Algorithm
Design and Analysis

Computational Geometry (Fundamentals and the Convex Hull)

Goals for today

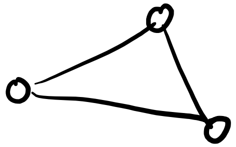
- Explore some fundamental tools for **computational geometry**
- Understand important tools/ideas such as:
  - **Dot and cross products**
  - The **line-side test**
- Define and solve the **convex hull** problem

Why geometry?

- Applications in robotics
- Applications to graphics
- Applications to algorithms (LPs!!)

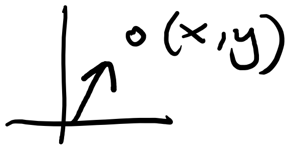
Representation and Model

How might we represent some of the following ideas?

Real number	Floating-point number	
Point	A pair of floating-point numbers	
Line	A pair of points	
Line Segment	A pair of points	
Triangle	A triple of points	

Concerns? Rounding ??

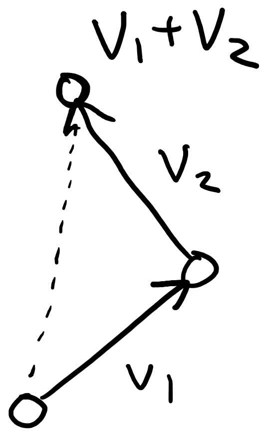
Fundamental Objects & Operations



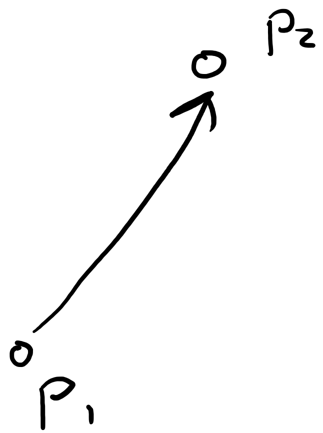
Representation (Point): A pair of real numbers

Representation (Vector): A pair of real numbers

We will use these interchangeably



Operation (Addition/subtraction):

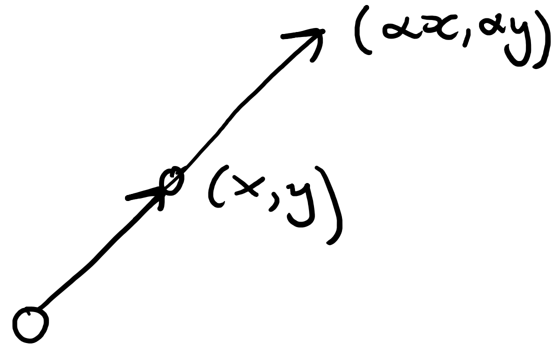


$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$$

Fundamental Operations (continued)

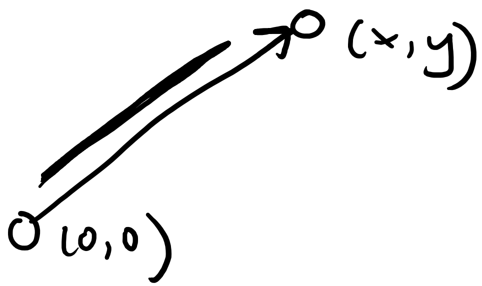
Operation (Scalar multiplication):

$$\alpha\,(x, y) = (\alpha x, \alpha y), \qquad \alpha \in R$$



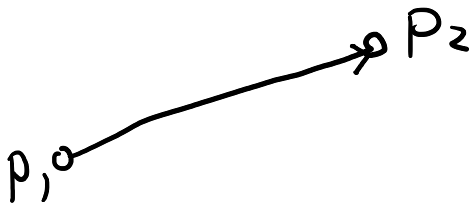
Operation (Length/magnitude):

$$||(x, y)|| = \sqrt{x^2 + y^2}$$



Application (Distance):

