被占据的元胞, 0 代表空置的元胞，'-888' 是表示禁止 (不能进入) 元胞. 作为一个约定, 模型中的车辆从矩阵 (矩阵) 的顶部向底部运动.

the tollbooths and after departing from them. Additionally, the larger system minimizes the impact of the boundary conditions on the traffic dynamics near the tollbooths (for example, a line will not become long enough to reach the boundary and interfere with application of the boundary conditions).

To allow for additional flexibility, the steepness of the merging and diverging are left as variables and can be modified from trial to trial. One would expect (as will be shown later) that bottlenecking will be severe for steep merging gradients. If tollbooth lanes end rapidly, vehicles must immediately converge to the central lanes and bottlenecking effects will intensify.

A special case of the plaza configuration occurs when the following is true:

$$\text{mod }(B-L,2)=1$$

where B is the number of tollbooths and L is the number of lanes in the original (and final) roadway. In short, this condition is satisfied when either B or L is even and the other is odd. When this occurs, it is no longer possible to create a symmetric plaza system. The number of stair-steps allocated for merging and diverging will be larger on one side than another. Figure 8 demonstrates such a

定的速度. 此外, 较大尺寸的系统最大限度地降低了边界条件对收费站附近交通动态的影响. (例如, 一个车辆队列不会变得足够长, 以致于接触到边界并干扰边界条件的实施.)

为了允许更多的灵活性, 合并和发散的倾斜度被设置为变量, 并允许任意尝试性地修改. 有人可能会预料到 (之后会展示), 瓶颈效应强列地依赖于合并倾斜度. 如果收费站车道 (比正常公路多出来的车道) 迅速消失, 车辆必须立即合并到中央的车道, 此时瓶颈效应将加剧.

当以下方程成立时, 将给出一种特殊的收费站结构:

其中 B 是收费亭的数量, L 是在原始 (最终) 公路的车道数. 总之, 当 B 和 L 一个为偶数, 另一个为奇数时, 这个条件就是成立的. 当这种情况发生时, 就不再可能创建一个对称的收费广场系统. 用来合并和发散车道的阶梯数目, 一侧将比另一侧大. 图8显示了这种情况.

case.

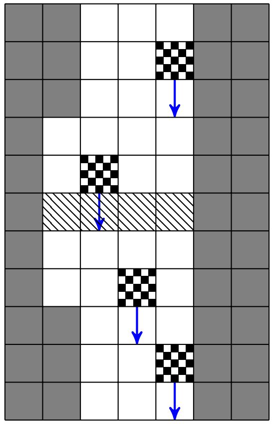

Figure 8: Schematic representation of asymmetric plaza. This example contains four booth lanes but only three normal travel lanes. We resolve issue by using one fewer stair-step on the right side. / 不对称收费广场示意图. 这个例子包含四个收费亭, 但却只有三条正常的车道. 我们通过在右侧少使用一个阶梯来解决这个问题.

As we will show later, the asymmetry of this design is not significant from a practical standpoint. Our results do not appear to be odd-even dependent.

The problem of plaza construction was generalized and implemented using the function 'create_plaza'. This MatLab function creates a plaza (no vehicles) of B tollbooth lanes and L normal travel lanes. For an explicit reproduction of this function, please consult Appendix C.

Governing Dynamics

With the establishment of a plaza, we next turn to the dynamics within it. As cars move through the toll plaza, we require a governing set of rules to dictate their behavior. The rules are applied to each vehicle to determine which action will be taken during the current time step. For any given toll plaza configuration or geometry, each vehicle has a specific number of options, each with an associated probability.

Just as crucial as the set of rules, however, is the order in which the rules

正如我们稍后将展示的, 从实际的角度来看, 这种设计的不对称性是不重要的. 我们的研究结果并不依赖于这种奇偶性.

虚拟广场构建问题是由函数 'create_plaza' 定义和实现的. 这个 Matlab 函数创建了一个含 B 条收费车道和 L 条正常车道的虚拟收费广场 (无车). 关于这个函数复本, 请参阅附录C.

动力学控制

在建立好一个虚拟广场后, 我们下一步的工作变为它的动力学控制. 当汽车通过收费广场时, 我们需要一套规则来控制他们的行为. 该规则被施加到每一辆车上, 以确定在当前时间步应该采取哪些行动. 对于任何给定的收费广场的配置或形状, 每辆车都有一个特定数量的选项, 每个选项都有一个与之相关的概率.

然而, 与这套规则同样重要的是, 决定哪些规则按什么样

are implemented. A fuzziness on rule application may give rise to confusing or misleading results. There are a number of decisions to consider. For instance, which car gets to act first? How does a vehicle decide to switch lanes? When two vehicles want to occupy the same cell, which is given preference? Such matters could be critical. On the other hand, one must be careful that systematically following a certain rule order does not create some obscure or unintended phenomenon. To perform any kind of meaningful simulation, these issues must be balanced.

的顺序被执行. 规则应用的模糊性可能会引起混淆或错误的结果. 有许多决定要考虑. 例如, 哪一辆车先运动? 车辆如何决定更换车道? 当两辆车要占用同一个元胞时, 哪一辆优先? 这些问题可能是关键的. 另一方面, 我们必须小心, 系统地按照一定顺序来执行规则, 不能造成一些模糊的或不确定的现象. 要进行任何有意义的模拟, 这些问题必须被解决.

A working cellular automata model is presented below. For each time step, the following rules are applied in sequential order:

下面给出的是一个有效的元胞自动机模型. 对于每一个时间步长, 下面的规则依次被执行:

1. Starting at the front of the traffic and moving backward (with respect to the flow), vehicles are advanced to the cell directly in front of them with probability p. If the next cell is not vacant, the vehicle does not advance and is flagged. Note: This probability is meant to simulate the stop-and-go nature of slowly moving traffic. One could think of p as a measure of driver attentiveness. $p = 1$ corresponds to the case where drivers are perfectly attentive and move forward at every opportunity. $p = 0$ would represent the extreme case where drivers have fallen asleep and they fail to move forward at all.

2. Using an influx distribution function, the appropriate number of new vehicles is randomly assigned to lanes at the initial boundary (see next section).

3. Starting at the front of traffic and moving backward, those vehicles flagged in step 1 are given the opportunity to switch lanes. For each row of traffic, the priority order for switching is determined by a random permutation of the number of lanes. Switching is attempted with probability q. If switching is attempted, left and right merges are given equal probability to be attempted first. If a merge in one direction (i.e. left or right) is impossible (meaning that the adjacent cell is not vacant), then the other direction is attempted. If both adjacent cells are unavailable, the vehicle is not moved.

4. Total waiting time for the current time step is computed by determining the

1. 从车流的最前面开始, 向后 (相对于流动方向) 依次检查每辆车, 每辆车以概率 P 前进至其前面的一个元胞. 如果下一个元胞不为空, 车辆将停止前进和并被标记. 注意：这个概率是用来模拟车辆在缓慢移动的交通中停停走走的行为. 也可以认为 P 是驾驶员注意力的一种衡量指标. $p = 1$ 对应的情况是: 司机非常专注, 不放过任何一个前进的机会. $p = 0$ 代表的极端情况是: 司机已经睡着了, 车辆根本无法前进.

2. 使用一个入流量分布函数, 适当数量的新车辆被随机分配到车道的初始边界上 (见下一小节).

3. 从车流的最前方, 向后依次检查每辆车, 那些在步骤 1 中被标记的车辆将有机会变更车道. 对于每一行的车辆, 更换车道的优先级顺序是由一个随机排列的车道顺序确定的. 尝试变更车道的概率为 q. 如果试图变更车道, 首先向左和右尝试换道的概率是相等的. 如果向某个方向（即左或右）的换道是做不到的 (意思是相邻元胞不是空置的), 则其尝试向另一方向换道. 如果两边相邻元胞都不可用, 则车辆不移动.

4. 当前时间步内总等待时间的计算是通过确定系统中的

number of cells in the system containing a vehicle.

5. The number of vehicles advancing through the far boundary (end of the simulation space) are tabulated and added to the total output. This number is later used to confirm conservation of traffic.

Other notes about the algorithm:

1. In step 1, those vehicles in the tollbooth position are not necessarily advanced. Rather, the value of the cell is incremented and the vehicle is not moved. This has the effect of delaying a vehicle being serviced at a tollbooth. The cell for such a vehicle is incremented each time step until it has waited enough time steps to constitute a "service delay". The vehicle is then released. The service delay is typically three to five time steps.

2. Most simulations are performed over what amounts to a 24-hr period to reflect the cyclic nature of daily traffic. This choice is examined later and its results are compared to those of a shorter simulation with a higher traffic influx (simulating only rush hour).

A simplified flow chart representing the above algorithm is presented in Figure 9

元胞含有车辆的数目.

5. 前进到远边界 (模拟空间的末端) 的车辆数被列入表中并添加到总输出车量. 这个数字后来被用来确认交通守恒.

关于算法的其他注释：

1. 在步骤1中, 那些正在收费亭位置的汽车不一定前进. 相反, 收费亭中车辆的元胞值是递增的, 并且车辆不动. 这样做的效果是: 延迟了正在收费亭中接受服务的车辆. 在每个时间步, 这样的一个车辆元胞的值都在递增, 直到等待了足够构成 "服务延迟" 的时间步数. 这个车辆随后被放行. 服务延迟通常是三到五个时间步.

