Algorithm Design and Analysis

Linear Programming (Part II)

Roadmap for today

- Linear program duality
- Weak and strong duality theorems
- Examples of duality

Review: Formal definition

Definition (Linear program): A linear program consists of

- n real-valued **variables** $x_1, x_2, \dots, x_n$
- A linear ***objective function***, e.g., minimize/maximize $2x_1 + 3x_2 + x_3$
- m linear **inequalities**, e.g., $3x_1 + 4x_2 \leq 6$, or $0 \leq x_1 \leq 3$

Goal: Find values for x's that satisfy the constraints and minimize/maximize the objective

Review: Standard form

- The same LP can be written in many ways
- It is convenient to have a "standard way" to write an LP

Definition (Standard Form): An LP with n variables $x_1, \dots, x_n$ and m constraints in **standard form** can be written with constants $c_1, \dots, c_n,\ \ b_1, \dots b_m,\ \ a_{11}, \dots, a_{mn}$

Objective must be max, not min

Constraints are all $\leq$ constant

All variables are non-negative. (These do not count towards the m number of constraints!)

$$\begin{aligned} \textbf{maximize}\ \ & c_1x_1 + \cdots + c_nx_n \\ \textbf{subject to}\ \ & a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + \cdots + a_{2n}x_n \leq b_2 \\ & \quad\vdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \\ & x_i \geq 0 \quad \text{for all } i \end{aligned}$$

$=$

$$\begin{aligned} \text{maximize}\ \ & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to}\ \ & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Review: Algorithms

- **Simplex**: Typically, the best in practice. Exponential worst case
- **Ellipsoid**: Polynomial time, bad in practice
- **Karmarkar**: Good in practice sometimes, polynomial time

Motivating problem: The carpenter

- You are a carpenter. You make tables, chairs, and shelves out of wood, nails, and paint.

Item	Wood	Nails	Paint	Sale Price
Table	8	20	5	$50
Chair	4	15	3	$30
Shelf	3	5	3	$20
Stock	**100**	**300**	**80**	

- How many of each item should you make for maximum profit (ignoring rounding errors)

Motivating problem: The carpenter

maximize $50x + 30y + 20z$

$$8x + 4y + 3z \leq 100$$
$$20x + 15y + 5z \leq 300$$
$$5x + 3y + 3z \leq 80$$

$$x, y, z \geq 0$$

Let x = #tables, y = #chairs, z = #shelves

Item	Wood	Nails	Paint	Sale Price
Table	8	20	5	$50
Chair	4	15	3	$30
Shelf	3	5	3	$20
Stock	**100**	**300**	**80**	

Solution: $x \approx 1.82$, $y \approx 14.55$, $z \approx 9.09$

$709.09

Along comes a merchant

- Along comes a traveling merchant willing to purchase your stock of wood, nails, and paint, for a fair price.
- What is a fair price for wood, nails, and paint?
- You are not willing to sell your materials for less than the amount you could make by turning them into items

Along comes a merchant

minimize $100w + 300s + 80p$

$8w + 20s + 5p \geq 50$

$4w + 15s + 3p \geq 30$

$3w + 5s + 3p \geq 20$

$w, s, p \geq 0$

Let w = \$wood, s = \$nails, p = \$paint

Item	Wood	Nails	Paint	Sale Price
Table	8	20	5	\$50
Chair	4	15	3	\$30
Shelf	3	5	3	\$20
Stock	**100**	**300**	**80**	

Solution: $w \approx 2.73$, $s \approx 0.73$, $p \approx 2.73$

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A tale of two LPs

$$\begin{aligned} \text{maximize} \quad & 50x + 30y + 20z \\ \text{subject to} \quad & 8x + 4y + 3z \le 100 \\ & 20x + 15y + 5z \le 300 \\ & 5x + 3y + 3z \le 80 \\ & x, y, z \ge 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & 100w + 300s + 80p \\ \text{subject to} \quad & 8w + 20s + 5p \ge 50 \\ & 4w + 15s + 3p \ge 30 \\ & 3w + 5s + 3p \ge 20 \\ & w, s, p \ge 0 \end{aligned}$$

$$A = \begin{bmatrix} 8 & 4 & 3 \\ 20 & 15 & 5 \\ 5 & 3 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 300 \\ 80 \end{bmatrix}, c = \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$$

$$\begin{aligned} \text{maximize} \quad & c^T \vec{x} \\ & A\vec{x} \le b \\ & \vec{x} \ge 0 \end{aligned} \longrightarrow \begin{aligned} \text{minimize} \quad & b^T \vec{y} \\ & A^T \vec{y} \ge c \\ & \vec{y} \ge 0 \end{aligned}$$

The dual program

Definition (Dual): Given a standard-form LP, its **dual** is

$$\begin{array}{ll} \text{maximize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array} \qquad\qquad \begin{array}{ll} \text{minimize} & \boldsymbol{b}^T\boldsymbol{y} \\ \text{subject to} & A^T\boldsymbol{y} \geq \boldsymbol{c} \\ & \boldsymbol{y} \geq 0 \end{array}$$

- The original problem is called the **primal problem**
- If the primal has n variables and m constraints, the dual has m variables and n constraints, i.e., variables and constraints swap roles!

