Hashing: Universal and Perfect Hashing

Hashing is a great practical tool, with an interesting and subtle theory too. In addition to its use as a dictionary data structure, hashing also comes up in many different areas, including cryptography and complexity theory. In this lecture we describe two important notions: *universal hashing* and *perfect hashing*.

Objectives of this lecture

In this lecture, we want to:

- Understand the formal definition and general idea of hashing
- Define and analyze *universal hashing* and its properties
- Analyze an algorithm for *perfect hashing*

Recommended study resources

- CLRS, *Introduction to Algorithms*, Chapter 11, Hash Tables
- DPV, *Algorithms*, Chapter 1.5, Universal Hashing

1 The word-RAM model of computation

In the previous lectures we've been focusing on the *comparison model* for analyzing our algorithms. However, as we've also seen, the comparison model exhibits many lower bounds proving that several important and fundamental problems have a hard limit on how efficiently they can be solved. To beat these lower bounds, we need to think outside the realm of the comparison model and make more assumptions about our inputs. Doing so will often lead us to faster algorithms in practice for a wide range of cases. Most often we will assume that we are dealing with algorithms with integer inputs.

We need to formalize exactly what operations we are going to permit and how much they will cost. When dealing with algorithms over integers, the most common model employed in theory is the *word RAM* model.

Definition: Word RAM model

In the word RAM model:

- We have unlimited constant-time addressable memory (called “registers”),
- Each register can store a w-bit integer (called a “word”),
- Reading/writing, arithmetic, logic, bitwise operations on a constant number of words takes constant time,
- With input size n, we need $w \geq \log n$.

The final assumption is needed because if our input contains n words, then we are surely going to need to be able to write down the integer n to index the input, and that requires $\log n$ bits.

This model is essentially just a more formal version of what you are probably used to from your previous classes when you analyzed an algorithm by counting "instructions". The only subtle part is the restriction on the word size w and the assumption that only operations on w-bit integers take constant time. Most of the time this is of no consequence, but there are some situations where it matters. Consider for example an algorithm that takes n integers as inputs each of which is written with w bits, and computes their product. Their product is an integer containing nw bits, which requires n registers to store! Computing this product would therefore take much more than $\Theta(n)$ time since multiplying a super-constant number of integers can not be done in a single instruction and would instead require an algorithm for multiplying large integers[1].

An alternative model is the unit-cost RAM model which does not place any restriction on the size of the integers. This might seem like an unimportant difference, but it turns out that this assumption allows you to implement some wild and crazy algorithms, such as being able to sort n integers in constant time![2] The w-bit assumption limits us to algorithms that are more likely to be realistic and work on a real computer, especially since real computers have exactly this requirement – the vast majority modern CPUs have registers that store 64-bit integers. In other words, a real computer can not multiply a pair of billion digit integers in one instruction, so we should not assume our algorithms can either!

2 The Dictionary problem

One of the main motivations behind the study and hashing and hash functions is the *Dictionary* problem. A dictionary D stores a set of *items*, each of which has an associated *key*. From an algorithmic point of view, items themselves are not typically important, they can be thought of as just data associated with a key, which is the important part for us as algorithm designers. The operations we want to support with a dictionary are:

[1] We might see an algorithm for this problem later in the course, but it is rather complicated, and certainly not constant time! It takes at least linear time in the number words needed to represent the integers.

[2] If you’re curious about the algorithm, its actually very cool, you can see Appendix A of *Computing with arbitrary and random numbers*, Michael Brand’s PhD thesis from Monash University.

Definition: Dictionary data type

A dictionary supports:

- `insert(item)`: add the given item (associated with its key)
- `lookup(key)`: return the item with the given key (if it exists)
- `delete(key)`: delete the item with the given key

In some cases, we don't care about adding and removing keys, we just care about fast query times—e.g., if we were storing a literal dictionary, the actual English dictionary does not change (or changes very gradually). This is called the *static case*. Another special case is when we just add keys: the *incremental case*. The general case is called the *dynamic case*.

