引力波天文学笔记

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目录

第一章　引力波

1.1 Linearized Gravity

[6]. 流形 $\mathbb{R}^4$. 任意坐标系 $\{x^\mu\}$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} s + \mathrm{O}(s^2)$, 得

$$R_{\mu\nu\lambda\sigma} = \partial_\sigma \partial_{[\mu} h_{\lambda]\nu} - \partial_\nu \partial_{[\mu} h_{\lambda]\sigma} + \mathrm{O}(s^2). \tag{1.1}$$

$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\sigma}h_{\lambda\sigma} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$.

$$-\frac{1}{2}\partial^\lambda\partial_\lambda \bar{h}_{\mu\nu} + \partial^\lambda \partial_{(\mu}\bar{h}_{\nu)\lambda} - \frac{1}{2}\eta_{\mu\nu}\partial^\lambda\partial^\sigma \bar{h}_{\lambda\sigma} + \mathrm{O}(s^2) = 8\pi T_{\mu\nu}. \tag{1.2}$$

存在 $\{x^\mu\}$, 使得 $\partial^\nu \bar{h}_{\mu\nu} + \mathrm{O}(s^2) = 0$ (Lorentz gauge). 令 $\{x^\mu\}$ 满足 $\partial^\nu \bar{h}_{\mu\nu} + \mathrm{O}(s^2) = 0$, 则

$$\partial^\lambda\partial_\lambda \bar{h}_{\mu\nu} + \mathrm{O}(s^2) = -16\pi T_{\mu\nu}. \tag{1.3}$$

略去 $\mathrm{O}(s^2)$ 条件: $h_{\mu\nu}$, $\partial_\lambda h_{\mu\nu}$…小.

1.2 Radiation Gauge

[6]. 存在 $\{x^\mu\}$, 使得 "无源处" $h + \mathrm{O}(s^2) = 0$ (TT gauge [7]) 且 $h_{0\mu} + \mathrm{O}(s^2) = 0$.

1.3 Quadrupole Approximation

[6]. 下略 $\mathrm{O}(s^2)$. 由(1.3)得

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4\int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|}\,\mathrm{d}V'. \tag{1.4}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega,\vec{r}) := \frac{1}{\sqrt{2\pi}}\int \bar{h}_{\mu\nu}(t,\vec{r})e^{i\omega t}\mathrm{d}t \tag{1.5}$$

$$= 4\int \frac{\hat{T}_{\mu\nu}(\omega,\vec{r}')}{|\vec{r}-\vec{r}'|}e^{i\omega|\vec{r}-\vec{r}'|}\,\mathrm{d}V'. \tag{1.6}$$

由 $\partial^\nu \bar{h}_{\mu\nu} = 0$,

$$-i\omega\hat{\bar{h}}_{0\mu} = \sum_i \frac{\partial\hat{\bar{h}}_{i\mu}}{\partial x^i}. \tag{1.7}$$

$|\vec{r}| \gg |\vec{r}'|$ 且 $\omega \ll 1/|\vec{r}'|$,

$$\hat{\bar{h}}_{ij}(\omega,\vec{r}) = 4\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{T}_{ij}(\omega,\vec{r}')\,\mathrm{d}V'. \tag{1.8}$$

$$\int \hat{T}_{ij}\,\mathrm{d}V' = \int \sum_k (\hat{T}_{kj}\frac{\partial x'^i}{\partial x'^k})\,\mathrm{d}V' \tag{1.9}$$

$$= \sum_k \left[\int \frac{\partial}{\partial x'^k}(\hat{T}_{kj}x'^i)\,\mathrm{d}V' - \int \frac{\partial\hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V'\right] \tag{1.10}$$

$$= \sum_k \int \partial'_k\,(\hat{T}_{kj}x'^i)\,\mathrm{d}V' - \sum_k \int \frac{\partial\hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V' \tag{1.11}$$

$$= \int \hat{T}_{kj}x'^i\,\mathrm{d}S' - \sum_k \int \frac{\partial\hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V' \tag{1.12}$$

$$= -\sum_k \int \frac{\partial\hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V' \tag{1.13}$$

$$= -\int (\sum_k \partial'_k\hat{T}_{kj})x'^i\,\mathrm{d}V' \tag{1.14}$$

$$= -\int (\partial_0\hat{T}_{0j})x'^i\,\mathrm{d}V' \tag{1.15}$$

$$= -i\omega\int \hat{T}_{0j}x'^i\,\mathrm{d}V' \tag{1.16}$$

$$= \int \hat{T}_{(ij)}\,\mathrm{d}V' \tag{1.17}$$

$$= -i\omega\int \hat{T}_{0(j}x'^{i)}\,\mathrm{d}V' \tag{1.18}$$

$$= -\frac{i\omega}{2}\int (\hat{T}_{0j}x'^i + \hat{T}_{0i}x'^j)\,\mathrm{d}V', \tag{1.19}$$