2. 大多数仿真模拟了超过 24 小时的时间, 以反映日常交通的周期性. 这种选择随后被验证, 它的结果与较短期的高交通入流量 (只模拟的高峰期) 的模拟结果进行了比较.

一个表示上述算法的简化流程图在图9中被显示.

Population Considerations

Recall that the daily (cyclic) influx distribution of cars, $F_{\text{in}}(t)$, was previously defined using a Fourier series (See Appendix A):

车辆数考虑

前面每天 (循环) 的车辆流入的分布函数 $F_{\text{in}}(t)$ 使用的是傅里叶级数定义 (见附录A):

$$F_{\text{in}}(t) = a_0 + \sum_{n=1}^{N} a_n \cos(n \cdot \omega \cdot t) + \sum_{n=1}^{N} b_n \sin(n \cdot \omega \cdot t)$$

This method is still valid for the automata model, but the influx values must be scaled to reflect the effective influx over a much smaller time interval (a single time step). The modified influx function, $F'_{\text{in}}(\tau)$, is computed as follows:

这种方法对自动机模型仍然是有效的, 但必须对入流量进行调整, 以反映在一个较小的时间间隔 (单个时间步长) 内有效的车流量. 修改后的流量函数为 $F'_{\text{in}}(\tau)$, 其计算过程如下:

$$F'_{\text{in}}(\tau) = \min\left[\text{round}\left(\frac{F_{\text{in}}(t)}{\eta}\right), L\right]$$

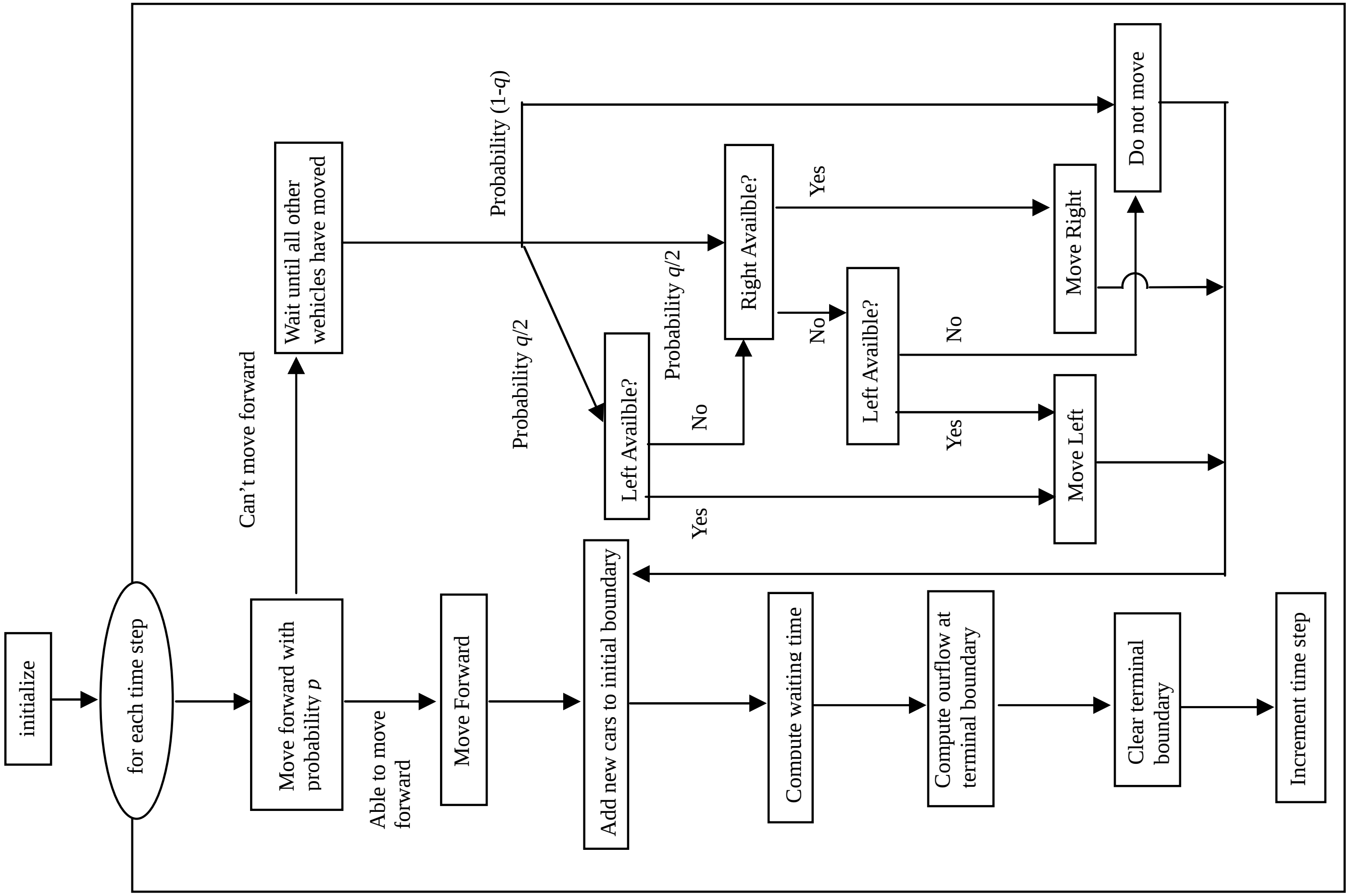

Figure 9: Flow-chart representation of the cellular automata algorithm/ 元胞自动机算法流程图

where η is a constant factor required for the conversion from units of t to those of τ and L is the number of initial travel lanes. A 'round( )' function is employed to round the argument to the nearest integer (recall that the influx per time step in the automata model must be an integer number of vehicles not exceeding the

其中 η 是一个将 t 的单位转化为 τ 的单位的常数因子, L 初始车道数量. 应用 'round( )' 函数来对参数四舍五入取整 (需要说明的是: 自动机模型中每个时间步的入流量必须是不超过可用车道数的整数辆车).

number of available lanes).

When this computation is carried out, we obtain the following distributions(see figure 10).

进行这种计算后, 我们得到了如下分布 (见图10).

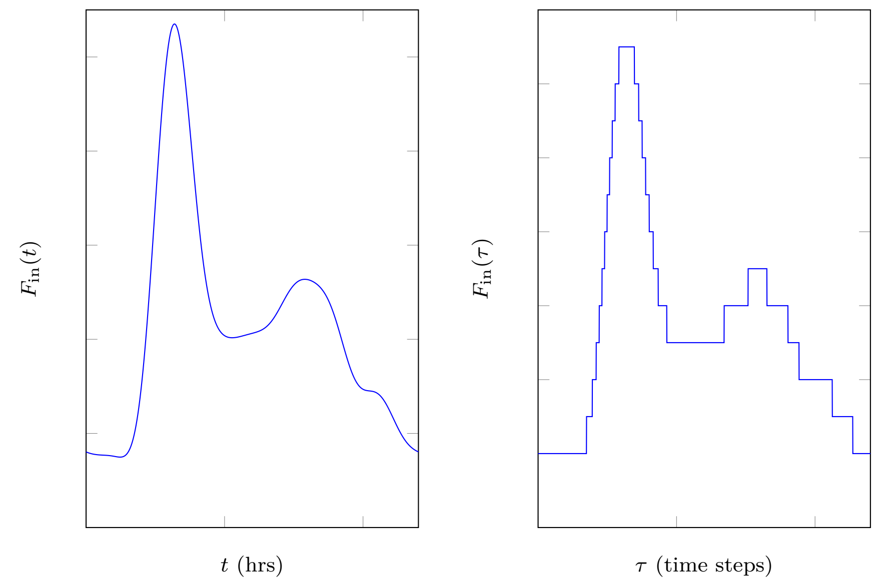

Figure 10: Plots of continuous influx ($F_{\text{in}}(t)$ – left) and normalized cellular automata model influx ($F'_{\text{in}}(\tau)$ – right). / 连续入流量 ($F_{\text{in}}(t)$-左) 以及标准化元胞自动机入流量 ($F'_{\text{in}}(\tau)$ – 右).

Boundary Conditions

To simulate the automata model, we must define boundary conditions for the plaza. Although plaza model is two-dimensional, the motion is essentially one-dimensional. The barriers on the sides of the plaza prevent the cars from occupying those cells. In a sense, thus, the boundary conditions for the sides of the plaza parallel to the roadway might be thought of as analogous to either Neumann or Dirichlet (with respect to the vehicles). That is, the vehicles may neither occupy the boundary nor traverse the boundary to leave or enter the system, respectively.

The boundary conditions for the boundaries perpendicular to the direction of

边界条件

为了执行自动机模型的模拟, 我们必须定义虚拟广场的边界条件. 虽然广场模型是二维的, 但车辆的运动基本上是一维的. 广场两侧的障碍 (禁止的元胞) 阻止了汽车占据这些元胞. 因此从某种意义上来讲, 广场平行于道路两侧的边界条件可以被认为是类似于诺伊曼条件或者狄利克雷条件 (相对于车辆). 也就是说, 车辆可能既不能占据边界, 也不能穿越边界离开或进入系统.