$$|| p_2 - p_1 ||$$



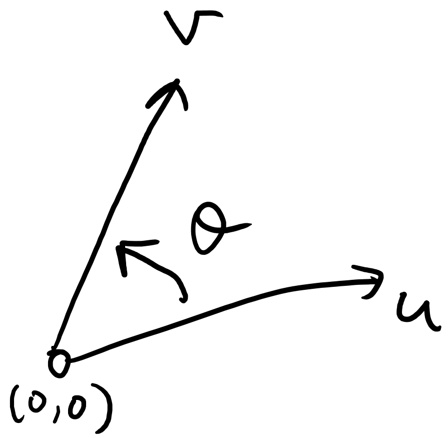
Fundamental Operations (continued)

Operation (The dot product):

$$(x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2$$

Useful theorem (The dot product angle formula):

$$u \cdot v = \|u\| \|v\| \cos(\theta)$$



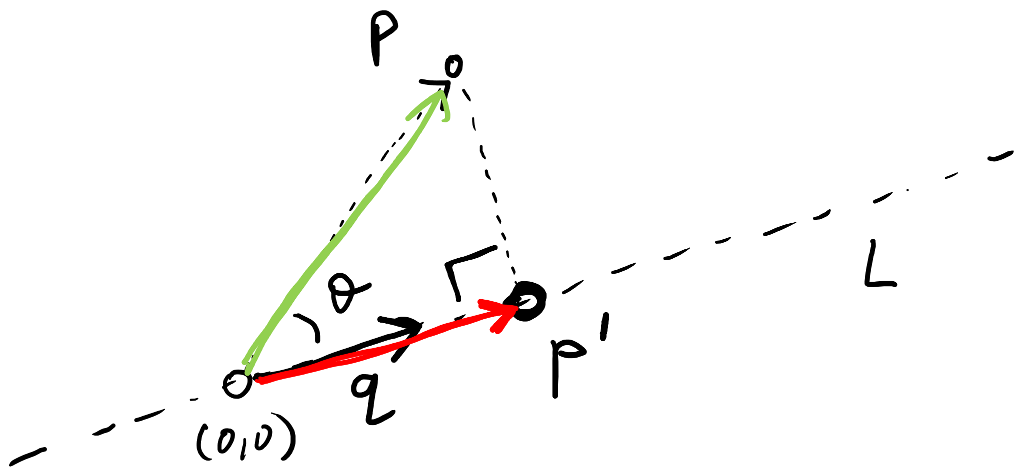
Application of the dot product

Application (Projection): Given a **point $\boldsymbol{p}$** and a **line $\boldsymbol{L}$** that goes through the origin in the direction of q (a unit vector), find the point p' on L that is closest to p

$$p \cdot q = \|p\| \|q\| \cos(\theta)$$

$$= \underbrace{\|p\| \cos(\theta)}$$

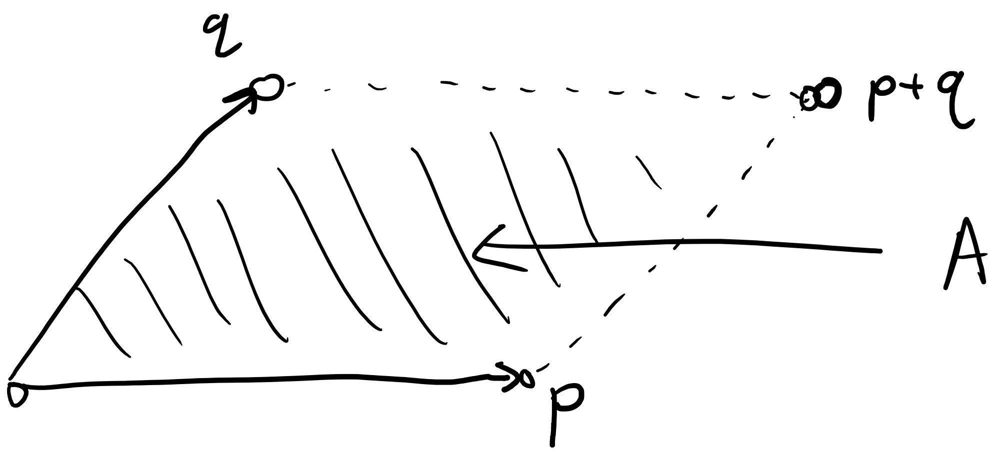
$$p' = \underbrace{(p \cdot q)}_{\text{length}} \underbrace{q}_{\text{unit vector}}$$



