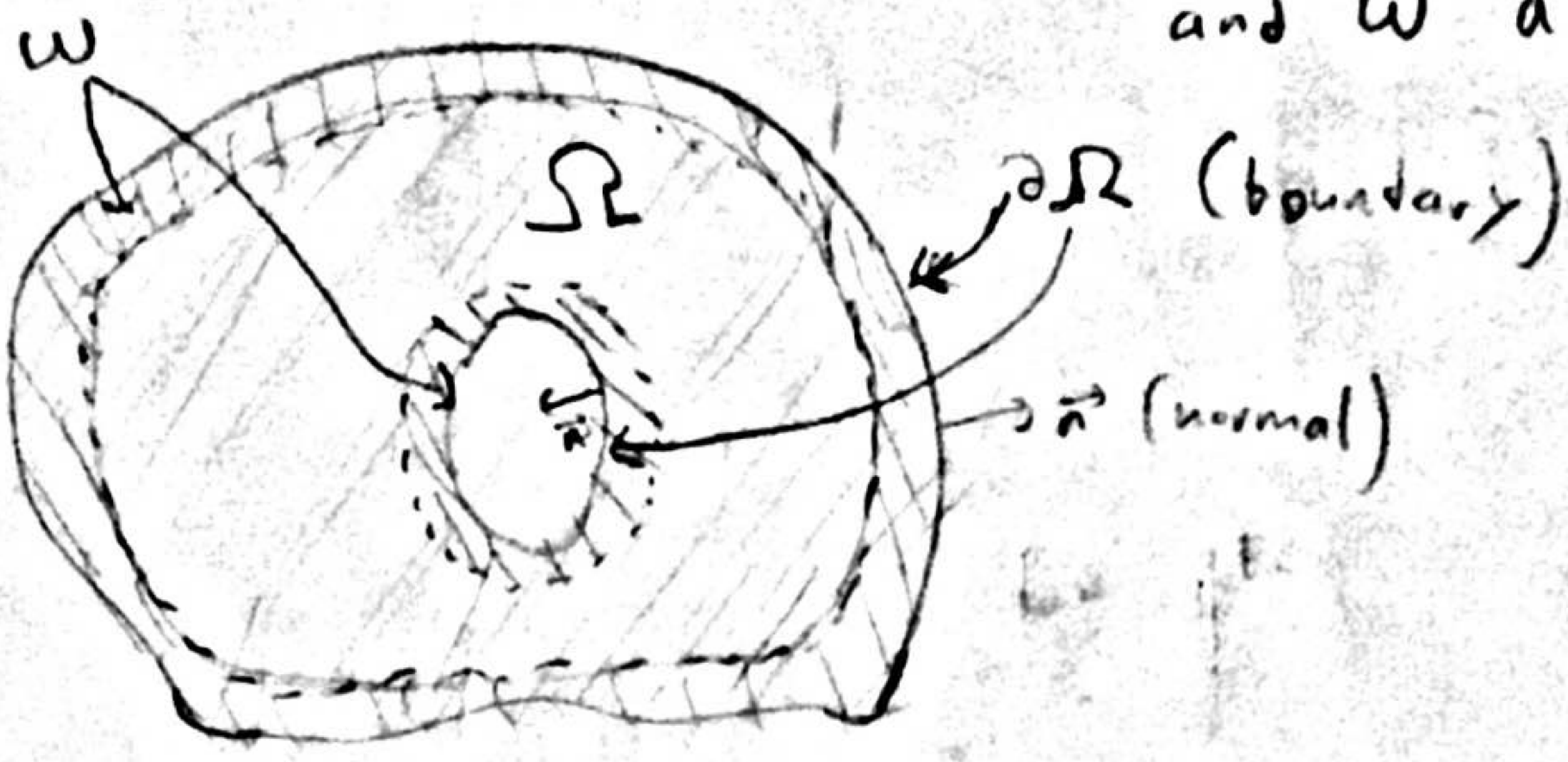
let Ω be a connected compact region (possibly with holes) as shown;
and ω a neighborhood of boundary



let $\phi^{BC} : \partial\Omega \to \mathbb{R}$ be a smooth function on boundary

let $\forall n \in \mathbb{N}$, $\varepsilon_n : \Omega \to \mathbb{R}$, $\phi_n : \Omega \to \mathbb{R}$ be sequences of <u>smooth</u> functions

