Algorithm Design and Analysis

Computational Geometry (Incremental Algorithms)

Goals for today

- Apply **randomized incremental algorithms** to geometry
- Give randomized incremental algorithms for two key problems:
  - The **closest pair** problem
  - The **smallest enclosing circle** problem

Model and assumptions

- Points are real-valued pairs (x, y)
- Arithmetic on reals is $O(1)$ again
- We can take the floor function of a real in $O(1)$ time
- Hashing is $O(1)$ time in expectation (see universal hashing)

Closest Pair

The closest pair problem

Problem (closest pair): Given n points P, define $CP(P)$ to be the closest distance, i.e.

$$CP(P) = \min_{p,q \in P} ||p - q||$$

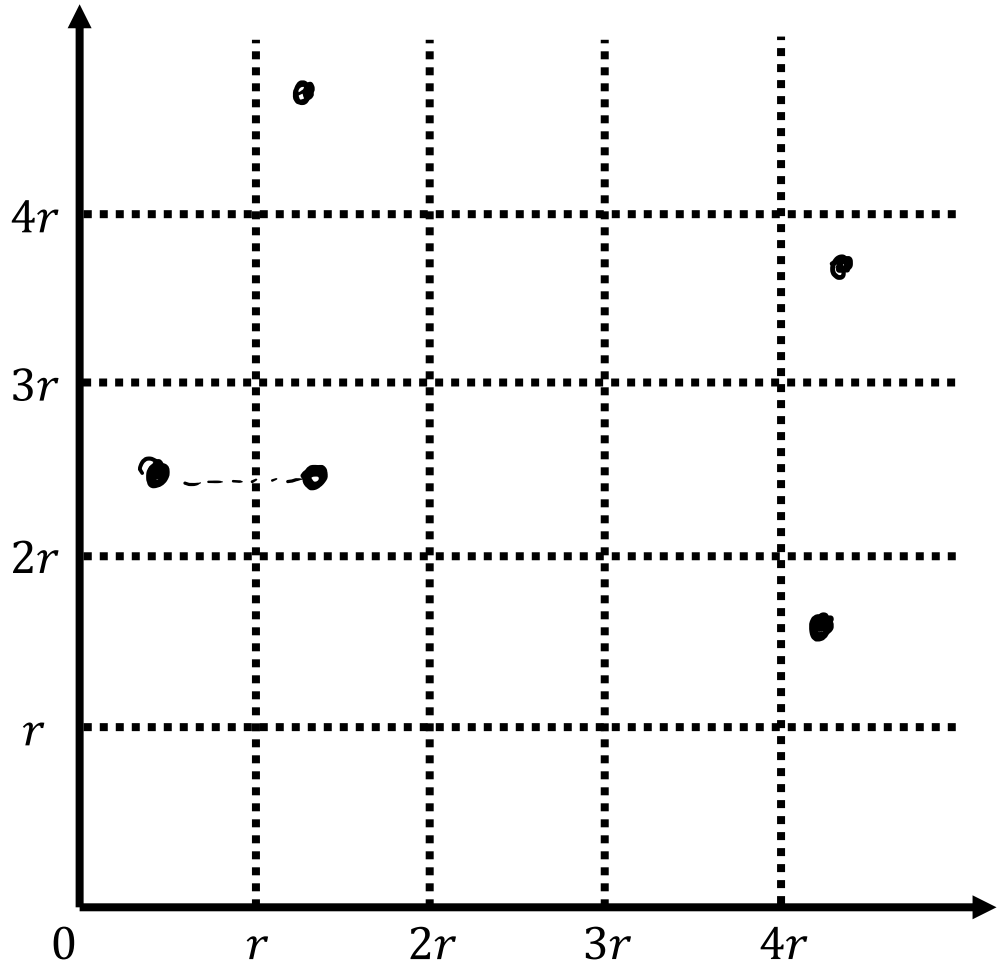
Goal is to compute $CP(P)$

Brute force: $O(n^2)$

A grid data structure

Let’s define a grid with size r

$$(x, y) \rightarrow \left(\left\lfloor \frac{x}{r} \right\rfloor, \left\lfloor \frac{y}{r} \right\rfloor \right)$$



How does this help?