Fundamental Operations (continued)

Operation (The cross product):

$$(x_1, y_1) \times (x_2, y_2) = \det\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\right) = x_1 y_2 - x_2 y_1$$



Area = p×q

"Signed area"

positive if q is left of p

negative otherwise

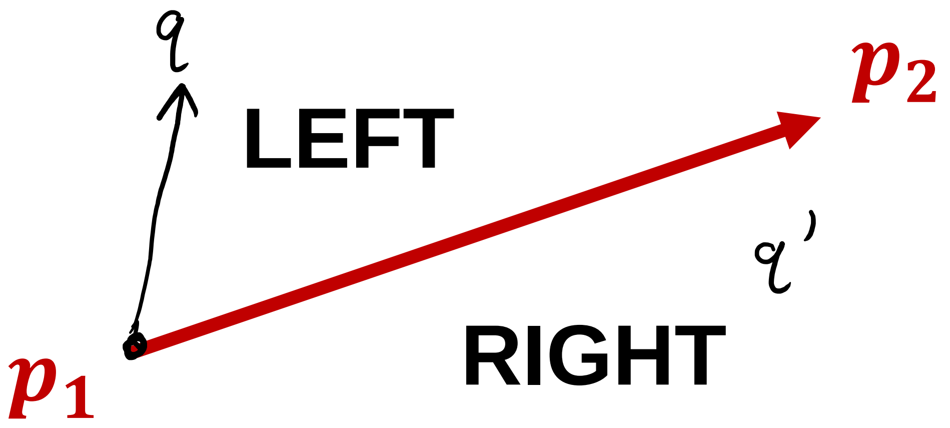
Line-side test (Important!)

Operation (Line-side test): Given points p_1, p_2, q, we want to know whether q is on the LEFT or RIGHT of the line from p_1 to p_2

$V_1 = p_2 - p_1$

$V_2 = q - p_1$

$$V_1 \times V_2 \begin{cases} > 0 & \text{LEFT} \\ < 0 & \text{RIGHT} \\ = 0 & \text{ON LINE} \end{cases}$$

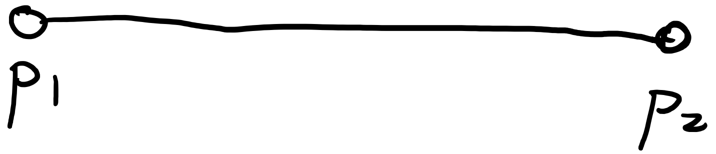


Convex Combinations

Definition (Convex combination): A ***convex combination*** of the points $p_1, p_2, \ldots, p_k$ is a point