For the static problem we could use a sorted array with binary search for lookups. For the dynamic we could use a balanced search tree. However, *hashtables* are an alternative approach that is often the fastest and most convenient way to solve these problems. You should hopefully already be familiar with the main ideas of hashtables from your previous studies.

3 Hashing and hashtables

To design and analyze hashing and hashing-based algorithms, we need to formalize the setting that we will work in.

The key space (the universe): The *keys* are assumed to come from some large *universe* U. Most often, when analyzed on the word RAM model, we will assume that $U = 0, \ldots, u-1$, where $u = 2^w$ is the universe size, i.e., the keys are word-sized integers.

The hashtable: There is some set $S \subseteq U$ of keys that we are maintaining (which may be static or dynamic). Let $n = |S|$. Think of n as much smaller than the size of U. We will perform inserts and lookups by having an array A of some size m, and a **hash function** $h : U \to \{0, \ldots, m-1\}$. Given an element x, the idea of a hashtable is that we want to store it in $A[h(x)]$. Note that if U was small then you could just store x in $A[x]$ directly, no need for hashing!. The problem is that U is big: that is why we need the hash function.

Collisions: Recall that hashtables suffer from *collisions*, and we need a method for resolving them. A *collision* is when $h(x) = h(y)$ for two different keys x and y. For this lecture, we will assume that collisions are handled using the strategy of *separate chanining*, by having each entry in A be a linked list. There are a number of other methods, but for the issues we will be focusing on here, this is the cleanest. To insert an element, we just put it at the top of the list. If h is a good hash function, then our hope is that the lists will be small.

One great property of hashing is that all the dictionary operations are straightforward to implement. To perform a lookup of a key x, simply compute the index $i = h(x)$ and then walk down the list at $A[i]$ until you find it (or walk off the list). To insert, just place the new element at the top of its list. To delete, one simply has to perform a delete operation on the associated linked list. The question we now turn to is: what do we need for a hashing scheme to achieve good performance?

Desired properties: The main desired properties for a good hashing scheme are:

1. The keys are nicely spread out so that we do not have too many collisions, since collisions affect the time to perform lookups and deletes.
2. $m = O(n)$: in particular, we would like our scheme to achieve property (1) without needing the table size m to be much larger than the number of elements n.
3. The function h is fast to compute. In our analysis today we will be viewing the time to compute $h(x)$ as a constant. However, it is worth remembering in the back of our heads that h shouldn't be too complicated, because that affects the overall runtime.

Given this, the time to lookup an item x is $O(\text{length of list } A[h(x)])$. The same is true for deletes. Inserts take time $O(1)$ if we don't check for duplicates, or the same time again if we do. So, our main goal is to be able to analyze how big these lists get.

Prehashing non-integer keys: One issue that we sweep under the rug in theory but that matters a lot in practice is dealing with non-integer keys. Hashtables in the real world are frequently used with data such as strings, so we want this to be applicable.

The way that we get around this in theory is to require non-integer key types to come equipped with a *pre-hash* function, i.e., a function that converts the keys reasonably uniformly into integers in the universe U. Then we can proceed as normal assuming integer keys.

Basic intuition: One way to spread elements out nicely is to spread them *randomly*. Unfortunately, we can't just use a random number generator to decide where the next element goes because then we would never be able to find it again. So, we want h to be something "pseudo-random" in some formal sense.

We now present some bad news, and then some good news.

Claim: Bad news

For any hash function h, if $|U| \geq (n-1)m + 1$, there exists a set S of n elements that all hash to the same location.

Proof. By the pigeonhole principle. In particular, to consider the contrapositive, if every location had at most $n-1$ elements of U hashing to it, then U could have size at most $m(n-1)$. □

So, this is partly why hashing seems so mysterious — how can one claim hashing is good if for any hash function you can come up with ways of foiling it? One answer is that there are a lot of simple hash functions that work well in practice for typical sets S. But what if we want to have a good *worst-case* guarantee?

