Homework 5: Probabilistic models

Student Name: Kuan-Lin Liu

Student ID: kll482

1. Logistic Regression

1.1 Equivalence of ERM and probabilistic approaches

(a)

$ERM = argmin_w \frac{1}{n} \sum_{i=1}^{n} log[1 + exp(-y_i w^T x_i)]$

$\hat{R}(w) = \frac{1}{n} \sum_{i=1}^{n} log[1 + exp(-y_i w^T x_i)]$

(b)

Let

$$P(y = 1|x; w) = f(w^T x) = \frac{1}{1 + exp(-w^T x)}$$

We know

$$P(Y = y|X = x) = f(w^T x)^y [1 - f(w^T x)]^{(1-y)}$$

Then,

$$L(w) = \prod_{i=1}^{n} P(Y = y_i|X = x_i) = \prod_{i=1}^{n} f(w^T x)^y [1 - f(w^T x)]^{(1-y)}$$

$$LL(w) = \log \left[\prod_{i=1}^{n} f(w^T x)^y [1 - f(w^T x)]^{(1-y)} \right] = \sum_{i=1}^{n} y_i \log f(w^T x_i) + (1 - y_i) \log(1 - f(w^T x_i))$$

$$NLL(w) = -\sum_{i=1}^{n} y_i \log f(w^T x_i) + (1 - y_i) \log(1 - f(w^T x_i))$$

(c)

Prove (a) and (b) are equal

When $y_i = 1 \text{ and } \hat{y_i} = 1,$

$$NLL(w) = \sum_{i=1}^{n} - \log f(w^T x_i) = \sum_{i=1}^{n} \log(1 + exp(-w^T x_i)) = n\hat{R(w)}$$

When $y_i = -1 \text{ and } \hat{y_i} = 0,$

$$NLL(w) = \sum_{i=1}^{n} - \log(1 - f(w^T x_i))$$
$$= \sum_{i=1}^{n} \log(1 - \frac{1}{1 + exp(-w^T x_i)})^{-1}$$
$$= \sum_{i=1}^{n} \log(\frac{exp(-w^T x_i)}{1 + exp(-w^T x_i)})^{-1}$$
$$= \sum_{i=1}^{n} \log(\frac{1 + exp(-w^T x_i)}{exp(-w^T x_i)})$$
$$= \sum_{i=1}^{n} \log(1 + exp(w^T x_i))$$
$$= n\hat{R(w)}$$

Since n is a constant, ERM and MLE will not be affected by the constant and will produce the same w.

1.2 Linearly Separable Data

1.2.1

According to the condition of that the data is linearly separable, we can find a decision boundary that predict y=1 if $w^T x \geq 0$ and y=0 if $w^T x < 0$. The decision boundary is $w^T x = 0$ with $P(y = 1 | x; w) = 0.5$.

1.2.2

$$\frac{\partial NLL(w;c)}{\partial c} = \frac{\partial - \sum_{i=1}^{n} y_i \log f(cw^T x_i) + (1-y_i) \log(1-f(cw^T x_i))}{\partial c}$$

Let $z_i = cw^T x_i$ and $f_i = f(cw^T x_i)$,

$$\frac{\partial NLL(w;c)}{\partial c} = \frac{\partial NLL}{\partial f_i} \cdot \frac{\partial f_i}{\partial z_i} \cdot \frac{\partial z_i}{\partial c}$$

$$= -\sum_{i=1}^{n} [\frac{y_i}{f_i} - \frac{1-y}{1-f}] \cdot f_i(1-f_i) \cdot w^T x_i$$

$$= -\sum_{i=1}^{n} [(1-f_i)y_i - f_i(1-y_i)] \cdot w^T x_i$$

$$= -\sum_{i=1}^{n} [y_i - f_i] \cdot w^T x_i$$

$$= \sum_{i=1}^{n} [f_i - y_i] \cdot w^T x_i$$

$$= \sum_{i=1}^{n} [f_i(cw^T x) - y_i] \cdot w^T x_i$$

$$\because \text{If } c \to \infty,\ f(cw^T x_i) \to 1$$

$\therefore$ the derivative of NLL on c is strictly positive

1.3 Regularized Logistic Regression

1.3.1

First, let's prove $\sum_{i=1}^{n} log(1 + exp(y_i w^T x_i)$ is convex.