- If the grid size is sufficiently large, closest pair will be in same cell, or in neighboring cells
- If the grid size is too large, there will be too many points per cell…

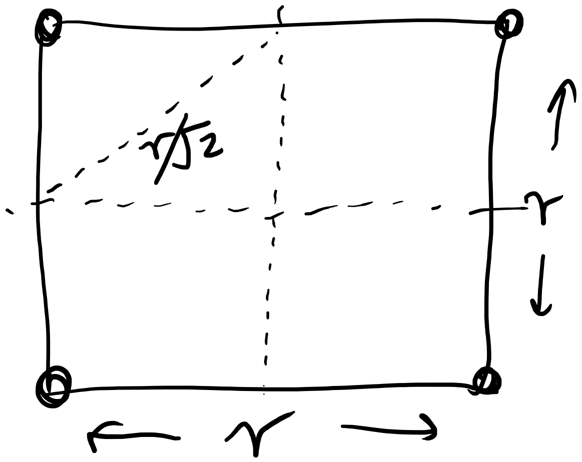
Goal: Choose the right grid size.

- Want few points per cell, so that looking in a cell is fast
- Want the closest pair to be in neighboring cells so we find them fast

The right grid size

Claim (the right grid size): Given a grid with points P and grid size $r = CP(P)$, no cell contains more than four points

Proof:



An incremental approach

Key idea (incremental): Add the points one at a time

- Check neighboring cells to see if there's a new closest pair
- If so, rebuild the grid with the new size
- Otherwise keep going

A grid data structure

Invariant (grid size): Given a grid containing a set of points P, we want the grid size r to always equal $CP(P)$

- MakeGrid(p, q): Make a grid containing p and q, with $r = ||p - q||$
- Lookup(G, p): Given a grid G and point p (not currently in the grid), we want to know whether p is part of a new closest pair
- Insert(G, p): Given a grid G and point p, inserts p and returns the grid size (which may have changed because of p)

Implementing the grid

Issue: The number of grid cells could be unbounded…

hashtable (dictionary)

key = (i, j) grid cell $\longrightarrow$ list of points

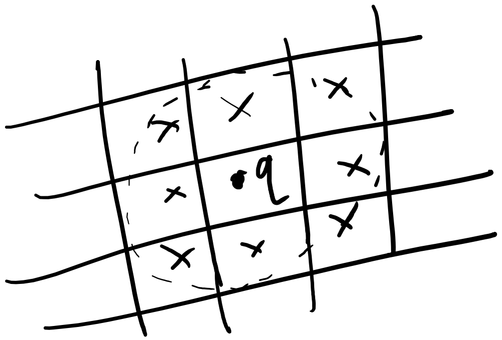
Implementing the grid

Implement MakeGrid(p, q):

$r = \|p-q\|$ insert p & q into grid

Implementing the grid

Implement Lookup(G, q):



Search at most 36 pts

$O(1)$ time

If $\|q-p\| < r$ for any p

return $\|q-p\|$

return NO

Implementing the grid

Implement Insert(G, q): First Lookup(G, q)

If distance doesn't change, insert into G, $O(1)$

Else

Make new grid, r = new distance

i pts so far, $O(i)$

Runtime

Claim (runtime): The worst-case runtime of the incremental grid algorithm is $O(n^2)$

Proof:

$$T = \sum_{i=1}^{n} i = O(n^2)$$

Randomization to the rescue!!!

Randomized runtime

Claim (randomized incremental is fast): Randomly shuffle the points, then run the incremental algorithm, it takes $O(n)$ time in expectation

Proof: $P_i = \langle p_{\pi_1}, p_{\pi_2}, \ldots, p_{\pi_i} \rangle$

$$X_i = \begin{cases} 1 & \text{if } CP(P_i) \neq CP(P_{i-1}) \\ 0 & \text{otherwise} \end{cases}$$

$$T = \sum_{i=2}^{n} (1 + X_i \cdot i)$$

$\Pr[X_i = 1]$

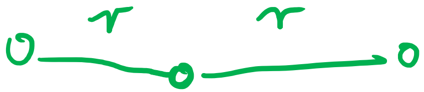
$$\mathbb{E}[T] = n + \sum_{i=2}^{n} \mathbb{E}[X_i] \cdot i$$

Randomized runtime (continued)