垂直于交通流方向的边界条件的定义不同. 正如在前

traffic flow are defined differently. As visited in the previous section, there is a time-dependent influx associated with the toll plaza. At the boundary where vehicles begin their trip through the plaza, new vehicles can be introduced at each time step. The number of cars introduced is given by the value of the influx function for the corresponding time step. Once the number of incoming vehicles is determined, the vehicles are distributed randomly among the available lanes. Thus, the number of incoming vehicles per time step can never exceed the number of highway lanes in the system.

面小节中见到的, 有一个与收费广场相关的依赖于时间的入流量. 车辆通过广场的过程开始于广场一端的边界, 在这个边界上, 新的车辆可以在每一个时间步被引入. 引入汽车的数量是由相应时间步长的入流量函数的值给出的. 一旦进入车辆的数量被确定, 车辆被随机分配到可用的车道中. 因此, 每一个时间步进入的车辆数不能超过系统中公路的车道数.

At the system boundary where vehicles exit the plaza, the boundary condition acts to remove all vehicles that entered the boundary row of cells since the previous time step. In other words, the final row of cells acts as a perfect absorber – every vehicle that reaches it is immediately removed and the output count is incremented. Recall that the output count is used to confirm the conservation of traffic principle. By removing vehicles before a new time step begins, the boundary clears the way for the next row of vehicles to leave the system. This prevents a buildup of vehicles at the system's departure zone.

在车辆离开广场的边界, 边界条件的作用是移除之前时间步进入边界行元胞的所有车辆. 换句话说, 最后一行的元胞作为一个理想的吸收体 – 每辆车到达边界就立即被删除, 并增加输出车量的计数. 需要提醒的是, 输出车量的计数是用来确认交通守恒原理. 通过在新的时间步开始前清除车辆, 该边界将为下一行车辆离开该系统扫清道路. 这可以防止车辆在系统的离开区域累积.

Computing Wait Time

Wait time is determined by looking through the entire matrix at each time step and noting the number of cells with positive values. Recall that the only cells containing positive values are those representing vehicles. Thus, by counting the number of vehicles in the plaza at any given time, we are also counting the amount of time spent by vehicles in the plaza (in units of time steps).

At time step i, total cumulative waiting time is computed as follows:

计算等待时间

等待时间的确定是通过检查每个时间步的整个矩阵, 并记录下具有正值的元胞数目. 需要注意的是: 只有那些包含正值的元胞才代表车辆. 因此, 通过计算任何特定时间广场内的车辆数, 我们也就计算了车辆在广场所花费的时间 (以时间步长为单位).

在第 i 时间步, 总累积等待时间计算如下:

$$\begin{aligned}&\forall x, \forall y\\ &W_i = W_{i-1} + 1\big(\text{plaza}(x,y) > 0\big)\end{aligned}$$

where '1( )' denotes an indicator function and 'plaza' denotes the matrix of cells.

其中 '1( )' 表示一个示性函数, "plaza" 指的是元胞矩阵.

10.5 Simulation and Results / 模拟和结果

To determine the optimal number of tollbooths for a given number of highway lanes, the cellular automata simulation is run for relevant combinations of the two.

为了确定一个给定高速公路车道数的最优收费亭数量, 元胞自动机模拟运行了两个相关情况的组合. 请注意我们的

Recall one of our general assumptions is that the number of tollbooths in any plaza is at least equal to the number of highway lanes that are feeding it. An optimal tollbooth number is selected for a given number of highway lanes when the system cost optimization method discussed earlier is applied.

一个基本假设是: 任何收费广场的收费亭数量至少等于进入广场公路的车道数量. 当前面讨论的系统成本优化方法被应用时, 给定车道数的最优收费亭数量被选出.

Recall the cost optimization method defines total cost as follows:

成本优化方法中定义总成本如下:

$$C_{\text{total}} = \alpha \cdot \gamma \cdot N \cdot W(B, L) + B \cdot Q$$

Using the cellular automata model, we compute waiting time as a function of both the number of lanes and the number of tollbooths. For a fixed L, we compare all values of C_{total} and choose the lowest one. The results of this method are presented in Table 6.

利用元胞自动机模型, 我们计算了等待时间, 等待时间是车道数量和收费亭数量的一个函数. 对于一个固定的 L, 我们比较所有值 C_{total}, 并选择其中最低的一个. 该方法的结果列在了表6中.

Table 6: Optimization for Cellular Automata Model / 元胞自动机优化结果

	Optimal # Booths	
Highway Lanes	Typical Day	Rush Hour
1	2	2
2	4	4
3	5	6
4	7	7
5	8	9
6	10	11
7	12	13
8	14	15
16	27	29

As indicated in Table 6, there is fairly good agreement between the recommended number of booths for a typical day and for peak hours. However, we note that the optimal booth number for a typical day never exceeds that for rush hour. Rush hour seems to require slightly more booths than a typical day in order for the plaza to operate most efficiently.

如表6所示, 对典型一天的收费亭数量的推荐值与对高峰时段的相当吻合. 然而, 我们注意到典型一天的最优收费亭数量都不会超过交通高峰时段的. 为了最有效的运行收费广场, 高峰时段似乎比典型一天需要稍多的收费亭.

Each value in Table 6 is representative of approximately 20 trials. Through these trials, we noted a remarkable stability in our model. Despite the stochastic

表6中的每个数值代表约 20 次模拟的平均值. 通过这些模拟, 我们注意到在我们的模型具有一个显著的稳定性. 尽

nature of our algorithm, each number of lanes was almost always optimized to the same number of tollbooths. There were a handful of exceptions; they occurred exclusively for small numbers of highway lanes (< 3 lanes). Integer values are presented in Table 6 only because fractional tollbooths have no physical meaning.

管我们的算法是随机的, 每种车道数情况下几乎总是优化至数量相同的收费亭. 也有少数例外情况, 它们只发生在高速公路车道数很小时 (< 3). 表6列出的只有整数值, 是因为非整数的收费亭数量是没有实际意义的.

Example

As an example of one optimization using the cellular automata model, let us consider the instance of six highway lanes. For comparison, the analogous optimization is carried out previously in models 1 and 2.

实例

作为一个使用元胞自动机模型优化的例子, 我们考虑了 6 条车道的高速公路实例. 为了进行比较, 前面模型 1 和模型 2 进行了类似的优化.

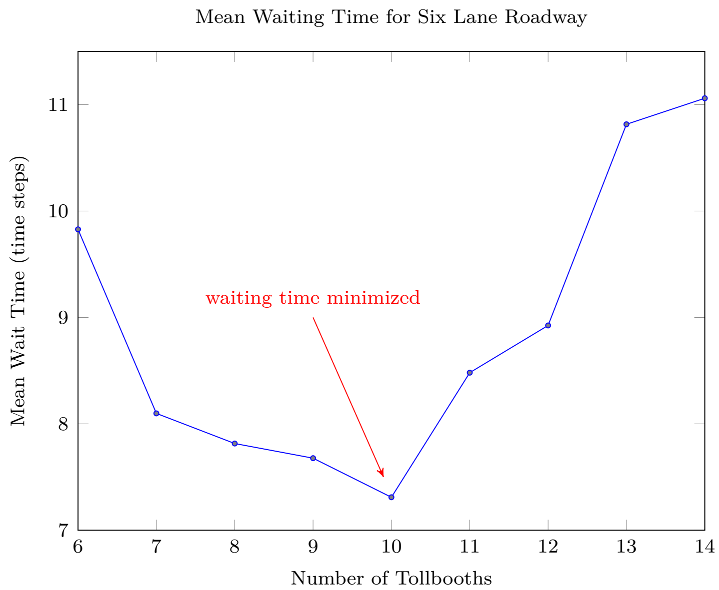

Figure 11: Minimization of mean waiting time for six lane roadway. Use of 10 tollbooths minimizes the mean wait for customers./ 六车道的高速公路的平均等待时间. 使用 10 个收费亭时能最大限度地减少客户的平均等待.

Figure 11 is created by running the simulation repeatedly for six lanes and varying the number of tollbooths. A choice of ten tollbooths provides the lowest mean wait time for vehicles. However, ten is not necessarily the optimal number of tollbooths for the system. To determine the optimal solution, we must refer to

图11是通过多次运行 6 车道的模拟并改变收费亭数量得到的. 选择 10 个收费亭时给出了最低的车辆平均等待时间. 然而, 10 不一定是系统的最优收费亭数量. 为了确定最优解, 我们必须参考之前建立的成本优化函数.

the cost optimization function developed previously.

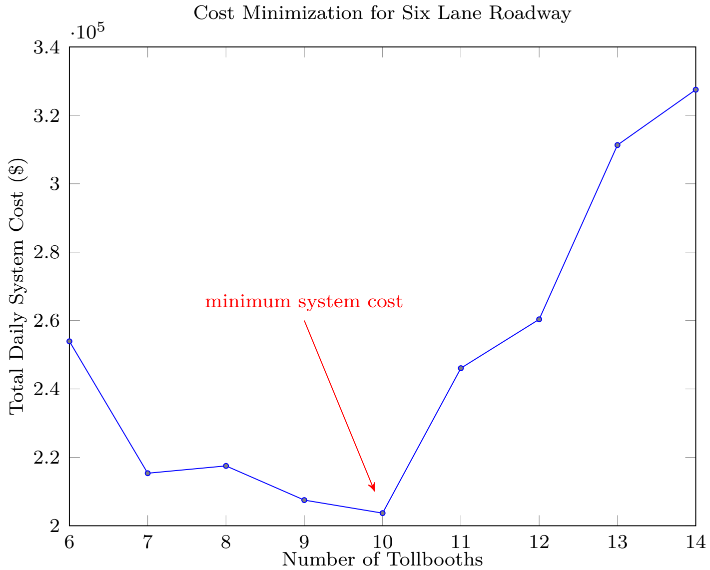

Figure 12: Minimization of total daily system cost for six lane roadway. Use of 10 tollbooths minimizes both the mean wait for customers and the system cost. / 六车道高速公路日均成本最小化. 使用 10 个收费亭时能最大限度地减少客户平均等待时间和系统成本.