$$p' = \sum_{i=1}^{k} \alpha_i \, p_i$$

such that $\sum \alpha_i = 1$ and $\alpha_i \geq 0$ for all i

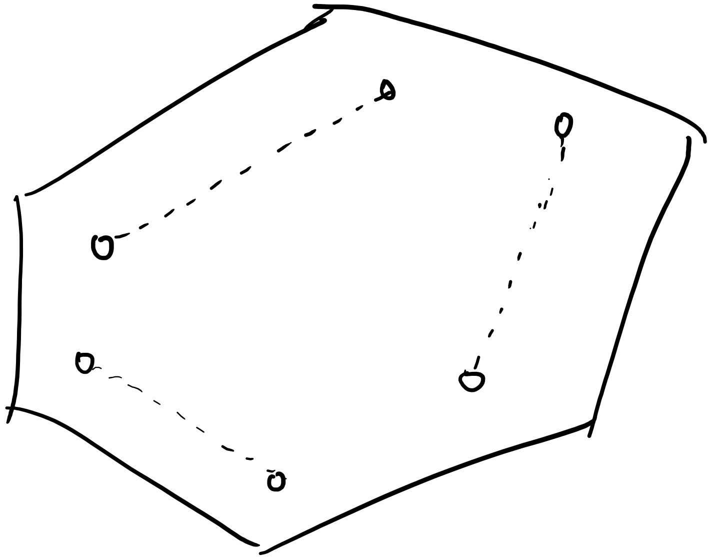


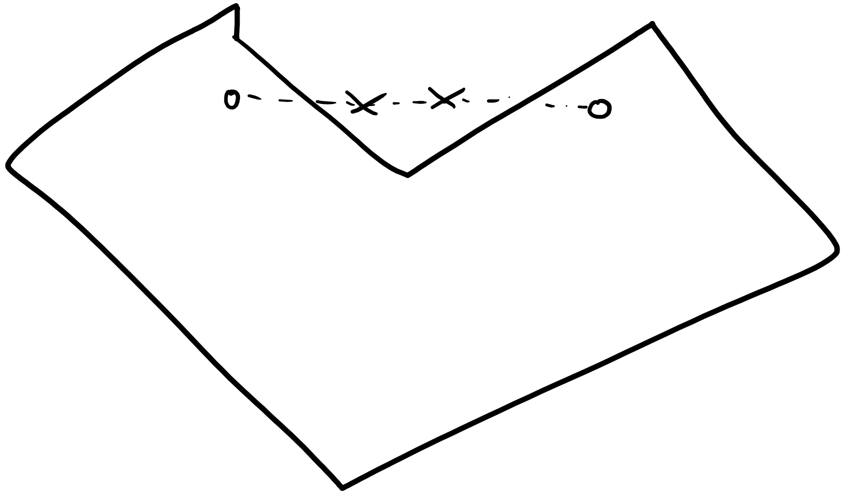
$p' = \alpha_1 p_1 + \alpha_2 p_2$

The Convex Hull

Convexity recap

Definition (Convex set): A set is convex if for any points p, q, any convex combination of p, q is also in the set

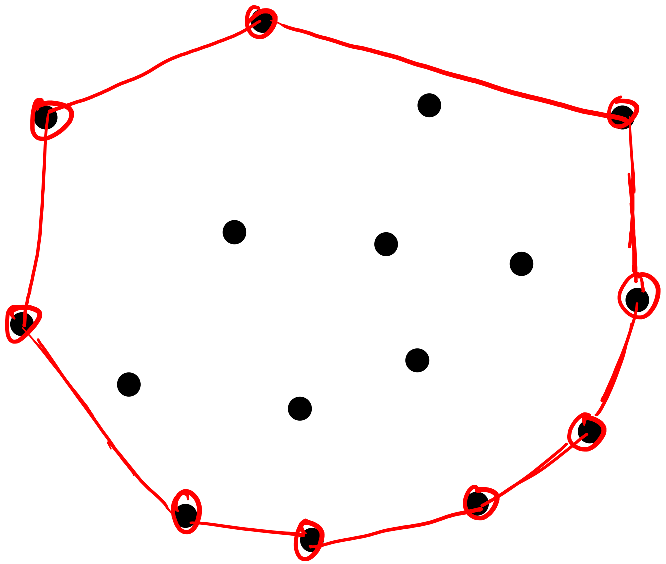




The Convex Hull

Definition (Convex hull): Given a set of points $p_1, \ldots, p_n$, the ***convex hull*** is the smallest convex polygon containing all of them

Goal: output the vertices of the hull in counterclockwise order



An $O(n^3)$-time algorithm

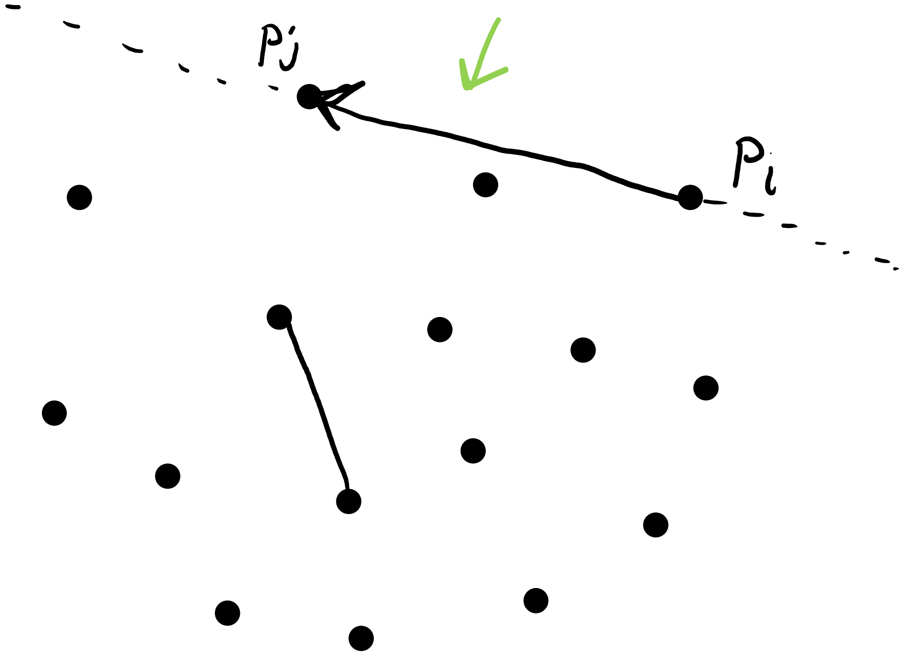
Observation (Hull edges): The edges of the convex hull must be pairs of points from the input

Claim (Hull edges): A segment (p_i, p_j) is on the convex hull if and only if…

all other points are LEFT

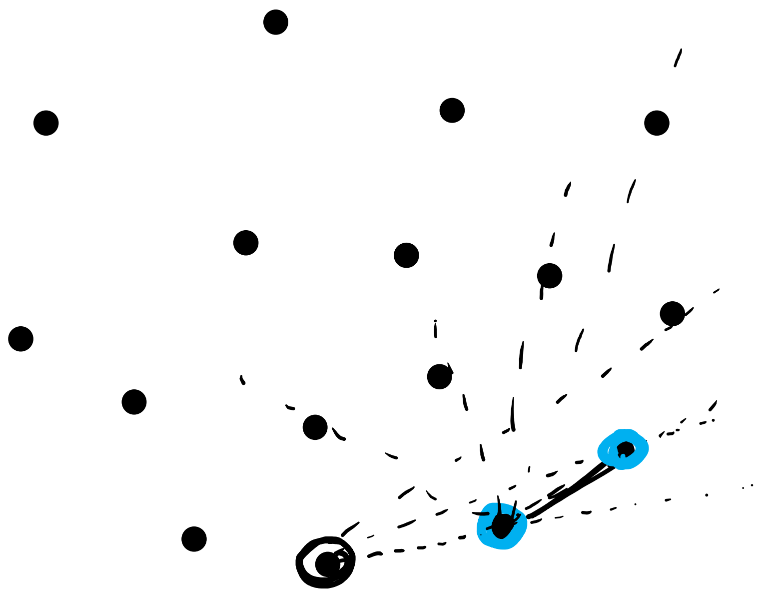
Brute force all (p_i, p_j)

LST all other p_k



Better: An $O(n^2)$-time algorithm

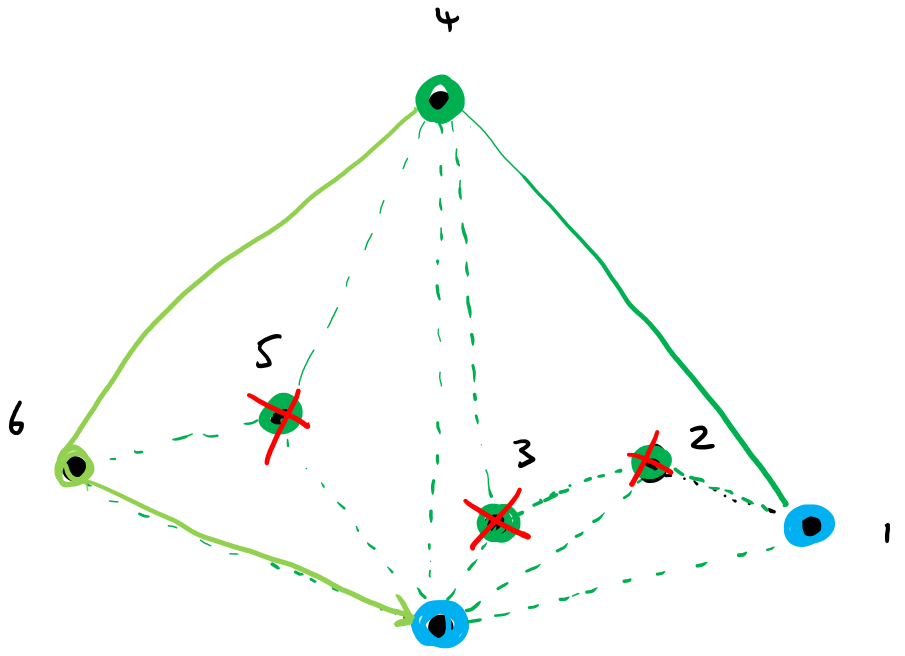
Observation (Order helps): The $O(n^3)$-time algorithm found the hull edges in an arbitrary order… What if we try to find them in CCW order



Find next pt as the one with least angle

Graham Scan: An $O(n \log n)$ algorithm

Observation (Order helps again): We went from $O(n^3)$ to $O(n^2)$ by finding the edges in order… but we still processed the points in an arbitrary order. Can we order the points and do better?



Graham Scan: An $O(n \log n)$ algorithm

Algorithm (Graham Scan):

Find lowest point p_0

Sort points $p_1, p_2, \ldots$ **counterclockwise by their angle with** $\boldsymbol{p_0}$

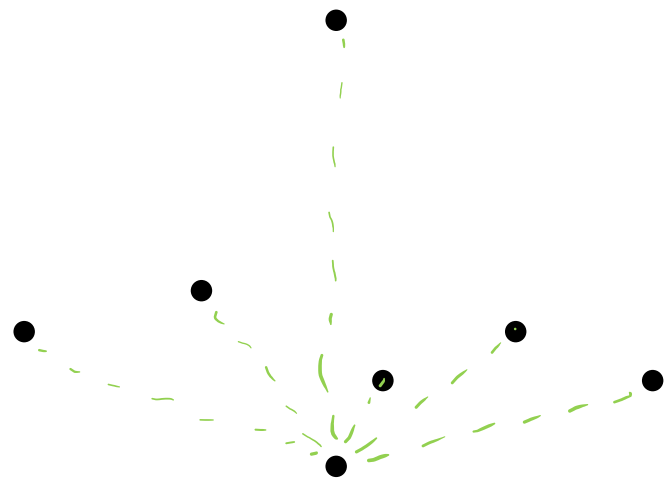
$\boldsymbol{H = [p_0, p_1]}$
for each point $i = 2 \ldots n-1$

while LST(H[-2], H[-1], p_i) == RIGHT

H.pop()

H.append(p_i)

return H



Graham Scan: Complexity

Theorem: Graham Scan runs in $O(n \log n)$ time

Proof:

Sorting takes $O(n \log n)$

Loops are $O(n) + O(n)$

$\square$

Lower Bound

Theorem: Any convex hull algorithm that uses line-side tests to find the hull requires $\Omega(n \log n)$ line-side tests (in a decision tree model)

Won't prove this

Take-home messages

- Computational geometry is all about using the right tools (and drawing good diagrams)
  - Dot product
  - Cross product
  - ***Line-side test***
  - ***Convex hull***