3.1 A Key Idea

Let's use randomization in our *construction* of h, in analogy to randomized quicksort. (The function h itself will be a deterministic function, of course). What we will show is that for *any* sequence of insert and lookup operations (we won't need to assume the set S of elements inserted is random), if we pick h in this probabilistic way, the performance of h on this sequence will be good in expectation. We can come up with different kinds of hashing schemes depending on what we mean by "good" in expectation. Essentially, the goal is to make the hash appear *as if it was a totally random function*, even though it isn't.

We will first develop the idea of *universal hashing*. Then, we will use it for an especially nice application called "perfect hashing".

4 Universal Hashing

Definition: Universal Hashing

A set of hash functions $\mathscr{H}$ where each $h \in \mathscr{H}$ maps $U \to \{0,\dots,m-1\}$ is called **universal** (or is called a *universal family*) if for all $x \neq y$ in U, we have

$$\Pr_{h\in\mathscr{H}}[h(x)=h(y)] \le 1/m. \tag{1}$$

Make sure you understand the definition! This condition must hold for *every pair* of distinct keys $x \neq y$, and the randomness is over the choice of the actual hash function h from the set $\mathscr{H}$. Here's an equivalent way of looking at this. First, count the number of hash functions in $\mathscr{H}$ that cause x and y to collide. This is

$$|\{h \in \mathscr{H} | h(x) = h(y)\}|.$$

Divide this number by $|H|$, the number of hash functions. This is the probability on the left hand side of (1). So, to show universality you want

$$\frac{|\{h \in \mathscr{H} \mid h(x) = h(y)\}|}{|H|} \le \frac{1}{m}$$

for every $x \neq y \in U$. Here are some examples to help you become comfortable with the definition.

Example

The following three hash families with hash functions mapping the set $\{a, b\}$ to $\{0,1\}$ are universal, because at most $1/m$ of the hash functions in them cause a and b to collide, were $m = |\{0,1\}|$.

	a	b
h_1	0	0
h_2	0	1

	a	b
h_1	0	1
h_2	1	0

	a	b
h_1	0	0
h_2	1	0
h_3	0	1

On the other hand, these next two hash families are not, since a and b collide with probability more than $1/m = 1/2$.

	a	b
h_1	0	0
h_3	1	1

	a	b	c
h_1	0	0	1
h_2	1	1	0
h_3	1	0	1

4.1 Using Universal Hashing

Theorem 1: Universal hashing

If $\mathcal{H}$ is universal, then for any set $S \subseteq U$ of size n, for any $x \in U$ (e.g., that we might want to lookup), if h is drawn randomly from $\mathcal{H}$, the **expected** number of collisions between x and other elements in S is less than n/m.

Proof. Each $y \in S$ ($y \neq x$) has at most a $1/m$ chance of colliding with x by the definition of universal. So,

- Let the random variable $C_{xy} = 1$ if x and y collide and 0 otherwise.
- Let C_x be the random variable denoting the total number of collisions for x. So,

$$C_x = \sum_{\substack{y \in S \\ y \neq x}} C_{xy}.$$

- We know $\mathbb{E}[C_{xy}] = \Pr(x \text{ and } y \text{ collide}) \leq 1/m$.
- So, by linearity of expectation,

$$\mathbb{E}[C_x] = \sum_{\substack{y \in S \\ y \neq x}} \mathbb{E}[C_{xy}] \leq \frac{|S|-1}{m} = \frac{n-1}{m},$$

  which is less than n/m.

$\square$

We now immediately get the following corollary.

Corollary

If $\mathcal{H}$ is universal then for any **sequence** of L insert, lookup, and delete operations in which there are at most m keys in the data structure at any one time, the expected total cost of the L operations for a random $h \in \mathcal{H}$ is only $O(L)$ (viewing the time to compute h as constant).

