

引力波天文学笔记

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第一章 引力波

1.1 Linearized Gravity

[4]. 流形 \mathbb{R}^4 . 任意坐标系 $\{x^\mu\}$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}s + O(s^2)$, 得

$$R_{\mu\nu\lambda\sigma} = \partial_\sigma \partial_{[\mu} h_{\lambda]\nu} - \partial_\nu \partial_{[\mu} h_{\lambda]\sigma} + O(s^2). \quad (1.1)$$

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\sigma}h_{\lambda\sigma} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h.$$

$$-\frac{1}{2}\partial^\lambda \partial_\lambda \bar{h}_{\mu\nu} + \partial^\lambda \partial_{(\mu} \bar{h}_{\nu)\lambda} - \frac{1}{2}\eta_{\mu\nu} \partial^\lambda \partial^\sigma \bar{h}_{\lambda\sigma} + O(s^2) = 8\pi T_{\mu\nu}. \quad (1.2)$$

存在 $\{x^\mu\}$, 使得 $\partial^\nu \bar{h}_{\mu\nu} + O(s^2) = 0$ (Lorentz gauge). 令 $\{x^\mu\}$ 满足 $\partial^\nu \bar{h}_{\mu\nu} + O(s^2) = 0$, 则

$$\partial^\lambda \partial_\lambda \bar{h}_{\mu\nu} + O(s^2) = -16\pi T_{\mu\nu}. \quad (1.3)$$

略去 $O(s^2)$ 条件: $h_{\mu\nu}, \partial_\lambda h_{\mu\nu} \dots$ 小.

1.2 Radiation Gauge

[4]. 存在 $\{x^\mu\}$, 使得 $h + O(s^2) = 0$ (TT gauge [5]) 且 $h_{0\mu} + O(s^2) = 0$.

1.3 Quadrupole Approximation

[4]. 下略 $O(s^2)$. 由(1.3)得

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4 \int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV'. \quad (1.4)$$

$$\hat{h}_{\mu\nu}(\omega, \vec{r}) := \frac{1}{\sqrt{2\pi}} \int \bar{h}_{\mu\nu}(t, \vec{r}) e^{i\omega t} dt \quad (1.5)$$

$$= 4 \int \frac{\hat{T}_{\mu\nu}(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega|\vec{r} - \vec{r}'|} dV'. \quad (1.6)$$

由 $\partial^\nu \bar{h}_{\mu\nu} = 0$,

$$-i\omega \hat{h}_{0\mu} = \sum_i \frac{\partial \hat{h}_{i\mu}}{\partial x^i}. \quad (1.7)$$

$|\vec{r}| \gg |\vec{r}'|$ 且 $\omega \ll 1/|\vec{r}'|$,

$$\hat{h}_{ij}(\omega, \vec{r}) = 4 \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{ij}(\omega, \vec{r}') dV'. \quad (1.8)$$

$$\int \hat{T}_{ij} dV' = \int \sum_k (\hat{T}_{kj} \frac{\partial x'^i}{\partial x'^k}) dV' \quad (1.9)$$

$$= \sum_k \left[\int \frac{\partial}{\partial x'^k} (\hat{T}_{kj} x'^i) dV' - \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \right] \quad (1.10)$$

$$= \sum_k \int \partial'_k (\hat{T}_{kj} x'^i) dV' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \quad (1.11)$$

$$= \int \hat{T}_{kj} x'^i dS' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \quad (1.12)$$

$$= - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \quad (1.13)$$

$$= - \int (\sum_k \partial'_k \hat{T}_{kj}) x'^i dV' \quad (1.14)$$

$$= - \int (\partial_0 \hat{T}_{0j}) x'^i dV' \quad (1.15)$$

$$= -i\omega \int \hat{T}_{0j} x'^i dV' \quad (1.16)$$

$$= \int \hat{T}_{(ij)} dV' \quad (1.17)$$

$$= -i\omega \int \hat{T}_{0(j} x'^i) dV' \quad (1.18)$$

$$= -\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) dV', \quad (1.19)$$

$$(1.20)$$

$$-\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) dV' = -\frac{i\omega}{2} \int \sum_k (\hat{T}_{0k} x'^i \frac{\partial x'^j}{\partial x'^k} + \hat{T}_{0k} \frac{\partial x'^i}{\partial x'^k} x'^j) dV' \quad (1.21)$$

