参数估计

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1　判断观测数据中有无信号

求 $\mathbb{P}(A_m|A_g)$

$$G_t(\omega) = \begin{cases} N_t(\omega) & \omega \in A_0 \\ N_t(\omega) + M_t(\omega) & \omega \in A_m \end{cases}$$

$$\mathbb{P}(A_m|A_g) = \frac{\mathbb{P}(A_g|A_m)\mathbb{P}(A_m)}{\mathbb{P}(A_g)}$$

$$\begin{aligned}
\mathbb{P}(A_g) &= \mathbb{P}(A_g|A_0)\mathbb{P}(A_0) + \mathbb{P}(A_g|A_m)\mathbb{P}(A_m) \\
&= \mathbb{P}(A_g|A_0)\mathbb{P}(A_0) + \mathbb{P}(A_m)\frac{\mathbb{P}(A_g \cap A_m)}{\mathbb{P}(A_m)} \\
&= \mathbb{P}(A_g|A_0)\mathbb{P}(A_0) + \mathbb{P}(A_m)\frac{\int \mathrm{d}\boldsymbol{\mu}\, p(A_g \cap A_{\boldsymbol{\mu}})}{\mathbb{P}(A_m)} \\
&= \mathbb{P}(A_g|A_0)\mathbb{P}(A_0) + \mathbb{P}(A_m)\int \mathrm{d}\boldsymbol{\mu}\, \frac{p(A_g \cap A_{\boldsymbol{\mu}})}{p(A_{\boldsymbol{\mu}})}\frac{p(A_{\boldsymbol{\mu}})}{\mathbb{P}(A_m)} \\
&= \mathbb{P}(A_g|A_0)\mathbb{P}(A_0) + \mathbb{P}(A_m)\int \mathrm{d}\boldsymbol{\mu}\, \mathbb{P}(A_g|A_{\boldsymbol{\mu}})p(\boldsymbol{\mu})
\end{aligned}$$

$$\mathbb{P}(A_m|A_g) = \frac{\Lambda}{\Lambda + \mathbb{P}(A_0)/\mathbb{P}(A_m)}$$

$$\Lambda := \int \mathrm{d}\boldsymbol{\mu}\, \lambda(\boldsymbol{\mu})$$

$$\lambda(\boldsymbol{\mu}) := p(\boldsymbol{\mu}) \frac{\mathbb{P}(A_g|A_{\boldsymbol{\mu}})}{\mathbb{P}(A_g|A_0)}$$

定理 1. $\mathbb{P}(A_g|A_{\boldsymbol{\mu}}) = \mathbb{P}(A_{g-m(\boldsymbol{\mu})}|A_0) \Leftrightarrow \{N = n\}$独立于$A_0, A_{\boldsymbol{\mu}}$.

证明. $G|_{A_{\boldsymbol{\mu}}} = N|_{A_{\boldsymbol{\mu}}} + m(\boldsymbol{\mu}) \Rightarrow \mathbb{P}(A_g|A_{\boldsymbol{\mu}}) = \frac{p(\{G=g\}\cap A_{\boldsymbol{\mu}})}{p(A_{\boldsymbol{\mu}})} = \frac{p(\{N=g-m(\boldsymbol{\mu})\}\cap A_{\boldsymbol{\mu}})}{p(A_{\boldsymbol{\mu}})}$.
$G|_{A_0} = N|_{A_0} \Rightarrow \mathbb{P}(A_{g-m(\boldsymbol{\mu})}|A_0) = \frac{\mathbb{P}(\{G=g-m(\boldsymbol{\mu})\}\cap A_0)}{\mathbb{P}(A_0)} = \frac{\mathbb{P}(\{N=g-m(\boldsymbol{\mu})\}\cap A_0)}{\mathbb{P}(A_0)}$.
$\frac{\mathbb{P}(\{N=n\}\cap A_0)}{\mathbb{P}(A_0)} = \frac{p(\{N=n\}\cap A_{\boldsymbol{\mu}})}{p(A_{\boldsymbol{\mu}})} := k$
$\Rightarrow \mathbb{P}(\{N = n\} \cap A_0) + \int \mathrm{d}\boldsymbol{\mu}\, p(\{N = n\} \cap A_{\boldsymbol{\mu}}) = k\mathbb{P}(A_0) + \int \mathrm{d}\boldsymbol{\mu}\, kp(A_{\boldsymbol{\mu}})$
$\Rightarrow \mathbb{P}(\{N = n\}) = k(\mathbb{P}(A_0) + \int \mathrm{d}\boldsymbol{\mu}\, p(A_{\boldsymbol{\mu}})) = k$.
依充分性的证明方法, 必要性是易证的. □