$$\tag{1.20}$$

$$-\frac{i\omega}{2}\int(\hat{T}_{0j}x'^i+\hat{T}_{0i}x'^j)\,\mathrm{d}V' = -\frac{i\omega}{2}\int\sum_k(\hat{T}_{0k}x'^i\frac{\partial x'^j}{\partial x'^k}+\hat{T}_{0k}\frac{\partial x'^i}{\partial x'^k}x'^j)\,\mathrm{d}V' \tag{1.21}$$

$$= -\frac{i\omega}{2}\sum_k\left[\int\frac{\partial}{\partial x'^k}(\hat{T}_{0k}x'^ix'^j)\,\mathrm{d}V'-\int\frac{\partial\hat{T}_{0k}}{\partial x'^k}x'^ix'^j\,\mathrm{d}V'\right] \tag{1.22}$$

$$= -\frac{i\omega}{2}\sum_k\int\partial'_k\,(\hat{T}_{0k}x'^ix'^j)\,\mathrm{d}V'+\frac{i\omega}{2}\sum_k\int\frac{\partial\hat{T}_{0k}}{\partial x'^k}x'^ix'^j\,\mathrm{d}V' \tag{1.23}$$

$$= -\frac{i\omega}{2}\sum_k\int\hat{T}_{0k}x'^ix'^j\,\mathrm{d}S'+\frac{i\omega}{2}\sum_k\int\frac{\partial\hat{T}_{0k}}{\partial x'^k}x'^ix'^j\,\mathrm{d}V' \tag{1.24}$$

$$= \frac{i\omega}{2}\sum_k\int\frac{\partial\hat{T}_{0k}}{\partial x'^k}x'^ix'^j\,\mathrm{d}V' \tag{1.25}$$

$$= \frac{i\omega}{2}\int(\sum_k\partial'_k\hat{T}_{0k})x'^ix'^j\,\mathrm{d}V' \tag{1.26}$$

$$= \frac{i\omega}{2}\int(\partial_0\hat{T}_{00})x'^ix'^j\,\mathrm{d}V' \tag{1.27}$$

$$= -\frac{\omega^2}{2}\int\hat{T}_{00}\,x'^ix'^j\,\mathrm{d}V'. \tag{1.28}$$

$$q_{ij}(t):=\int T_{00}\,x'^ix'^j\,\mathrm{d}V', \tag{1.29}$$

$$\hat{\bar{h}}_{ij}(\omega,\vec{r})=-2\omega^2\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\hat{q}_{ij}(\omega), \tag{1.30}$$

$$\bar{h}_{ij}(t,\vec{r})=\frac{2}{|\vec{r}|}\frac{\mathrm{d}^2}{\mathrm{d}t^2}q_{ij}(t-|\vec{r}|). \tag{1.31}$$

1.4 + Mode and × Mode

寻新标架 $(e'^1)_a=(e^+)_a$, $(e'^2)_a=(e^\times)_a$, $(e'^3)_a=(e^r)_a$, $\bar{h}_{ij}(e^i)_a(e^j)_b=\bar{h}'_{ij}(e'^i)_a(e'^j)_b$, 取 x, y 分量后去迹, $h_+=\frac{1}{2}(\bar{h}'_{11}-\bar{h}'_{22})$, $h_\times=\bar{h}'_{12}=\bar{h}'_{21}$? [5] [2], $\vec{n}:=\frac{\vec{r}}{|\vec{r}|}$,

$$h_{ij}^{\mathrm{TT}}=\frac{2}{|\vec{r}|}\mathcal{P}_{ijkm}\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q^{km}(t-|\vec{r}|), \tag{1.32}$$

$$\mathcal{P}_{ijkm} := (\delta_{ik} - \vec{n}_i\vec{n}_k)(\delta_{jm} - \vec{n}_j\vec{n}_m) - \frac{1}{2}(\delta_{ij} - \vec{n}_i\vec{n}_j)(\delta_{km} - \vec{n}_k\vec{n}_m), \quad (1.33)$$

$$Q^{km}(t) := \int T_{00}\left(x'^k x'^m - \frac{1}{3}\delta^{km}\sum_n x'^n x'^n\right) \mathrm{d}V' \quad (1.34)$$

