Chu-Isbell duality for Koopman operators

Preprint · March 2024

DOI: 10.13140/RG.2.2.30886.33609

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Chu-Isbell duality and reflexivity for Koopman operators

Gary Nan Tie, Mar 12, 2024

Abstract Koopman operator adjointness can be viewed as a form of Isbell duality. It is then natural to ask when reflexivity is attained. For appropriately defined Yoneda and co-Yoneda embeddings we see the Isbell conjugates are reflexive when the Koopman operator is unitary.

Notation: dynamic $f : \mathcal{X} \to \mathcal{X}$, $\mathcal{X} \subseteq \mathbb{R}^n$

reproducing kernel $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ with

corresponding RKHS $H(k)$. Assume $g \circ f \in H$, $\forall g \in H$.

Koopman operator
$$\begin{aligned} \mathbb{K} : H &\longrightarrow H \\ g &\longmapsto g \circ f \end{aligned}$$

Let $k_y(\cdot) \triangleq k(\cdot, y) \in H$

Note that $\forall F \in H$, $F(x) = \langle F, k_x \rangle_H$

Define <u>Yoneda embedding</u>
$$\begin{aligned} y : \mathcal{X} &\longrightarrow H(k) \subseteq [\mathcal{X}, \mathbb{R}] \\ x &\longmapsto k_{f(x)} \end{aligned}$$

and <u>co-Yoneda embedding</u>
$$\begin{aligned} z : \mathcal{X} &\longrightarrow H(k) \subseteq [\mathcal{X}, \mathbb{R}] \\ x &\longmapsto \mathbb{K} k_x \end{aligned}$$

Define Isbell conjugates $H \underset{\#}{\overset{*}{\rightleftarrows}} H$ as follows:

$$F^* : \mathcal{X} \longrightarrow \mathbb{R}$$
$$x \longmapsto \langle F, y(x) \rangle = \langle F, k_{f(x)} \rangle = F \circ f\,(x)$$

So $F^* = F \circ f = K F$.

$$G^\# : \mathcal{X} \longrightarrow \mathbb{R}$$
$$x \longmapsto \langle z(x), G \rangle = \langle K k_x, G \rangle = \langle k_x, K^* G \rangle = K^* G(x)$$

So $G^\# = K^* G$.

Hence, $\langle F^*, G \rangle = \langle K F, G \rangle = \langle F, K^* G \rangle = \langle F, G^\# \rangle$,

a kind of Isbell duality.

$$(F^*)^\# = (\mathcal{K} F)^\# = \mathcal{K}^* \mathcal{K} F$$

$$(G^\#)^* = (\mathcal{K}^* G)^* = \mathcal{K} \mathcal{K}^* G$$

If Koopman operator $\mathcal{K} : H(k) \to H(k)$

is unitary ie $\mathcal{K}\mathcal{K}^* = I = \mathcal{K}^*\mathcal{K}$

then $(F^*)^\# = F$ and $(G^\#)^* = G$

ie $*$ and $\#$ are <u>Isbell reflexive.</u>

$\square$

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