Proof. For any given operation in the sequence, its expected cost is constant by Theorem 1, so the expected total cost of the L operations is $O(L)$ by linearity of expectation. □

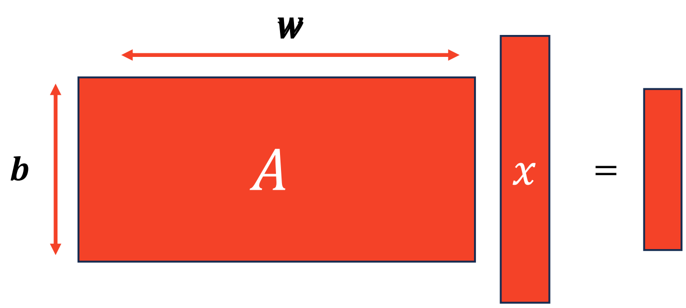
Can we actually construct a universal $\mathcal{H}$? If not, this this is all pretty vacuous. Luckily, the answer is yes.

4.2 Constructing a universal hash family: the matrix method

Definition: The matrix method for universal hashing

We assume that keys are w-bits long, so $U = 0, \ldots, 2^w - 1$. We require that the table size m is a power of 2, so an index is b-bits long with $m = 2^b$. We pick a random b-by-w 0/1 matrix A, and define $h(x) = Ax$, where we do addition mod 2. Here, x is interpreted as a 0/1 vector of length w, and $h(x)$ is a 0/1 vector of length b, denoting the bits of the result.

These matrices are short and fat. For instance:



Claim: The matrix method is universal

Let $\mathcal{H}$ be the hash family generated by the matrix method. For all $x \neq y$ from U, we have

$$\Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \frac{1}{2^b} = \frac{1}{m}$$

Proof. First of all, what does it mean to multiply A by x? We can think of it as adding some of the columns of A (doing vector addition mod 2) where the 1 bits in x indicate which ones to add. E.g., if $x = (1010\cdots)^\intercal$, Ax is the sum of the 1st and 3rd columns of A.

Now, take an arbitrary pair of keys x, y such that $x \neq y$. They must differ someplace, so say they differ in the ith coordinate and for concreteness say $x_i = 0$ and $y_i = 1$. Imagine we first choose all the entries of A but those in the ith column. Over the remaining choices of ith column, $h(x) = Ax$ is fixed, since $x_i = 0$ and so Ax does not depend on the ith column of A. However, each of the 2^b different settings of the ith column gives a different value of $h(y)$ (in particular, every time we flip a bit in that column, we flip the corresponding bit in $h(y)$). So there is exactly a $1/2^b$ chance that $h(x) = h(y)$.

More verbosely, let $y' = y$ but with the ith entry set to zero. So $Ay = Ay' +$ the ith column of A. Now Ay' is also fixed now since $y'_i = 0$. Now if we choose the entries of the ith column of A, we get $Ax = Ay$ exactly when the ith column of A equal $A(x - y')$, which has been fixed by the choices of all-but-the-ith-column. Each of the b random bits in this ith column must come out right, which happens with probability $(1/2)$ each. These are independent choices, so we get probability $(1/2)^b$. $\square$

Okay great, so its universal! But how efficient is it? Well, if we manually compute the matrix product, since it is an $b \times w$ matrix, this will take us $O(bw) = O(w \lg m)$ time, which is not amazing, since this is actually worse than using a binary search tree. However, this is assuming that we compute the result bit-by-bit. If we take advantage of the word RAM and use the fact that the key and rows are w-bit integers, we can compute each row-vector product in constant time with a single multiplication instruction and improve the performance to $O(\lg m)$ time, which is about the same as a balanced binary search tree since we assume $m = O(n)$.

5 More powerful hash families

Recall that our overarching goal with universal hashing was to produce a hash function that behaved *as if it was totally random*. We can try to be more specific about what we mean. In the case of universal hashing, if we took any two distinct keys x, y from our universe, and then hashed them using our hash function from a universal family, then the probability of collision was at most $1/m$, which is the probability that we would get if the hash function was totally random! We can therefore think of universal hashing as hashing that appears to behave totally randomly if all we care about is pairwise collisions.

