Algorithm Design and Analysis

Computational Geometry (Fundamentals and the Convex Hull)

Goals for today

- Explore some fundamental tools for **computational geometry**
- Understand important tools/ideas such as:
  - **Dot and cross products**
  - The **line-side test**
- Define and solve the **convex hull** problem

Why geometry?

- Applications in robotics
- Applications to graphics
- Applications to algorithms (LPs!!)

Representation and Model

How might we represent some of the following ideas?

Real number	Floating-point number
Point	A pair of floating-point numbers
Line	A pair of points
Line Segment	A pair of points
Triangle	A triple of points

Concerns?

Fundamental Objects & Operations

Representation (Point): A pair of real numbers

Representation (Vector): A pair of real numbers

We will use these interchangeably

Operation (Addition/subtraction):

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$$

Fundamental Operations (continued)

Operation (Scalar multiplication):

$$\alpha\,(x, y) = (\alpha x, \alpha y), \qquad \alpha \in R$$

Operation (Length/magnitude):

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

Application (Distance):

Fundamental Operations (continued)

Operation (The dot product):

$$(x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2$$

Useful theorem (The dot product angle formula):

$$u \cdot v = \|u\| \|v\| \cos(\theta)$$

Application of the dot product

Application (Projection): Given a **point $\boldsymbol{p}$** and a **line $\boldsymbol{L}$** that goes through the origin in the direction of q (a unit vector), find the point p' on L that is closest to p

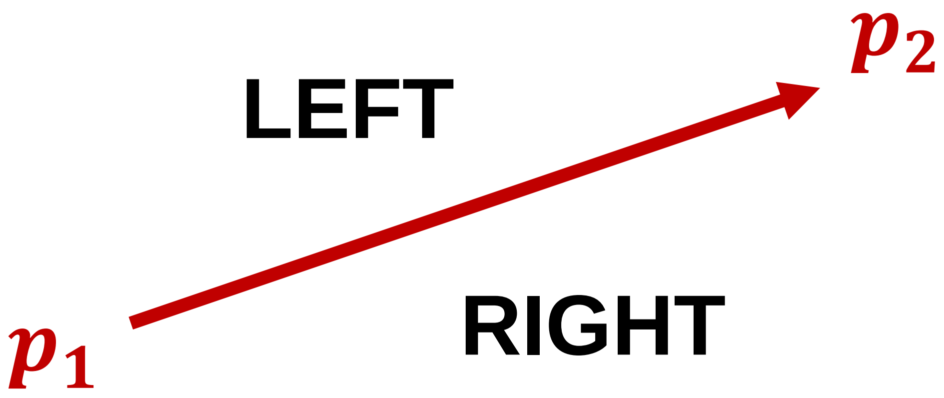
Fundamental Operations (continued)

Operation (The cross product):

$$(x_1, y_1) \times (x_2, y_2) = \det\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\right) = x_1 y_2 - x_2 y_1$$

Line-side test (Important!)

Operation (Line-side test): Given points p_1, p_2, q, we want to know whether q is on the LEFT or RIGHT of the line from p_1 to p_2



Convex Combinations

Definition (Convex combination): A ***convex combination*** of the points $p_1, p_2, \ldots, p_k$ is a point

$$p' = \sum_{i=1}^{k} \alpha_i \, p_i$$

such that $\sum \alpha_i = 1$ and $\alpha_i \geq 0$ for all i

The Convex Hull

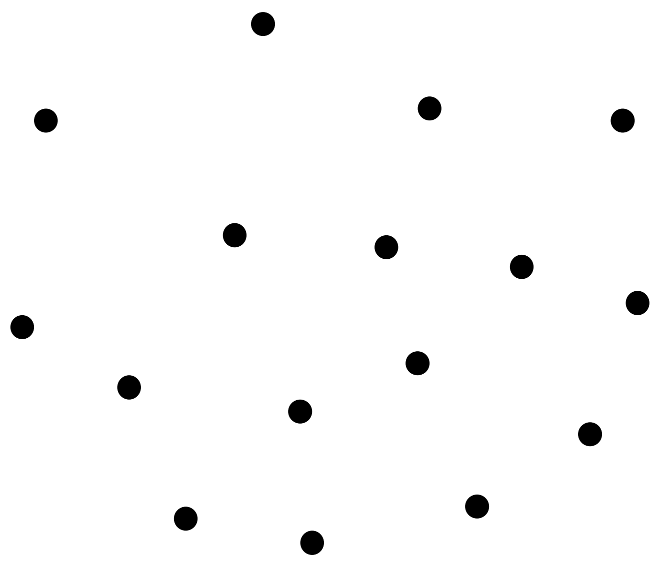
Convexity recap

Definition (Convex set): A set is convex if for any points p, q, any convex combination of p, q is also in the set

The Convex Hull

Definition (Convex hull): Given a set of points $p_1, \ldots, p_n$, the ***convex hull*** is the smallest convex polygon containing all of them

