Algorithm Design and Analysis

Linear Programming (Part I)

Roadmap for today (and beyond)

- Definition of linear programming and examples
- Linear programs for flows
- Linear programs for two-player zero-sum games
- Overview of algorithms for linear programming
- **LP Part II** (Tuesday): Duality
- **LP Part III** (next Thursday): Simplex and integrality

Formal definition

Definition (Linear program): A linear program consists of

- n real-valued **variables** $x_1, x_2, \dots, x_n$
- A linear ***objective function***, e.g., minimize/maximize $2x_1 + 3x_2 + x_3$
- m linear **inequalities**, e.g., $3x_1 + 4x_2 \leq 6$, or $0 \leq x_1 \leq 3$

Goal: Find values for x's that satisfy the constraints and minimize/maximize the objective

Example: Plant production problem

Example Problem (Plant production): You run 4 production plants for making cars. Each is different and requires a certain amount of **labor** per car produced, **materials** per car produced, and **pollution** caused per car produced.

- Need to produce at least 400 cars at plant 3 (contract)
- We have 3300 hours of labor available
- We have 4000 units of materials available
- We are allowed to produce 12000 units of pollution
- Goal is to maximize the number of cars produced

	labor	materials	pollution
plant 1	2	3	15
plant 2	3	4	10
plant 3	4	5	9
plant 4	5	6	7

Variables: $x_1 \quad x_2 \quad x_3 \quad x_4$

x_i = #cars made at i

Objective: maximize $x_1 + x_2 + x_3 + x_4$

Constraints:

$x_i \geq 0 \quad \forall i$

$x_3 \geq 400$

$2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 3300$

$3x_1 + 4x_2 + 5x_3 + 6x_4 \leq 4000$

$15x_1 + 10x_2 + 9x_3 + 7x_4 \leq 12000$

Note: We are not guaranteed to get an integer solution! The optimal solution might be to produce 213.5 cars at a plant.

Important notes

- Linear programs ***can not*** contain strict inequalities, e.g., $x_1 < 3$
- Linear equalities are okay, e.g., $x_1 + x_2 = 3$
- **Super Important:** Linear programs ***can not*** guarantee integer-valued solutions! Variables can be assigned fractional values.
- Objective and inequalities ***must be linear***, e.g., can't have $x_1 x_2 \leq 3$
- Sometimes you don't need an objective function. Any solution that satisfies the constraints is good. This is called the "*feasibility problem*"

Classification of solutions

There are three possible outcomes:

