ELEC5470/IEDA6100A Convex Optimization - Homework #2

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All your answers must be appropriately justified.
Due date: **Sept. 28th 2020 before 6pm**. Please, submit a zip or pdf file through canvas containing code, plots, and justifications.
Observation: make sure to write your code in a modular, readable way. Code organization will be taken into account for the grading.
In case you do not have access to canvas, submit your solutions, as a zip or pdf file, to jvdmc@connect.ust.hk.
Late submissions, or submissions that are not zip or pdf files won't be considered.
Total marks: 100

Instructions: you **must** use the discipline convex programming language CVX to code the problems below. There are several interfaces for CVX in popular programming languages including Python[1], R[2], Julia[3], and MATLAB[4]. Choose your favorite one!

Problem #1 (25 marks)

Risk Parity Portfolio (RPP): The vanilla risk parity portfolio can be computed as the solution to the following optimization problem

$$\underset{\boldsymbol{x}>\boldsymbol{0}}{\text{minimize}} \quad \frac{1}{2}\boldsymbol{x}^\top \Sigma \boldsymbol{x} - \boldsymbol{b}^\top \log(\boldsymbol{x}), \tag{1}$$

[1] https://www.cvxpy.org/
[2] https://cvxr.rbind.io/
[3] https://github.com/jump-dev/Convex.jl
[4] http://cvxr.com/cvx/

where $\log(\boldsymbol{x})$ is to be understood as applied elementwise, i.e., $\log(\boldsymbol{x}) = (\log(x_1), \log(x_2), \ldots, \log(x_n))^\top$, $\boldsymbol{b}$ is a given risk budget vector and Σ is a given covariance matrix of the returns.
Assuming $\boldsymbol{x}^\star$ is the solution to the above problem, then the RPP weights are simply
$\boldsymbol{w} = \dfrac{\boldsymbol{x}^\star}{\mathbf{1}^\top \boldsymbol{x}^\star}$.
Is this problem convex?

Implement code to solve the above problem and print its solution $\boldsymbol{w}^\star$.

Use

$$\Sigma = \begin{bmatrix} 1.0 & 0.0015 & -0.02 \\ 0.0015 & 1.0 & -0.1 \\ -0.02 & -0.1 & 1.0 \end{bmatrix}, \tag{2}$$

and $\boldsymbol{b} = [0.1594, 0.0126, 0.8280]^\top$.

Problem #2 (25 marks)

Portfolio Optimization: In Modern Portfolio Theory, one is interested in allocating a certain amount of money B into a set of N stocks. This allocation process is formulated as the following optimization problem

$$\begin{aligned} \underset{\boldsymbol{w}}{\text{minimize}} \quad & \boldsymbol{w}^\top \Sigma \boldsymbol{w} - \lambda \boldsymbol{\mu}^\top \boldsymbol{w} \\ \text{subject to} \quad & \boldsymbol{w} \geq \mathbf{0}, \boldsymbol{w}^\top \mathbf{1} = 1, \end{aligned} \tag{3}$$

where Σ is the given covariance matrix of the stock returns and $\boldsymbol{\mu}$ is the given vector of expected stock returns, $\boldsymbol{w}^\top \Sigma \boldsymbol{w}$ is called portfolio variance, $\boldsymbol{\mu}^\top \boldsymbol{w}$ is called portfolio expected return, and λ is a trade-off parameter that controls how much the investor values the portfolio expected return over the portfolio variance. The optimal solution to the above problem, $\boldsymbol{w}^\star$, is often called the "Markowitz portfolio" or the "Mean-variance portfolio".

Is this problem convex?

Write a piece of code to solve the above problem for the following values of $\lambda = \{0, 10^{-4}, 2 \cdot 10^{-4}, 3 \cdot 10^{-4}, 4 \cdot 10^{-4}, 5 \cdot 10^{-4}, 6 \cdot 10^{-4}, 7 \cdot 10^{-4}, 8 \cdot 10^{-4}, 9 \cdot 10^{-4}, 10^{-3}\}$, with the following values of Σ and $\boldsymbol{\mu}$:

$$\Sigma = \begin{bmatrix} 1.0 & 0.0015 & -0.02 \\ 0.0015 & 1.0 & -0.1 \\ -0.02 & -0.1 & 1.0 \end{bmatrix}, \quad \boldsymbol{\mu} = [0.001, 0.05, 0.005]^\top \tag{4}$$

For each solution $\boldsymbol{w}^\star(\lambda)$, **compute** the following quantities

$$\mathsf{PortfolioExpectedReturn}(\lambda) = \boldsymbol{\mu}^\top \boldsymbol{w}^\star(\lambda) \tag{5}$$

$$\mathsf{PortfolioVolatility}(\lambda) = \sqrt{\boldsymbol{w}^{\star\top}(\lambda) \Sigma \boldsymbol{w}^\star(\lambda)} \tag{6}$$

Create a curve plot $\mathsf{PortfolioExpectedReturn} \times \mathsf{PortfolioVolatility}$.

Problem #3 (25) marks

Write code to solve the following constrained least-squares problem:

$$\begin{array}{ll} \underset{\boldsymbol{\beta}\in\mathbb{R}^n}{\text{minimize}} & \|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}\|_2^2 \\ \text{subject to} & \beta_i \leq \beta_{i+1}, \quad i = 1, 2, \ldots, n-1, \end{array} \tag{7}$$

generate $\boldsymbol{X} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{y} \in \mathbb{R}^m$ randomly for resonable values of m and n. Show the value of the objective function at the global optimum $\boldsymbol{\beta}^\star$ and plot $\boldsymbol{\beta}^\star$.

Problem # 4 (25) marks

Gaussian graphical model: Write code to solve the following convex optimization problem, which represents the Maximum Likelihood Estimator of the precision matrix of a Gaussian graphical model:

$$\underset{\boldsymbol{\Theta}\succ\boldsymbol{0}}{\text{minimize}} \quad \text{tr}(\boldsymbol{S}\boldsymbol{\Theta}) - \log\det(\boldsymbol{\Theta}\boldsymbol{S}) + \alpha\|\boldsymbol{\Theta}\|_1, \tag{8}$$

where $\boldsymbol{\Theta}$ is a positive definite matrix variable of size $p \times p$ and $\boldsymbol{S} = \frac{1}{n}\boldsymbol{X}^\top\boldsymbol{X}$ is the sample covariance matrix. Generate $\boldsymbol{X} \in \mathbb{R}^{n\times p}$, $n > p$, randomly such that each column of $\boldsymbol{X}$ is Gaussian distributed with zero mean-vector and as identity matrix as the covariance matrix. $\|\boldsymbol{\Theta}\|_1 = \sum_{ij} |\boldsymbol{\Theta}_{ij}|$.

Plot the solution $\boldsymbol{\Theta}^\star(\alpha)$ for five different values of α including $\alpha = 0$.

Plot the quantity $\|\boldsymbol{\Theta}^\star(\alpha)\|_1$ as a function of α.