1.5 电磁—引力对比

$$A_\mu(t, \vec{r}) = \frac{\mu_0}{4\pi}\int \frac{J_\mu(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|}\mathrm{d}V' \quad (1.35)$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4G\int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|}\mathrm{d}V' \quad (1.36)$$

$$A_\mu(t, \vec{r}) = \frac{1}{\sqrt{2\pi}}\int \hat{A}_\mu(\omega, \vec{r})e^{-i\omega t}\mathrm{d}t \quad (1.37)$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = \frac{1}{\sqrt{2\pi}}\int \hat{\bar{h}}_{\mu\nu}(\omega, \vec{r})e^{-i\omega t}\mathrm{d}t \quad (1.38)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\hat{J}_\mu(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|}e^{i\omega|\vec{r} - \vec{r}'|}\mathrm{d}V' \quad (1.39)$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G\int \frac{\hat{T}_{\mu\nu}(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|}e^{i\omega|\vec{r} - \vec{r}'|}\mathrm{d}V' \quad (1.40)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{J}_\mu(\omega, \vec{r}')e^{-i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}')}\mathrm{d}V' \quad (1.41)$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{T}_{\mu\nu}(\omega, \vec{r}')e^{-i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}')}\mathrm{d}V' \quad (1.42)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{J}_\mu(\omega, \vec{r}')\left[1 - i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}') - \dots\right]\mathrm{d}V' \quad (1.43)$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{T}_{\mu\nu}(\omega, \vec{r}')\left[1 - i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}') - \dots\right]\mathrm{d}V' \quad (1.44)$$

1.5.1 电偶极—引力对比

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{J}_i \,\mathrm{d}V' \tag{1.45}$$

$$\hat{\bar{h}}_{ij} = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{T}_{ij} \,\mathrm{d}V' \tag{1.46}$$

$$\int \hat{J}_i \,\mathrm{d}V' = -i\omega \int \hat{J}_0 x'^i \,\mathrm{d}V' \tag{1.47}$$

$$\int \hat{T}_{ij} \,\mathrm{d}V' = -\frac{\omega^2}{2}\int \hat{T}_{00}\, x'^i x'^j \,\mathrm{d}V' \tag{1.48}$$

$$\hat{p}_i = \int \hat{J}_0 x'^i \,\mathrm{d}V' \tag{1.49}$$

$$\hat{q}_{ij} = \int \hat{T}_{00}\, x'^i x'^j \,\mathrm{d}V' \tag{1.50}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega\hat{p}_i) \tag{1.51}$$

$$\hat{\bar{h}}_{ij} = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-\frac{\omega^2}{2}\hat{q}_{ij}) \tag{1.52}$$

$$A_i = \frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|}\frac{\mathrm{d}}{\mathrm{d}t}p_i(t-|\vec{r}|) \tag{1.53}$$

$$\bar{h}_{ij} = 4G\frac{1}{|\vec{r}|}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}q_{ij}(t-|\vec{r}|) \tag{1.54}$$

1.5.2 电四极—引力对比

$$\hat{A}_i(\omega,\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\;(-i\omega)\int \hat{J}_i(\omega,\vec{r}')(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}')\,\mathrm{d}V' \tag{1.55}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega)\int \hat{J}'_i n^j x'_j \,\mathrm{d}V' \tag{1.56}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega)\int n^j x'_j \hat{J}'_i \,\mathrm{d}V' \tag{1.57}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega)n^j\left[\int x'_{(j}\hat{J}'_{i)}\,\mathrm{d}V'\right] \tag{1.58}$$

$$\begin{aligned}
\int x'_{(j}\hat{J}'_{i)}\,\mathrm{d}V' &= \frac{1}{2}\int(\hat{J}'_j x'_i + \hat{J}'_i x'_j)\,\mathrm{d}V' &(1.59)\\
&= \frac{1}{2}\int\sum_k(\hat{J}'_k x'^i\frac{\partial x'^j}{\partial x'^k} + \hat{J}'_k\frac{\partial x'^i}{\partial x'^k}x'^j)\,\mathrm{d}V' &(1.60)\\
&= \frac{1}{2}\sum_k\left[\int\frac{\partial}{\partial x'^k}(\hat{J}'_k x'^i x'^j)\,\mathrm{d}V' - \int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V'\right] &(1.61)\\
&= \frac{1}{2}\sum_k\int\partial'_k\,(\hat{J}'_k x'^i x'^j)\,\mathrm{d}V' - \frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' &(1.62)\\
&= \frac{1}{2}\sum_k\int\hat{J}'_k x'^i x'^j\,\mathrm{d}S' - \frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' &(1.63)\\
&= -\frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' &(1.64)\\
&= -\frac{1}{2}\int(\sum_k\partial'_k\hat{J}'_k)x'^i x'^j\,\mathrm{d}V' &(1.65)\\
&= -\frac{1}{2}\int(\partial_0\hat{J}'_0)x'^i x'^j\,\mathrm{d}V' &(1.66)\\
&= -\frac{i\omega}{2}\int\hat{J}'_0 x'^i x'^j\,\mathrm{d}V' &(1.67)
\end{aligned}$$

$$\hat{D}_{ij} = \int\hat{J}'_0\,x'^i x'^j\,\mathrm{d}V' \tag{1.68}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-\frac{\omega^2}{2}n^j\hat{D}_{ij}) \tag{1.69}$$

$$A_i = \frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|}n^j\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}D_{ij}(t-|\vec{r}|) \tag{1.70}$$

