

引力波天文学笔记

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目录

第一章 引力波	5
1.1 Linearized Gravity	5
1.2 Radiation Gauge	5
1.3 Quadrupole Approximation	6
1.4 + Mode and \times Mode	7
1.5 电磁—引力对比	8
1.5.1 电偶极—引力对比	9
1.5.2 电四极—引力对比	9
第二章 能量	11
第三章 双星系统	13
3.1 基本公式	13
3.2 Post-Newtonian Approximation	13
3.3 Stationary Phase Approximation	13
第四章 宇宙学效应	15
第五章 电磁引力	17
5.1 时空张量转化为空间张量	17
5.2 电磁空间矢量	18
5.3 引力空间张量	19
第六章 Varying G	21
6.1 Modification of Amplitude	21
6.2 Modification of Phase	27

第一章 引力波

1.1 Linearized Gravity

[7]. 流形 \mathbb{R}^4 . 任意坐标系 $\{x^\mu\}$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}s + O(s^2)$, $g^{\mu\nu} = \eta^{\mu\nu} + \gamma^{\mu\nu}s + O(s^2)$, 则 $\delta^\mu_\lambda = \eta^{\mu\nu}\eta_{\nu\lambda} + \gamma^{\mu\nu}\eta_{\nu\lambda}s + \gamma^{\mu\nu}\eta_{\nu\lambda}s + O(s^2)$, 所以 $\gamma^{\mu\nu} = \eta^{\mu\nu}$, $\gamma^{\mu\nu} = \eta^{\mu\sigma}\delta^\nu_\sigma = \eta^{\mu\sigma}\eta_{\sigma\lambda}\eta^{\lambda\nu} = -\eta^{\mu\sigma}\gamma_{\sigma\lambda}\eta^{\lambda\nu} = -\eta^{\mu\sigma}\gamma_{\sigma\lambda}\eta^{\lambda\nu} = -\gamma^{\mu\nu}$, 所以 $g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu}s + O(s^2) = \eta^{\mu\nu} - h^{\mu\nu} + O(s^2)$.

$$R_{\mu\nu\lambda\sigma} = \partial_\sigma\partial_{[\mu}h_{\lambda]\nu} - \partial_\nu\partial_{[\mu}h_{\lambda]\sigma} + O(s^2). \quad (1.1)$$

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\sigma}h_{\lambda\sigma} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h.$$

$$-\frac{1}{2}\partial^\lambda\partial_\lambda\bar{h}_{\mu\nu} + \partial^\lambda\partial_{(\mu}\bar{h}_{\nu)\lambda} - \frac{1}{2}\eta_{\mu\nu}\partial^\lambda\partial^\sigma\bar{h}_{\lambda\sigma} + O(s^2) = 8\pi T_{\mu\nu}. \quad (1.2)$$

存在 $\{x^\mu\}$, 使得 $\partial^\nu\bar{h}_{\mu\nu} + O(s^2) = 0$ (Lorentz gauge). 设 $x'^\mu = x^\mu - \xi^\mu = x^\mu - \zeta^\mu s - O(s^2)$, 则 $\frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial x'^\mu} = \frac{\partial}{\partial x^\lambda}(\delta^\lambda_\mu + \frac{\partial\xi^\lambda}{\partial x'^\mu}) = \frac{\partial}{\partial x^\mu} + O(s^2)$, $g'_{\mu\nu} = g_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} = g_{\lambda\sigma}(\delta^\lambda_\mu + \frac{\partial\xi^\lambda}{\partial x'^\mu})(\delta^\sigma_\nu + \frac{\partial\xi^\sigma}{\partial x'^\nu}) = g_{\mu\nu} + g_{\mu\sigma}\frac{\partial\xi^\sigma}{\partial x'^\nu} + g_{\lambda\nu}\frac{\partial\xi^\lambda}{\partial x'^\mu} = g_{\mu\nu} + (\eta_{\mu\sigma} + O(s))(\frac{\partial\xi^\sigma}{\partial x^\nu} + O(s^2)) + (\eta_{\lambda\nu} + O(s))(\frac{\partial\xi^\lambda}{\partial x^\mu} + O(s^2)) = g_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + O(s^2)$, 所以 $h'_{\mu\nu} = g'_{\mu\nu} - \eta_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + O(s^2) = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + O(s^2)$, 因此存在 ξ^μ , 使得 $\partial^\nu\bar{h}'_{\mu\nu} + O(s^2) = 0$ [7]. 令 $\{x^\mu\}$ 满足 $\partial^\nu\bar{h}_{\mu\nu} + O(s^2) = 0$, 则

$$\partial^\lambda\partial_\lambda\bar{h}_{\mu\nu} + O(s^2) = -16\pi T_{\mu\nu}. \quad (1.3)$$

略去 $O(s^2)$ 条件: $h_{\mu\nu}$, $\partial_\lambda h_{\mu\nu}$...小.

1.2 Radiation Gauge

[7]. 存在 $\{x^\mu\}$, 使得 “无源处” $h + O(s^2) = 0$ (TT gauge [8]) 且 $h_{0\mu} + O(s^2) = 0$.