Since we know $\sum_{i=1}^{n} \lambda ||w||^2$ is also convex, so $J_{logistic}(w)$ is also convex.

1.3.2

In [6]:

```
import numpy as np
from scipy.optimize import minimize
```

In [63]:

```python
def f_objective(theta, X, y, l2_param=1):
    '''
    Args:
        theta: 1D numpy array of size num_features
        X: 2D numpy array of size (num_instances, num_features)
        y: 1D numpy array of size num_instances
        l2_param: regularization parameter

    Returns:
        objective: scalar value of objective function
    '''
    # y should be in {-1, 1}
    n = X.shape[0]
    x1 = 0 # exp(0) = 1
    x2 = np.array([-y[i]*v for i, v in enumerate(np.dot(X, theta))])
    return np.sum(np.logaddexp(x1, x2))/n + l2_param*np.sum(np.square(theta))
```

In [126]:

```python
def f_objective(theta, X, y, l2_param=1):
    n = X.shape[0]
    summ = 0
    for i in range(n):
        summ += np.logaddexp(0, -y[i]*np.dot(theta, X[i]))
    return summ/n + l2_param*sum(theta**2)
```

1.3.3

In [164]:

```python
def fit_logistic_reg(X, y, objective_function, l2_param=1):
    '''
    Args:
        X: 2D numpy array of size (num_instances, num_features)
        y: 1D numpy array of size num_instances
        objective_function: function returning the value of the objective
        l2_param: regularization parameter

    Returns:
        optimal_theta: 1D numpy array of size num_features
    '''
    theta_0 = np.zeros(X.shape[1])
    minimizer = minimize(objective_function, theta_0, args=(X, y, l2_param))

    # return the optimal theta
    return minimizer.x
```

Remember to do preprocessing and add the bias vector in the following question.

1.3.4

In [128]:

```
### function for reading the txt file ###
def read_txt(file_path):
    data = []
    with open(file_path, "r") as f:
        for row in f:
            row_float = [float(i) for i in row.strip().split(",")]
            data.append(row_float)
    return np.array(data)
```

In [148]:

```
### input X_train, X_val, y_train, y_val ###
relative_path = "code/logistic-code/"
X_train = read_txt(relative_path+"X_train.txt")
X_val = read_txt(relative_path+"X_val.txt")
y_train = read_txt(relative_path+"y_train.txt")
y_val = read_txt(relative_path+"y_val.txt")

### revise the domain of y_train and y_val to be in {-1, 1} ###
y_train = y_train.reshape(y_train.shape[0])
y_train = [1 if row > 0.5 else -1 for row in y_train]

y_val = y_val.reshape(y_val.shape[0])
y_val = [1 if row > 0.5 else -1 for row in y_val]
```

In [149]:

```
### add bias ###
bias_train = np.ones((X_train.shape[0], 1))
bias_val = np.ones((X_val.shape[0], 1))
X_train = np.hstack((X_train, bias_train))
X_val = np.hstack((X_val, bias_val))
```

In [150]:

```
### standardize ###
from sklearn.preprocessing import StandardScaler
std_scaler = StandardScaler()
X_train_scaled = std_scaler.fit_transform(X_train)
X_val_scaled = std_scaler.fit_transform(X_val)
```

In [151]:

```
### fit ###
weight = fit_logistic_reg(X_train_scaled, y_train, f_objective)
```

In [152]:

```
weight
```

Out[152]:

```
array([ 9.55816228e-04, -2.98337651e-04,  3.02827133e-03,  1.05326768e-01,
       -3.58817577e-03, -1.35868486e-03, -3.85249746e-03, -7.90075397e-04,
       -1.14383982e-03, -7.17818640e-02,  6.54805982e-03, -4.51114274e-03,
        1.12493000e-02, -3.86482133e-03, -2.71206254e-03,  1.50358193e-03,
       -2.78418333e-03, -9.19055703e-03, -6.82303734e-03, -1.02758348e-02,
       -9.05911182e-09])
```

> From the question 1.1 above, we have proved $NLL(w)$ is equal to $n\hat{R}(w)$. Therefore, $LL(w)$ is equal to $-n\hat{R}(w)$. Just for reminding, $\hat{R}(w)$ is my objective function without the regularizer.

