ELEC Convex Optimization - Homework #1

Professor Daniel P. Palomar
TAs: Vinícius and Rui
Department of Electronic & Computer Engineering (ECE)
Department of Industrial Engineering & Decision Analytics (IEDA)
The Hong Kong University of Science and Technology
Clear Water Bay, Hong Kong, SAR China

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All your answers must be appropriately justified.
Due date: **Sept. 21st 2020**. Please, submit a pdf file through canvas.
In case you do not have access to canvas, submit your solutions, as a pdf file, to `jvdmc@connect.ust.hk`.
Late submissions, or submissions that are not pdf files won't be considered.
Total marks: 100 + 20 (bonus).

Problem #1 (20 marks)

Assume $\boldsymbol{x} \in \mathbb{R}^n$, unless otherwise noted. Show whether the following functions are convex.

(a) $f(\boldsymbol{x}) = \log\left(\sum_{i=1}^{n} e^{x_i}\right)$.

(b) $f(\boldsymbol{x}) = \|\boldsymbol{x}\|^p, p \geq 1$.

(c) $f(\boldsymbol{x}) = \dfrac{1}{g(\boldsymbol{x})}$, where g is concave and $g(\boldsymbol{x}) > 0\ \forall\ \boldsymbol{x} \in \mathbb{R}^n$.

(d) $f(\boldsymbol{x}) = \alpha g(\boldsymbol{x}) + \beta$, where $g : \mathbb{R}^n \mapsto \mathbb{R}$ is a convex function, and α and β are scalars such that $\alpha \geq 0$.

(e) $f(\boldsymbol{x}) = \exp\left(\beta \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x}\right)$ where $\boldsymbol{A}$ is a positive semidefinite symmetric $n \times n$ matrix and β is a positive scalar.

(f) $f(\boldsymbol{x}) = g(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b})$, where $g : \mathbb{R}^m \mapsto \mathbb{R}$ is a convex function, $\boldsymbol{A}$ is an $m \times n$ matrix, and $\boldsymbol{b}$ is a vector in $\mathbb{R}^m$.

(g) $f(\boldsymbol{x}) = \sum_{i=1}^{n-1} |x_{i+1} - x_i|$. Is this function a norm?

Problem #2 (20) marks

Explain on your own words what are: (i) vector spaces, (ii) linear subspaces of a vector space, (iii) convex sets, (iv) null space, (v) column space, and (vi) row space. Next, prove whether linear subspaces of a vector space are convex sets.

Problem #3 (20) marks

Let C be a nonempty convex subset of $\mathbb{R}^n$.

(a) Let $f : C \mapsto \mathbb{R}$ be a convex function and $g : \mathbb{R} \mapsto \mathbb{R}$ be a function that is convex and monotonically nondecreasing over a convex set that contains the set of values that f can take, i.e., $\{f(x) : x \in C\}$. Show that the function h defined by $h(x) = g(f(x))$ is convex over C. In addition, show that if g is monotonically increasing and f is strictly convex, then h is strictly convex.

(b) Let $f = (f_1, \ldots, f_m)$, where each $f_i : C \mapsto \mathbb{R}$ is a convex function, and let $g : \mathbb{R}^m \mapsto \mathbb{R}$ be a function that is convex and monotonically nondecreasing over a convex set that contains the set $\{f(x) : x \in C\}$, in the sense that for all u, v in this set such that $u \leq v$, we have $g(u) \leq g(v)$. Show that the function h defined by $h(x) = g(f(x))$ is convex over $C \times \cdots \times C$.

Hint: use the definition of convex function.

Problem #4 (20) marks

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a convex function. Use the definition of convexity to show that f is "turning upwards" in the sense that if x_1, x_2, and x_3 are three scalars such that $x_1 < x_2 < x_3$, then

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2} \tag{1}$$

Problem #5 (20) marks

Suppose we are given a $p \times p$ symmetric, positive-definite matrix Σ, and a vector $\boldsymbol{b} \in \mathbb{R}^p_{++}$, such that $\mathbf{1}^\top \boldsymbol{b} = 1$, where $\mathbf{1} = (1, 1, ..., 1)^\top$.

Consider the risk parity feasibility problem

$$\begin{array}{rl} \text{find} & \boldsymbol{w} \\ \text{subject to} & \boldsymbol{w} > 0,\ w_i(\Sigma \boldsymbol{w})_i = b_i \left(\boldsymbol{w}^\top \Sigma \boldsymbol{w}\right), i = 1, 2, ..., p. \end{array} \tag{2}$$

Is the above problem, as given, convex?

Now, define the function $h : \mathbb{R}^p_{++} \to \mathbb{R}$ given by $h(\boldsymbol{w}) = -\sum_{i=1}^{p} b_i \log(w_i)$ and consider the problem

$$\begin{array}{ll} \underset{\boldsymbol{w}>0}{\text{minimize}} & h(\boldsymbol{w}), \\ \text{subject to} & \boldsymbol{w}^\top \Sigma \boldsymbol{w} \leq 1. \end{array} \tag{3}$$

Is the above problem convex?

Bonus Problem (20) marks

The convex hull of a finite set S is defined as the smallest convex set that contains S. Argue that the convex hull of a finite set S is the intersection of all convex sets that contain S.