We need to bound $\Pr[X_i = 1]$... *(i.e.,* $\Pr[CP(P_i) \neq CP(P_{i-1})]$*)*

points P_i. q is "critical" if $CP(P_i \setminus \{q\}) \neq CP(P_i)$

- No critical pts $\rightarrow$ $\Pr = 0$
- If one critical pt $\rightarrow$ $\Pr = 1/i$



- If two critical pt $\rightarrow$ $\Pr = 2/i$

- Three critical pts? $\times$

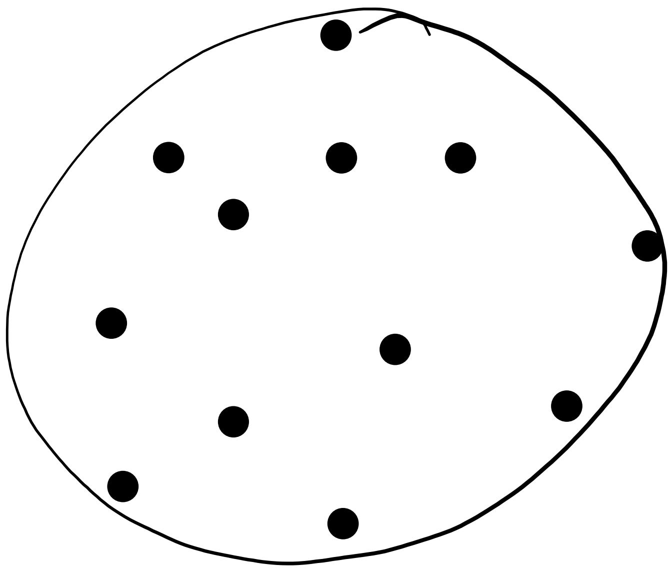
$\Pr[X_i = 1] \leq 2/i$

$\square$

Smallest enclosing circle

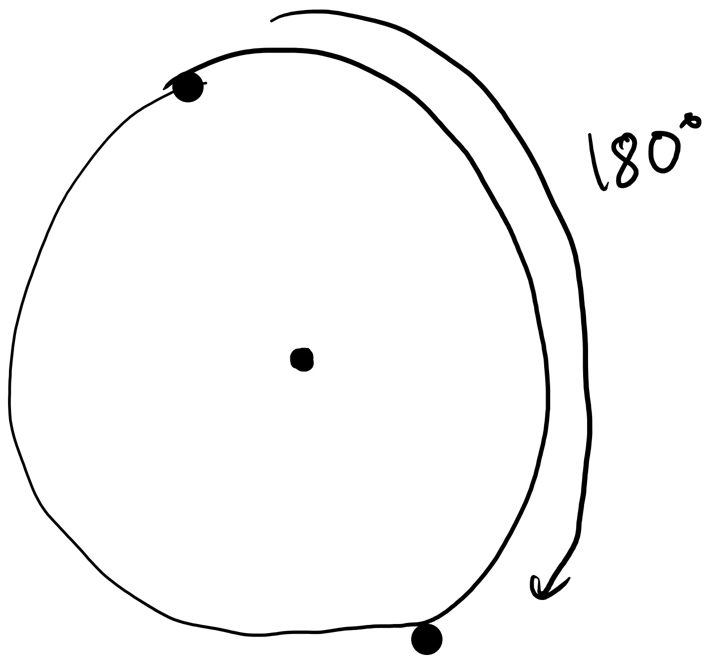
The smallest enclosing circle

Problem (Smallest enclosing circle): Given $n \geq 2$ points in two dimensions, find the smallest circle that contains all of them



