Algorithm Design and Analysis

Hashing: Universal and Perfect Hashing

Roadmap for today

- Breaking out of the comparison model, the ***word-RAM***
- Review the ***dictionary problem*** and motivate hashing
- See ***universal hashing*** and how to prove that a family is universal
- See an algorithm for ***static perfect hashing***

Quick announcement:

- ***Helpful hints on Piazza for Homework 1, Problem 3:***

Danny Sleator 3 minutes ago

In response to questions from office hours and on Piazza, here is some more scaffolding for problem 3:

For the induction hypothesis, we've said that the edges that have been said to exist should form a tree. Call this part 1 of the induction hypothesis.

As in the proof of connectivity, there will be another part of the induction hypothesis that deals with what we know about the pairs of nodes within the tree. Call this part 2. There will be a third part of the induction hypothesis that deals with what we know about the pairs of nodes (x,y) where x is in one tree and y is in a different tree.

For part 2, there may exist certain pairs within a tree which we must ensure have been probed already. Because for these pairs, whether or not an edge exists between them cannot change the bipartiteness of the graph, no matter how the remaining unqueried pairs are instantiated (as edges or non-edges). You need to define this part of the induction hypothesis by figuring out what those pairs are.

Part 3 is about the pairs of nodes (x, y) where x is in one tree T, and y is in a different tree T'. The trick here is classify all such pairs into two categories (in a specific way that you have to figure out), and you will want it to be the case that for both categories, there is an unprobed pair. If there is a probe to the last remaining pair of a category then you cannot answer "no" because that would violate the induction hypothesis. So you have to answer "yes" and merge the two trees together. (Caveat: the invariant will have to handle the special case when both trees are single nodes.)

Model of Computation

Formal model of computation

- We're leaving the comparison model today. We want to take advantage of integer-ness for more performance!!

- ***Model (word-RAM)***:
  - We have unlimited constant-time addressable memory ("registers")
  - Each register can store a w-bit integer (a "word")
  - Reading/writing, arithmetic, logic, bitwise operations on a constant number of words takes constant time
  - With input size n, we need $w \geq \log n$.

- This is just a more formal version of the "counting instructions" model you've probably been using in your prerequisite courses

Implications of the word-RAM

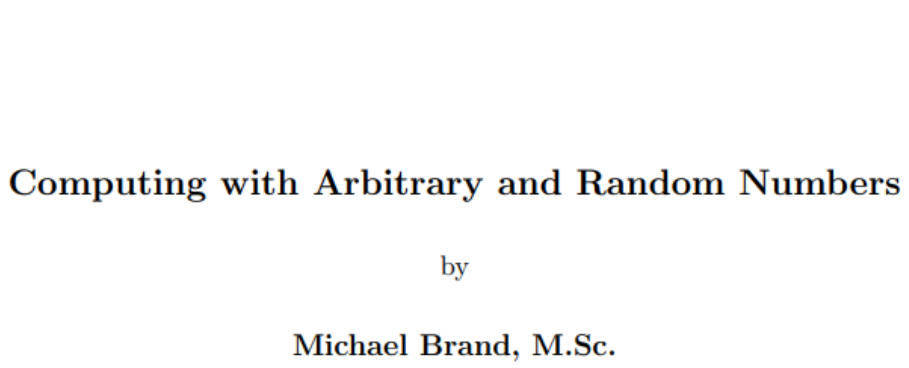
- Adding two b-bit integers gives a $(b+1)$-bit integer.
- Multiplying two b-bit integers gives a $2b$-bit integer.
- A constant number of these is therefore okay since the result fits in a constant number of registers.

- What if **multiply** n w-bit integers? We get a $\Theta(nw)$-bit answer! This ***does not fit*** in a single/constant number of registers!!
- Such an algorithm would therefore take ***more than*** $\Theta(n)$ time.

Take-home message: Be careful when doing lots of arithmetic. If your integers become more than a constant-factor larger in bits, it takes more than constant time.

Do we *really* need to restrict w?