$$= -\frac{i\omega}{2} \sum_k \left[\int \frac{\partial}{\partial x'^k} (\hat{T}_{0k} x'^i x'^j) dV' - \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \right] \quad (1.22)$$

$$= -\frac{i\omega}{2} \sum_k \int \partial'_k (\hat{T}_{0k} x'^i x'^j) dV' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \quad (1.23)$$

$$= -\frac{i\omega}{2} \sum_k \int \hat{T}_{0k} x'^i x'^j dS' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \quad (1.24)$$

$$= \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \quad (1.25)$$

$$= \frac{i\omega}{2} \int (\sum_k \partial'_k \hat{T}_{0k}) x'^i x'^j dV' \quad (1.26)$$

$$= \frac{i\omega}{2} \int (\partial_0 \hat{T}_{00}) x'^i x'^j dV' \quad (1.27)$$

$$= -\frac{\omega^2}{2} \int \hat{T}_{00} x'^i x'^j dV'. \quad (1.28)$$

$$q_{ij}(t) := \int T_{00} x'^i x'^j dV', \quad (1.29)$$

$$\hat{h}_{ij}(\omega, \vec{r}) = -2\omega^2 \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \hat{q}_{ij}(\omega), \quad (1.30)$$

$$\bar{h}_{ij}(t, \vec{r}) = \frac{2}{|\vec{r}|} \frac{d^2}{dt^2} q_{ij}(t - |\vec{r}|). \quad (1.31)$$

1.4 + Mode and \times Mode

寻新标架 $(e'^1)_a = (e^+)_a$, $(e'^2)_a = (e^\times)_a$, $(e'^3)_a = (e^r)_a$, $\bar{h}_{ij}(e^i)_a (e^j)_b = \bar{h}'_{ij}(e'^i)_a (e'^j)_b$, 取 x, y 分量后去迹, $h_+ = \frac{1}{2}(\bar{h}'_{11} - \bar{h}'_{22})$, $h_\times = \bar{h}'_{12} = \bar{h}'_{21}$? [3]
[1], $\vec{n} := \frac{\vec{r}}{|\vec{r}|}$,

$$h_{ij}^{\text{TT}} = \frac{2}{|\vec{r}|} \mathcal{P}_{ijkm} \frac{d^2}{dt^2} Q^{km}(t - |\vec{r}|), \quad (1.32)$$

$$\mathcal{P}_{ijkm} := (\delta_{ik} - \vec{n}_i \vec{n}_k) (\delta_{jm} - \vec{n}_j \vec{n}_m) - \frac{1}{2} (\delta_{ij} - \vec{n}_i \vec{n}_j) (\delta_{km} - \vec{n}_k \vec{n}_m), \quad (1.33)$$

$$Q^{km}(t) := \int T_{00} \left(x'^k x'^m - \frac{1}{3} \delta^{km} \sum_n x'^n x'^n \right) dV' \quad (1.34)$$

1.5 电磁—引力对比

$$A_\mu(t, \vec{r}) = \frac{\mu_0}{4\pi} \int \frac{J_\mu(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV' \quad (1.35)$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4G \int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV' \quad (1.36)$$

$$A_\mu(t, \vec{r}) = \frac{1}{\sqrt{2\pi}} \int \hat{A}_\mu(\omega, \vec{r}) e^{-i\omega t} d\omega \quad (1.37)$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = \frac{1}{\sqrt{2\pi}} \int \hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) e^{-i\omega t} d\omega \quad (1.38)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\hat{J}_\mu(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega |\vec{r} - \vec{r}'|} dV' \quad (1.39)$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G \int \frac{\hat{T}_{\mu\nu}(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega |\vec{r} - \vec{r}'|} dV' \quad (1.40)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega |\vec{r}|}}{|\vec{r}|} \int \hat{J}_\mu(\omega, \vec{r}') e^{-i\omega (\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}')} dV' \quad (1.41)$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G \frac{e^{i\omega |\vec{r}|}}{|\vec{r}|} \int \hat{T}_{\mu\nu}(\omega, \vec{r}') e^{-i\omega (\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}')} dV' \quad (1.42)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega |\vec{r}|}}{|\vec{r}|} \int \hat{J}_\mu(\omega, \vec{r}') \left[1 - i\omega \left(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}' \right) - \dots \right] dV' \quad (1.43)$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G \frac{e^{i\omega |\vec{r}|}}{|\vec{r}|} \int \hat{T}_{\mu\nu}(\omega, \vec{r}') \left[1 - i\omega \left(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}' \right) - \dots \right] dV' \quad (1.44)$$

