

Def of invariant

$P: A \rightarrow \text{Bool}$

$P: A \hookrightarrow \mathbb{N}$

If $w \rightarrow \checkmark$

then $\boxed{P_{cw}) = P_{c\checkmark})}$

Exle: 1) $\text{Odd}(w) : A \rightarrow \text{Bool}$

2) $\text{Even}(w) : A \rightarrow \text{Bool}$

3) $A \rightarrow \mathbb{N}$

$w \mapsto \#a \bmod 2$

Assume: $aab \rightarrow$ only this rule

Then: $\#a \bmod 3$ is an invariant

For instance: $aaaaabb \rightarrow aabb$

$\underbrace{}_{}$ $\overbrace{}^{}$

$\#a \bmod 3$

2

$\#a \bmod 3$

2

does not change

therefore $\#a \bmod 3$ is an invariant

Back to the exercise
from the lecture
on invariants

aa \rightarrow

b \rightarrow

first delete b's
then delete aa's

Q: In what sense are
Even / Odd the "only"
invariants?

Even is a complete
invariant

Invariants allow you to
make distinctions

How do we know that

abaababba $\not\leftrightarrow$ baababba?

1) only works
if normal forms
are unique

reduce to different normal forms

2) use the invariant

$$\begin{array}{l} ab \rightarrow ba \\ ba \rightarrow ab \end{array} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \begin{array}{l} \#a \text{ is different} \\ (6a \text{ left}, 5a \text{ right}) \end{array}$$

What if we add

$$\begin{array}{l} aa \rightarrow \\ b \rightarrow \end{array}$$

$\#a \bmod 2$ is 0 left
 $\#a \bmod 2$ is 1 right

Intuitively: An invariant is complete if

it allows you to make all
distinctions.

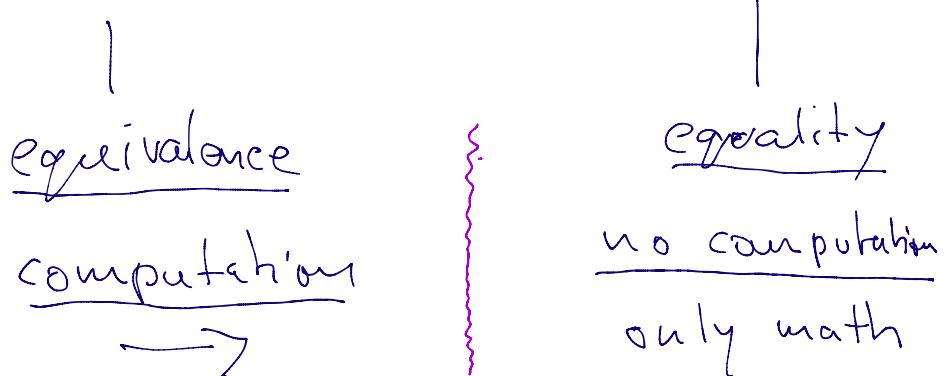
Exe: In our example

$$\begin{array}{l} * \text{ Even} \\ * \text{ Odd} \\ * \#a \bmod 2 \end{array} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad \begin{array}{l} \text{each of them} \\ \text{is a} \\ \text{complete} \\ \text{invariant} \end{array}$$

If I ask to find all invariants
I am looking for one
complete invariant

Def An invariant I is
complete if

$$w \xleftrightarrow{*} v \iff I(w) = I(v)$$



Recall the uninitiated chess board :

algorithms for tiling the plane } vs } # black tiled = # white tiled

One invariant

vs
A set of invariants ?

We can always combine invariants into one.

The generic way of doing this is :

Assume we have:

$$I_1 : A \rightarrow \mathbb{N}$$

$$I_2 : A \rightarrow \text{Bool}$$

Combine:

$$A \rightarrow \mathbb{N} \times \text{Bool}$$

$$a \mapsto (I_1(a), I_2(a))$$

How can we know that

Even

is complete

for

$$ab \rightarrow ba$$

$$ba \rightarrow ab$$

$$aa \rightarrow$$

$$b \rightarrow$$

?

$$w \leftrightarrow v \quad (\Rightarrow) \quad \text{Even}(w) = \text{Even}(v)$$

the same



✓
(invariant)

If $\text{Even}(w)$ and not $\text{Even}(v)$

then not $w \leftrightarrow v$.