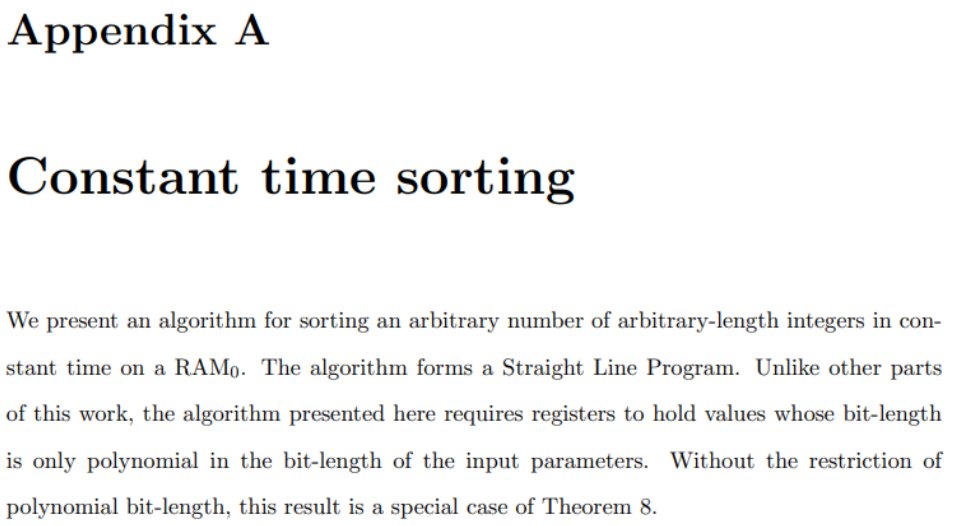
- Suppose we allow reading/writing/instructions on arbitrarily long integers
- This is usually called the unit-cost RAM (as opposed to the word-RAM)



Computing with Arbitrary and Random Numbers

by

Michael Brand, M.Sc.



Appendix A

Constant time sorting

We present an algorithm for sorting an arbitrary number of arbitrary-length integers in constant time on a RAM_0. The algorithm forms a Straight Line Program. Unlike other parts of this work, the algorithm presented here requires registers to hold values whose bit-length is only polynomial in the bit-length of the input parameters. Without the restriction of polynomial bit-length, this result is a special case of Theorem 8.

Dictionaries & Hashing

The dictionary problem

The dictionary data type stores ***items*** that have associated unique ***keys***

unique key ➡

STUDENT	
id:	*integer*
name:	*string*
grade:	*character*

Dictionary Interface

insert*(item)*: Insert the given item (associated with its key)

lookup*(key)*: Return the item with the given key if it exists

delete*(key)*: Delete the item with the given key if it exists

Python equivalent

d[key] = item

*item = d[key] (throws **KeyError** if not present)*

*d.pop(key) (throws **KeyError** if not present)*

Formal setup for hashing/hash tables

- The keys come from $U = [0 \ldots u - 1]$ (the ***universe*** of keys)
- We want to store items in a table A of size m. Assume $u \gg m$, so we can not just store key x at $A[x]$
- ***Key idea (Hash function):*** Define a function

$$h : U \to \{0,1, \ldots, m - 1\}$$

- Try to store item with key x at $A[h(x)]$

Handling collisions

Approach #1 (Open addressing): When a collision occurs, cleverly find a **different location in the table** for the new item

- Very hard to analyze, bad performance if not implemented well
- Amazing performance if done well! **All state-of-the-art hashtables do this**

Approach #2 (Chaining): Instead of storing a single item in each slot, store a **list of items**. Add all items that hash to that slot to the list

- Simple to analyze and implement
- Decent performance in practice, used by the C++ standard library
- Much easier to parallelize

Prehashing non-integer keys

Idea (prehashing): For non-integer keys, we want to convert them into some representative integer.

Example (strings): Strings can be interpreted as integers by interpreting each character as a digit, in base alphabet size

B A C Z

1 0 2 25

$$= \mathbf{1} \cdot \mathbf{26}^{\mathbf{3}} + \mathbf{0} \cdot \mathbf{26}^{\mathbf{2}} + \mathbf{2} \cdot \mathbf{26} + \mathbf{25}$$

$$= \mathbf{17653}$$

Choosing a hash function h

- **Main goal:** We want it to be unlikely that $h(x) = h(y)$ for $x \neq y$
- We want $m = O(n)$, where n is the number of keys in the table
  - We could just pick $m = u$ then there are no collisions!!
  - But this is an unacceptable amount of memory if $u \gg n$
- We also want $h(x)$ to be fast to compute. Ideally $O(1)$ time
- How long does a hashtable operation take using chaining?

Can we just pick… the best hash function?