1.5.1 电偶极—引力对比

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_i dV' \quad (1.45)$$

$$\hat{\bar{h}}_{ij} = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{ij} dV' \quad (1.46)$$

$$\int \hat{J}_i dV' = -i\omega \int \hat{J}_0 x'^i dV' \quad (1.47)$$

$$\int \hat{T}_{ij} dV' = -\frac{\omega^2}{2} \int \hat{T}_{00} x'^i x'^j dV' \quad (1.48)$$

$$p_i = \int \hat{J}_0 x'^i dV' \quad (1.49)$$

$$q_{ij} = \int \hat{T}_{00} x'^i x'^j dV' \quad (1.50)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega p_i) \quad (1.51)$$

$$\hat{\bar{h}}_{ij} = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \left(-\frac{\omega^2}{2} q_{ij}\right) \quad (1.52)$$

$$A_i = \frac{\mu_0}{4\pi} \frac{1}{|\vec{r}|} \frac{d}{dt} p_i(t - |\vec{r}|) \quad (1.53)$$

$$\bar{h}_{ij} = 4G \frac{1}{|\vec{r}|} \frac{1}{2} \frac{d^2}{dt^2} q_{ij}(t - |\vec{r}|) \quad (1.54)$$

1.5.2 电四极—引力对比

$$\hat{A}_i(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int \hat{J}_i(\omega, \vec{r}') \left(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}'\right) dV' \quad (1.55)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int \hat{J}'_i n^j x'_j dV' \quad (1.56)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int n^j x'_j \hat{J}'_i dV' \quad (1.57)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) n^j \left[\int x'_{(j} \hat{J}'_{i)} dV' \right] \quad (1.58)$$

$$\int x'_{(j} \hat{J}'_{i)} dV' = \frac{1}{2} \int (\hat{J}'_j x'_i + \hat{J}'_i x'_j) dV' \quad (1.59)$$

$$= \frac{1}{2} \int \sum_k (\hat{J}'_k x'^i \frac{\partial x'^j}{\partial x'^k} + \hat{J}'_k \frac{\partial x'^i}{\partial x'^k} x'^j) dV' \quad (1.60)$$

$$= \frac{1}{2} \sum_k \left[\int \frac{\partial}{\partial x'^k} (\hat{J}'_k x'^i x'^j) dV' - \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \right] \quad (1.61)$$

$$= \frac{1}{2} \sum_k \int \partial'_k (\hat{J}'_k x'^i x'^j) dV' - \frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \quad (1.62)$$

$$= \frac{1}{2} \sum_k \int \hat{J}'_k x'^i x'^j dS' - \frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \quad (1.63)$$

$$= -\frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \quad (1.64)$$

$$= -\frac{1}{2} \int (\sum_k \partial'_k \hat{J}'_k) x'^i x'^j dV' \quad (1.65)$$

$$= -\frac{1}{2} \int (\partial_0 \hat{J}'_0) x'^i x'^j dV' \quad (1.66)$$

$$= -\frac{i\omega}{2} \int \hat{J}'_0 x'^i x'^j dV' \quad (1.67)$$

$$D_{ij} = \int \hat{J}'_0 x'^i x'^j dV' \quad (1.68)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \left(-\frac{\omega^2}{2} n^j D_{ij} \right) \quad (1.69)$$

$$A_i = \frac{\mu_0}{4\pi} \frac{1}{|\vec{r}|} n^j \frac{1}{2} \frac{d^2}{dt^2} D_{ij}(t - |\vec{r}|) \quad (1.70)$$

第二章 电磁引力

[2].