1.6 Varying G

$$T_{ab} = 2U_{(a}J_{b)} + U^c J_c U_a U_b \tag{1.71}$$

$$J_b = -U^a T_{ab} \tag{1.72}$$

$$\partial^a \bar{h}_{ab} = 0 \tag{1.73}$$

$$\partial^c \partial_c \bar{h}_{ab} = -16\pi \frac{G_0}{c_0^3} T_{ab} \tag{1.74}$$

$$\Gamma^c{}_{ab} = \frac{1}{2}\eta^{cd}(2\partial_{(a}h_{b)d} - \partial_d h_{ab}) \tag{1.75}$$

$$U^a \partial_a U^c + \Gamma^c{}_{ab} U^a U^b = 0 \tag{1.76}$$

$$U^a \partial_a U^c = -\frac{1}{2}\eta^{cd}(2\partial_{(a}h_{b)d} - \partial_d h_{ab}) U^a U^b \tag{1.77}$$

$$A_b = -\frac{1}{4} U^a \bar{h}_{ab} \tag{1.78}$$

$$A_0 = -\frac{1}{4} c_0 \bar{h}_{00} = -\frac{1}{2} c_0 (\bar{h}_{00} - \frac{1}{2}\eta_{00}\eta^{00}\bar{h}_{00}) = -\frac{1}{2} c_0 h_{00} \tag{1.79}$$

$$A_i = -\frac{1}{4} c_0 \bar{h}_{0i} = -\frac{1}{4} c_0 h_{0i} \tag{1.80}$$

$$U^\mu \partial_\mu U^i = -\frac{1}{2}\eta^{i\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) U^\mu U^\nu \tag{1.81}$$

$$\begin{aligned}
-\frac{1}{2}\eta^{i\sigma}(\partial_0 h_{0\sigma} + \partial_0 h_{0\sigma} - \partial_\sigma h_{00}) U^0 U^0 &= \frac{1}{2} c_0^2 \eta^{i\sigma} \partial_\sigma h_{00} && (1.82)\\
&= \frac{1}{2} c_0^2 \partial^i h_{00} && (1.83)\\
&= c_0 \partial^i A_0 && (1.84)\\
&= -E^i && (1.85)
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{2}\eta^{i\sigma}(\partial_0 h_{j\sigma} + \partial_j h_{0\sigma} - \partial_\sigma h_{0j}) U^0 U^j &= -\frac{1}{2} c_0 \eta^{i\sigma}(\partial_j h_{0\sigma} - \partial_\sigma h_{0j}) v^j && (1.86)\\
&= -\frac{1}{2} c_0 \eta^{ik}(\partial_j h_{0k} - \partial_k h_{0j}) v^j && (1.87)\\
&= 2\eta^{ik}(\partial_j A_k - \partial_k A_j) v^j && (1.88)\\
&= 2(\partial_j A^i - \partial^i A_j) v^j && (1.89)\\
&= -2\varepsilon^i{}_{jk} v^j B^k && (1.90)
\end{aligned}$$

$$-\frac{1}{2}\eta^{i\sigma}(\partial_j h_{k\sigma} + \partial_k h_{j\sigma} - \partial_\sigma h_{jk}) U^j U^k = 0 \tag{1.91}$$

$$a^i = -E^i - 4\varepsilon^i{}_{jk} v^j B^k \tag{1.92}$$

$$\partial^i \left(\frac{c_0}{16\pi G_0} E_i\right) = \rho \tag{1.93}$$

$$\partial^i B_i = 0 \tag{1.94}$$

$$\varepsilon^i{}_{jk} \partial^j E^k = -\partial_t B^i \tag{1.95}$$

$$\varepsilon^i{}_{jk} \partial^j \left(\frac{c_0^3}{16\pi G_0} B^k\right) = j^i + \partial_t \left(\frac{c_0}{16\pi G_0} E^i\right) \tag{1.96}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\mu \vec{H}) \tag{1.97}$$

$$\vec{\nabla} \times \vec{H} = +\frac{\partial}{\partial t}(\varepsilon \vec{E}) \tag{1.98}$$

$$\vec{\nabla} \cdot (\varepsilon \vec{E}) = 0 \tag{1.99}$$

$$\vec{\nabla} \cdot (\mu \vec{H}) = 0 \tag{1.100}$$

$$\vec{\nabla}^2 \vec{E} - \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \varepsilon\mu \frac{\partial^2}{\partial t^2} \vec{E} = 0 \tag{1.101}$$