- For any hash function you choose, I can find a set of n items that hash to the same location…
- There's no such thing as a hash function that works for every input.
- ***Big idea (randomization):*** We need to employ randomization to build a hash function that doesn't have a horrible worst-case behaviour
- **Specifically, we want to choose a random hash function from some big set of possible hash functions**

Random hash families

- ***Definition (totally random hash)***: A set $\mathcal{H}$ of hash functions is ***totally random*** if for all $x \in U$, $t \in \{0, \ldots m-1\}$, independent of all $y \in U$

$$\Pr_{h \in \mathcal{H}}[h(x) = t] = \frac{1}{m}$$

- Essentially equivalent to "*Simple uniform hashing*" (if you know it)

- Totally random hashing has all nice properties, but its not possible to do practically…

Less random, but still random

- **Goal**: We need a hash function that is still "pretty random", but not totally random, since that's too expensive

Definition (Universal Hashing): A set $\mathcal{H}$ of hash functions $h : U \to \{0, \dots, m-1\}$ is called ***universal*** if for all $x \neq y$

$$\Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \frac{1}{m}$$

Can compute probability by counting:

$$\Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \frac{|h(x) = h(y)|_{h \in \mathcal{H}}}{|\mathcal{H}|}$$

Examples: Universal or not?

$$|U| = 2, \qquad M = 2$$

	a	b
h_1	0	0
h_2	0	1

	a	b
h_1	0	0
h_2	1	1

	a	b
h_1	0	1
h_2	1	0

	a	b
h_1	0	1
h_2	1	0
h_3	0	1

More examples

$|U| = 3, \qquad M = 2$

	a	b	c
h_1	0	0	1
h_2	1	1	0
h_3	1	0	1

$|U| = 3, \qquad M = 3$

	a	b	c
h_1	0	0	0
h_2	0	1	2
h_3	1	2	0
h_4	2	0	1

Analysis of Universal Hashing

***Theorem*: If $\mathcal{H}$ is a universal family, then for any set $S \subseteq U$ with $|S| = n$, for any $x \in S$, if h is chosen at random from $\mathcal{H}$, then the **expected** number of collisions between x and other elements is at most n/m.**

Corollary

Definition (Load Factor): The quantity n/m is called the ***load factor***

Corollary: Using separate chaining, given a universal family $\mathcal{H}$, the expected cost of each operation is $O(1 + n/m)$

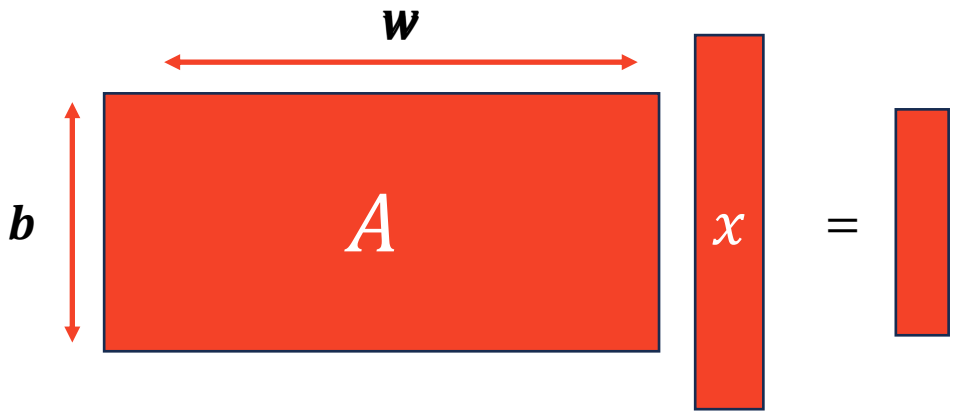
Corollary: Using separate chaining, given a universal family $\mathcal{H}$, if the load factor is always at most 1, for any sequence of L insert, lookup, delete operations, the expected cost of the L operations over a random $h \in \mathcal{H}$ is $O(L)$.

- Assumes h can be computed in $O(1)$ time

Okay… how do we construct one?

- Universal families sound great. How do we make one?

- ***Construction (Random binary matrix)***: Assume $|U| = 2^w, m = 2^b$
  - Let A be a random $w \times b$ matrix of zeros and ones
  - Interpret $x \in U$ as a w length vector of its bits
  - Let $h(x) = Ax \bmod 2$, again interpreting $h(x)$ as a b length vector of bits



Analysis of random binary matrix

Theorem: Its universal, i.e., for $x \neq y$, $\Pr_{h \in \mathcal{H}}[h(x) = h(y)] = \frac{1}{m}$

Wait, that's not constant time!