- **Infeasible**: There is no solution that satisfies the constraints

  $x \leq 1 \qquad x \geq 2$

- **Feasible and bounded**: There is a solution of maximum objective value

  $\max \; x \qquad x \leq 1$

- **Feasible and unbounded**: There are solutions of arbitrarily large value

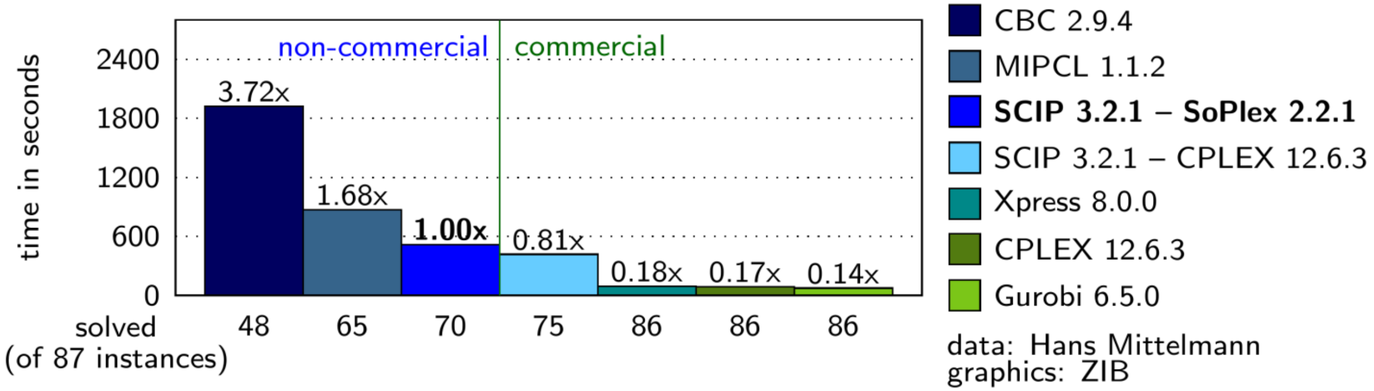
  $\max \qquad x \geq 1$

Linear programming in practice

- Linear programming is the basis of tremendously ***many real-world optimization problems***
- Business schools teach entire courses on linear programming, and have entire departments dedicated to optimization / OR.
- Linear programming underpins more general tools like ***integer programming***, ***constraint programming***

Linear programming in practice

- Commercial tools for linear/integer/constraint programming cost 10s of thousands of $$$
  - Cplex, Gurobi, Xpress
- Lots of open-source tools exist, but they are substantially slower than the commercial alternatives
  - SCIP, MIPCL, Coin-OR Branch-and-Cut (CBC), ***Google OR-Tools***
- Commercial solvers are ***5x faster*** than open source…



Example: Max flow

- How would we model the maximum flow problem as an LP?

Variables: f_{uv} for each (u,v)

Objective: $\sum_u f_{su} - \sum_u f_{us}$ (maximize)

Constraints: $\sum_u f_{uv} = \sum_u f_{vu} \quad \forall v \notin \{s,t\}$

$0 \le f_{uv} \le c(u,v) \quad \forall (u,v)$

Example: Minimum-cost Max flow

- Sounds tricky, there's two objectives! (Min cost, and max flow)
- **Solution #1**: Maximize flow first

Variables: f_{uv}

Objective: $\sum \$(u,v)\, f_{uv}$ (minimize)

Constraints: $\sum_u f_{su} - \sum_u f_{us} = F \leftarrow$ solve max flow

(same as before)

Example: Minimum-cost Max flow

- Sounds tricky, there's two objectives! (Min cost, and max flow)
- **Solution #2**: Reduce to "minimum-cost circulation"

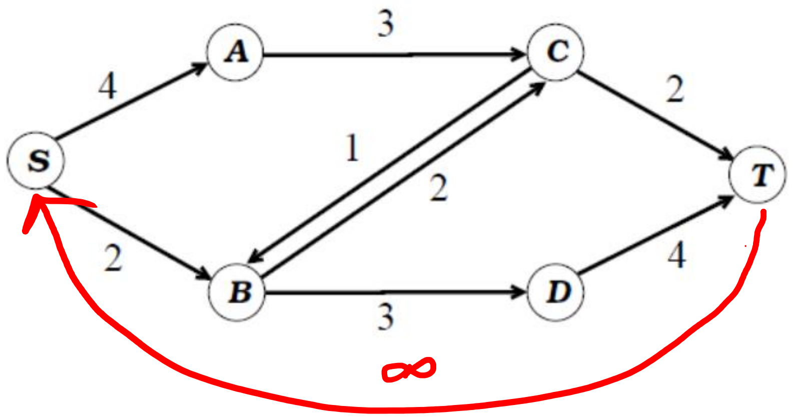
Variables: f_{uv}

Objective: minimize $\sum \$(u,v) f_{uv}$

Constraints: $0 \le f_{uv} \le c(u,v)$

$$\sum_u f_{uv} = \sum_u f_{vu} \quad \forall v$$

includes $\{t, s\}$



\$ – BIG

Example: 2-player zero-sum games

Variables: $p_1 \cdots p_n$, v (value)

Objective: maximize v

Constraints: $\sum p_i m_{ij} \geq v \quad \forall j$ (columns)

$\sum p_i = 1$

$0 \leq p_i \leq 1$

m	column player		
row player	20	-10	5
	5	10	-10
	-5	0	10
		j	

$$\max_{p} \underbrace{\min_{q} V_R(p, q)}$$

$\sum p_i m_{ij} \geq v$

$\forall j$ columns

Standard form

- The same LP can be written in many ways
- It is convenient to have a "standard way" to write an LP