$$\vec{\nabla}^2 \vec{H} - \vec{\nabla}(\vec{\nabla} \cdot \vec{H}) - \varepsilon\mu \frac{\partial^2}{\partial t^2} \vec{H} = 0 \tag{1.102}$$

$$\varepsilon \vec{\nabla} \cdot \vec{E} + \vec{\nabla}\varepsilon \cdot \vec{E} = 0 \tag{1.103}$$

$$\mu \vec{\nabla} \cdot \vec{H} + \vec{\nabla}\mu \cdot \vec{H} = 0 \tag{1.104}$$

$$\vec{\nabla}^2 \vec{H} - \vec{\nabla}(\vec{\nabla} \cdot \vec{H}) - \frac{\partial^2}{\partial (v_{\mathrm{p}} t)^2} \vec{H} = 0 \tag{1.105}$$

$$\mu \vec{\nabla} \cdot \vec{H} + \vec{\nabla}\mu \cdot \vec{H} = 0 \tag{1.106}$$

$$\vec{\nabla}\mu \cdot \vec{H} = 0 \tag{1.107}$$

$$\vec{\nabla}^2 \vec{H} - \frac{\partial^2}{\partial (v_{\mathrm{p}} t)^2} \vec{H} = 0 \tag{1.108}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{1.109}$$

$$4\pi r_1^2 (\vec{E} \times \vec{H})|_{t=0, r=r_1} = 4\pi r_2^2 (\vec{E} \times \vec{H})|_{t=\int_{r_1}^{r_2} \frac{\mathrm{d}r}{v_{\mathrm{p}}}, r=r_1} \tag{1.110}$$

$$4\pi r_1^2 (\vec{S})|_{t=0, r=r_1} = 4\pi r_2^2 (\vec{S})|_{t=\int_{r_1}^{r_2} \frac{\mathrm{d}r}{v_{\mathrm{p}}}, r=r_1} \tag{1.111}$$

$$S_{\mathrm{G}} \propto \dot{h}^2 \propto \omega^2 h^2 \tag{1.112}$$

$$S_{\mathrm{G}} \propto \frac{c^3}{G}\omega^2 h^2 \tag{1.113}$$

双星系统引力辐射本为

$$h = \frac{\mathcal{M}[\pi\mathcal{M}F(t)]^{2/3}}{r} Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi F(t)\,\mathrm{d}t] \tag{1.114}$$

设双星系统常量 c^*, G^*, 一观者临近双星系统且与双星系统相对静止, 其与双星系统距离为 r, 测得强度 h_r, 频率 F_r, 则[1]

$$h_r = \frac{\mathcal{M}[\pi\mathcal{M}F_r(t)]^{2/3}}{r/c^*} Q(\theta,\phi,\psi,\iota) \tag{1.115}$$

与双星系统距离为 r 的观者测得的引力辐射光度 $L_r \propto 4\pi r^2({c^*}^3/G^*)F_r^2 h_r^2$, 设地球观者与双星系统距离为 d, 双星系统红移为 z, 测得强度 h_d, 频率 F_d, 则地球观者测得的引力辐射光度正比于 $L_d \propto 4\pi d^2(c^3/G)F_d^2 h_d^2$, 且有 $L_d = L_r/(1+z)^2$, 所以 $r^2({c^*}^3/G^*)F_r^2 h_r^2/(1+z)^2 = d^2(c^3/G)F_d^2 h_d^2$, 又有 $F_d = F_r/(1+z)$, 所以 $r^2({c^*}^3/G^*)h_r^2 = d^2(c^3/G)h_d^2$, 则

$$h_d = \sqrt{\frac{{c^*}^3/G^*}{c^3/G}}\frac{r^2}{d^2}h_r \tag{1.116}$$

$$= \sqrt{\frac{{c^*}^3/G^*}{c^3/G}}\frac{\mathcal{M}[\pi\mathcal{M}F_r(t)]^{2/3}}{d/c^*} Q(\theta,\phi,\psi,\iota) \tag{1.117}$$

所以地球观者测得

$$h = \sqrt{\frac{{c^*}^3/G^*}{c^3/G}}\frac{\mathcal{M}[\pi\mathcal{M}F_r(t)]^{2/3}}{d/c^*} Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi\frac{F_r(t)}{1+z}\,\mathrm{d}t] \tag{1.118}$$