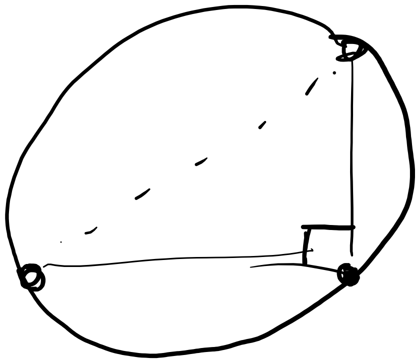
Base cases

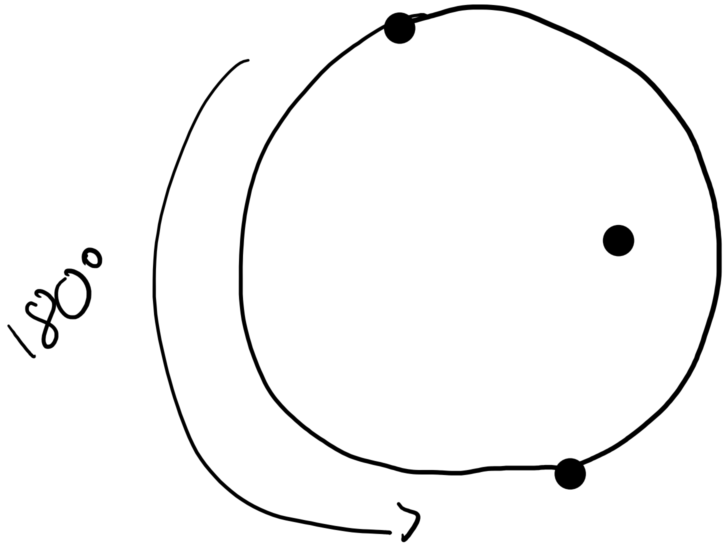
Base case (two points):



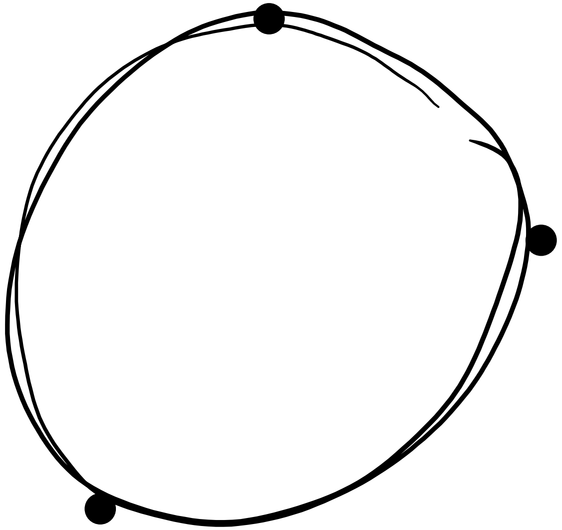
Base cases

Base case (three points):





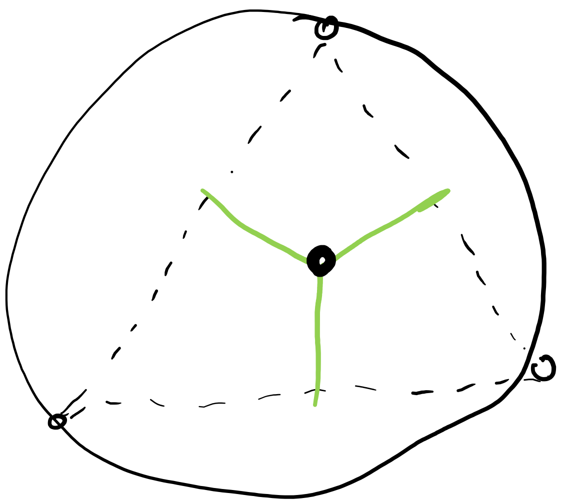
Case 1: Obtuse angle

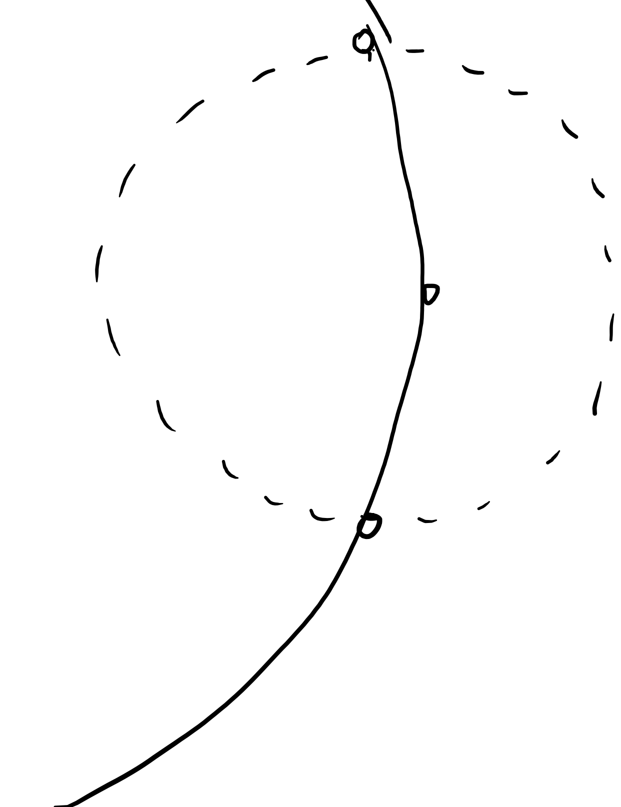


Case 2: Acute angle

Three points and a circle

Fact (unique circle): Given three non-colinear points, there is a unique circle that goes through them





The general case

Given $n > 3$ points, how many circles do we need to consider?