$$\lambda(\boldsymbol{\mu}) := p(\boldsymbol{\mu}) \frac{\mathbb{P}(A_{g-m(\boldsymbol{\mu})}|A_0)}{\mathbb{P}(A_g|A_0)}$$

$$S_n(f) = 2\mathcal{F}(C_n(\tau))$$

$$\mathbb{P}(A_{\{g_i\}}|A_0) = \frac{\exp[-\frac{1}{2}\sum C_{jk}^{-1} g_j g_k]}{[(2\pi)^N \det(C_{jk})]^{1/2}}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}t_j\, e^{2\pi i f t_j} \int_{-\infty}^{\infty} \mathrm{d}t_l\, C_n(t_j - t_l) C^{-1}(t_l, t_k) \\
&= \int_{-\infty}^{\infty} \mathrm{d}t_j \int_{-\infty}^{\infty} \mathrm{d}t_l\, C_n(t_j - t_l) C^{-1}(t_l, t_k) e^{2\pi i f t_j} \\
&= \int_{-\infty}^{\infty} \mathrm{d}t_l \int_{-\infty}^{\infty} \mathrm{d}t_j\, C_n(t_j - t_l) C^{-1}(t_l, t_k) e^{2\pi i f t_j} \\
&= \int_{-\infty}^{\infty} \mathrm{d}t_l \int_{-\infty}^{\infty} \mathrm{d}\tau\, C_n(\tau) C^{-1}(t_l, t_k) e^{2\pi i f t_l} e^{2\pi i f \tau} \quad (t_j - t_l = \tau) \\
&= \int_{-\infty}^{\infty} \mathrm{d}t_l\, C^{-1}(t_l, t_k) e^{2\pi i f t_l} \int_{-\infty}^{\infty} \mathrm{d}\tau\, C_n(\tau) e^{2\pi i f \tau}
\end{aligned}$$

定理 2. $\int_{-\infty}^{\infty} \mathrm{d}t\, p(t)q(t) = \int_{-\infty}^{\infty} \mathrm{d}f\, \mathcal{F}(p)(f)\mathcal{F}(q)(f)^* \Leftrightarrow q \in \mathbb{R}$.

证明. $\int_{-\infty}^{\infty} \mathrm{d}t\, p(t)q(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}f\, p(t)\mathcal{F}(q)(f)e^{-2\pi ift}$.
$\int_{-\infty}^{\infty} \mathrm{d}f\, \mathcal{F}(p)(f)\mathcal{F}(q)(f)^* = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}f\mathrm{d}t\, p(t)\mathcal{F}(q)(f)^* e^{2\pi ift}$,
$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}f\mathrm{d}t\, p(t)\mathcal{F}(q)(f)^* e^{2\pi ift} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}f\mathrm{d}t\, p(t)\mathcal{F}(q)(-f)^* e^{-2\pi ift}$.
取 $\int_{-\infty}^{\infty} \mathrm{d}t\, p(t)e^{-2\pi ift} = \delta(f-f_0)$, 即 $p(t) = e^{2\pi if_0t}$,
得 $\mathcal{F}(q)(f) = \mathcal{F}(q)(-f)^* \Rightarrow q \in \mathbb{R}$.
依充分性的证明方法, 必要性是易证的. □

$$\mathbb{P}(A_m|A_g) = \frac{\Lambda}{\Lambda + \mathbb{P}(A_0)/\mathbb{P}(A_m)}$$
$$\Lambda := \int \mathrm{d}\boldsymbol{\mu}\, \lambda(\boldsymbol{\mu})$$
$$\lambda(\boldsymbol{\mu}) := p(\boldsymbol{\mu}) \exp[2\langle g(t), m_{\boldsymbol{\mu}}(t)\rangle - \langle m_{\boldsymbol{\mu}}(t), m_{\boldsymbol{\mu}}(t)\rangle]$$
$$\langle p(t), q(t)\rangle = \int_{-\infty}^{\infty} \mathrm{d}f\, \frac{\tilde{p}(f)\tilde{q}(f)^*}{S_n(f)}$$