As seen in Figure 12, ten tollbooths minimizes system cost as well as mean waiting time (Figure 11). Thus, ten tollbooths is the optimal number for a toll plaza with six incoming lanes (given our selection of parameters). Although the curves in Figures 11 and 12 look very similar, they are indeed more than scalar multiples. One notes that the differences between the two curves is most pronounced at the two ends.

如图12所示, 与平均等待时间 (图11) 一样, 10 个收费亭时系统成本被最小化. 因此, 10 个收费亭是 6 车道收费站最佳数量 (在我们给定的参数下). 虽然图11中的曲线和图12的非常相似, 但它们也确实不仅仅是等比例的缩放. 有人可能会意识到, 两条曲线之间的差异在两端最明显.

10.6 Discussion of Cellular Automata Traffic Model / 元胞自动机交通模型的讨论

Evaluation of Assumptions

Let us now consider the assumptions made in the development of the cellular automata traffic model. In what way have these assumptions been either confirmed

假设评价

现在让我们来考虑建立元胞自动机交通模型的假设. 在什么情况下这些假设被证实或推翻? 是否有假设已经被证明

or discredited? Has there been an assumption which has proven to be particularly limiting?

- We first assumed that the plaza contains only three types of cells – occupied, vacant, and forbidden. In fact, there are two other kinds of cells (flagged cells and incrementing booth cells). These arose as artifacts of the nature of the computer program and did not affect the dynamics of the system. Although treating the plaza in this simplified manner may have neglected some details, it was necessary to develop the framework for a cellular automata based simulation. The model could possibly be improved by adding detail (via additional cell types), but new features – unless dramatic – would probably not change the fundamental behavior of the system.
- Our next assumption was that cells represent a physical space that may accommodate a standard vehicle and a comfortable buffer region on either side. Again, this was a simplifying assumption designed to accommodate the use of only three cell types. However, it is not a bad assumption if when we recall that vehicles are typically required to move slowly within toll plazas (< 15 mph) and probably maintain reasonable buffer zones between adjacent vehicles. We have no clear way of evaluating whether this assumption limited our model in some important way, but we suspect not.
- Next, we assumed the presence of only one type of vehicle within the system. More specifically, we assumed that all vehicles are the same size. The use of larger vehicles that occupied multiple cells was not explored and represents one possible extension of this study. We doubt that larger vehicles interspersed in the possibility exists.

Sensitivity Analysis

In this section, we briefly evaluate the sensitivity of our cellular automata model. Does perturbing the value of any of our model parameters have a notable effect on the resulting solution?

To answer this question, we make changes to three key variables in our system and observe the corresponding influence on the optimized number of tollbooths. Ideally, we would hope that the optimal number of tollbooth is unchanged for even modest variations in these parameters.

是特别限制的?

- 首先我们假设广场只包含三种类型的元胞 – 被占用, 空置和禁止的元胞. 事实上, 还有其他两种元胞 (无法前进而被标记的元胞, 和值被增加的收费亭元胞). 这些元胞的出现是计算机程序自然的产物, 对系统的动力学并没有影响. 虽然这种对广场简单的处理方式可能忽略一些细节, 但就仿真而言, 建立这样一套元胞自动机框架是有必要的. 该模型也许可以通过添加细节 (通过增加元胞类型) 来得到改善, 但新的特征 – 除非显著的 – 否则可能不会改变系统的基本行为.
- 我们的下一个假设是: 元胞代表可容纳标准车辆的物理空间和车辆前后适当的缓冲区域. 再次, 这是一个简化假设, 目的是要允许只使用三种类型的元胞. 但是, 如果我们还记得车辆通常需要在收费广场内缓慢移动 (<15 英里每小时), 并可能维持与相邻车辆之间合适的缓冲区域, 就会知道它不是一个坏的假设. 我们没有明确的办法评估这种假设是否在某些重要方面限制了我们的模型, 但我们猜测没有.
- 接下来, 我们又假设系统内只存在一种类型的车辆. 更具体地说, 我们假设所有车辆具有相同大小的尺寸. 作为本文研究的一种可能的扩展, 使用占用多个元胞的较大车辆没有被探讨. 我们怀疑, 更大尺寸的车辆穿插其中的可能性是存在的.

灵敏度分析

在本节中, 我们简要地评估一下我们的元胞自动机模型的灵敏度. 扰动我们模型中任何参数的值对所得到的结果是否有显着影响?

要回答这个问题, 我们对我们系统中的三个关键变量做了改变, 并考查对收费亭优化数量的相应影响. 理想地, 我们希望即使适度改变这些参数, 收费亭的最佳数量也不会改变.

The parameters we choose to modify are p (probability of advancement), 'delay' (number of time steps required to serve a vehicle in a tollbooth), and q (the probability that a flagged vehicle opts to attempt a turn). The results of this analysis are presented in Table 7. Since we have used six lanes as our standard test case, we continue with this choice here.

我们选择修改的参数是 p(前进的概率), '时间延迟'(车辆在收费亭中接收服务所需的时间步数), 和 q(一个被标记车辆选择尝试变更车道的概率). 这个分析的结果见表7. 既然我们已经使用 6 车道作为我们的标准测试算例, 在此我们将继续选择这个算例.

Table 7: Sensitivity Analysis for Cellular Automata Model ($L = 6$) / 元胞自动机模型灵敏度分析

p	q	Delay	Optimal # Booths
0.9	0.95	4	10
0.8	0.95	4	10
1.0	0.95	4	10
0.9	0.90	4	10
0.9	1.00	4	10
0.9	0.95	5	11
0.9	0.95	3	10

As indicated in Table 7, our cellular automata model is relatively insensitive to both p and q. Changes of $\pm$ 11% and $\pm$5.2% in p and q, respectively, had no effect on the optimal number of tollbooths for a six lane highway. On the other hand, increasing the delay time by 25% shifted the optimal number of booths from 10 to 11 (10%). Decreasing the delay by 25% had no effect on the solution. Perhaps additional work could lead to an elucidation of the relation between delay and optimal booth number that could help stabilize the cellular automata model.

如表7所示, 我们的元胞自动机模型对 p 和 q 相对不敏感. p 和 q 分别变化 $\pm$11% 和 $\pm$ 5.2%, 对一个 6 车道的高速公路上收费亭的最佳数量没有影响. 另一方面, 增加 25% 的延迟时间将会使收费亭的最佳数量从 10 变为 11(10%). 降低 25% 的延迟对结果没有影响. 也许额外的研究工作可以澄清延迟时间和最佳收费亭数量之间的关系, 这可能有助于提升元胞自动机模型的稳定性.

Rush Hour Dynamics

An important test of our model was a simulation of rush hour only. Previously, we have considered traffic behavior over the course of individual days. By doing this, we may risk obscuring important behavior that occurs only during peak hours. With this in mind, we use the cellular automata model to simulate a three hour period in which the influx of traffic nearly saturated the lanes.

The results of the rush hour analysis seem to confirm the robustness of our model. Despite the extreme influx of vehicles, the optimal number of tollbooths never differed by more than one from our corresponding normal case values.

高峰期

我们模型的一个重要测试是只模拟高峰期. 此前, 我们已经考虑了一天过程中的交通行为. 这样做, 我们可能会面临掩盖一些只发生在高峰时段的重要行为的风险. 考虑到这一点, 我们使用元胞自动机模型来模拟 3 小时的高峰期, 这种高峰期内车道中的入流量接近饱和.

高峰期的分析结果似乎证实了我们模型的鲁棒性. 尽管在极端的入流量情况下, 收费亭的最佳数量从未与我们的正常情况下相应的值相差一个以上.

Conservation of Traffic

With a computational model such as the one presented here, one must be careful that small logic errors or programming bugs do not give rise to unexpected phenomena. As one check of the cellular automata traffic algorithm, we call upon the principle of conservation of traffic. As its name implies, this rule dictates that no vehicles are gained or lost during the course of our simulation. Assuming all vehicles are allowed to traverse the plaza, we must count the same number of vehicles leaving the plaza as we permitted to enter the plaza. Indeed, for all 24-hour and rush hour simulations in this trial, it was confirmed that the corresponding input and output vehicle counts were equal. This confirmation lends credibility to our model.

交通守恒

对于像这里给出的计算模型, 必须小心逻辑错误或编程错误不会引起意想不到的现象. 作为对元胞自动机算法的一个检查, 我们称之为交通守恒原则. 正如其名称所暗示的, 这条原则要求在我们的模拟过程中不能凭空得到或失去任何车辆. 假设所有车辆都可以穿过广场, 我们统计出离开广场的车辆数必须与允许进入广场的数量相同. 事实上, 对于所有 24 小时和高峰时段的模拟测试, 这个守恒原则都得到确认: 相应的输入和输出车辆计数是相等的. 这种确认增加了我们模型的可信度.