In [162]:

```
### get loglikelilood ###
def get_logll_function(X, y, theta, l2_param):
    my_objective = f_objective(theta, X, y, l2_param)
    n = X.shape[0]

    return -n*(my_objective-l2_param*np.sum(theta**2))
```

In [163]:

```
### function for plotting the LL for a list of regularization parameter ###
import matplotlib.pyplot as plt

def plot_logll(l2_param_list, logll_list):
    plt.plot(l2_param_list, logll_list, marker="o")
    plt.xlabel("l2 param")
    plt.ylabel("log likelihood")
    plt.title("The curve of log likelihood")
    plt.show()
```

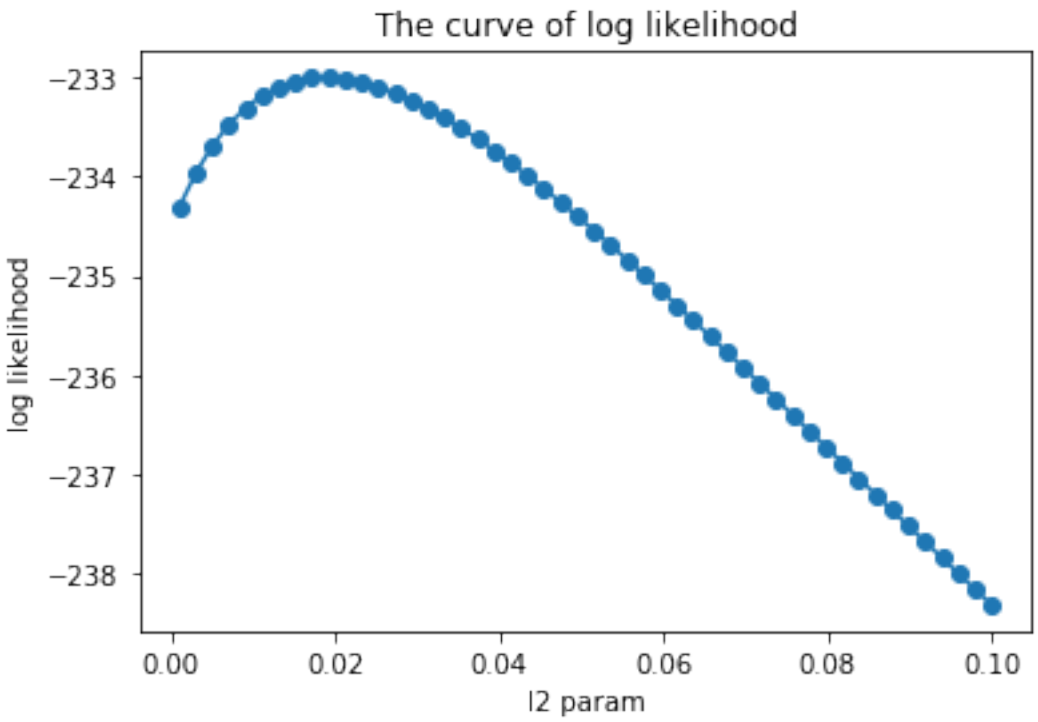
In [165]:

```
### find the minimal LL ###
def train_logistic_reg(X_train, y_train, X_val, y_val, objective_function, l2_pa
ram_list):
    logll_list = []
    for l2 in l2_param_list:
        best_theta = fit_logistic_reg(X_train, y_train, objective_function, l2)
        logll_list.append(get_logll_function(X_val, y_val, best_theta, l2))

    # return the logll list
    return logll_list
```

In [166]:

```
### plot the result ###
l2_param_list = np.linspace(0.001, 0.1, 50)
logll_list_result = train_logistic_reg(X_train_scaled, y_train, X_val_scaled, y_
val, f_objective, l2_param_list)
plot_logll(l2_param_list, logll_list_result)
```



In [167]:

```
### Get maximal log-likelihood from the plot ###
max_logll_l2 = max(zip(l2_param_list, logll_list_result), key=lambda x: x[1])
print("The minimal log-likelihood I found is: {}".format(max_logll_l2[1]))
print("The value of the l2 parameter is: {}".format(max_logll_l2[0]))
```

```
The minimal log-likelihood I found is: -233.00235581647405
The value of the l2 parameter is: 0.019183673469387756
```

2. Bayesian Logistic Regression with Gaussian Priors

2.1

$P(w|\mathcal{D}) = \frac{P(w \cap \mathcal{D})}{P(\mathcal{D}} = \frac{P(w) \cdot P(\mathcal{D}|w)}{P(\mathcal{D})}$