Definition (Standard Form): An LP with n variables $x_1, \dots, x_n$ and m constraints in **standard form** can be written with constants $c_1, \dots, c_n$, $b_1, \dots b_m$, $a_{11}, \dots, a_{mn}$

Objective must be max, not min

Constraints are all $\leq$ constant

All variables are non-negative. (These do not count towards the m number of constraints!)

$$
\begin{aligned}
\textbf{maximize}\quad & c_1x_1 + \cdots + c_nx_n \\
\textbf{subject to}\quad & a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + \cdots + a_{2n}x_n \leq b_2 \\
& \qquad\vdots \\
& a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \\
& x_i \geq 0 \quad \text{for all } i
\end{aligned}
$$

$$=$$

$$
\begin{aligned}
\text{maximize}\quad & \boldsymbol{c}^T\boldsymbol{x} \\
\text{subject to}\quad & A\boldsymbol{x} \leq \boldsymbol{b} \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

Converting to standard form

Claim: Every LP that is not written in standard form can be converted to an equivalent LP in standard form

- ***How to convert minimization to maximization?***

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- ***How to convert a $\geq$ constraint?***

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- ***How to convert an $=$ constraint?***

$\leq \quad \& \quad \geq$

- ***How to convert a variable x_i which could negative?***

$x_i = x_i^+ - x_i^-$

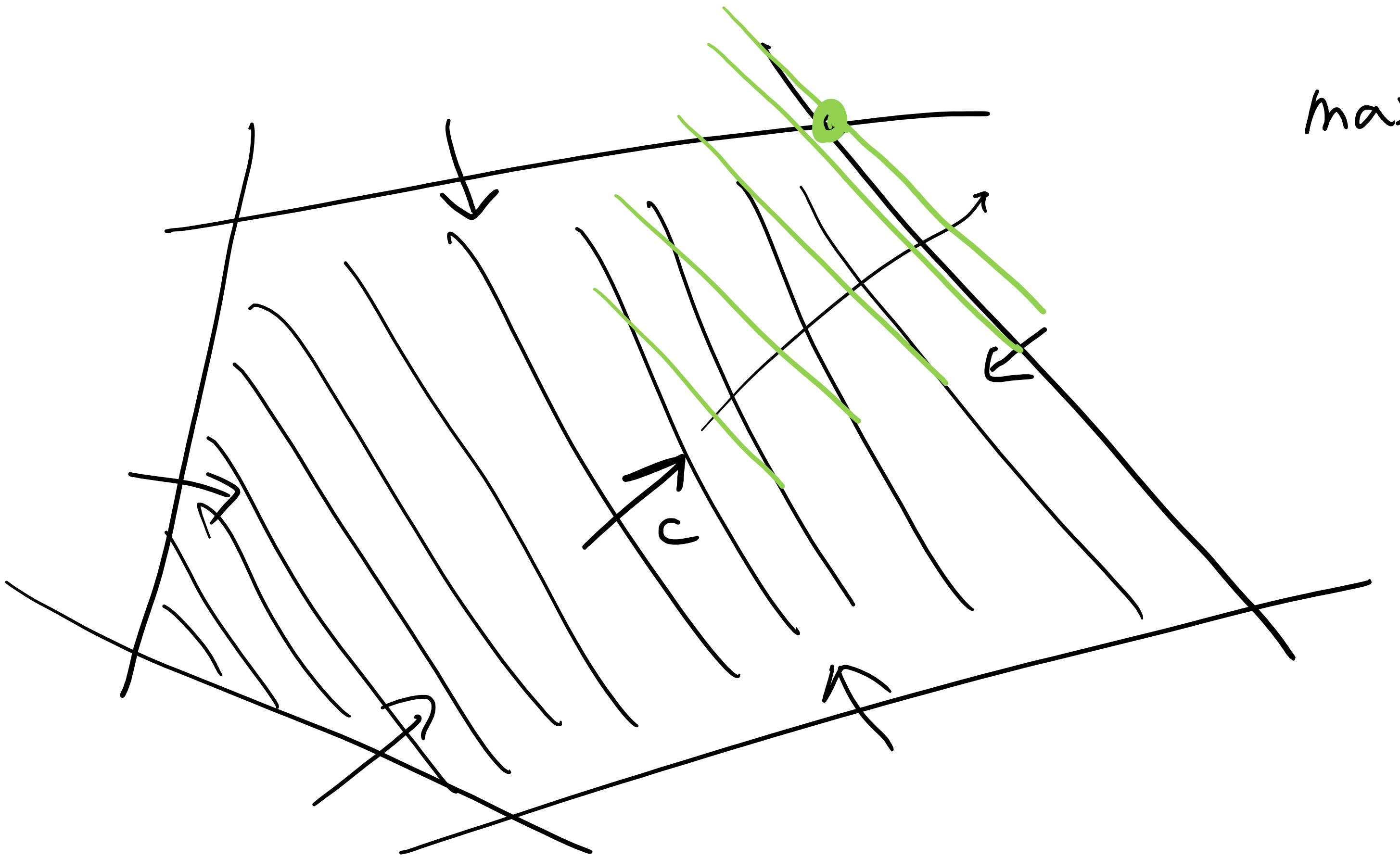
Objective must be max, not min

Constraints are all $\leq$ constant

All variables are non-negative.

The geometry of linear programming

- What does a linear program look like geometrically?



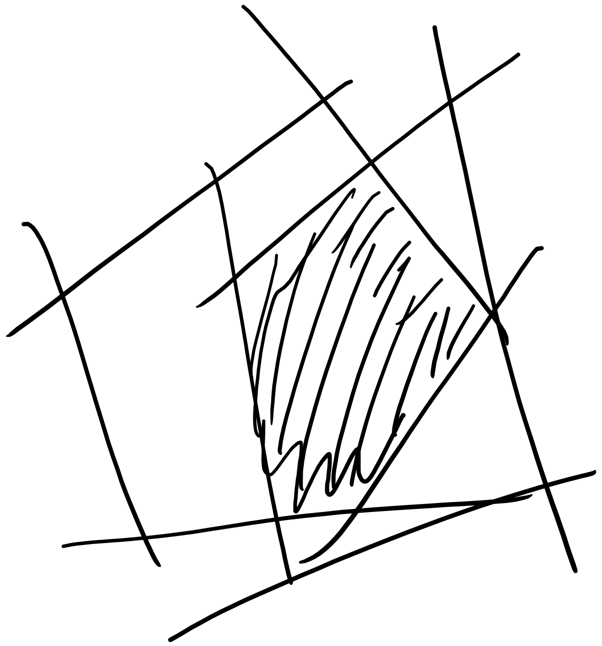
max $c^T x$

The geometry of linear programming

- Each constraint is a ***halfspace***: It cuts $\mathbb{R}^n$ into two halves
- The intersection of all the halfspaces is the ***feasible region***
- If the feasible region is empty, the LP is ***infeasible***
- The feasible region can be unbounded
- Maximizing $c^T x$ moves a hyperplane with normal vector c until it is tangent to the feasible region

Convexity

- The feasible region $Ax \leq b, \; x \geq 0$ is a convex set
  - (If p and q are feasible, then so is any point on the line segment pq)
  - Why? A halfspace is convex, and the intersection of convex sets is convex