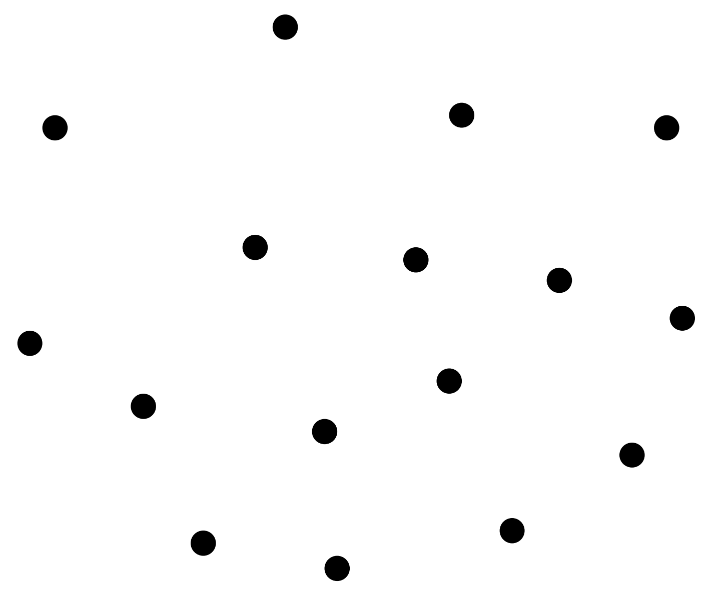
Goal: output the vertices of the hull in counterclockwise order



An $O(n^3)$-time algorithm

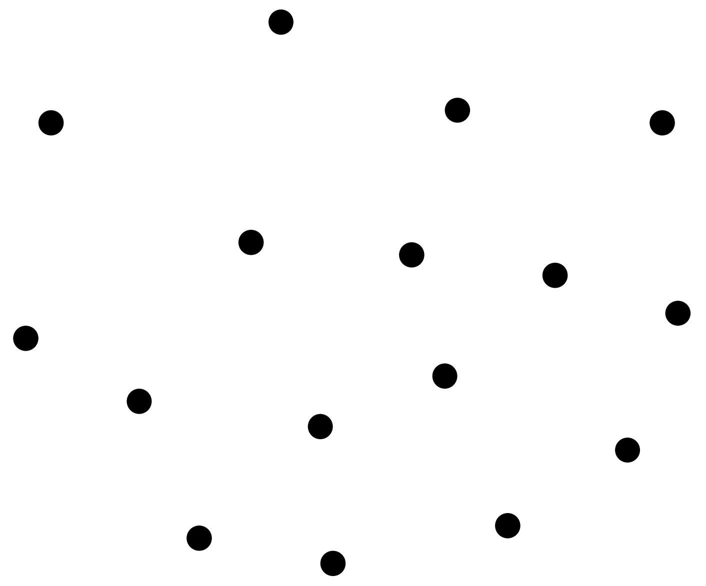
Observation (Hull edges): The edges of the convex hull must be pairs of points from the input

Claim (Hull edges): A segment (p_i, p_j) is on the convex hull if and only if…



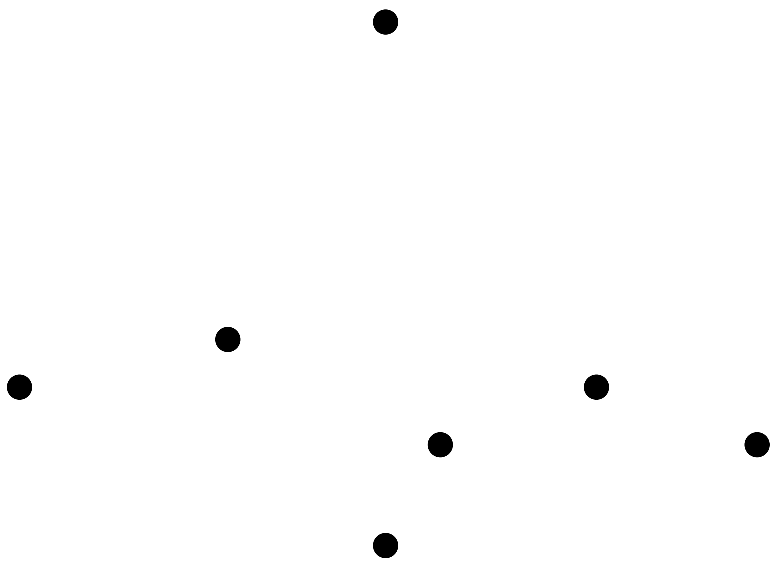
Better: An $O(n^2)$-time algorithm

Observation (Order helps): The $O(n^3)$-time algorithm found the hull edges in an arbitrary order… What if we try to find them in CCW order



Graham Scan: An $O(n \log n)$ algorithm

Observation (Order helps again): We went from $O(n^3)$ to $O(n^2)$ by finding the edges in order… but we still processed the points in an arbitrary order. Can we order the points and do better?



Graham Scan: An $O(n \log n)$ algorithm

Algorithm (Graham Scan):

Find lowest point p_0

Sort points $p_1, p_2, \ldots$ **counterclockwise by their angle with $\boldsymbol{p_0}$**

$\boldsymbol{H = [p_0, p_1]}$
for each point $i = 2 \ldots n-1$

Graham Scan: Complexity

Theorem: Graham Scan runs in $O(n \log n)$ time

Proof:

Lower Bound

Theorem: Any convex hull algorithm that uses line-side tests to find the hull requires $\Omega(n \log n)$ line-side tests (in a decision tree model)

Won't prove this

Take-home messages

- Computational geometry is all about using the right tools (and drawing good diagrams)
  - Dot product
  - Cross product
  - ***Line-side test***
  - ***Convex hull***