$\propto P(w) \cdot P(\mathcal{D}|w)$

$\propto P(w) \cdot L(w)$

$\propto P(w) \cdot exp(LL(w))$

$\propto P(w) \cdot exp(-NLL(w))$

2.2

$L(w) = \prod_{i=1}^{n} P(Y = y_i | X = x_i) = \prod_{i=1}^{n} f(w^T x)^y [1 - f(w^T x)]^{(1-y)}$

From the equation above, we know the likelihood of logistic regression is a Bernoulli distribution, which is the beta family. However, w has a normal distribution. Therefore, $P(w)$ is not a conjugate prior to the likelihood of logistic regression.

2.3

Given,

$$\mathcal{N}(0, \Sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{w^T w}{2\Sigma})$$

Solving,

$$\begin{aligned}
-\log P(w|\mathcal{D}) &\propto -\log[P(w) \cdot exp(-NLL(w))] \\
&\propto -\log P(w) + NLL(w), \ \ \text{given } w \sim \mathcal{N}(0, \Sigma) \\
&\propto -[\log(2\pi\Sigma)^{-\frac{1}{2}} - \frac{1}{2} w^T w \Sigma^{-1}] + n\hat{R(w)} \\
&\propto \frac{1}{2} log(2\pi\Sigma) + \frac{1}{2} w^T w \Sigma^{-1} + n\hat{R(w)} \\
&\propto \frac{1}{2} w^T w \Sigma^{-1} + n\hat{R(w)} \ \ \because \frac{1}{2} log(2\pi\Sigma) \ \text{ is constant}
\end{aligned}$$

Let,

$$\Sigma = \frac{1}{2n\lambda} I$$

Then we can get,

$$-\log P(w|\mathcal{D}) \propto n\hat{R(w)} + n\lambda ||w||^2$$

2.4

Continuing from Q2.3, if $\Sigma = I$, then we set $\lambda = \frac{1}{2n}$ so that the minimizer is equal to the mode of the posterior distribution of w.

3. Coin Flipping with Partial Observability

3.1

$P(x = H|\theta_1, \theta_2) = \Sigma_{\acute{z} \in \{H,T\}} P(x = H, z = \acute{z} \,|\theta_1\theta_2)$

Solve,

$P(x = H, z = \acute{z} \,|\theta_1\theta_2)$

$= \frac{P(x=H, z=\acute{z}, \theta_1\theta_2)}{P(\theta_1\theta_2)}$

$= \frac{P(x=H|z=\acute{z},\ \theta_1\theta_2) \cdot P(z=\acute{z},\ \theta_1\theta_2)}{P(\theta_1\theta_2)}$

$= P(x = H|z = \acute{z},\ \theta_2) \cdot P(z = \acute{z} \,|\theta_1\theta_2) \quad \because \theta_1$ is independent to x given by $z,\ \theta_2$

$= P(x = H|z = \acute{z},\ \theta_2) \cdot P(z = \acute{z} \,|\theta_1) \quad \because \theta_2$ is independent to z given by θ_1

Since $P(x = T|z = T) = 1$, we only care about the condition of $\acute{z} = H$.

Therefore,

$$P(x = H|\theta_1, \theta_2) = P(x = H|z = H, \theta_2) \cdot P(z = H|\theta_1) = \theta_1\theta_2$$

3.2

$P(x|\theta_1, \theta_2) = P(x = H|\theta_1, \theta_2) \cdot P(x = T|\theta_1, \theta_2)$

$P(\mathcal{D}_r|\theta_1, \theta_2) = (\frac{N_r}{n_h+n_t})(\theta_1\theta_2)^{n_h}(1 - \theta_1\theta_2)^{n_t}$

$-\log P(\mathcal{D}_r|\theta_1, \theta_2) \propto -n_h \log\theta_1\theta_2 - n_t \log(1 - \theta_1\theta_2)$

By MLE, we want to minimize $-\log P(\mathcal{D}|\theta_1, \theta_2)$

(a)

$\frac{\partial -\log P(\mathcal{D}|\theta_1,\theta_2)}{\partial\theta_1} = \frac{-n_h}{\theta_1} + \frac{n_t\theta_2}{1-\theta_1\theta_2} = 0$

(b)

$\frac{\partial -\log P(\mathcal{D}|\theta_1,\theta_2)}{\partial\theta_2} = \frac{-n_h}{\theta_2} + \frac{n_t\theta_1}{1-\theta_1\theta_2} = 0$

$\Rightarrow \frac{n_t\theta_1\theta_2}{n_h} = 1 - \theta_1\theta_2$

Substitute $\frac{n_t\theta_1\theta_2}{n_h}$ with $1 - \theta_1\theta_2$ in (a), we will get $\frac{-n_h}{\theta_1} + \frac{n_h n_t}{n_t\theta_1} = 0.$

So, θ_1, θ_2 can not be estimated using MLE.

