Merton Model Results

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January 25, 2021

1 Theory Overview

The model for corporate default is based on the assumption that firm value, V, is the sum of equity value, E, and debt value, D. V is further assumed to follow a GBM:

$$V_t = E_t + D_t, \qquad dV = \mu_V V dt + \sigma_V V dW \tag{1}$$

Given the **face value** of debt, F, and the time-to-maturity, T, the Distance-to-Default, DD, and associated Probability-of-Default, $\pi_{\mathrm{t,T}}$, are then defined via:

$$\text{Distance-to-Default} \;=\; DD \;=\; \frac{\ln\left(\frac{V}{F}\right) + \left(\mu_V - 0.5\sigma_V^2\right)T}{\sigma_V\sqrt{T}},$$

$$\text{Probability-of-Default} \;=\; \pi_{\mathrm{t,T}} \;=\; \mathrm{Prob}\{V_T < F\} \;=\; \Phi(-DD) \tag{2}$$

In reality, V, μ_V and σ_V are unobservable. Various authors have suggested numerous ways of estimating these parameters – see Literature Review for details.

As a quick summary of the possible solution techniques:

- The original solution assumes a Black-Scholes style framework and the value of equity is a call option on the firm value. Given E_t and σ_E, the following non-linear equations can be solved either **simultaneously** or **iteratively** for V and σ_V:

$$E_t = \mathrm{V}\ \Phi(d_1) - e^{-rT}\ \mathrm{F}\ \Phi(d_2) \tag{3}$$

$$\sigma_E \;=\; \left(\frac{\mathrm{V}}{E}\right)\Phi(d_1)\sigma_{\mathrm{V}} \tag{4}$$

- Alternatively, one can make simplifying assumptions on the dynamics of V. For example, one can say that $V_t \approx E_t + F_t$ and/or derive estimates of μ_V and σ_V using historical data, GARCH-type forecasts, risk-neutral pricing or option-implied data.
- One can simply assume that $\sigma_V \approx \sigma_E$ and use σ_E to solve for the implied value of V in Equation (3). This alleviates the problems associated with any simultaneous or iterative methods.

2 Candidate Probability of Default Measures

2.1 Merton (1974) Simultaneous Equations Method

This is the original method and requires an initial estimate, $\hat{\mu}_V$, of the firm drift rate in Equation (2). V_t and σ_V can then be estimated by simultaneously solving Equations (3) and (4). The PD is then a function of:

$$DD_{\text{Merton}} = \frac{\ln\left(\frac{\hat{V}}{F}\right) + \left(\hat{\mu}_V - 0.5\hat{\sigma}_V^2\right)T}{\hat{\sigma}_V\sqrt{T}}, \tag{5}$$

2.2 KMV/Moody's Iterative Method

An alternative approach – developed by KMV and also used by Bharath and Shumway (2008), Duffie et al. (2007), amongst others – is a calculation-intensive iterative procedure. In this process an initial guess value of σ_V is used in Equation (3) in order to infer the market value of the assets V for the firm on a daily basis in the prior year. This generates a time series whose volatility is an updated guess of σ_V, which is used to compute a new time series of the firm's assets. The procedure is repeated until the volatility used to calculate the time series converges to the volatility of the calculated values. Then, the last time series is used to infer the values of σ_V and μ_V which are used in Equation (2) of the model.

$$\begin{aligned}
\hat{\mathbf{V}}_t &= (\hat{V}_{t-252}, \hat{V}_{t-251}, \ldots, \hat{V}_{t-1}, \hat{V}_t) \\
\hat{\mu}_V &= \overline{\hat{\mathbf{V}}} \\
\hat{\sigma}_V &= \sigma(\hat{\mathbf{V}})
\end{aligned}$$

$$DD_{\text{KMV}} = \frac{\ln\left(\frac{E+F}{F}\right) + \left(\hat{\mu}_V - 0.5\hat{\sigma}_V^2\right)T}{\hat{\sigma}_V\sqrt{T}}, \tag{6}$$

2.3 Bharath & Shumway (2008) Naïve Method

Rather than treating equity value as a call option, choose an estimate for μ_V and further assume:

$$\begin{aligned}
\hat{\sigma}_D &= 0.05 + 0.25 \times \hat{\sigma}_E \\
\hat{\sigma}_V &= \frac{E_t}{E_t + F_t} \times \hat{\sigma}_E + \frac{F_t}{E_t + F_t} \times \hat{\sigma}_D
\end{aligned}$$

$$DD_{\text{BhSh}} = \frac{\ln\left(\frac{E+F}{F}\right) + \left(\hat{\mu}_V - 0.5\hat{\sigma}_V^2\right)T}{\hat{\sigma}_V\sqrt{T}}, \tag{7}$$

2.4 Afik et al. (2016) Simplified Method

As an even simpler approach, just assume that $\sigma_V \approx \sigma_E$. Then estimate the mean (μ_E) and volatility (σ_E) of equity. The PD is a function of:

$$DD_{\text{Afik}} = \frac{\ln\left(\frac{E+F}{F}\right) + \left(\hat{\mu}_E - 0.5\hat{\sigma}_E^2\right)T}{\hat{\sigma}_E\sqrt{T}}, \tag{8}$$

2.5 Charitou et al. (2013) Simplified Method

As the market price of debt is unobservable, proxy the total firm value using the sum of the market price of equity and face value of debt. From this assumption, we can calculate the sample moments for the firm value and derive a PD measure from:

$$\begin{aligned}
V_t &= E_t + F_t, \\
\hat{\mu}_V &= \overline{V}_t \;=\; \overline{E_t + F_t} \\
\hat{\sigma}_V &= \sigma(V_t) \;=\; \sigma(E_t + F_t)
\end{aligned}$$

$$DD_{\text{Charitou}} = \frac{\ln\left(\frac{E+F}{F}\right) + \left(\hat{\mu}_V - 0.5\hat{\sigma}_V^2\right)T}{\hat{\sigma}_V\sqrt{T}}, \tag{9}$$

3 Estimating μ_E, σ_E (and inferring μ_V, σ_V)

The aforementioned PD models all require estimates for μ_V and σ_V. Usually, such estimates are derived from μ_E and σ_E. Tables 1 & 2, summarize a few different ways of estimating the moments of the equity return distribution.

Name	**Symbol**	**Description**
Historical Return	$\mu_{\text{historical}}$	1-year daily sample mean return (annualized)
Risk-free Return	r_{rf}	Ken French risk-free rate (annualized)
Bounded Return	$\max(\mu_{\text{historical}}, r_{rf})$	Take maximum to prevent negative firm drift rate
CAPM Return	μ_{CAPM}	$E(r) = r_{rf} + \beta \times (E(r_{market}) - r_{rf})$

Table 1: Equity drift rate estimates, $\hat{\mu}_E$.

Name	**Symbol**	**Description**
Historical Volatility	$\sigma_{\text{historical}}$	1-year daily sample std dev (annualized)
Historical Mean Absolute Deviation	σ_{MAD}	1-year daily absolute deviation (annualized)
GARCH(1,1) Volatility	σ_{GARCH}	Volatility estimate from fitted GARCH(1,1) model
Option-implied Volatility	σ_{implied}	BS implied volatility from ATM puts/calls

Table 2: Equity volatility estimates, $\hat{\sigma}_E$.

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