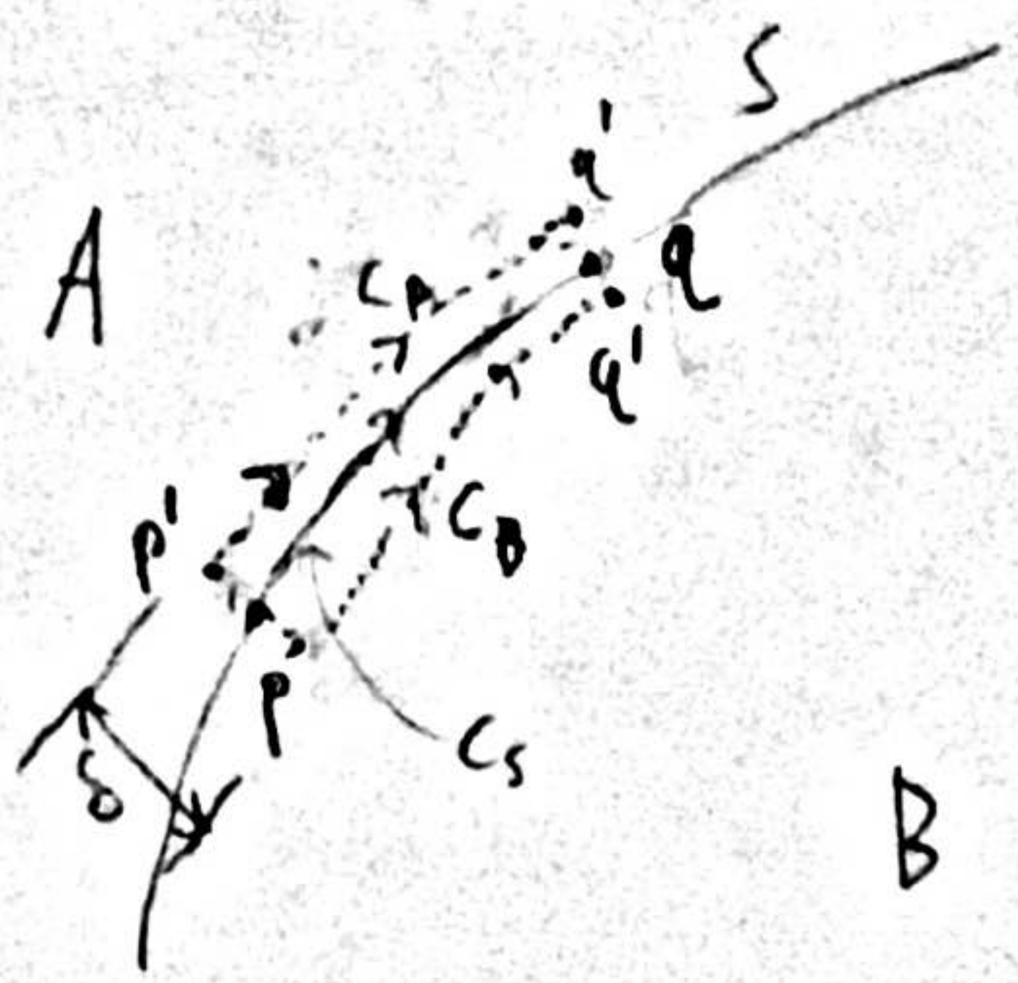
let A, B be regions separated by boundary S as shown



let $\phi_A : A \to \mathbb{R}$, $\phi_B : B \to \mathbb{R}$ smooth

suppose ϕ_A and ϕ_B can be continuously extended to S.

Suppose $\forall p \in S$, $\vec{T}$ tangent to S at p, $\vec{\nabla}\phi_A(p) \cdot \vec{T} = \vec{\nabla}\phi_B(p) \cdot \vec{T}$

then let $p, q \in S$ be arbitrary as shown above

let C_A C_B be the paths on either side of C_S separated by width δ where C_S goes from p to q

then $\phi_A(q) - \phi_A(p) = \phi_A(q') - \phi_A(p') + O(\delta)$

$$= \int_{C_A} \nabla\phi_A \cdot d\vec{\ell} + O(\delta)$$

$$= \int_{C_S} \vec{\nabla}\phi_A \cdot \vec{T}\, |d\vec{\ell}| + O(\delta)$$

similarly for B

thus $\forall \delta > 0$, $\left(\phi_A(q) - \phi_A(p)\right) - \left(\phi_B(q) - \phi_B(p)\right) = \int_{C_S} \left(\vec{\nabla}\phi_A \cdot \vec{T} - \nabla\phi_B \cdot \vec{T}\right) |d\vec{\ell}| + O(\delta)$

taking $\delta \to 0$ limit, $\left(\phi_A(q) - \phi_B(q)\right) - \left(\phi_A(p) - \phi_B(p)\right) = \int_{C_S} \left(\vec{\nabla}\phi_A \cdot \vec{T} - \vec{\nabla}\phi_B \cdot \vec{T}\right)$

$= 0$ by assumption

thus $\forall p, q \in S$, $\phi_A(q) - \phi_B(q) = \phi_A(p) - \phi_B(p)$

thus $\exists!\ \phi_0 \in \mathbb{R}$ s.t. $\phi(x) = \begin{cases} \phi_A(x), & x \in A \cup S \\ \phi_B(x) + \phi_0, & x \in B \end{cases}$ is continuous