记 $F_{\mathrm{obs}}(t) = F_r(t)/(1+z)$, $\mathcal{M}_{\mathrm{obs}} = \mathcal{M}(1+z)$, 光度距离 $d_{\mathrm{L}} = d(1+z)$, 则

$$h = \sqrt{\frac{{c^*}^3/G^*}{c^3/G}}\frac{\mathcal{M}[\pi\mathcal{M}F_r(t)]^{2/3}}{d(1+z)/c^*} Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi F_{\mathrm{obs}}(t)\,\mathrm{d}t] \tag{1.119}$$

$$= \sqrt{\frac{{c^*}^3/G^*}{c^3/G}}\frac{\mathcal{M}_{\mathrm{obs}}[\pi\mathcal{M}_{\mathrm{obs}}F_{\mathrm{obs}}(t)]^{2/3}}{d_{\mathrm{L}}/c^*} Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi F_{\mathrm{obs}}(t)\,\mathrm{d}t] \tag{1.120}$$

$$= \sqrt{\frac{{c^*}^5/G^*}{c^5/G}}\frac{\mathcal{M}_{\mathrm{obs}}[\pi\mathcal{M}_{\mathrm{obs}}F_{\mathrm{obs}}(t)]^{2/3}}{d_{\mathrm{L}}/c} Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi F_{\mathrm{obs}}(t)\,\mathrm{d}t] \tag{1.121}$$

---

[1] $\mathcal{M}$ 和 c^*, G^* 简并, 所以可以笼统地仍记作 $\mathcal{M}$.

用引力波测距测得 $d_{\mathrm{L,G}}$, 则

$$d_{\mathrm{L,G}} = d_{\mathrm{L}} \sqrt{\frac{c^5/G}{c^{*5}/G^*}} \tag{1.122}$$

[4]

$$h(t) = \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{\xi\, d_{\mathrm{L}}} Q(\text{angles}) \cos\Phi(t) \tag{1.123}$$

$$\tilde{h}(f) = \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{\xi\, d_{\mathrm{L}}} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4}]} \tag{1.124}$$

问题转化为估计 ξ

$$p(\mu) \propto p^{(0)}(\mu) \exp[-\frac{1}{2}\Gamma_{ab}(\mu^a - \hat{\mu}^a)(\mu^b - \hat{\mu}^b)] \tag{1.125}$$

$$p^{(0)}(\mu) \propto \exp[-\frac{1}{2}\Gamma^{(0)}_{ab}(\mu^a - \bar{\mu}^a)(\mu^b - \bar{\mu}^b)] \tag{1.126}$$

设待估参数为 $\mu = (\ln\xi, \ln(d_{\mathrm{L}}/d_{\mathrm{L0}}), \ln Q, \ldots)$, $\ldots$ 为其他参数 (如 $\mathcal{M}$), 则 $\tilde{h}_{,\ln\xi} = \tilde{h}_{,\ln(d_{\mathrm{L}}/d_{\mathrm{L0}})} = -\tilde{h}_{,\ln Q} = -\tilde{h}$, $\tilde{h}$ 对其他参数求偏导皆为纯虚数, 则由 $\Gamma_{ab} = \langle h_{,a} | h_{,b} \rangle$ 和 $\mathrm{SNR} := \rho = \sqrt{\langle h|h \rangle}$ 得

$$\Gamma_{ab} = \begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 & 0 & \ldots \\ \rho^2 & \rho^2 & -\rho^2 & 0 & \ldots \\ -\rho^2 & -\rho^2 & \rho^2 & 0 & \ldots \\ 0 & 0 & 0 & ? & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.127}$$

又设

$$\Gamma^{(0)}_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ 0 & 1/\sigma^2_{\ln d_{\mathrm{L}}} & 0 & 0 & \ldots \\ 0 & 0 & 1/\sigma^2_{\ln Q} & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.128}$$

则由 $\Sigma_{ab} = (\Gamma^{(0)}_{ab} + \Gamma_{ab})^{-1}$ 得

$$\Sigma_{ab} = \begin{bmatrix} \begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 \\ \rho^2 & \rho^2 + 1/\sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L0}})} & -\rho^2 \\ -\rho^2 & -\rho^2 & \rho^2 + 1/\sigma^2_{\ln Q} \end{bmatrix}^{-1} & 0 \\ 0 & [?]^{-1} \end{bmatrix} \tag{1.129}$$

而

$$\begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 \\ \rho^2 & \rho^2 + 1/\sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L}0})} & -\rho^2 \\ -\rho^2 & -\rho^2 & \rho^2 + 1/\sigma^2_{\ln Q} \end{bmatrix}^{-1} \tag{1.130}$$

$$= \begin{bmatrix} 1/\rho^2 + \sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L}0})} + \sigma^2_{\ln Q} & -\sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L}0})} & \sigma^2_{\ln Q} \\ -\sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L}0})} & \sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L}0})} & 0 \\ \sigma^2_{\ln Q} & 0 & \sigma^2_{\ln Q} \end{bmatrix} \tag{1.131}$$