If $\text{Even}(w) = \text{Even}(v)$

then $w \leftrightarrow v$

Proof: a a b a b a a a b b b

eliminate
all b

↓
x
a a a a

↓
x
a a

eliminate
all a a

↓
[]

[] empty string

This proves $w \leftrightarrow v$ if w, v have even #a.

A similar argument works in case

w, v have odd #a :

a a a a b a b a a b a a a

↓
x

a a a a a

↓

a a a

a

$$P : A \rightarrow \text{Bool}$$

Def 1

I added a section on weak invariants to the lecture notes.

$$w \rightarrow v \Rightarrow P(w) = \text{true}$$

(\Rightarrow)

$$P(v) = \text{true}$$

Def 2

We may call this strong invariants

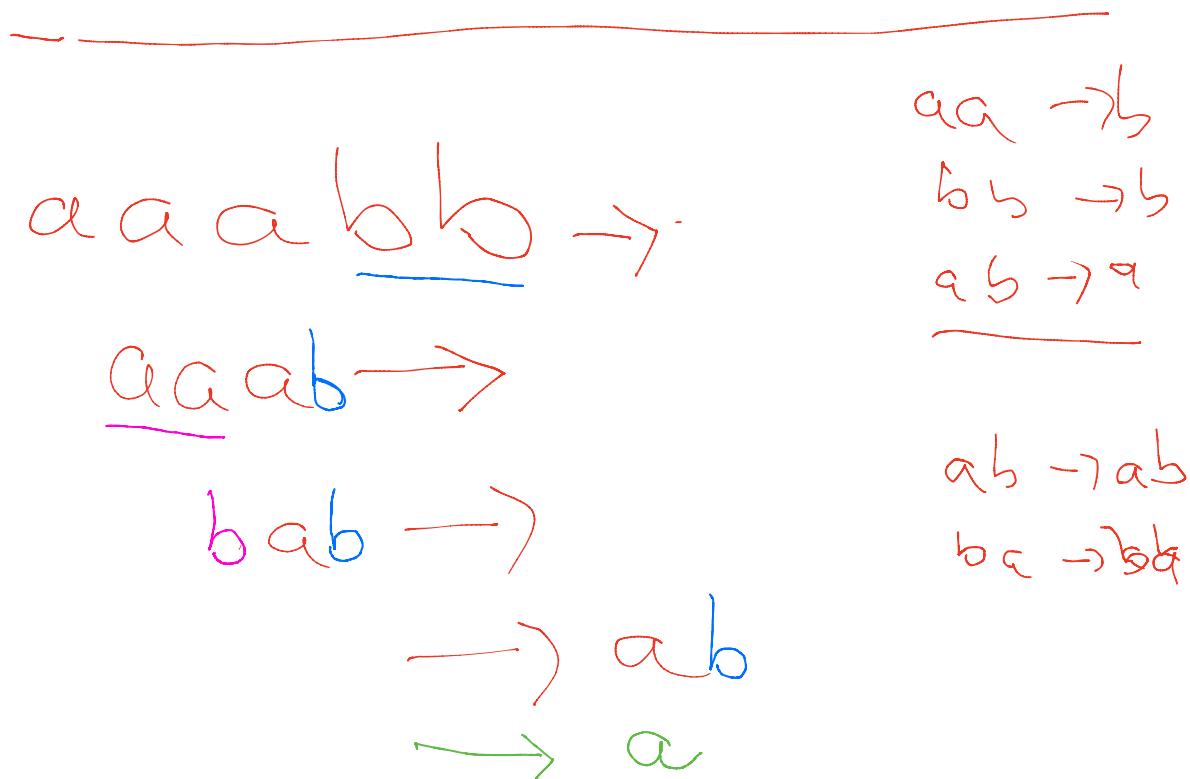
$$w \rightarrow v \Rightarrow P(w) = P(v)$$

The orange highlights how
s

$$P(w) = \text{true} \quad (\Leftrightarrow) \quad P(v) = \text{true}$$

The question of whether E or σ on its own is a complete invariant depends on whether we use

Def 1 or Def 2



$a^2 a^3 a^2 b^2 b^2 \xrightarrow{*}$ replace all a 's
by b

$a^2 b^2 b^2 b^2 b^2 \xrightarrow{*}$ the a eats
all b 's

a

1) only even a 's result b

2) only b 's result b

3) odd a 's + some b 's
result a

4) even a 's + some b 's

The lecture stopped here. If you finish
item 4) you can solve some other items.

This will suggest a complete invariant.

At the end you will have two different characterisations of the equivalence classes.

Normal Form	Invariant
$[]$	# a even
a	# a odd