11 Comparison of Results from the Models / 几种模型结果的比较

Again, we found the following results for optimal number of booths given 1, 2, 3, 4, 5, 6, 7, 8, or 16 lanes approaching a toll plaza(see table 8).

再次，对于给定的 1, 2, 3, 4, 5, 6, 7, 8, 或 16 条进入收费广场的车道, 我们得到了如下最优收费亭数量 (见表8).

Table 8: Comparison of final recommendations for three models / 三种模型最终推荐结果的比较

Lanes	Car-Tracking	Macroscopic	Automata
1	4	3	2
2	5	5	4
3	7	6	5
4	8	7	7
5	10	9	8
6	12	11	10
7	13	12	12
8	16	14	14
16	29	27	27

From descriptions of our models, we first know that the Basic Car-Tracking model serves as an upper bound for the optimal number of booths, due to its omission of bottlenecking. As we can see in the data above, each value of B under the Car- Tracking column is greater than or equal to its counterparts in the

从我们模型的描述中, 首先我们知道: 由于忽略了瓶颈效应, 基本的车辆追踪模型的结果只能作为最佳收费亭数量的一个上限. 正如上面我们看到的数据, ‘Car-Tracking’ 列下的每一个 B 值下都是大于或等于成本最小化的宏观模型和元

Macroscopic Model for Cost Minimization and Cellular Automata Model. In this respect, our consideration of the simple and somewhat uninteresting Car-Tracking model is complete.

The Macroscopic Model for Cost Minimization and the Cellular Automata Model provide us with very similar results for an optimal number of tollbooths, considering a given number of lanes entering a toll plaza. Despite the overwhelming differences in methodology and theory behind these two models, the number of times they coincide or differ by only one booth is astounding. It almost goes so far as to say that precision implies accuracy here.

We conjecture that the reason the Macroscopic Model recommends a greater number of tollbooths than the Cellular Automata Model due to consideration of bottlenecking. Whereas the Macroscopic Model takes bottlenecking into account, we use it merely to say that there is an arbitrary threshold above which bottlenecking occurs. While we use reasonable factors to create the threshold of outflux over which bottlenecking occurs, this value is likely to be an overestimate, thus underestimating the amount of bottlenecking. Since the Macroscopic Model undervalues bottlenecking, it will recommend a higher number of booths as optimal.

On the other hand, the Cellular Automata Model maintains the strictest observation of the complex process of bottlenecking. Since this model details the toll plaza geometry, especially the lane merging after the tollbooths, it has the makings of a very accurate merging demonstration. Furthermore, the Cellular Automata Model uses such microscopic detail to count exactly the amount of time spent waiting in line by each car. Due to its examination of each car and each period waited, we lean more toward the Cellular Automata Model for a more accurate representation of number of booths versus lanes than toward the Macroscopic Model for Cost Minimization.

胞自动机模型的相应值. 对于简单并且有些无趣的车辆跟踪模型, 我们在这方面的考虑还是全面的.

对于一个给定进入收费广场的车道数量, 基于成本最小化的宏观模型和元胞自动机模型为我们提供的最佳收费亭数量的结果非常相似. 尽管这两种模型背后的方法和理论完全不同, 它们结果一致或仅相差一个收费亭的次数是令人意外的. 几乎可以说, 这里的精度就意味着准确度.

宏观模型建议的收费亭数量比元胞自动机模型的大, 我们推测其原因是宏观模型考虑了瓶颈效应. 然而我们在宏观模型中考虑瓶颈时, 只是说有一个主观的阈值, 超过它就会发生瓶颈. 虽然我们使用了合理的量来构造瓶颈发生时的出流量阈值, 但该值可能被高估了, 从而低估了瓶颈效应的程度. 由于宏观模型低估了瓶颈, 它会推荐较多的收费亭数量作为最佳.

另一方面，元胞自动机模型对瓶颈效应的复杂过程进行了严格的观察. 由于该模型细化了收费广场的几何形状, 特别是收费站后面的车道合并, 它有一种非常精确的合并演示的特质. 此外, 元胞自动机模型使用这样的微观细节, 准确地计算出每辆车花费在排队等待的时间. 由于其在每个等待时刻对每辆车的检查, 我们更多的倾向于: 元胞自动机模型比基于成本最小化的宏观模型更能准确地表示出收费亭的数量.

Table 9: Comparison of Linear Fit Parameters for our Models / 本文模型的线性拟合参数的比较

Model	Slope	Intercept	R^2
Basic Car Tracking	1.699	1.738	0.997
Macroscopic	1.598	1.215	0.998
Cellular Automata	1.672	0.228	0.998

Table 9 shows linear fit parameters for all three models. Note that all three models are well described by a linear equation.

表9显示的是所有三种模型的线性拟合参数. 请注意, 所有这三个模型都能由一个线性方程很好地描述.

12 One Tollbooth for Each Lane / 每个车道一个收费亭

General Consideration

Now, we consider the implications of introducing no extra tollbooth lanes to the plaza. Each highway lane is leads to a single tollbooth. This scenario is represented for four lanes in Figure 13.

综合考虑

现在, 我们来考虑广场上没有额外收费车道的情况. 每条高速公路车道都只通向一个收费亭. 图13展示了四条车道的这种情况.

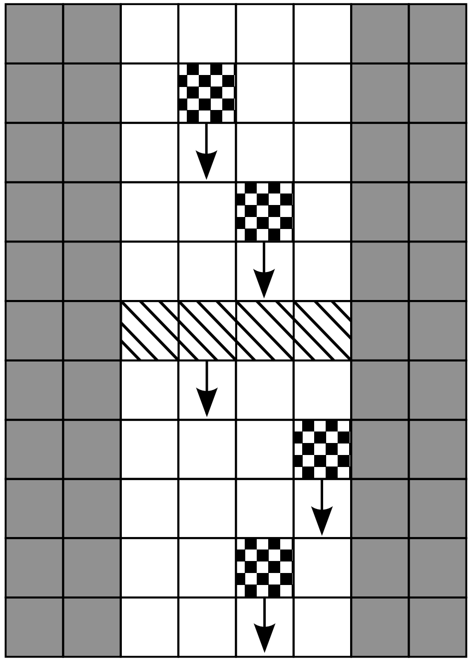

Figure 13: Schematic representation of four lane case with 1:1 ratio of tollbooths to highway lanes. / 收费亭与车道比为 1:1 的四车条道情况的示意图

In the case of a 1:1 ratio of tollbooths to lanes, we observe that the plaza does not fan out in the center. The toll plaza has no extra tollbooth lanes to

在收费亭数量与车道数量比为 1:1 的情况下, 我们观察到收费广场在中心区域并没有呈扇形展开. 收费广场没有额

relieve traffic congestion. What are the consequences of this scenario? Certainly, it would appear that traffic congestion would increase and a line would build up behind the tollbooths. In this case, one notes that the waiting time by customers comes exclusively from waiting in line. That is, once vehicles pass through the toll plazas, there is no waiting time associated with bottlenecking. There can be no bottlenecking because the lanes do not merge. Thus, we see that although one component of the customers' waiting time is increased (waiting in line), another component is completely eliminated (bottlenecking).

The previous discussion focused on times in which the influx to the toll plaza was too great for the tollbooths to prevent formation of a line. What if vehicles enter less frequently? In this case, we note that line formation is not an issue (and bottlenecking remains a non-factor). Thus, it would seem that a 1:1 ratio of booths to lanes is ideal when the highway traffic is quite low.

What other factors may allow a simple 1:1 configuration to be more effective than its multi-boothed counterparts? Since bottlenecking is eliminated in the 1:1 case, any change that has the effect of reducing line length will tend to favor the 1:1 case over a case with bottlenecking.

Line length can be reduced in a number of ways:

- Increase the rate at which tollbooths are able to service vehicles. If vehicles move through a tollbooth more quickly, it will take a greater influx of vehicles to generate a line.
- Increase driver attentiveness. Recall that in the cellular automata model, drivers are each assigned a probability p of moving forward during each time step. If p is increased, drivers will move forward with greater regularity and congestion is likely to be a factor near the lines.

Changing the above parameters in the opposite ways would have the effect of favoring the 'current' case (> 1 tollbooth per incoming lane) over the basic 1:1 case.

Quantitative Estimation

For the cellular automata model, an immediate reduction in cost often is realized upon adding only a single extra tollbooth to a 1:1 plaza configuration (by 1:1, we mean one tollbooth for each incoming lane). This cost reduction is

外收费车道来缓解交通拥堵. 这种情况的后果是什么? 当然, 这将导致交通拥堵增加, 并在收费亭后形成一条队列. 在这种情况下, 人们注意到, 客户的等待时间完全来自排队等待的时间. 那是因为, 一旦车辆通过收费站, 没有与瓶颈相关的等待时间. 因为车道没有合并, 所以没有瓶颈. 因此, 我们看到虽然顾客的一部分等待时间 (排队) 增加了, 另一部分 (瓶颈) 则完全消除了.

进入收费广场的车流对于收费亭来说过大以至于不能阻止形成队列, 前面的讨论集中在这种队列中的等待时间. 但如果车辆进入较少会怎么样呢? 在这种情况下, 我们注意到队列的形成将不再是问题 (并且没有瓶颈效应). 因此, 当公路交通车流比较低时, 收费亭数与车道数比为 1:1 似乎是理想的.

