ELEC 5470 HW 1

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Problem 1

(1) $f(x) = \log(\sum_{i=1}^{n} e^{x_i})$

Ans: $\nabla f(x) = \frac{1}{\sum_{i=1}^{n} e^{x_i}} [e^{x_1}, e^{x_2}, \cdots, e^{x_n}]$

$$\nabla^2 f(x) = \frac{1}{(\sum_{i=1}^{n} e^{x_i})^2} \begin{bmatrix} e^{x_1}(e^{x_2}+e^{x_3}\cdots e^{x_n}) & -e^{x_1+x_2} & \cdots & -e^{x_1+x_n} \\ -e^{x_1+x_2} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -e^{x_1+x_n} & \cdots & \cdots & e^{x_n}(e^{x_1}+\cdots+e^{x_{n-1}}) \end{bmatrix}$$

As $\nabla^2 f(x)$ is symmetric matrix, all eigenvalues are positive, $\therefore \nabla^2 f(x) \geq 0$

Then $f(x)$ is convex function

(2) $f(x) = \|x\|^p \quad p \geq 1$

Ans: for $v, w \in \mathbb{R}^n$, $\theta \in [0,1]$. By triangle inequality

$\|\theta v + (1-\theta) w\|^p \leq \|\theta v\|^p + \|(1-\theta) w\|^p = \theta \|v\|^p$

which implies $f(\theta v + (1-\theta) w) \leq \theta f(v) + (1-\theta) f(w)$

$\therefore f(x) = \|x\|^p$ is convex.

(3) $f(x) = \frac{1}{g(x)}$

Ans: $f(x) = h(-g(x))$ where $h(t) = -\frac{1}{t}$ for $t < 0$

h is convex and monotonically increasing for $\{t \mid t<0\}$

while $g(x)$ is concave

Thus $f(x) = \frac{1}{g(x)}$ is convex

(4) $f(x) = \alpha g(x) + \beta$, $g(x)$ is convex

Ans $h(t) = \alpha t + \beta$ is convex and monotonically nondecreasing with $\alpha \geq 0$

and $g(x)$ is convex function

$\therefore$ $f(x) = h(g(x)) = \alpha g(x) + \beta$ is convex

(5) $f(x) = \exp(\beta x^T A x^T)$

Ans: Denote $g(x) = \beta x^T A x^T$, $\nabla^2 g(x) = \beta(A + A^T) \geq 0$

for $A \geq 0$ and $\beta > 0$ $\therefore$ $g(x)$ is convex

$\because$ $h(x) = e^x$ is convex and nondecreasing

$\therefore$ $f(x) = h(g(x))$ is convex

(b) $f(x) = g(Ax + b)$

Ans for $v, w \in R^n$, $\theta \in [0,1]$

$$f(\theta v + (1-\theta) w) = g(\theta Ax + (1-\theta)Ay + b)$$
$$= g(\theta(Av+b) + (1-\theta)(Aw+b))$$
$$\le \theta g(Av+b) + (1-\theta) g(Aw+b)$$
$$= \theta f(v) + (1-\theta) f(w)$$

$\therefore$ $f(x)$ is convex

(7) $f(x) = \sum_{i=1}^{n-1} |x_{i+1} - x_i|$

Ans ① $\forall$ $v \in R^n$

$$f(v) = \sum_{i=1}^{n-1} |v_{i+1} - v_i| \ge 0$$

② $\forall$ $v \in R^n$, $\lambda \in R$

$$f(\lambda v) = \sum_{i=1}^{n-1} |\lambda (v_{i+1} - v_i)| = |\lambda| \sum_{i=1}^{n-1} |x_{i+1} - x_i|$$
$$= |\lambda| f(\lambda)$$

③ $\forall$ $v, w \in R^n$

$$f(v+w) = \sum_{i=1}^{n-1} | v_{i+1} + w_{i+1} - v_i - w_i |$$
$$\le \sum_{i=1}^{n-1} | v_{i+1} - v_i | + \sum_{i=1}^{n-1} |w_{i+1} - w_i| = f(v) + f(w)$$

$\therefore$ $f(x)$ is norm, and it is convex

Problem 2.

(i) vector space: a set that is closed under finite vectors' linear combination.