Exercise: Show that the dual of the dual is the primal. This shows that which one you call the primal and which you call the dual is arbitrary

Theorems

Primal LP

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} \leq \boldsymbol{b} \leftarrow \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{b}^T \boldsymbol{y} \\ \text{subject to} \quad & A^T\boldsymbol{y} \geq \boldsymbol{c} \leftarrow \\ & \boldsymbol{y} \geq 0 \end{aligned}$$

Theorem (Weak Duality): If x is any feasible solution to the primal LP and y is any feasible solution to the dual LP

$$c^T x \leq b^T y$$

Proof.

$$c^T x \leq (A^T y)^T x = (y^T A) x = y^T (Ax) \leq y^T b = b^T y \qquad \square$$

Theorems

Primal LP

maximize $\boldsymbol{c}^T\boldsymbol{x}$

subject to $A\boldsymbol{x} \leq \boldsymbol{b}$

$\boldsymbol{x} \geq 0$

Dual LP

minimize $\boldsymbol{b}^T\boldsymbol{y}$

subject to $A^T\boldsymbol{y} \geq \boldsymbol{c}$

$\boldsymbol{y} \geq 0$

Theorem (Strong Duality): If the primal problem is feasible and bounded, then the dual is feasible and bounded. If x^* is an optimal solution to the primal LP and y^* is an optimal solution to the dual LP

$$c^Tx^* = b^Ty^*$$

Proof: Too long.

Consequences

- Suppose the primal problem is unbounded, what can we say about the dual?

primal $\leq$ dual

$\rightarrow \infty \qquad \leftarrow -\infty$

- ***Consequence***: It is impossible for both the primal and dual to be unbounded.

Consequences

- Which combinations are possible?

Dual

Primal

	Infeasible	Feasible & Bounded	Unbounded
Infeasible	✓	✗	✓
Feasible & Bounded	✗	✓	✗
Unbounded	✓	✗	✗

Exercise: Find an LP that is infeasible such that its dual is also infeasible.

Zero-sum games, again

Variables: $p_1, \ldots, p_n$ and v.
Objective: Maximize v.
Constraints:

- $p_i \geq 0$ for all $1 \leq i \leq n$,
- $\sum_{i=1}^{n} p_i = 1.$ (the p_i form a probability distribution)
- $\sum_{i=1}^{n} p_i R_{ij} \geq v$ for all columns $1 \leq j \leq m$

- Let R denote the payoff matrix
- Assume WLOG $R_{ij} \geq 0$
- Solution of this LP is

$$\mathrm{lb}^* = \max_{\boldsymbol{p}} \min_{j} \sum_{i} p_i R_{ij}$$

$$v - \sum_{i=1}^{n} p_i R_{ij} \leq 0 \quad \forall j$$

$$\sum_{i=1}^{n} p_i \leq 1$$

$$\boldsymbol{x} = \begin{pmatrix} v \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \quad \boldsymbol{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \end{pmatrix} \quad \boldsymbol{A} = \begin{bmatrix} 1 & \\ 1 & -R^T \\ 1 & \\ 1 & \\ 0 & 1 \; 1 \; 1 \; \ldots \; 1 \end{bmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Zero-sum games, the dual

$$\boldsymbol{x} = \begin{bmatrix} v \\ p_1 \\ \vdots \\ p_n \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \boldsymbol{A} = \begin{bmatrix} 1 & & -R^T & \\ 1 & & & \\ \vdots & & & \\ 0 & 1 & \cdots & 1 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Dual LP

minimize $\boldsymbol{b}^T\boldsymbol{y}$
subject to $A^T\boldsymbol{y} \geq \boldsymbol{c}$ ←
$\boldsymbol{y} \geq 0$

Call the dual variables $\boldsymbol{y} = [q_1, q_2, \ldots, q_m, v']^T$

$$A^T = \left[\begin{array}{ccc|c} 1 & 1 & \cdots & 0 \\ \hline & -R & & 1 \\ & & & \vdots \end{array}\right]$$

minimize v'
$\sum q_i \geq 1$
$v' - \sum q_j R_{ij} \geq 0 \quad \forall i$

$$UB^* = \begin{cases} \text{minimize } v' \\ \sum q_i = 1 \\ \sum_{j=1}^{m} q_j R_{ij} \leq v' \quad \text{for all rows } 1 \leq i \leq n \\ q_i \geq 0 \end{cases}$$

Corollary: Minimax theorem

Theorem (Minimax): Given a finite 2-player zero-sum game with row payoff matrix R

$$\mathrm{lb}^* = \max_{\boldsymbol{p}} \min_{j} \sum_{i} p_i R_{ij} = \min_{\boldsymbol{q}} \max_{i} \sum_{j} q_i R_{ij} = \mathrm{ub}^*$$

Proof: Strong duality of the LPs from the last two slides!

Take-home messages

- ***Duality*** gives us a powerful tool to prove see a problem in an equivalent but different form
- The ***strong and weak duality theorems*** tell us about the relationship between the primal and dual problem
- Duality can be used to prove equivalence between two problems (e.g., proving the ***minimax theorem***)