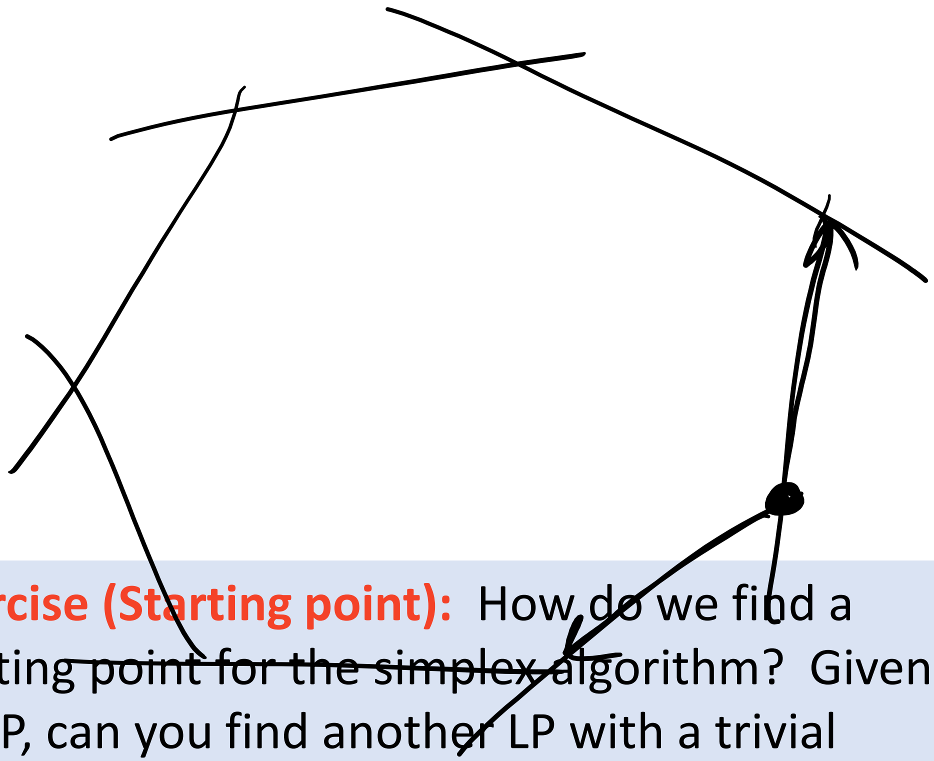


Algorithms for Linear Programming

- ***The simplex algorithm***
  - The first, most famous, and very practical. Exponential worst-case time.
- ***Ellipsoid algorithm***
  - First discovered polynomial time algorithm. Slow in practice.
- ***Karmarkar's Algorithm (interior-point method)***
  - More efficient polynomial-time algorithm

The Simplex Algorithm

- The feasible region is a polytope in $\mathbb{R}^n$
- Pick a starting ***vertex*** of the polytope
- Move to highest-cost neighbouring vertex
- Repeat until maximum
- Worst-case complexity is exponential.

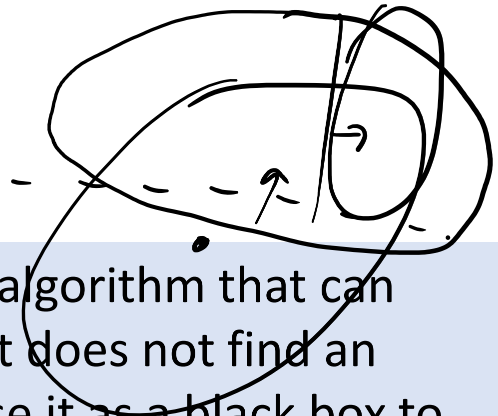


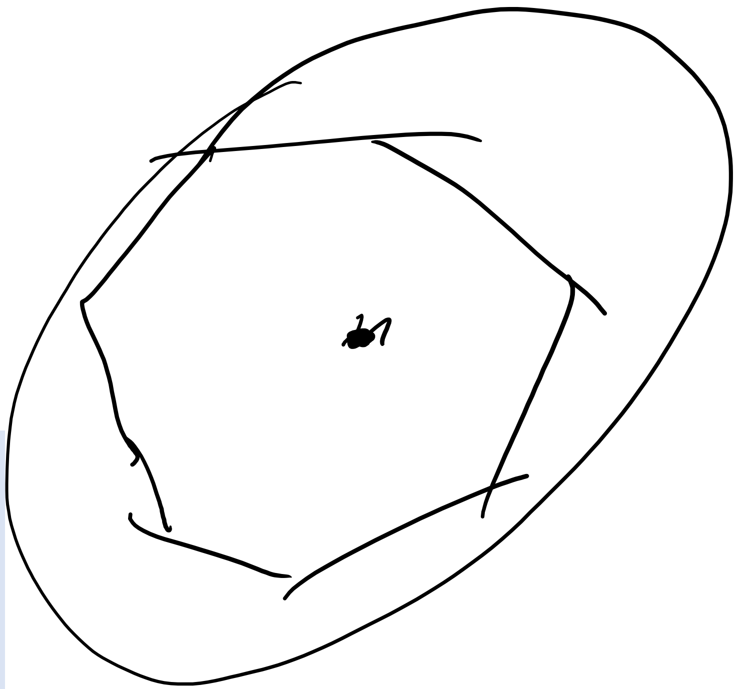
Fun fact: The typical simplex implementation doesn't use standard form. It uses so-called "equational form", where you write maximize $\boldsymbol{c^T x}$ subject to $\boldsymbol{Ax = b}$, where $\boldsymbol{x \geq 0}$