(ii) Linear subspace of a vector space:
a vector space that is a subset of larger vector space.

(iii) Convex set:
a subset that intersect every line into a single linear segment

(iv) null space:
a set of vectors in the domain of mapping to be mapped to the zero vector

(v) column space:
a set of column vectors' linear combination

(vi) row space:
a set that is closed under row vectors' linear combination

(vii) Proof: for $A \subset B$, where B is a vector space.

$\therefore$ A is closed under linear combination

$\therefore$ $\forall\ v, w \in A$, $\theta v + (1-\theta) w \in A$

$\therefore$ A is convex set

Problem 3

(a)

Ans: $h(x) = g(f(x))$

$\forall\ v, w \in C\ ,\ \theta \in [0,1]$

$$h(\theta v + (1-\theta) w) = g(f(\theta v + (1-\theta) w))$$
$$\leq g(\theta f(v) + (1-\theta) f(w)) \quad \text{---} \ ①$$
$$\leq \theta g(f(v)) + (1-\theta) g(f(w))$$
$$= \theta h(v) + (1-\theta) h(w)$$

$\therefore$ $h(x)$ is convex over C

if g is monotonically increasing and f is strictly convex

the first inequality is strict

$\therefore$ $h(x)$ is strict convex over C

(b) let $v, w \in R^n$, $\theta \in [0,1]$

$$h(\theta v + (1-\theta) w) = g(f(\theta v + (1-\theta) w))$$
$$= g(f_1(\theta v + (1-\theta) w), \cdots, f_m(\theta v + (1-\theta) w))$$
$$\leq g(\theta f_1(v) + (1-\theta) f(w), \cdots, \theta f_m(v) + (1-\theta) f_m(w))$$
$$= g(\theta f(v) + (1-\theta) f(w))$$
$$\leq \theta g(f(v)) + (1-\theta) g(f(w))$$

$= \alpha h(x) + (1-\alpha) h(x)$

$\therefore h(x)$ is convex

Problem 4

Ans: By definition of convexity

$\forall\ v, w \in R^n,\ \theta \in [0,1]$

$f(\theta v + (1-\theta) w) \le \theta f(v) + (1-\theta) f(w)$

for $x_1 < x_2 < x_3$

Denote $x_2 = \frac{x_3 - x_2}{x_3 - x_1} x_1 + \frac{x_2 - x_1}{x_3 - x_1} x_3$

$\therefore f(x_2) = f(\frac{x_3 - x_2}{x_3 - x_1} x_1 + \frac{x_2 - x_1}{x_3 - x_1} x_3)$

$\le \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3)$

And $f(x_2) = \frac{x_3 - x_2}{x_3 - x_1} f(x_2) + \frac{x_2 - x_1}{x_3 - x_1} f(x_2)$

$\therefore (x_3 - x_2) f(x_2) + (x_2 - x_1) f(x_2) \le (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3)$

$\therefore$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

Problem 5

① Ans: denote $\Sigma = [\Sigma_1, \Sigma_2, \cdots, \Sigma_p]^T$

$w = [w_1, w_2, \cdots, w_p]^T$

$\because w_i (\Sigma w)_i = w_i \Sigma_i w$

$b_i (w^T \Sigma w) = b_i (w^T \Sigma w)$

$\therefore$ The constraint of $w_i (\Sigma w)_i = b_i w^T \Sigma w$

is equal to $w_i \Sigma_i = b_i (w^T \Sigma)$

$\Leftrightarrow w_i \Sigma_i = b_i (w_1 \Sigma_1 + w_2 \Sigma_2 + \cdots + w_i \Sigma_i + \cdots + w_p \Sigma_p)$

which is not a convex set

$\therefore$ This problem is not convex problem

③ Ans As $-\log w$ is convex, $b \geq 0$

$\therefore h(w)$ is convex

And $w^T \Sigma w^T \leq 1$ is a convex set

$\therefore$ This is a convex problem

Bonus Problem

Ans: The argument is true, which can be proved by contradiction,

Denote the intersection of All convex sets that contain S as M

By definition of intersection.

$S \in M$

if $\exists\, m \in M$ but $m \notin S$

then S is not smallest convex set that contains S, which is a contradiction to the definition