In some cases (for some algorithms), though, this is not good enough. Although universal hashing looks good if all we care about are collisions, there are scenarios where universal hashes appear totally not random. Lets consider an example. Suppose we are maintaining a hash table of size $m = 2$, and an evil adversary would like to cause a collision by inserting just two items. If our hash was totally random, then the adversary would have a 50/50 chance of success just by pure chance. Suppose that we use the following universal family for our hash table.

	a	b	c
h_1	0	0	1
h_2	1	0	1

In this case, the evil adversary can just first insert a, and now we are in trouble. If a goes into slot 0, then the adversary knows we have h_1 and can hence select b to insert next, causing a guaranteed collision. Otherwise, if a goes into slot 1, then the adversary can select c and cause a guaranteed collision. So, even though we used a universal hash family, it wasn't as good as a totally random hash, because the adversary was able to figure out which hash function had been selected by just knowing the hash of one element. The problem at a high level was that although this family makes collisions unlikely, it doesn't do anything to prevent the hashes of different elements from correlating. In this family, the adversary can deduce the hash values of b and c by just knowing the hash of a.

To fix this, there is a closely-related concept called pairwise independence.

Definition: Pairwise independence

A hash family $\mathscr{H}$ is *pairwise independent* if for all pairs of distinct keys $x_1, x_2 \in U$ and every pair of values $v_1, v_2 \in \{0, \ldots, m-1\}$, we have

$$\Pr_{h \in \mathscr{H}}[h(x_1) = v_1 \text{ and } h(x_2) = v_2] = \frac{1}{m^2}$$

Intuitively, pairwise independence guarantees that if we only ever look at pairs of keys in our universe, then their hash values appear to behave totally randomly! In other words, if the adversary ever learns the hash value of one key, it can not deduce any information about the hash values of the other keys, they appear totally random. Of course, it is possible that by learning the hash values of *two* elements, the adversary may be able to deduce information about other elements. To improve this, we can generalize the definition of pairwise independence to arbitrary-size sets of keys.

Definition: k-wise independence

A hash family $\mathscr{H}$ is k*-wise independent* if for all k distinct keys $x_1, x_2, \ldots, x_k$ and every set of k values $v_1, v_2, \ldots, v_k \in \{0, \ldots, m-1\}$, we have

$$\Pr_{h \in \mathscr{H}}[h(x_1) = v_1 \text{ and } h(x_2) = v_2 \text{ and } \ldots \text{ and } h(x_k) = v_k] = \frac{1}{m^k}$$

Intuitively, if a hash family is k-wise independent, then the hash values of sets of k elements appear totally random, or, if an adversary learns the hash values of $k-1$ elements, it can not deduce any information about the hash values of any other elements.

6 Perfect Hashing

The next question we consider is: if we fix the set S (the dictionary), can we find a hash function h such that *all* lookups are constant-time? The answer is *yes*, and this leads to the topic of *perfect hashing*. We say a hash function is **perfect** for S if all lookups involve $O(1)$ deterministic work-case cost (though lookup must be deterministic, randomization is still needed to actually construct the hash function). Here are now two methods for constructing perfect hash functions for a given set S.

6.1 Method 1: an $O(n^2)$-space solution

Say we are willing to have a table whose size is quadratic in the size n of our dictionary S. Then, here is an easy method for constructing a perfect hash function. Let $\mathcal{H}$ be universal and $m = n^2$. Then just pick a random h from $\mathcal{H}$ and try it out! The claim is there is at least a 50% chance it will have no collisions.