还有其他什么因素会让一个简单的 1:1 配置比其相应的多收费亭更有效? 由于 1:1 情况下瓶颈是完全被消除的, 相比于有瓶颈的情况, 任何会减少排队长度的改变倾向于青睐 1:1 的情况.

可以减少队列长度方法有多种:

- 增加收费站能够服务车辆的速率. 如果车辆更快地通过收费亭, 这将带走大量进入收费站形成列队的车辆.
- 提高司机的注意力. 在元胞自动机模型中, 每一个时间步中, 每个车辆的前进都分配了一个概率 p. 如果 p 增加了, 车辆将更有规律地向前移动, 队列附近的交通拥堵很可能是一个因素.

相对于 1:1 基本的情况, 以相反的方式改变上述参数将倾向于 '当前" 情况 (每条进入的车道对应着 > 1 的收费亭).

定量估计

对元胞自动机模型, 在 1:1 的广场配置 (所谓 1:1, 我们指的是每条进入的车道只对应一个收费亭) 中只增加了一个额外的收费亭往往就能立即减少成本. 这种成本通常持续降低,

typically continued until a local minimum is reached – the number of tollbooths corresponding to this minimum is frequently the optimal number of tollbooths (which we will designate B^*). As additional tollbooths are added to the plaza configuration, the cost function typically increases at approximately the same rate as it fell before reaching B^*. For cases such as this, we may write a simple expression for the number of tollbooths at which the cost function reaches a value comparable to that for the 1:1 configuration, which we designate Ψ:

直到达到一个局部最小值 – 对应于最低成本的收费亭数量通常是最优的收费亭数量 (这个收费亭数量我们将定义为 B^*). 随着更多的收费亭被加入到广场中, 成本函数通常会以大致相同于达到 B^* 之前下降的速率增加. 在这样的情况下, 我们可以写出一个简单的表达式, 来给出成本函数达到与 1:1 配置相仿的收费亭数量, 我们定义 Ψ:

$$\Psi = 2 \cdot B^* - L$$

where L is the number of lanes feeding the toll plaza. In this expression, Ψ represents the number of tollbooths at which the benefit of added tollbooths in terms of line length is balanced by the corresponding increase in bottlenecking. In other words, if the number of tollbooths exceeds Ψ, the cost of the system increases beyond what it would have been if no tollbooths had been added.

其中 L 是进入收费广场的车道数. 在这个表达式中，Ψ 表示“增加的收费亭在减小队列长度方面的贡献与增加瓶颈方面的损失相抵消”时的收费亭数量. 换句话说，如果收费亭的数量超过 Ψ, 系统成本增加将超过了没有增加额外收费亭的情况 (即 1:1 的情况).

Certain data from the macroscopic model indicate that the effect of bottlenecking is never sufficient to counterbalance the initial cost reduction from adding tollbooths. Such an example is provided in Figure 14.

宏观模型的某些数据表明, 瓶颈的影响不足以抵消通过增加收费亭降低的初始成本. 图14提供了一个这样的例子.

Briefly, both of our bottlenecking models agree that the first few tollbooths added to a plaza with a 1:1 configuration will reduce waiting time and system costs. However, much of our data suggest that these initial gains achieved by reducing line length are not ever counterbalanced by the effect of bottlenecking. There exist some counterexamples in which Ψ may be estimated using the above method.

简而言之，本文的两个瓶颈模型都认为在一个 1:1 配置广场最初增加的几个收费亭将降低等待时间和系统成本. 然而，我们大部分的数据表明，这些最初通过减少队列长度的收益 (节省的时间) 从未被瓶颈效应抵消. 但也存在一些反例, 在这些反例中 Ψ 可以使用上述方法估计.

13 Conclusion / 结论

We used three models – the Basic Car-Tracking Model Without Bottlenecks, the Macroscopic Model for Total Cost Minimization, and the Cellular Automata Model - in order to determine the optimal (per our definition) number B of tollbooths required in a toll plaza of L lanes.

为了确定 L 条车道的收费广场的最佳 (按我们的定义) 收费亭数量 B, 我们应用了三个模型 – 不考虑瓶颈的基本车辆追踪模型, 总成本最小化的宏观模型和元胞自动机模型.

In short, the Basic Car-Tracking Model uses a simple orderly lineup of cars approaching tollbooths and ignores bottlenecking after the tollbooths. While a quick model, it does omit bottlenecks, and provides us with a strong upper bound on B for any given L. Cost analysis on this model was not as effective as threshold anal-

简而言之, 基本车辆追踪模型使用一个简单的驶向收费站车辆的有序队列, 并忽略了收费亭后的瓶颈. 作为一个快速的模型, 它忽略了瓶颈, 对于任何给定的 L, 它为我们提供了一个 B 的有效上限. 对于这个模型, 成本分析不如阈值分

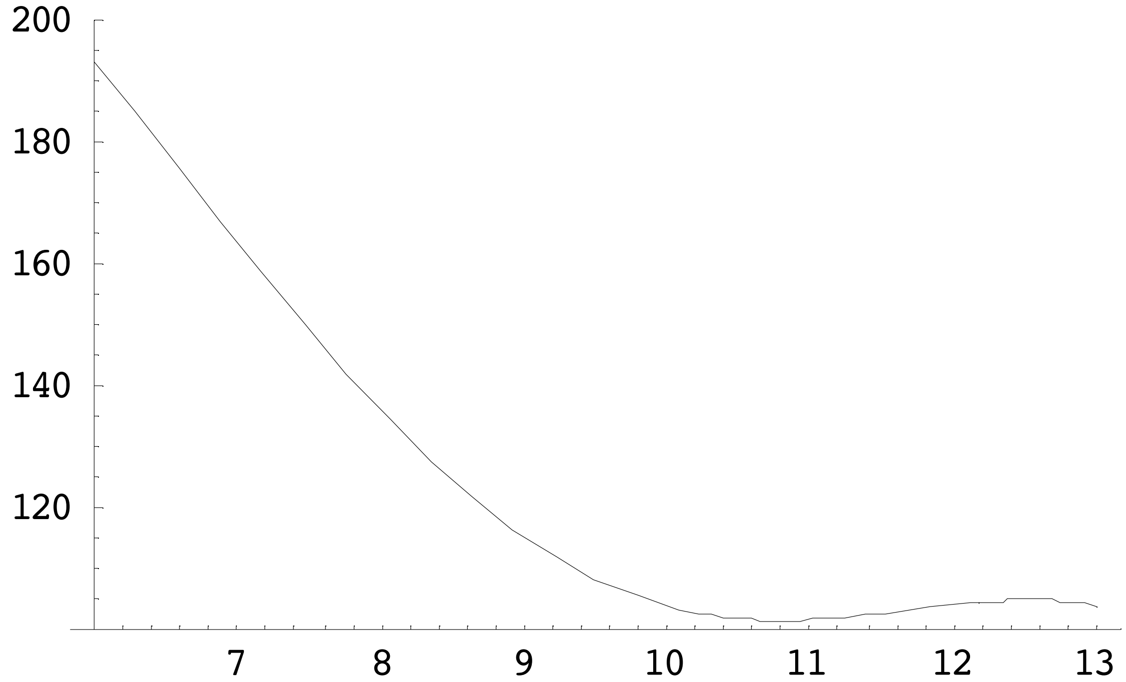

Figure 14: Total wait time vs number tollbooths. Data from macroscopic model illustrate that bottlenecking is not sufficient to incur same cost as 1:1 configuration (Note $L = 6$). 总等待时间 vs 收费亭数量. 宏观模型的结果显示瓶颈不足以导致与 1:1 配置相同的成本 (注意这里 $L = 6$).

ysis, and we determined an optimal B by recognizing when additional tollbooths did not decrease waiting time significantly.

The Macroscopic Model looks at the motion of traffic as a whole, rather than individual models. It tabulates waiting time in line before the tollbooths by considering times when traffic influx into the toll plaza is greater than tollbooth service time. It also finds bottlenecking time by assuming there exists a threshold of outflux, above which bottlenecks will occur, and notices when outflux is greater than said threshold. This is a much more accurate model than the Car-Tracking Model, and it provides us with reasonable solutions for B in terms of L.

析有效, 我们通过识别当额外增加的收费亭不再明显减少等待时间时, 来确定一个最优的 B 值.

宏观模型着眼于整个交通的运动, 而不是着眼于个体运动的模型. 宏观模型通过考查当进入收费广场的交通流量超过收费站服务效率的时间, 来求出车辆在收费亭前排队等待时间. 通过假设存在一个出流量阈值, 宏观模型也能求出瓶颈所浪费的时间, 当出流量超出这个阈值时将出现瓶颈, 并能识别出什么时候出流量会高于所谓的阈值. 这是一个比车辆追踪模型更精确的模型, 根据 L, 它为我们提供了合理 B 值的解.

The Cellular Automata Model looks at individual vehicles, and their "per lane length" motion on a toll plaza made up of cells. With a probabilistic model of how drivers advance and change lanes, this model far better details the waiting time in line and the bottlenecking after the tollbooths than the previous models. It was through this detail that we decide that this model is likely to be most accurate.

Thus we decide to recommend values closer to those provided by the automata model than the macroscopic one. In order to write B explicitly in terms of L, we invoke the linearity of the chart shown in the Comparison of Results. Also, in order to preserve integral values for B, we use the floor function and determine that $B = \lfloor 1.65L + 0.9 \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than x.