3.3

$P(\mathcal{D}_r, \mathcal{D}_c|\theta_1, \theta_2) = P(D_c|\theta_1)P(D_r|\theta_1, \theta_2)$

$-\log P(\mathcal{D}_r, \mathcal{D}_c|\theta_1, \theta_2) = -\log P(D_c|\theta_1)P(D_r|\theta_1, \theta_2)$

$\propto \Big[-c_h \log\theta_1 - c_t \log(1-\theta_1)\Big] + \Big[-n_h \log\theta_1\theta_2 - n_t \log(1-\theta_1\theta_2)\Big]$

By MLE, we want to minimize $\propto \Big[-c_h \log\theta_1 - c_t \log(1-\theta_1)\Big] + \Big[-n_h \log\theta_1\theta_2 - n_t \log(1-\theta_1\theta_2)\Big]$

(a)

$\frac{\partial -\log P(\mathcal{D}_r,\mathcal{D}_c|\theta_1,\theta_2)}{\partial\theta_1} = -\Big[\frac{c_h}{\theta_1} - \frac{c_t}{1-\theta_1}\Big] - \Big[\frac{n_h}{\theta_1} - \frac{\theta_2 n_t}{1-\theta_1\theta_2}\Big] = 0$

(b)

$\frac{\partial -\log P(\mathcal{D}_r,\mathcal{D}_c|\theta_1,\theta_2)}{\partial\theta_2} = -\Big[\frac{n_h}{\theta_2} - \frac{n_t\theta_1}{1-\theta_1\theta_2}\Big] = 0$

$\Rightarrow 1-\theta_1\theta_2 = \frac{n_t\theta_1\theta_2}{n_h}$

Substitute $1-\theta_1\theta_2$ with $\frac{n_t\theta_1\theta_2}{n_h}$ in (a), we will get

$\theta_1 = \frac{c_h}{c_h+c_t}$

$\theta_2 = \frac{n_h}{(n_h+n_t)\theta_1} = \frac{n_h(c_h+c_t)}{(n_h+n_t)c_h}$

3.4

Given $g(\theta_1) = \theta_1^{h-1}(1-\theta_1)^{t-1}$

$\theta_{1,MAP} = argmax_{\theta_1} g(\theta_1)L(\theta_1, \theta_2)$

Let $\acute{L}(\theta_1, \theta_2) = g(\theta_1)L(\theta_1, \theta_2)$

$$\acute{L}L(\theta_1, \theta_2) = \left[(h-1)\log\theta_1 + (t-1)\log(1-\theta_1)\right] + \left[c_h \log\theta_1 + c_t \log(1-\theta_1)\right] + \left[n_n \log(\theta_1, \theta_2) + n$$

(a)

$$\frac{\partial N\acute{L}L}{\partial\theta_1} = -\left[\frac{h-1}{\theta_1} - \frac{t-1}{1-\theta_1}\right] - \left[\frac{n_h}{\theta_1} - \frac{\theta_2 n_t}{1-\theta_1\theta_2}\right] - \left[\frac{c_h}{\theta_1} - \frac{c_t}{1-\theta_1}\right] = 0$$

(b)

$$\frac{\partial N\acute{L}L}{\partial\theta_2} = \frac{n_h}{\theta_2} - \frac{\theta_1 n_t}{1-\theta_1\theta_2} = 0$$

$$\Rightarrow \frac{n_h}{\theta_1 n_t} = \frac{\theta_2}{1-\theta_1\theta_2}$$

From (a) and (b), we can obtain

$$\theta_1 = \frac{c_h+h-1}{c_h+c_t+h+t-2}$$

$$\theta_2 = \frac{n_h}{(n_h+n_t)\theta_1} = \frac{n_h(c_h+c_t+h+t-2)}{(n_h+n_t)(c_h+h-1)}$$

In [ ]: