Algorithm Design and Analysis

Minimum-cost Flows

Roadmap for today

- Another flow problem, ***minimum-cost flows***
- The ***cheapest augmenting paths*** algorithm
- The ***cycle cancelling*** algorithm

Network Flow recap recap

- A ***flow network*** is a directed graph with:
  - ***capacities*** $c(u, v)$
  - A ***source vertex*** s and ***sink vertex*** t
- A ***flow*** is an assignment of values to edges:
  - ***Capacity constraint:*** $0 \leq f(u, v) \leq c(u, v)$
  - ***Conservation constraint:*** "flow in = flow out" for all vertices except s, t

$$\sum_{v \in V} f(u, v) = \sum_{v \in V} f(v, u)$$

- The value of a flow is the net flow out of the source (can prove via conservation that is = net flow into sink)

Network Flow recap

- The ***maximum flow*** problem is to find a flow of maximum value
- We learned the ***Ford-Fulkerson*** algorithm:
  - Define the ***residual network:***

$$c_f(u,v) = c(u,v) - f(u,v)$$

$$c_f(v,u) = c(v,u) + f(u,v)$$

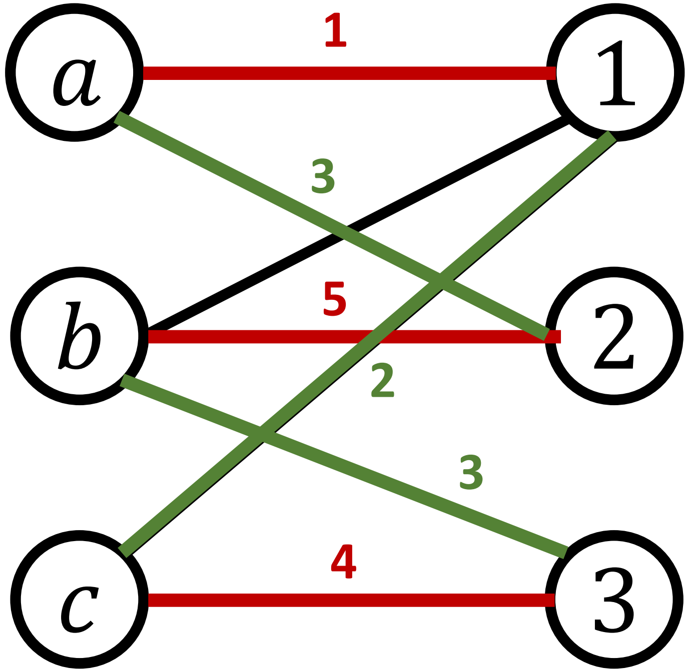
Then the algorithm is:

while there exists a path in the residual network:
  add flow to that path.

- Last lecture we saw ***Edmonds-Karp*** and ***Dinic's***.

Motivation

- There can be multiple maximum flows in a particular network
- What if we want to preference some over others?
- Example: Bipartite matching allows us to find whether a matching is possible. If there are multiple, can we also have preferences so that we get the “best” matching?



Minimum-cost flows

- We consider the same setting as before: A directed graph with capacities.
- Edges now also have ***costs***. Edge e costs $\$(e)$
- The cost of an edge is **per unit of flow**. The total cost is

- ***Goal***: Find maximum flow of minimum cost
- ***Note***: Other variants of the problem exist. E.g., you might want the minimum possible cost, regardless of the flow value (not maximum)

Assumptions

- Negative costs are allowed!
- Negative cycles are also allowed!!
  - However, some algorithms don't work.
  - Assume that there is no infinite capacity negative cycle (or the cost is $-\infty$)

The residual network

- The residual network is a powerful tool. Let's keep using it
- If $f(u,v) > 0$, we define the *residual capacities and residual costs*

$$c_f(u,v) = c(u,v) - f(u,v) \qquad \$_f(u,v) =$$
$$c_f(v,u) = c(v,u) + f(v,u) \qquad \$_f(v,u) =$$

- Note: If the input graph contains anti-parallel edges (u,v) and (v,u), then we need to have $\$(u,v) = -\$(v,u)$

An augmenting path algorithm

- Ford-Fulkerson finds a maximum flow (ignoring costs completely)
- What is a natural way to choose the augmenting paths?

- Find a ***cheapest augmenting path***.
- Use Bellman-Ford to find the augmenting paths (why not Dijkstra?)
- Requires no negative cycles in the input network!
- Assume integer capacities as well for termination

Does it work?

- We need two things:
  - ***Question 1:*** Does the algorithm actually terminate?
  - ***Question 2:*** Does it give a minimum-cost flow?

To answer Question 1, we need to prove that G_f never contains a negative-cost cycle! (Or the cheapest path would be undefined).

A powerful lemma

Theorem: Given a network G and flow f such that G_f contains no negative-cost cycles, if we augment a cheapest path, then the result still has no negative-cost cycles.

Lemma: Augmenting a cheapest path does not **decrease** the cost of the cheapest $s - t$ path in the residual network.

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Does it work?

- We need two things:
  - ~~***Question 1:*** Does the algorithm actually terminate?~~
  - ***Question 2:*** Does it give a minimum-cost flow?

To answer Question 2, we need some more definitions!

Optimality criteria

Definition (cost optimal): A flow is ***cost optimal*** if it is the cheapest possible flow of all flows of the same value.

- Recall that a flow is not maximum if and only if there exists an augmenting path
- It would be great if there was an analogy for costs: A flow is not cost optimal if and only if there exists…

Augmenting cycles!

Cost optimality

Theorem (cost optimality): A flow f is cost optimal if and only if there are no negative-cost cycles in G_f

(The cost of a cycle is the sum of the costs of the edges in the cycle)

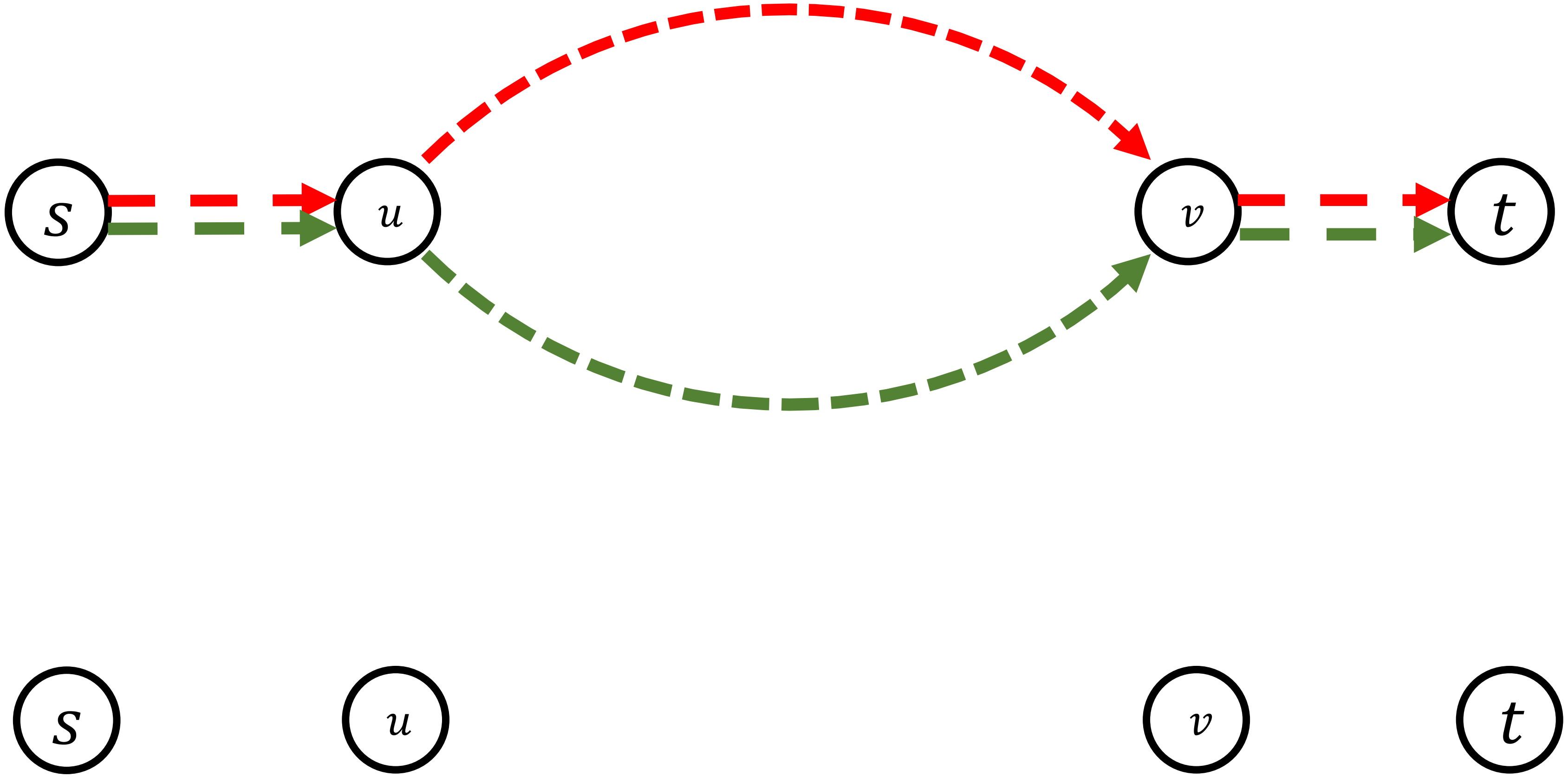
Exists negative cycle $\Rightarrow$ ***not cost optimal***

Cost optimality

Not cost optimal $\Rightarrow$ ***exists a negative cycle***

Since f is not cost optimal, there exists f' of **same value but lower cost**

Cost optimality



Cost optimality

- $f' - f$ is called a *circulation*. It's a collections of cycles, a flow of value zero.

Claim: $f' - f$ is feasible in G_f

$$\text{cost}(f' - f) =$$

Completing the analysis

- Cheapest augmenting paths never creates a negative cycle
- Therefore, the flow is always cost optimal
- Augmenting until there are no augmenting paths implies the flow is maximum.
- Therefore, it outputs a ***minimum-cost maximum flow***

Runtime:

Another algorithm

- Cheapest augmenting paths started with a flow that was cost optimal (the zero flow) then incrementally made it more maximum
- Can we start with a maximum flow and incrementally make it cheaper?

Theorem (cost optimality): A flow f is cost optimal if and only if there are no negative-cost cycles in G_f

(The cost of a cycle is the sum of the costs of the edges in the cycle)

Cycle cancelling

find a maximum flow (e.g., Ford-Fulkerson, or Dinic’s)
while there exists a negative cost cycle in G_f
    augment flow along the negative cost cycle

- Works when the input has negative-cost cycles!
- Assumes integer costs

Runtime:

Summary of today

- The ***minimum-cost flow problem***, and two algorithms
- ***Cheapest augmenting paths***
    - Ford-Fulkerson but always use cheapest cost augmenting path
    - Works for integer-capacity, negative-cycle-free networks
    - Runs in $O(nmF)$ worst-case time
- ***Cycle cancelling***
    - Find max flow and use augmenting cycles until no negative cycles left
    - Works for integer-cost networks. Negative cycles permitted!
    - Runs in $O(nm^2UC)$ worst-case time