2.1 时空张量转化为空间张量

$$h_{ab} := g_{ab} + Z_a Z_b. \quad (2.1)$$

$$h_a{}^b = \delta_a{}^b + Z_a Z^b. \quad (2.2)$$

$$Z^a h_{ab} = 0. \quad (2.3)$$

$$V_{\langle a} := h_a{}^b V_b. \quad (2.4)$$

$$Z^a V_{\langle a} = 0. \quad (2.5)$$

$$T_{\langle ab \rangle} := h_{(a}{}^c h_{b)}{}^d T_{cd} - \frac{1}{3} h_{cd} T^{cd} h_{ab}. \quad (2.6)$$

$$Z^a (h_a{}^c h_b{}^d T_{cd}) = 0. \quad (2.7)$$

$$Z^a (h_b{}^c h_a{}^d T_{cd}) = 0. \quad (2.8)$$

$$Z^a (h_{(a}{}^c h_{b)}{}^d T_{cd}) = 0. \quad (2.9)$$

$$Z^a (h_{cd} T^{cd} h_{ab}) = 0. \quad (2.10)$$

$$Z^a T_{\langle ab \rangle} = 0. \quad (2.11)$$

$$T_{\langle (ab) \rangle} = T_{\langle ab \rangle}. \quad (2.12)$$

$$h^{ab} T_{\langle ab \rangle} = 0. \quad (2.13)$$

$$\varepsilon_{abc} := \varepsilon_{abcd} Z^d. \quad (2.14)$$

$$\varepsilon_{0123} := -\sqrt{|g|}. \quad (2.15)$$

$$T_a := \frac{1}{2} \varepsilon_{abc} T^{[bc]}. \quad (2.16)$$

$$[U, V]_a := \varepsilon_{abc} U^b V^c. \quad (2.17)$$

$$[S, T]_a := \varepsilon_{abc} g_{de} S^{bd} T^{ce}. \quad (2.18)$$

$$D_t T^{a\dots}_{b\dots} := Z^c \nabla_c T^{a\dots}_{b\dots}. \quad (2.19)$$

$${}^3\nabla_a T^{b\dots}_{c\dots} := h_a{}^p h^b{}_q \dots h_c{}^r \dots \nabla_p T^{q\dots}_{r\dots}. \quad (2.20)$$

$$(\operatorname{div} V) := {}^3\nabla^a V_a. \quad (2.21)$$

$$(\operatorname{curl} V)_a := \varepsilon_{bca} {}^3\nabla^b V^c. \quad (2.22)$$

$$(\operatorname{div} T)_a := {}^3\nabla^b T_{ab}. \quad (2.23)$$

$$(\operatorname{curl} T)_{ab} := \varepsilon_{cd(a} {}^3\nabla^c g_{b)e} T^{ed}. \quad (2.24)$$

2.2 电磁空间矢量

$${}^*F_{ab} := \frac{1}{2} \varepsilon_{abcd} F^{cd} \quad (2.25)$$

$$E_a := F_{ab} Z^b = E_{\langle a \rangle}. \quad (2.26)$$

$$B_a := {}^*F_{ab} Z^b = B_{\langle a \rangle}. \quad (2.27)$$

$$\rho = -Z^a J_a. \quad (2.28)$$

$$j_a = h_a{}^b J_b. \quad (2.29)$$

$$\nabla_{[a} F_{bc]} = 0. \quad (2.30)$$

$$\nabla^a F_{ab} = \mu J_b. \quad (2.31)$$

$$(\operatorname{div} E) = \mu\rho - \dots \quad (2.32)$$

$$(\operatorname{div} B) = + \dots \quad (2.33)$$

$$(\operatorname{curl} E)_a + \dots = -D_t B_{\langle a} - \dots \quad (2.34)$$

$$(\operatorname{curl} B)_a + \dots = \mu j_a + D_t E_{\langle a} + \dots \quad (2.35)$$

2.3 引力空间张量

$${}^*C_{abcd} := \frac{1}{2}\varepsilon_{abef}C^ef_{cd}. \quad (2.36)$$

$$E_{ab} := C_{acbd}Z^cZ^d = E_{\langle ab \rangle}. \quad (2.37)$$

$$B_{ab} := {}^*C_{acbd}Z^cZ^d = B_{\langle ab \rangle}. \quad (2.38)$$

$$(\operatorname{div} E)_a = \kappa \frac{1}{3} \nabla_a \rho - \dots \quad (2.39)$$

$$(\operatorname{div} B)_a = \kappa(\rho + p)\omega_a + \dots \quad (2.40)$$

$$(\operatorname{curl} E)_{ab} + \dots = -D_t B_{\langle ab \rangle} - \dots \quad (2.41)$$

$$(\operatorname{curl} B)_{ab} + \dots = \kappa \frac{1}{2}(\rho + p)\sigma_{ab} + D_t E_{\langle ab \rangle} + \dots \quad (2.42)$$

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