2 参数估计 (MLE)

求 $\hat{\boldsymbol{\mu}}$ 和 $V(P)$

$$p(A_{\boldsymbol{\mu}}|A_g) = \frac{\lambda(\boldsymbol{\mu})}{\Lambda + \mathbb{P}(A_0)/\mathbb{P}(A_m)}$$

$$p(A_{\boldsymbol{\mu}}|A_g \cap A_m) = \frac{\lambda(\boldsymbol{\mu})}{\Lambda}$$

$$\frac{\partial \ln p(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}}|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}} + 2\left\langle \frac{\partial m_{\boldsymbol{\mu}}}{\partial \boldsymbol{\mu}}|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}}(t), g(t) - m_{\boldsymbol{\mu}}|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}}(t)\right\rangle = 0$$

$$P = \int_{\boldsymbol{\mu}\in V(P)} \mathrm{d}\boldsymbol{\mu}\, p(A_{\boldsymbol{\mu}}|A_g)$$
$$V(P) = \{\boldsymbol{\mu} \,|\, p(A_{\boldsymbol{\mu}}|A_g) \geq \kappa^2\}$$

3 灵敏度 (准确度)

3.1 精确结果

求 $p(A_{\tilde{\boldsymbol{\mu}}}|A_{\hat{\boldsymbol{\mu}}})$

$$A_{\hat{\boldsymbol{\mu}}} = \bigcup_{g \Rightarrow \hat{\boldsymbol{\mu}}} A_g$$

$$\nu_i = 2 \left\langle N_t, \frac{\partial m_{\boldsymbol{\mu}}}{\partial \mu_i}|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}}(t) \right\rangle \quad (A_{\hat{\boldsymbol{\mu}}} \to \mathbb{R})$$

$$\langle N_t, h(t) \rangle = \int_{-\infty}^{\infty} \mathrm{d}f \, \frac{\mathcal{F}(N)(f)\mathcal{F}(h)(f)^*}{S_n(f)}$$

$$\mathcal{F}(N)(f) = \int_{-\infty}^{\infty} \mathrm{d}t \, N_t e^{2\pi i f t}$$

定理 3. 常数乘任意期望为0的正态随机变量后是期望为0的正态随机变量.

定理 4. 任意两个期望为0的正态随机变量的和是期望为0的正态随机变量.

定理 5. 任意期望为0的正态随机变量序列, 若其极限存在, 则此极限是期望为0的正态随机变量.

定理 6.

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}t \left(\int_{-\infty}^{\infty} \mathrm{d}f \, W(f) e^{-2\pi i f t} \right) \left(\int_{-\infty}^{\infty} \mathrm{d}f' \, W'(f') e^{-2\pi i f' t} \right) \\
&= \int_{-\infty}^{\infty} \mathrm{d}f \, W(f) \left(\int_{-\infty}^{\infty} \mathrm{d}f' \, W'(f') \delta(f + f') \right).
\end{aligned}$$

证明.

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}t \left(\int_{-\infty}^{\infty} \mathrm{d}f \, W(f) e^{-2\pi i f t} \right) \left(\int_{-\infty}^{\infty} \mathrm{d}f' \, W'(f') e^{-2\pi i f' t} \right) \\
&= \int_{-\infty}^{\infty} \mathrm{d}t \, \mathcal{F}(W)(t) \mathcal{F}(W')(t).
\end{aligned}$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}f\, W(f) \left(\int_{-\infty}^{\infty} \mathrm{d}f'\, W'(f')\delta(f+f') \right) \\
&= \int_{-\infty}^{\infty} \mathrm{d}f\, W(f) W'(-f).
\end{aligned}
$$

$$
\int_{-\infty}^{\infty} \mathrm{d}t\, \mathcal{F}(W)(t)\mathcal{F}(W')(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}f\, W(f)\mathcal{F}(W')(t)e^{-2\pi i f t}.
$$

$$
\int_{-\infty}^{\infty} \mathrm{d}f\, W(f)W'(-f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}f\mathrm{d}t\, W(f)\mathcal{F}(W')(t)e^{2\pi i(-f)t}.
$$