- How efficient is computing $h(x)$?

- Thankfully, there exists universal families whose hash functions can be computed in constant time (but they are harder to analyze).

Example (The multiplication method): Suppose $|U| = 2^w$ and choose a power of two table size $m = 2^r$ and a **random odd integer a**

$$h(x) = [(ax) \bmod 2^w] \gg (w - r)$$

Even more randomness!

- Can we make a hash family that is "more random" than universal, but still less than totally random? Yes!

- ***Definition (pairwise independent)*: A hash family $\mathcal{H}$ is called ***pairwise independent*** if for every pair $x_1 \neq x_2$ of distinct keys and every pair of values $v_1, v_2 \in \{0, \ldots, m-1\}$ (not necessarily distinct),**

$$\Pr_{h \in \mathcal{H}}[h(x_1) = v_1 \text{ and } h(x_2) = v_2] = \frac{1}{m^2}$$

Intuitively, for every pair of distinct keys (x_1, x_2), all pairs of values (v_1, v_2) are equally likely to occur (there are m^2 possible pairs of values).

Example

	a	b	c
h_1	0	0	0
h_2	0	1	1
h_3	1	0	1
h_4	1	1	0

	a	b
h_1	0	0
h_2	0	1
h_3	1	0
h_4	1	1

c
0
1
1
0

$= h(a) \oplus h(b)$

	a	c
h_1	0	0
h_2	0	1
h_3	1	1
h_4	1	0

b
0
1
0
1

$= h(a) \oplus h(c)$

	b	c
h_1	0	0
h_2	1	1
h_3	0	1
h_4	1	0

a
0
0
1
1

$= h(b) \oplus h(c)$

Even more randomness!

- ***Definition (k-wise independent)*:** A hash family $\mathcal{H}$ is called k-wise independent if for every set of k distinct keys $x_1, \dots, x_k$ and k values $v_1, \dots, v_k$ (not necessarily distinct) we have

$$\Pr_{h \in \mathcal{H}}[h(x_1) = v_1 \text{ and } \dots \text{ and } h(x_k) = v_k] = \frac{1}{m^k}$$

Question break

Static perfect hashing

Problem: Suppose we ***know the n keys in advance*** want deterministic constant query time in the worst case? Is this possible?

Idea: Reduce collision probability by making the table really really big!

Theorem: Given a universal family $\mathcal{H}$, taking $m = n^2$ gives us

$$\Pr_{h \in \mathcal{H}}[\text{no collisions}] \geq \frac{1}{2}$$

Some analysis

Theorem: Given a universal family $\mathcal{H}$, taking $m = n^2$ gives us

$$\Pr_{h \in \mathcal{H}}[\text{no collisions}] \geq \frac{1}{2}$$

That's a bit too much

- Okay, no collisions is nice, but n^2 space is way too much.
- Can we achieve the same with only $O(n)$ space?
- ***Idea***: The number of collisions per element is usually small anyway. Squaring those numbers might not be too big