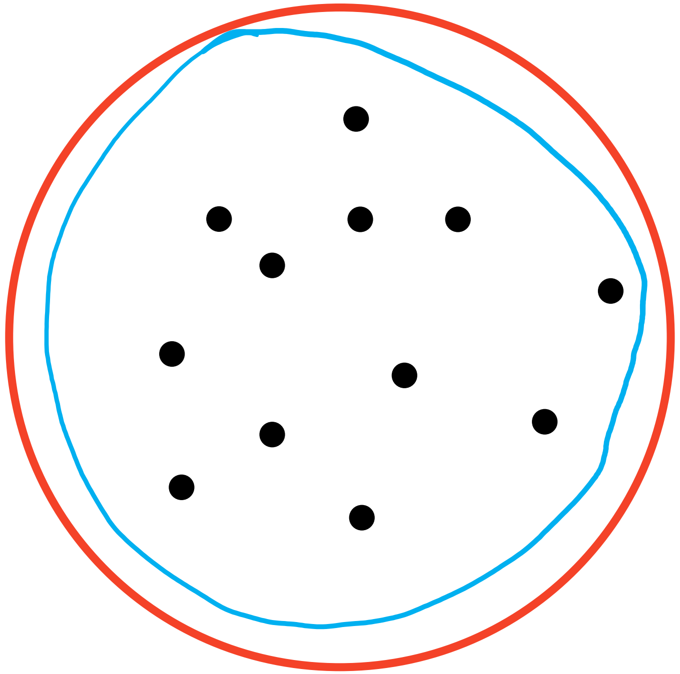
Theorem (three points is always enough): For any set of points, the smallest enclosing circle either touches two points p_i, p_j at a diameter, or touches three points p_i, p_j, p_j forming an ***acute*** triangle

In other words: For any set of points, there exists i, j, k, such that

$$SEC(p_1, \dots, p_n) = SEC(p_i, p_j, p_k)$$

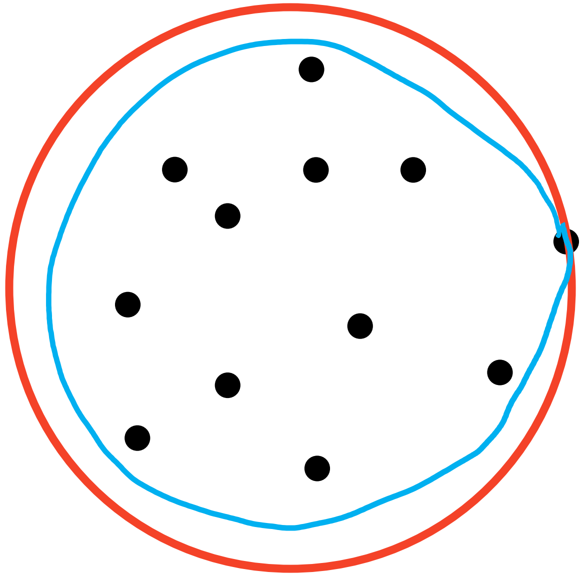
Proof of theorem

Case 1 (no points):