元胞自动机模型着眼于个体车辆, 并且在由元胞构成的收费广场上一步一个格子地移动. 结合司机如何前进和变更车道的概率模型, 元胞自动机比之前的模型更好地细画排队等待时间及收费亭后的瓶颈. 正是由于这个细节, 我们认为元胞自动机模型可能是最准确的.

相比于宏观模型, 我们决定推荐接近于自动机模型给出的值. 为了用 L 明确表达出 B, 我们对结果比较中表格里的数据进行线性拟合. 同时，为了限定 B 为整数值，我们调用了向下取整函数并确定 $B = \lfloor 1.65L + 0.9 \rfloor$, 其中 $\lfloor x \rfloor$ 是小于 x 的最大整数.

Table 10: Final Results and Recommendations / 最终结果和建议

Lanes	Car-Tracking	Macroscopic	Automata	Recommendation
1	4	3	2	2
2	5	5	4	4
3	7	6	5	5
4	8	7	7	6
5	10	9	8	9
6	12	11	10	10
7	13	12	12	12
8	16	14	14	14
16	29	27	27	27

14 Potential Extension and Further Consideration

Our models assume that each booth is identical to any other. In recent years, however, systems such as E-ZPass, which allow a driver to electronically pay a toll from an in-car device without ever slowing down to stop at a booth window, have been increasingly prevalent. If all E-ZPass booths also double as regular teller-operated booths, much of our models remain the same, except the average service rate might be increased. The same effects would be similar upon introduction of electronic coin collectors for those drivers with exact change. The trouble comes

我们的模型假设每个收费亭与其它任何收费亭都是相同的. 然而, 近年来, 诸如电子收费这样的系统已越来越普遍, 电子收费系统允许司机无需在收费亭窗口减速停止, 而直接从车辆内置设备电子支付通行费用. 如果所有的电子收费亭也兼作普通人工收费亭, 我们的模型基本保持不变, 除了可能需要增加平均服务率. 同样的效果将类似于引入电子收费器向那些司机收取零钱. 当所有收费亭不完全一样时, 麻烦

when all the booths are not the same and drivers may need to change lanes upon entering the plaza. This directed lane changing was not implemented in any of the models presented here, but could easily become a part of the automata model. Exclusive E-ZPass booths also would drastically reduce the operating cost for the booth, since an operator's salary would not need to be paid (from \$180,000 to \$16,000 annually) [4].

来了: 当司机们进入广场时, 可能需要更换车道. 本文提出的模型中没有实现这种有针对性地更换车道, 但这可以很容易地成为自动机模型的一个部分. 完全使用电子收费亭也将大大减少收费站的运营成本, 因为不需要向操作员支付工资 (每年在 180,000 美元到 16,000 美元之间) [4].

References

[1] Chris Tampère, Serge P Hoogendoorn, and Bart Van Arem. Capacity funnel explained using the human-kinetic traffic flow model. In *Traffic and Granular Flow' 03*, pages 189–197. Springer, 2005.

[2] E Gelenbe SI and G Pujolle. Introduction to queueing networks, 1987.

[3] Jess S Boronico and Philip H Siegel. Capacity planning for toll roadways incorporating consumer wait time costs. *Transportation Research Part A: Policy and Practice*, 32(4):297–310, 1998.

[4] R Sullivan. Fast lane. *Forbes*, 154:112–115, 1994.

Appendices

A Fourier Series

Hour	Influx (cars/min)	Fourier Approx of Influx	% Error
0.5	15.44	15.16272478	1.796
1.5	15.32	15.42467822	0.683
2.5	15.16	15.18796896	0.184
3.5	19.90	19.81853474	0.409
4.5	47.09	47.22251986	0.281
5.5	89.95	89.61825869	0.369
6.5	105.9	106.4828683	0.550
7.5	85.52	84.72959878	0.924
8.5	54.68	55.57942216	1.645
9.5	43.11	42.42662327	1.585
10.5	40.16	40.49538486	0.835
11.5	40.85	40.83544106	0.036
12.5	41.72	41.63346483	0.207
13.5	44.54	44.44085865	0.223
14.5	48.88	49.29448007	0.848
15.5	53.20	52.55619485	1.210
16.5	51.61	52.21058951	1.164
17.5	48.38	48.16410937	0.446
18.6	39.72	39.50374966	0.544
19.5	30.51	31.11397219	1.980
20.5	29.48	28.86864636	2.074
21.5	26.82	27.19686700	1.405
22.5	21.21	21.26085220	0.240
23.5	17.22	16.91795178	1.754

The equation for the Fourier Series approximation of the data table above is as follows:

$$\begin{aligned}F(t) = 41.68 - {} & 16.38\cos(t\omega) - 18.59\cos(2t\omega) + 3.572\cos(3t\omega) + 7.876\cos(4t\omega)\cdots\\ & - 0.5048\cos(5t\omega) - 2.970\cos(6t\omega) + 0.2518\cos(7t\omega) + 0.5785\cos(8t\omega)\cdots\\ & + 12.53\sin(t\omega) + 0.6307\sin(2t\omega) - 13.67\sin(3t\omega) + 0.4378\sin(4t\omega)\cdots\\ & + 6.930\sin(5t\omega) + 0.4869\sin(6t\omega) - 1.554\sin(7t\omega) - 0.5871\sin(8t\omega)\end{aligned}$$

and it has an R^2 value of 0.9997, which represents an excellent fit to the data. We can also see that this is true because only one of our 24 percent error values is above 2%.

The following chart shows the values of the coefficients a_0 through a_8 and b_1 through b_8. In the latter two columns, we can see the lower and upper bounds for a 95% confidence level for the coefficients. The tightness of the bounds again shows how accurate this Fourier fit is.

Coefficient	Value	Lower Bound*	Upper Bound*
a_0	41.68	41.59	41.76
a_1	-16.38	-16.5	-16.26
a_2	-18.59	-18.71	-18.47
a_3	3.572	3.419	3.725
a_4	7.876	7.758	7.995
a_5	-0.5048	-0.6503	-0.3594
a_6	-2.970	-3.088	-2.852
a_7	0.2518	0.1312	0.3725
a_8	0.5785	0.4601	0.6968
b_1	12.53	12.41	12.55
b_2	0.6307	0.4814	0.7799
b_3	-13.67	-13.79	-13.55
b_4	0.4378	0.2991	0.5765
b_5	6.93	6.811	7.049
b_6	0.4869	0.3613	0.6126
b_7	-1.554	-1.673	-1.435
b_8	-0.5871	-0.7069	-0.4674
ω	0.2513	0.2513	0.2514

B Basic Car-Tracking Model Code

```
t = 86400/5+1; %number of 5 second periods in a day
lanes =6; %number of lanes, assume lanes ≤ booths
booths = 10; %number of booths... assume booths << cars
%number of cars... 61565 found through integration of the influx equation
i = ceil(61565 * lanes/4);

inrate = 0;
outrate = 0;
S=0;
A=0;
L=0;
B=0;
Start=0;
N=0;
E=0;
last=0;
L2=0;
L3=0;

S(1:i) = 0; %service duration time for car i
A(1:i) = 0; %arrival time of car i
L(1:i) = 0; %leaving time of car i
B(1:i) = 0; %tollbooth used by car i
Start(1:i) = 0; %service start time for car i
last(booths) = 0; %last time at which a car left a particular booth
inrate(1:t) = 0; %influx at time t
L2(1:t) = 0; %number of cars leaving tollbooths at time t
L3(1:t) = 0; %smoothed our version of L2
outrate(1:24) = 0; %rate of outflux
N(1:t) = 0; %number of cars in line at time t

h = linspace(0,24,17281); %indexing vector for each 5 second time period
a0 = 41.68;
a = [-16.38, -18.59, 3.572, 7.876, -0.5048, -2.97, 0.2518, 0.5785];
b = [12.53, 0.6307, -13.67, 0.4378, 6.93, 0.4869, -1.554, -0.5871];
omega = 0.2513; %all Fourier Coefficients
inrate = a0 * ones(size(h));
```

```
for n = 1:8
    inrate = inrate + a(n).*cos(n.*h.*omega) + b(n).*sin(n.*h.*omega);
end

inrate = inrate * lanes / 4; %scaling lanes by appropriate amount
A(1) = 12/inrate(1); %arrival time of car 1 in terms of inrate
for j = 2:i
    k = floor(A(j-1));
    if k == 0
       k = 1;
    end
    A(j) = A(j-1) + 12/inrate(k); %arrival time of car i in terms of inrate
end

mu = 2.4; %mean service duration time in 5-second periods
S = exprnd(mu,1,i); %service time as an exponential random variable
for j = 1:i
    for k = 1:booths
        if (last(k) == min(last))
           B(j) = k; %find booth that was/will be emptied soonest
        end
    end
    if A(j) > last(B(j)) % if there is an empty booth, then...
       Start(j) = A(j); % start right away
       L(j) = Start(j) + S(j);
       last(B(j)) = L(j);
    else %if not...
       Start(j) = last(B(j)); %start once the soonest one becomes available
       L(j) = Start(j) + S(j);
       last(B(j)) = L(j);
    end
end

for j = 1:i
    k = ceil(L(j));
    if k > t
       k = t;
```

```
    end
    for m = ceil(A(j)):k
        N(m) = N(m) + 1; %counts the number of people in line
    end
end

W = L - (A + S); %waiting time is line time - (arrival time + service time)

for k = 1:i
    if L(k) ≤ t
       L2(ceil(L(k))) = L2(ceil(L(k))) + 1; %creation of L2, outflux from tollbooths
    end
end
k = length(W);

for j = 1:i
    if W(j) == 0
       k = k-1;
    end
end

totalavgwait = sum(W)/i/12/60;
carsavgwait = sum(W)/k/12/60;
maxwait = max(W)/12/60;
```