Claim

If $\mathcal{H}$ is universal and $m = n^2$, then

$$\Pr_{h \in \mathcal{H}}(\text{no collisions in } S) \geq 1/2.$$

Proof. How many pairs (x, y) in S are there? **Answer:** $\binom{n}{2}$. For each pair, the chance they collide is $\leq 1/m$ by definition of universal. Therefore,

$$\Pr(\text{exists a collision}) \leq \frac{\binom{n}{2}}{m} = \frac{n(n-1)}{2m} \leq \frac{n^2}{2n^2} = \frac{1}{2}$$

□

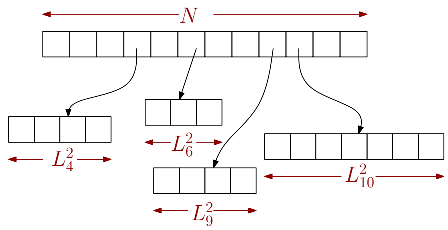
This is like the other side to the "birthday paradox". If the number of days is a lot *more* than the number of people squared, then there is a reasonable chance *no* pair has the same birthday.

So, we just try a random h from $\mathcal{H}$, and if we got any collisions, we just pick a new h. On average, we will only need to do this twice. Now, what if we want to use just $O(n)$ space?

6.2 Method 2: an $O(n)$-space solution

The question of whether one could achieve perfect hashing in $O(n)$ space was a big open question for some time, posed as "should tables be sorted?" That is, for a fixed set, can you get constant lookup time with only linear space? There was a series of more and more complicated attempts, until finally it was solved using the nice idea of universal hash functions in a 2-level scheme.

The method is as follows. We will first hash into a table of size n using universal hashing. This will produce some collisions (unless we are extraordinarily lucky). However, we will then rehash each bin using Method 1, squaring the size of the bin to get zero collisions. So, the way to think of this scheme is that we have a first-level hash function h and first-level table A, and then n second-level hash functions $h_1, \ldots, h_n$ and n second-level tables $A_1, \ldots, A_n$. To lookup an element x, we first compute $i = h(x)$ and then find the element in $A_i[h_i(x)]$. (If you were doing this in practice, you might set a flag so that you only do the second step if there actually were collisions at index i, and otherwise just put x itself into $A[i]$, but let's not worry about that here.)



Say hash function h hashes L_i elements of S to location i. We already argued (in analyzing Method 1) that we can find $h_1, \ldots, h_n$ so that the total space used in the secondary tables is $\sum_i (L_i)^2$. What remains is to show that we can find a first-level function h such that $\sum_i (L_i)^2 = O(n)$. In fact, we will show the following:

Theorem

If we pick the initial h from a universal family $\mathscr{H}$, then

$$\Pr\left[\sum_i (L_i)^2 > 4n\right] < \frac{1}{2}.$$

Proof. We will prove this by showing that $\mathbb{E}\left[\sum_i (L_i)^2\right] < 2n$. This implies what we want by Markov's inequality. (If there was even a 1/2 chance that the sum could be larger than $4n$ then that fact by itself would imply that the expectation had to be larger than $2n$. So, if the expectation is less than $2n$, the failure probability must be less than 1/2.)

Now, the neat trick is that one way to count this quantity is to count the number of ordered pairs that collide, including an element colliding with itself. E.g, if a bucket has `{d,e,f}`, then `d` collides with each of `{d,e,f}`, `e` collides with each of `{d,e,f}`, and `f` collides with each of `{d,e,f}`, so we get 9. So, we have:

$$\begin{aligned}
\mathbb{E}\left[\sum_i (L_i)^2\right] &= \mathbb{E}\left[\sum_x \sum_y C_{xy}\right] && (C_{xy} = 1 \text{ if } x \text{ and } y \text{ collide, else } C_{xy} = 0) \\
&= n + \sum_x \sum_{y \neq x} \mathbb{E}[C_{xy}] \\
&\leq n + \frac{n(n-1)}{m} && \text{(where the } 1/m \text{ comes from the definition of universal)} \\
&< 2n. && \text{(since } m = n\text{)}
\end{aligned}$$

□

So, we simply try random h from $\mathscr{H}$ until we find one such that $\sum_i L_i^2 < 4n$, and then fixing that function h we find the n secondary hash functions $h_1, \ldots, h_n$ as in Method 1.

Exercises: Hashing

Problem 1. Show that any pairwise independent (2-universal) hash family is also a universal hash family.

Problem 2. Show that the matrix method as defined above, which was universal, is **not** pairwise independent.