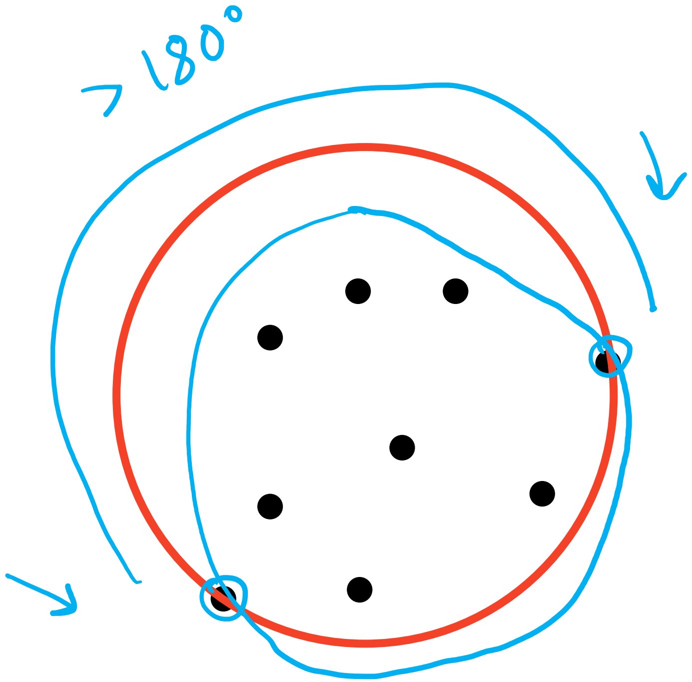
Proof of theorem

Case 2 (one point):



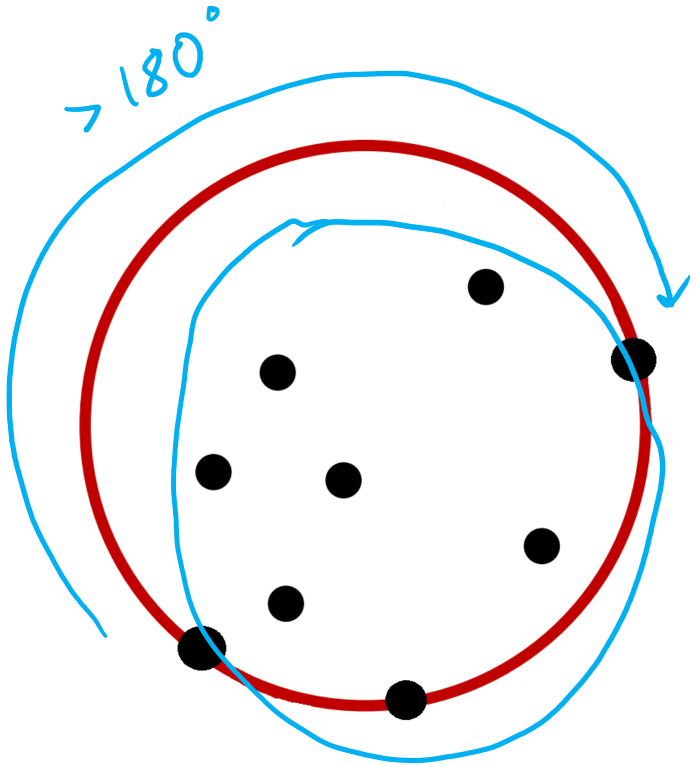
Proof of theorem

Case 3 (two points, not on a diameter):



Proof of theorem

Case 4 (three points, no acute angle):



We just proved

Theorem: For any set of points, there exists i, j, k, such that

$$SEC(p_1, \dots, p_n) = SEC(p_i, p_j, p_k)$$

- Either two points at a diameter, or
- Three points forming an acute triangle

Brute force algorithms

Algorithm 1 (brute force): Try all triples of points and find their smallest enclosing circle. Check whether this circle contains every point. Returns the smallest such circle.

$O(n^4)$

Algorithm 2 (better brute force): Try all triples of points and find their smallest enclosing circle. Return the **largest** such circle.