C Cellular Automata Model Code

cellular.m

```
clear all;
W = 0;
for j = 0:7
    B = 6+j; %number booths
    L = 6; %number lanes in highway before and after plaza
    T = 1; % # hrs to simulate
```

```
    global plazalength;
    plazalength = 101;
    plaza = create_plaza(B,L);
    entry_vector = create_entry(T,L);
    waiting_time = 0;
    output = 0;
    for i = 1:T*1440
        plaza = move_forward(plaza); %move cars forward
        plaza = new_cars(plaza, entry_vector(i)); %allow new cars to enter
        plaza = switch_lanes(plaza); %allow lane changes

        %compute waiting time during timestep i
        waiting_time = waiting_time + compute_wait(plaza);
        output = output + compute_output(plaza);
        plaza = clear_boundary(plaza);
    end
    plaza;
    W(j+1) = waiting_time;
end
```

move_forward.m

```
function new = move_forward(old)
new = old; %create new plaza looking same as old
[L, W] = size(new); %get its dimensions
prob = .7;
delay = 3;
%%DOWNSTREAM OF TOLL BOOTHS %%
for i = (L-1):-1:((L + 1)/2 + 1)
    for j = 1:W
        if new(i,j) == 1
            if new(i+1, j) ≠ 0
               new(i,j) = -2;
            end
            if new(i+1, j) == 0
               if prob ≥ rand
                  new(i,j) = 0;
```

```
                    new(i+1, j) = 1;
                end
            end
        end
    end
end
%%AT TOLL BOOTHS %%
for i = (L+1)/2
    for j = 1:W
        if new(i,j) > 0
            if new(i,j) == delay
                new(i,j) = 0;
                new(i+1,j) = 1;
            end
            if new(i,j) ≠delay
               if new(i,j) ≠0
                  new(i,j) = new(i,j) + 1;
               end
            end
        end
    end
end
%% UPSTREAM OF TOLL BOOTHS %%
for i = (L-1)/2:-1:1
    for j = 1:W
        if new(i,j) == 1
            if new(i+1, j) ≠ 0
               new(i,j) = -2;
            end
            if new(i+1, j) == 0
               if prob ≥ rand
                  new(i,j) = 0;
                  new(i+1, j) = 1;
                end
            end
        end
    end
```

```
end
```

new__cars.m

```
function plaza = new_cars(B, L, old, entry)

new = old;
if entry > 0
   if entry ≤ L
      x = randperm(L);
      y = ceil((B-L)/2+1);
      for i = 1:entry
          new(1, (y + x(i))) = 1;
      end
   end
   if entry > L
      y = ceil((B-L)/2+1);
      for i = 1:L
          new(1,(y + i)) = 1;
      end
   end
end
```

compute__wait.m

```
function time = compute_wait(plaza)
[a,b] = size(plaza);
time = 0;
for i = 1:a
    for j = 1:b
        time = time + (plaza(i,j) > 0);
    end
end
```

creat__plaza.m

```
function plaza = create_plaza(B, L)

global plazalength;
topgap = 5;
bottomgap = 1;

plaza = zeros(plazalength,B+2);

if mod(B-L,2)==0
    for row = 1:plazalength
        plaza(row,1) = -888;
        plaza(row,2+B) = -888;
    end
    for col = 2:B/2 - L/2 + 1
        for row = 1:(plazalength-1)/2 - topgap * (col-1)
            plaza(row,col) = -888;
            plaza(row,B+3-col) = -888;
        end
    end
    for col = 2:B/2 - L/2 + 1
        for row = (plazalength+3)/2 + bottomgap*(col-1):plazalength
            plaza(row,col) = -888;
            plaza(row,B+3-col) = -888;
        end
    end
else
    for row = 1:plazalength
        plaza(row,1) = -888;
        plaza(row,3+B) = -888;
    end
    for col = 2:(B+1)/2 - L/2 + 1
        for row = 1:(plazalength-1)/2 - topgap * (col-1)
            plaza(row,col) = -888;
            plaza(row,B+4-col) = -888;
        end
    end
    for col = 2:(B+1)/2 - L/2 + 1
```

```
        for row = (plazalength+3)/2 + bottomgap*(col-1):plazalength
            plaza(row,col) = -888;
            plaza(row,B+4-col) = -888;
        end
    end
    for row = 1:plazalength
        plaza(row,2+B) = -888;
    end
end
```

switch_lanes.m

```
function new = switch_lanes(old)
new = old;
prob = 0.8;
x = rand;
y = rand;
[L,W] = size(new);
for i = (L-1):-1:1
    for j = 2:(W-1)
        if new(i,j) == -2
           if x < prob %chance turn will be made
              if y > 0.5 %will attempt left
                 if new(i, j-1) == 0
                    new(i, j-1) = 1;
                    new(i, j) = 0;
                 elseif new(i, j+1) == 0
                    new(i, j+1) = 1;
                    new(i,j) = 0;
                 elseif new(i,j) == -2
                    new(i,j) = 1;
                 end
              end
              if y ≤ 0.5 %will attempt right
                 if new(i, j+1) == 0
                    new(i,j+1) = 1;
                    new(i,j) = 0;
```

```
                    elseif new(i, j-1) == 0
                        new(i, j-1) = 1;
                        new(i,j) = 0;
                    elseif new(i,j) == -2
                        new(i,j) = 1;
                    end
                end
            end
            if x ≥ prob
                new(i,j) = 1;
            end
        end
    end
end
```

compute_output.m

```
function count = compute_output(plaza)
count = 0;
[a, b] = size(plaza);
for j = 1:b
    count = count + (plaza(a,j) > 0);
end
```

clear_boundary.m

```
function plaza = clear_boundary(input)
plaza = input;
[a,b] = size(plaza);
for i = 1:b
    if plaza(a,i) > 0
        plaza(a,i) = 0;
    end
end
```

creat_entry.m

```
function entry = create_entry(T,L)
k = linspace(0,T,T.*60.*24);
a0 = 41.68;
entry = a0.*ones(size(k));
a = [-16.38, -18.59, 3.572, 7.876, -.5048, -2.97, 0.2518, 0.5785];
b = [12.53, 0.6307, -13.67, 0.4378, 6.93, 0.4869, -1.554, -0.5871];
omega = 0.2513;
for n = 1:8
    entry = entry + a(n).*cos(n.*k.*omega) + b(n).*sin(n.*k.*omega);
end
k = k.*1440;
entry = entry./24;
entry = round(entry);
%%% FOR RUSH HOUR SIMULATION %%%
% k = linspace(0,T,T.*60.*24);
% entry = zeros(size(k));
% entry(1:2:length(k)) = L;
```

D Model 2 Outflux [cars/min] for $L = 6$

Time	$B = 6$	$B = 7$	$B = 8$	$B = 9$	$B = 10$	$B = 11$	$B = 12$	$B = 13$
0	7.57	7.58	7.6	7.57	7.57	7.57	7.55	7.6
1	7.7	7.68	7.65	7.72	7.7	7.72	7.7	7.68
2	7.67	7.68	7.68	7.63	7.65	7.67	7.68	7.6
3	10.38	10.37	10.4	10.38	10.42	10.38	10.4	10.4
4	23.73	23.78	23.88	23.85	23.83	23.87	23.87	23.93
5	30.7	34.28	39.53	41.58	44.05	44.15	44.07	44.03
6	30.75	34.67	40.28	45.95	51.03	52.32	52.43	52.38
7	30	35.13	39.56	44.87	43.75	42.27	42.33	42.32
8	29.98	33.83	39.25	34.68	28.25	28.37	28.23	28.22
9	29.85	34.77	29.87	21.47	21.45	21.38	21.43	21.47
10	30.23	33.03	20.27	20.27	20.32	20.27	20.33	20.3
11	30.57	23.62	20.48	20.4	20.38	20.5	20.45	20.43
12	30.88	20.83	20.83	20.85	20.85	20.82	20.75	20.85
13	29.53	22.25	22.25	22.23	22.3	22.22	22.32	22.25
14	24.57	24.65	24.6	24.63	24.57	24.62	24.62	24.63
15	26.2	26.13	26.2	26.2	26.17	26.15	26.18	26.17
16	26.05	26.08	26.02	26.05	26.07	26.13	26.07	26.03
17	23.87	24.02	24.02	23.97	23.95	23.93	23.93	23.92
18	19.9	19.73	19.73	19.8	19.85	19.83	19.75	19.9
19	15.75	15.73	15.73	15.72	15.7	15.7	15.75	15.7
20	14.45	14.45	14.48	14.43	14.45	14.47	14.45	14.43
21	13.47	13.48	13.47	13.52	13.48	13.48	13.5	13.45
22	10.67	10.72	10.72	10.68	10.7	10.68	10.72	10.77
23	8.55	8.5	8.5	8.52	8.55	8.53	8.5	8.52