Exercise (Starting point): How do we find a starting point for the simplex algorithm? Given an LP, can you find another LP with a trivial starting point whose solution gives a feasible point in the original LP?

The Ellipsoid Algorithm

- Determines whether an LP is feasible (doesn’t solve for optimal)
- Find an ellipse that contains the feasible region
- Check if the center is a feasible point
- If not, shrink the ellipse and repeat
- Complexity is $O(n^6 w)$

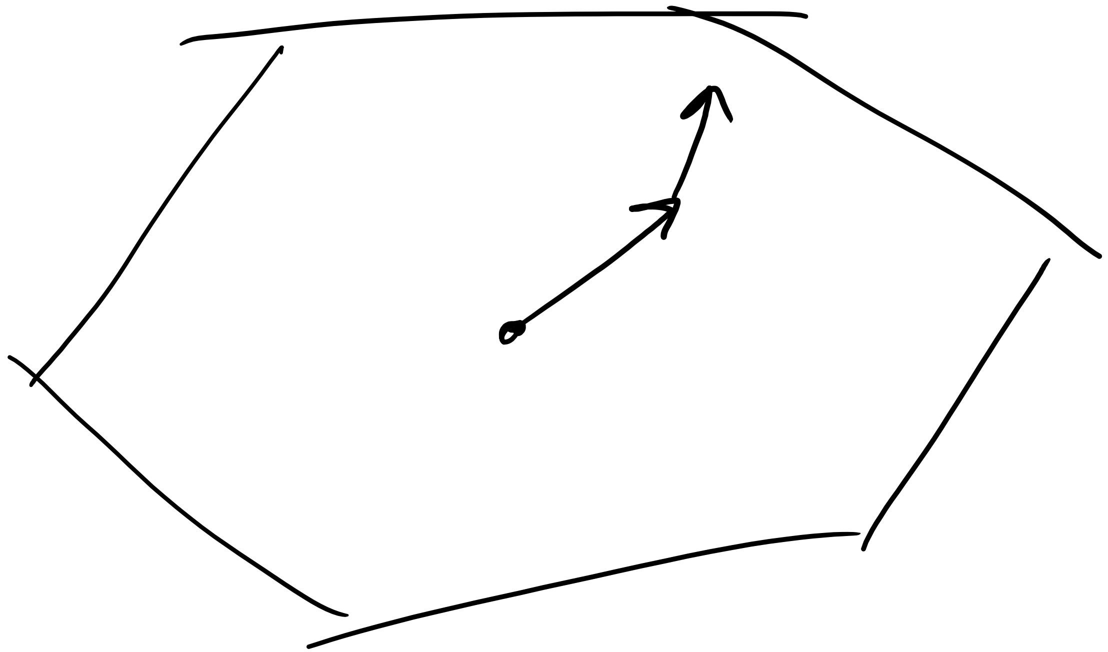




Exercise (Optimizing): Given an algorithm that can determine if an LP is feasible (but does not find an optimal solution), how can we use it as a black box to find an optimal solution of an LP?

Karmarkar's Algorithm

- Works with feasible points but doesn't go corner to corner
- Moves in interior of the feasible region – "interior point method"
- Complexity is $O(n^{3.5}w)$



Take-home messages

- LPs have tremendously many applications, many theoretically interesting, and many lucrative.
- Many interesting problems we learned about earlier in the course can be expressed/solved using LPs (flows, games)