Suppose $\forall n \in \mathbb{N}$, $\varepsilon_n > 0$ and $\vec{\nabla} \cdot (\varepsilon_n \vec{\nabla} \phi_n) = 0$ and $\phi_n \big|_{\partial\Omega} = \phi^{BC}$ (dielectric BVP)

Suppose
- $\lim_{n\to\infty} \varepsilon_n = \varepsilon$ pointwise almost everywhere, where $\varepsilon|_\omega$ is smooth
- $\lim_{n\to\infty} \phi_n = \phi$ pointwise on ω for some $\phi : \omega \to \mathbb{R}$ <u>smooth</u>
- convergence of $\varepsilon_n, \phi_n, \nabla\phi_n$ is uniform on ω.
- for some $\varepsilon_{min}, \varepsilon_{max} \in \mathbb{R}$, $\forall n$, $0 < \varepsilon_{min} \le \varepsilon_n \le \varepsilon_{max}$

then $\forall n$,
$$\vec{\nabla} \cdot (\phi_n \varepsilon_n \vec{\nabla} \phi_n) = (\vec{\nabla}\phi_n) \cdot (\varepsilon_n \vec{\nabla} \phi_n) + \phi_n \underbrace{(\nabla \cdot (\varepsilon_n \nabla \phi_n))}_{0 \text{ by assumed PDE}}$$
$$= \varepsilon_n |\vec{\nabla}\phi_n|^2$$

note since Ω compact, smooth functions are integrable

thus $\forall n$, $\int_\Omega \varepsilon_n |\nabla\phi_n|^2 = \int_\Omega \vec{\nabla} \cdot (\phi_n \varepsilon_n \vec{\nabla} \phi_n) = \int_{\partial\Omega} (\phi_n \varepsilon_n \vec{\nabla}\phi_n) \cdot \vec{n}$

by divergence theorem.

clearly $\forall n,\ \varepsilon_{min}|\nabla\phi_n|^2 \le \varepsilon_n|\nabla\phi_n|^2$

thus $\forall n,\ \varepsilon_{min}\int_\Omega|\nabla\phi_n|^2 = \int \varepsilon_{min}|\nabla\phi|^2 \le \int_\Omega \varepsilon_n|\nabla\phi_n|^2$

also $\int_{\partial\Omega}\phi_n\varepsilon_n\vec\nabla\phi_n\cdot\vec n \le \int_{\partial\Omega}|\phi_n\varepsilon_n\vec\nabla\phi_n\cdot\vec n|$

$\le \int_{\partial\Omega}|\phi_n|\varepsilon_n|\vec\nabla\phi_n|$ since $|\vec\nabla\phi_n\cdot\vec n| \le |\nabla\phi||\cancel{\vec n}|^{1}$

$= \int_{\partial\Omega}|\phi^{BC}|\cdot\varepsilon_n\cdot|\nabla\phi_n|$ since $\forall n,\ \phi_n|_{\partial\Omega} = \phi^{BC}$

$\le \varepsilon_{max}\int_{\partial\Omega}|\phi^{BC}|\cdot|\nabla\phi_n|$

thus $\forall n, \int_\Omega|\nabla\phi_n|^2 \le \dfrac{\varepsilon_{max}}{\varepsilon_{min}}\int_{\partial\Omega}|\phi^{BC}||\nabla\phi_n|$

since on ω $\phi_n\to\phi$ uniformly and $\nabla\phi_n$ also converges uniformly (by assumption)
it follows $\nabla\phi_n\to\nabla\phi$ (on ω, and thus on $\partial\Omega$)

also, since ϕ smooth and $\omega, \partial\Omega$ are compact,
it follows $\nabla\phi$ and $\nabla\phi_n$ are bounded
i.e. $\exists L\in\mathbb{R}: \forall n, \forall x\in\partial\Omega, |\nabla\phi_n(x)| < L$

thus $\int_\Omega|\nabla\phi_n|^2 \le \dfrac{\varepsilon_{max}}{\varepsilon_{min}}L\int_{\partial\Omega}\phi^{BC}$ where $\int_{\partial\Omega}\phi^{BC} < \infty$

since ϕ^{BC} also bounded
(and $\partial\Omega$ compact / $\int_{\partial\Omega}1 < \infty$)

thus $\int_\Omega|\nabla\phi_n|^2$ is a <u>bounded</u> sequence in n.

thus ϕ_n <u>cannot</u> converge to a step-like discontinuity, since
if it did, then $\nabla\phi_n\to\delta$ (delta function) and
$\int_\Omega|\nabla\phi_n|^2 \longrightarrow \infty$.