$O(n^3)$

Beating brute force: incremental

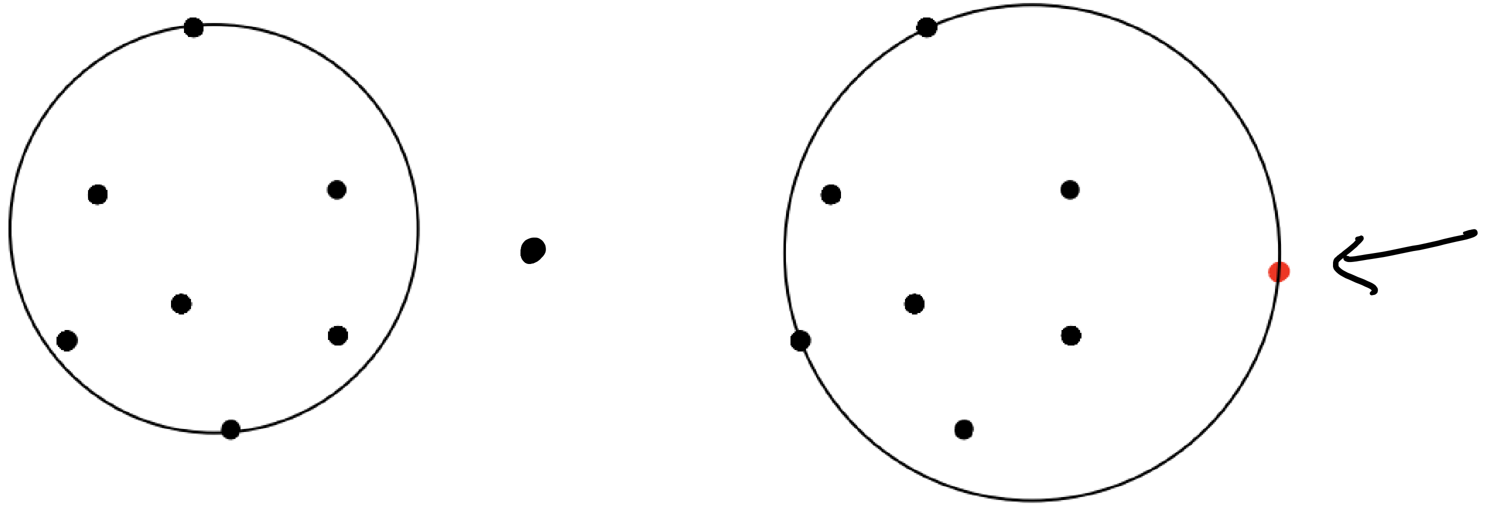
Incremental approach: Insert points one by one and maintain the smallest enclosing circle

When inserting p_i:

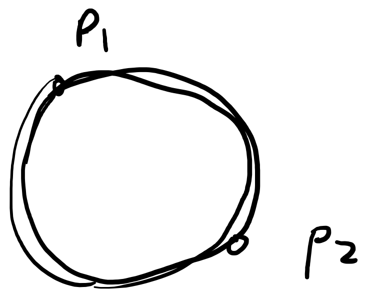
- ***Case 1:*** p_i is inside the current circle. Great, do nothing!
- ***Case 2:*** p_i is outside the current circle. Need to find the new one

Making incremental fast

Observation: When we add p_i, if it is not in the current circle, then it is on the boundary of the new circle



Incremental algorithm



SEC($[p_1, p_2, \ldots, p_n]$) = {

Let C be the smallest circle enclosing p_1 and p_2

for *i = 3 to n do* {

if p_i is not inside C **then** C = SEC1($[p_1, p_2, \ldots, p_{i-1}]$, p_i)

Lock in p_i

}

return C

}

Incremental algorithm continued

Locked in

SEC1($[p_1, p_2, \dots, p_k], q$) = {

Let C be the smallest circle enclosing p_1 and q

for *i = 2 to k do* {

if p_i is not inside C **then** $C =$ SEC2($[p_1, p_2, \dots, p_{i-1}], p_i, q$)

locked in

}

return C

}

Incremental algorithm deeper again

locked in

SEC2($[p_1, p_2, \ldots, p_k], \overbrace{q_1, q_2}$) = {

Let C be the smallest circle enclosing q_1 and q_2

for *i = 1 to k do* {

if p_i is not inside C **then** $C =$ SEC of (q_1, q_2, p_i)

}

return C

}

Runtime

Runtime (SEC2): SEC2 runs in $\underbrace{O(k)}$ time

Runtime (SEC1): In the worst case, SEC1 runs in $O(k^2)$ time

Runtime (SEC): In the worst case, SEC runs in $O(n^3)$ time

$$\sum_{i=1}^{n} i^2 = O(n^3)$$

Randomization to the rescue!!!

Claim (randomized SEC is fast): If we randomly shuffle the points in SEC and SEC1, then SEC1 runs in $O(k)$ expected time and SEC runs in $O(n)$ expected time

$$\mathbb{P}r[\text{point } i \text{ changes answer}] \leq \frac{3}{i}$$

$\rightarrow$ Same math as before

Summary

- **Randomized incremental algorithms** are pretty great. We can turn slow brute force algorithms into expected linear-time algorithms!
- We got $O(n)$ time for **closest pair** and **smallest enclosing circle**