□

$$
\begin{aligned}
&S_n(f) \\
&= 2\int_{-\infty}^{\infty} \mathrm{d}\tau\, C_n(\tau) e^{2\pi i f\tau} \\
&= 2\int_{-\infty}^{\infty} \mathrm{d}\tau\, \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}t\, n(t)n(t+\tau) \right] e^{2\pi i f\tau} \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty}\int_{-T}^{T} \mathrm{d}\tau\mathrm{d}t\, n(t)n(t+\tau) e^{2\pi i f\tau}
\end{aligned}
$$

$$
\begin{aligned}
&\left|\mathcal{F}(n)(f)\right|^2 \\
&= \left[\int_{-\infty}^{\infty} \mathrm{d}t\, n(t)e^{-2\pi i f t} \right] \left[\int_{-\infty}^{\infty} \mathrm{d}t'\, n(t')e^{2\pi i f t'} \right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}t'\mathrm{d}t\, n(t)n(t') e^{2\pi i f(t'-t)}
\end{aligned}
$$

$$
\nu_i(\omega) = -2\left\langle m_{\tilde{\boldsymbol{\mu}}(\omega)}(t) - m_{\hat{\boldsymbol{\mu}}}(t), \frac{\partial m_{\boldsymbol{\mu}}}{\partial \mu_i}|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}}(t) \right\rangle - \frac{\partial \ln p(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}}|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}}
$$

$$
p(A_{\boldsymbol{\nu}}|A_{\hat{\boldsymbol{\mu}}}) = \frac{\exp[-\frac{1}{2}\sum \mathcal{C}_{ij}\nu_i\nu_j]}{[(2\pi)^N \det(\mathcal{C}_{ij}^{-1})]^{1/2}}
$$

$$
\mathcal{C}_{ij}^{-1} = 2\left\langle \frac{\partial m_{\boldsymbol{\mu}}}{\partial \mu_i}|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}}(t), \frac{\partial m_{\boldsymbol{\mu}}}{\partial \mu_j}|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}}(t) \right\rangle
$$

$$P=\int_{V(P)}\mathrm{d}\tilde{\boldsymbol{\mu}}\,p(A_{\tilde{\boldsymbol{\mu}}}|A_{\hat{\boldsymbol{\mu}}})=\int_{\boldsymbol{\nu}(V(P))}\mathrm{d}\boldsymbol{\nu}\,p(A_{\boldsymbol{\nu}}|A_{\hat{\boldsymbol{\mu}}})$$
$$V(P)=\{\tilde{\boldsymbol{\mu}}|p(A_{\boldsymbol{\nu}(\tilde{\boldsymbol{\mu}})}|A_{\hat{\boldsymbol{\mu}}})\geq\mathcal{K}^2\}$$

3.2 近似结果

$$\delta\mu_i(\omega)=-\sum\mathcal{C}_{ij}\left[\nu_j(\omega)+\frac{\partial\ln p(\boldsymbol{\mu})}{\partial\mu_j}|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}}\right]$$

$$\overline{\left(\sum\mathcal{C}_{ik}\nu_k\right)\left(\sum\mathcal{C}_{jl}\nu_l\right)}=\overline{\sum\mathcal{C}_{ik}\nu_k\nu_l\mathcal{C}_{lj}}=\sum\mathcal{C}_{ik}\overline{\nu_k\nu_l}\mathcal{C}_{lj}=\sum\mathcal{C}_{ik}\mathcal{C}^{-1}_{kl}\mathcal{C}_{lj}=\mathcal{C}_{ij}$$

$$p(A_{\hat{\boldsymbol{\mu}}+\delta\boldsymbol{\mu}}|A_{\hat{\boldsymbol{\mu}}})=\frac{\exp[-\frac{1}{2}\sum\mathcal{C}^{-1}_{ij}(\delta\mu_i-\overline{\delta\mu_i})(\delta\mu_j-\overline{\delta\mu_j})]}{[(2\pi)^N\det(\mathcal{C}_{ij})]^{1/2}}$$
$$\overline{\delta\mu_i}=-\sum\mathcal{C}_{ij}\frac{\partial\ln p(\boldsymbol{\mu})}{\partial\mu_j}|_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}}$$