第二章　能量

第三章　双星系统

3.1　基本公式

$$\mathcal{M} := \mu^{3/5} M^{2/5} \tag{3.1}$$

$$h_+ = \frac{4\mathcal{M}}{D}[\pi \mathcal{M} F(t)]^{2/3} \frac{1+\cos^2\iota}{2} \cos\Phi(t) \tag{3.2}$$

$$h_\times = \frac{4\mathcal{M}}{D}[\pi \mathcal{M} F(t)]^{2/3} \cos\iota \sin\Phi(t) \tag{3.3}$$

$$h = F_+ h_+ + F_\times h_\times \tag{3.4}$$

3.2　Post-Newtonian Approximation

3.3　Stationary Phase Approximation

[4], if $\zeta(t)$ varies slowly near $t = t_0$ where the phase has a stationary point: $\phi'(t_0) = 0$,

$$\begin{aligned}
\int \zeta(t) e^{i\phi(t;f)}\,\mathrm{d}t &= \int \zeta(t) e^{i[\phi(t_0)+\phi'(t_0)(t-t_0)+\frac{1}{2}\phi''(t_0)(t-t_0)^2+\dots]}\,\mathrm{d}t && (3.5)\\
&\simeq e^{i\phi(t_0)} \int \zeta(t) e^{i[\frac{1}{2}\phi''(t_0)(t-t_0)^2]}\,\mathrm{d}t && (3.6)\\
&\simeq e^{i\phi(t_0)} \int \zeta(t_0) e^{\frac{-\sqrt{-i\phi''(t_0)}^2(t-t_0)^2}{2}}\,\mathrm{d}t && (3.7)\\
&= \frac{\sqrt{2\pi}}{\sqrt{-i\phi''(t_0)}} \zeta(t_0) e^{i\phi(t_0)}. && (3.8)
\end{aligned}$$

$$
\begin{aligned}
h &= \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\cos\Phi(t) && (3.9)\\
&= \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}[e^{i\Phi(t)}+e^{-i\Phi(t)}] && (3.10)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h}(f) &= \int h(t)e^{i2\pi ft}\,\mathrm{d}t && (3.11)\\
&= \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}[e^{i\Phi(t)}+e^{-i\Phi(t)}]e^{i2\pi ft}\,\mathrm{d}t && (3.12)\\
&= \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}\{e^{i[2\pi ft+\Phi(t)]}+e^{i[2\pi ft-\Phi(t)]}\}\,\mathrm{d}t && (3.13)\\
&\simeq \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}e^{i[2\pi ft-\Phi(t)]}\,\mathrm{d}t && (3.14)\\
&= \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F]^{2/3}Q\frac{1}{2}e^{i[2\pi ft(F)-\Phi(F)]}\frac{\mathrm{d}t}{\mathrm{d}F}\,\mathrm{d}F && (3.15)\\
&\simeq \frac{\sqrt{2\pi}}{\sqrt{-i[2\pi ft(F)-\Phi(F)]''_{F=f}}} && (3.16)\\
&\left[\frac{\mathcal{M}}{D}(\pi\mathcal{M}F)^{2/3}Q\frac{1}{2}\frac{\mathrm{d}t}{\mathrm{d}F}\right]_{F=f} e^{i[2\pi ft(f)-\Phi(f)]} && (3.17)\\
&\simeq \frac{\sqrt{2\pi}}{\sqrt{-i\left\{2\pi f\left[-\frac{5}{256}\mathcal{M}(\pi\mathcal{M}F)^{-8/3}\right]-\left[\frac{1}{16}(\pi\mathcal{M}F)^{-5/3}\right]\right\}''_{F=f}}} && (3.18)\\
&\left\{\frac{\mathcal{M}}{D}(\pi\mathcal{M}F)^{2/3}Q\frac{1}{2}\left[\frac{5\pi\mathcal{M}^2}{96}(\pi\mathcal{M}F)^{-11/3}\right]\right\}_{F=f} e^{i[2\pi ft(f)-\Phi(f)]} && (3.19)\\
&= \frac{\sqrt{30}}{48\pi^{2/3}}\frac{\mathcal{M}^{5/6}Q}{D}f^{-7/6}e^{i[2\pi ft(f)-\Phi(f)-\frac{\pi}{4}]} \quad (\texttt{pnspa.py}) && (3.20)
\end{aligned}
$$

或 [1], $h(t)=2A(t)\cos\phi(t)$, $\mathrm{d}\ln A/\mathrm{d}t \ll \mathrm{d}\phi/\mathrm{d}t$ 且 $|\mathrm{d}^2\phi/\mathrm{d}t^2| \ll (\mathrm{d}\phi/\mathrm{d}t)^2$.

第四章　宇宙学效应

第五章　电磁引力

[3].