FKS Hashing

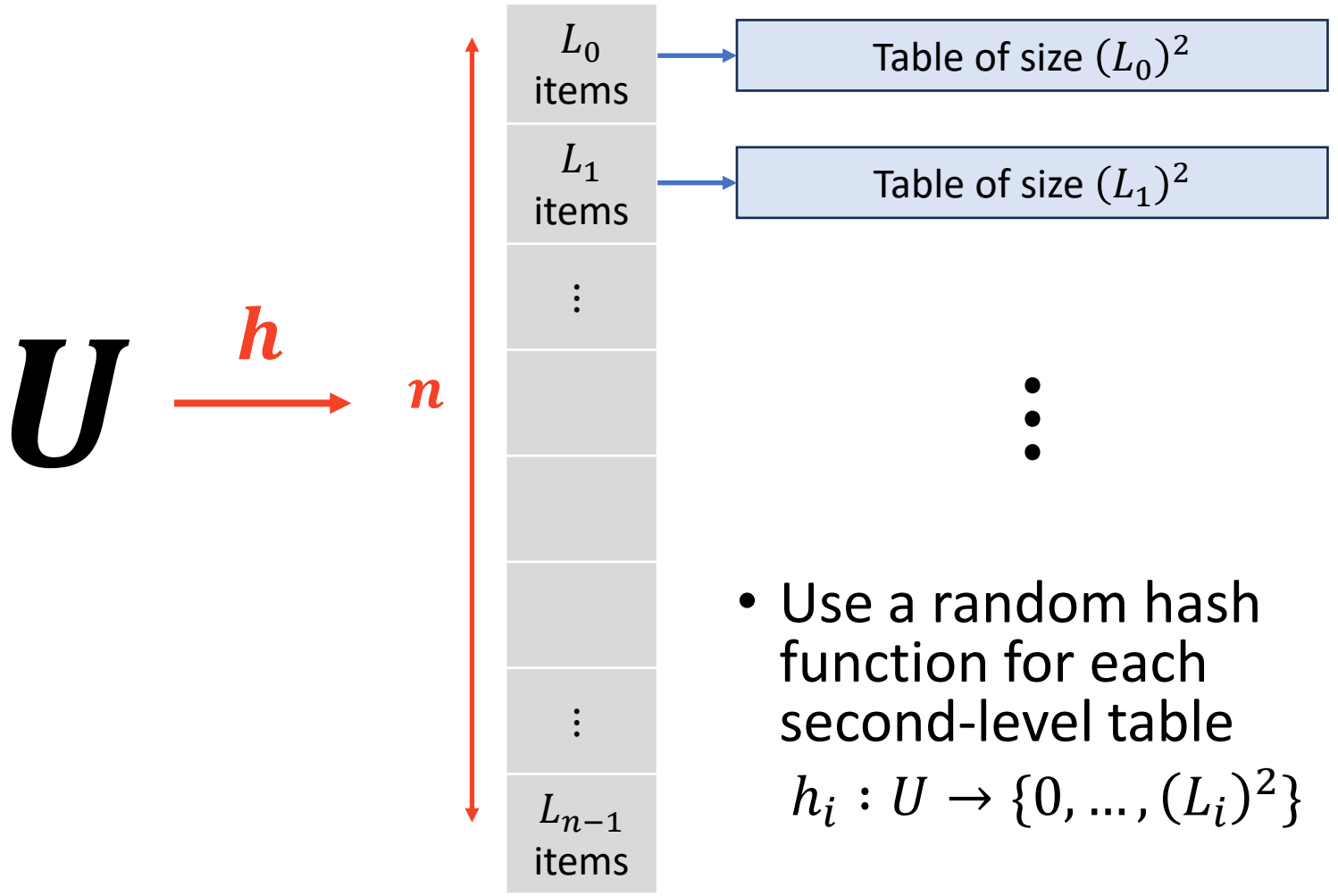
- Choose a hash function $h \in \mathcal{H}$ (universal)

$$h: U \to \{0, \dots, n-1\}$$

- Let L_i be the number of keys x such that

$$h(x) = i$$

- Store the L_i items at position i in a second-level table of size $(L_i)^2$



- Use a random hash function for each second-level table

$$h_i : U \to \{0, \dots, (L_i)^2\}$$

Analysis of second-level tables

- We know that for each second-level table, we have a $\geq 1/2$ probability that there are no collisions
- There are n such tables, so there are bound to be ***some*** with collisions

- ***Solution***: If there are collisions in a second-level table, just pick another random hash from the family until there isn't.

Analysis of top level

Theorem: If h is chosen from a universal family $\mathcal{H}$, then

$$\Pr_{h \in \mathcal{H}} \left[\sum L_i^2 > 4n \right] \leq \frac{1}{2}$$

Analysis continued…

Lemma: Define $C_{xy} = 1$ if $h(x) = h(y)$, else $C_{xy} = 0$

$$\sum (L_i)^2 = \sum\sum C_{xy}$$

Analysis continued continued…

Lemma: If h is chosen from a universal family $\mathcal{H}$, then

$$\mathbb{E}\left[\sum (L_i)^2\right] < 2N$$

Completing the analysis

Theorem: If h is chosen from a universal family $\mathcal{H}$, then

$$\Pr_{h \in \mathcal{H}} \left[\sum L_i^2 > 4n \right] \leq \frac{1}{2}$$

Summary of today

- Universal hashing gives us “enough” randomness to get nice results
  - Operations on a hash table with separate chaining run in $O(1 + n/m)$ time.
  - Static FKS hashing gives deterministic lookup in constant worst-case time.
- For “more randomness”, we can employ pairwise independent, or k-wise independent hashing.