5.1　时空张量转化为空间张量

$$h_{ab} := g_{ab} + Z_a Z_b. \tag{5.1}$$

$$h_a{}^b = \delta_a{}^b + Z_a Z^b. \tag{5.2}$$

$$Z^a h_{ab} = 0. \tag{5.3}$$

$$V_{\langle a\rangle} := h_a{}^b V_b. \tag{5.4}$$

$$Z^a V_{\langle a\rangle} = 0. \tag{5.5}$$

$$T_{\langle ab\rangle} := h_{(a}{}^c h_{b)}{}^d T_{cd} - \frac{1}{3} h_{cd} T^{cd} h_{ab}. \tag{5.6}$$

$$Z^a (h_a{}^c h_b{}^d T_{cd}) = 0. \tag{5.7}$$

$$Z^a (h_b{}^c h_a{}^d T_{cd}) = 0. \tag{5.8}$$

$$Z^a (h_{(a}{}^c h_{b)}{}^d T_{cd}) = 0. \tag{5.9}$$

$$Z^a (h_{cd} T^{cd} h_{ab}) = 0. \tag{5.10}$$

$$Z^a T_{\langle ab\rangle} = 0. \tag{5.11}$$

$$T_{(\langle ab\rangle)} = T_{\langle ab\rangle}. \tag{5.12}$$

$$h^{ab} T_{\langle ab\rangle} = 0. \tag{5.13}$$

$$\varepsilon_{abc} := \varepsilon_{abcd} Z^d. \tag{5.14}$$

$$\varepsilon_{0123} := -\sqrt{|g|}. \tag{5.15}$$

$$T_a := \frac{1}{2} \varepsilon_{abc} T^{[bc]}. \tag{5.16}$$

$$[U, V]_a := \varepsilon_{abc} U^b V^c. \tag{5.17}$$

$$[S, T]_a := \varepsilon_{abc} g_{de} S^{bd} T^{ce}. \tag{5.18}$$

$$\mathrm{D}_t T^{a\ldots}{}_{b\ldots} := Z^c \nabla_c T^{a\ldots}{}_{b\ldots}. \tag{5.19}$$

$${}^3\nabla_a T^{b\ldots}{}_{c\ldots} := h_a{}^p h^b{}_q \ldots h_c{}^r \ldots \nabla_p T^{q\ldots}{}_{r\ldots}. \tag{5.20}$$

$$(\operatorname{div} V) := {}^3\nabla^a V_a. \tag{5.21}$$

$$(\operatorname{curl} V)_a := \varepsilon_{bca} {}^3\nabla^b V^c. \tag{5.22}$$

$$(\operatorname{div} T)_a := {}^3\nabla^b T_{ab}. \tag{5.23}$$

$$(\operatorname{curl} T)_{ab} := \varepsilon_{cd(a} {}^3\nabla^c g_{b)e} T^{ed}. \tag{5.24}$$

5.2 电磁空间矢量

$${}^*F_{ab} := \frac{1}{2} \varepsilon_{abcd} F^{cd} \tag{5.25}$$

$$E_a := F_{ab} Z^b = E_{\langle a \rangle}. \tag{5.26}$$

$$B_a := {}^*F_{ab} Z^b = B_{\langle a \rangle}. \tag{5.27}$$

$$\rho = -Z^a J_a. \tag{5.28}$$

$$j_a = h_a{}^b J_b. \tag{5.29}$$

$$\nabla_{[a} F_{bc]} = 0. \tag{5.30}$$

$$\nabla^a F_{ab} = \mu J_b. \tag{5.31}$$

$$(\operatorname{div} E) = \mu\rho - \dots. \tag{5.32}$$

$$(\operatorname{div} B) = + \dots. \tag{5.33}$$

$$(\operatorname{curl} E)_a + \dots = -\mathrm{D}_t B_{\langle a\rangle} - \dots. \tag{5.34}$$

$$(\operatorname{curl} B)_a + \dots = \mu j_a + \mathrm{D}_t E_{\langle a\rangle} + \dots. \tag{5.35}$$

5.3 引力空间张量

$${}^{*}C_{abcd} := \frac{1}{2}\varepsilon_{abef}C^{ef}{}_{cd}. \tag{5.36}$$

$$E_{ab} := C_{acbd}Z^cZ^d = E_{\langle ab\rangle}. \tag{5.37}$$

$$B_{ab} := {}^{*}C_{acbd}Z^cZ^d = B_{\langle ab\rangle}. \tag{5.38}$$

$$(\operatorname{div} E)_a = \kappa\frac{1}{3}{}^3\nabla_a\rho - \dots. \tag{5.39}$$

$$(\operatorname{div} B)_a = \kappa(\rho + p)\omega_a + \dots. \tag{5.40}$$

$$(\operatorname{curl} E)_{ab} + \dots = -\mathrm{D}_t B_{\langle ab\rangle} - \dots. \tag{5.41}$$

$$(\operatorname{curl} B)_{ab} + \dots = \kappa\frac{1}{2}(\rho + p)\sigma_{ab} + \mathrm{D}_t E_{\langle ab\rangle} + \dots. \tag{5.42}$$

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