1.3 Quadrupole Approximation

[7]. 下略 $O(s^2)$. 由(1.3)得

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4 \int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV'. \quad (1.4)$$

$$\hat{h}_{\mu\nu}(\omega, \vec{r}) := \frac{1}{\sqrt{2\pi}} \int \bar{h}_{\mu\nu}(t, \vec{r}) e^{i\omega t} dt \quad (1.5)$$

$$= 4 \int \frac{\hat{T}_{\mu\nu}(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega |\vec{r} - \vec{r}'|} dV'. \quad (1.6)$$

由 $\partial^\nu \bar{h}_{\mu\nu} = 0$,

$$-i\omega \hat{h}_{0\mu} = \sum_i \frac{\partial \hat{h}_{i\mu}}{\partial x^i}. \quad (1.7)$$

$|\vec{r}| \gg |\vec{r}'|$ 且 $\omega \ll 1/|\vec{r}'|$,

$$\hat{h}_{ij}(\omega, \vec{r}) = 4 \frac{e^{i\omega |\vec{r}|}}{|\vec{r}|} \int \hat{T}_{ij}(\omega, \vec{r}') dV'. \quad (1.8)$$

$$\int \hat{T}_{ij} dV' = \int \sum_k (\hat{T}_{kj} \frac{\partial x'^i}{\partial x'^k}) dV' \quad (1.9)$$

$$= \sum_k \left[\int \frac{\partial}{\partial x'^k} (\hat{T}_{kj} x'^i) dV' - \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \right] \quad (1.10)$$

$$= \sum_k \int \partial'_k (\hat{T}_{kj} x'^i) dV' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \quad (1.11)$$

$$= \int \hat{T}_{kj} x'^i dS' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \quad (1.12)$$

$$= - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \quad (1.13)$$

$$= - \int (\sum_k \partial'_k \hat{T}_{kj}) x'^i dV' \quad (1.14)$$

$$= - \int (\partial_0 \hat{T}_{0j}) x'^i dV' \quad (1.15)$$

$$= -i\omega \int \hat{T}_{0j} x'^i dV' \quad (1.16)$$

$$= \int \hat{T}_{(ij)} dV' \quad (1.17)$$

$$= -i\omega \int \hat{T}_{0(j} x'^{i)} dV' \quad (1.18)$$

$$= -\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) dV', \quad (1.19)$$

$$-\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) dV' = -\frac{i\omega}{2} \int \sum_k (\hat{T}_{0k} x'^i \frac{\partial x'^j}{\partial x'^k} + \hat{T}_{0k} \frac{\partial x'^i}{\partial x'^k} x'^j) dV' \quad (1.20)$$

$$= -\frac{i\omega}{2} \sum_k \left[\int \frac{\partial}{\partial x'^k} (\hat{T}_{0k} x'^i x'^j) dV' - \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \right] \quad (1.21)$$

$$= -\frac{i\omega}{2} \sum_k \int \partial'_k (\hat{T}_{0k} x'^i x'^j) dV' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \quad (1.22)$$

$$= -\frac{i\omega}{2} \sum_k \int \hat{T}_{0k} x'^i x'^j dS' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \quad (1.23)$$

$$= \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \quad (1.24)$$

$$= \frac{i\omega}{2} \int (\sum_k \partial'_k \hat{T}_{0k}) x'^i x'^j dV' \quad (1.25)$$

$$= \frac{i\omega}{2} \int (\partial_0 \hat{T}_{00}) x'^i x'^j dV' \quad (1.26)$$

$$= -\frac{\omega^2}{2} \int \hat{T}_{00} x'^i x'^j dV'. \quad (1.27)$$

$$q_{ij}(t) := \int T_{00} x'^i x'^j dV', \quad (1.28)$$

$$\hat{h}_{ij}(\omega, \vec{r}) = -2\omega^2 \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \hat{q}_{ij}(\omega), \quad (1.29)$$

$$\bar{h}_{ij}(t, \vec{r}) = \frac{2}{|\vec{r}|} \frac{d^2}{dt^2} q_{ij}(t - |\vec{r}|). \quad (1.30)$$

1.4 + Mode and \times Mode

寻新标架 $(e^1)_a = (e^+)_a$, $(e^2)_a = (e^\times)_a$, $(e^3)_a = (e^r)_a$, $\bar{h}_{ij}(e^i)_a(e^j)_b = \bar{h}'_{ij}(e'^i)_a(e'^j)_b$, 取 x, y 分量后去迹, $h_+ = \frac{1}{2}(\bar{h}'_{11} - \bar{h}'_{22})$, $h_\times = \bar{h}'_{12} = \bar{h}'_{21}$. (?)

[6]

$$[2], \vec{n} := \frac{\vec{r}}{|\vec{r}|},$$

$$h_{ij}^{\text{TT}} = \frac{2}{|\vec{r}|} \mathcal{P}_{ijkm} \frac{d^2}{dt^2} Q^{km}(t - |\vec{r}|), \quad (1.31)$$

$$\mathcal{P}_{ijkm} := (\delta_{ik} - \vec{n}_i \vec{n}_k) (\delta_{jm} - \vec{n}_j \vec{n}_m) - \frac{1}{2} (\delta_{ij} - \vec{n}_i \vec{n}_j) (\delta_{km} - \vec{n}_k \vec{n}_m), \quad (1.32)$$

$$Q^{km}(t) := \int T_{00} \left(x'^k x'^m - \frac{1}{3} \delta^{km} \sum_n x'^n x'^n \right) dV' \quad (1.33)$$

1.5 电磁—引力对比

$$A_\mu(t, \vec{r}) = \frac{\mu_0}{4\pi} \int \frac{J_\mu(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV' \quad (1.34)$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4G \int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV' \quad (1.35)$$

$$A_\mu(t, \vec{r}) = \frac{1}{\sqrt{2\pi}} \int \hat{A}_\mu(\omega, \vec{r}) e^{-i\omega t} d\omega \quad (1.36)$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = \frac{1}{\sqrt{2\pi}} \int \hat{h}_{\mu\nu}(\omega, \vec{r}) e^{-i\omega t} d\omega \quad (1.37)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\hat{J}_\mu(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega|\vec{r} - \vec{r}'|} dV' \quad (1.38)$$

$$\hat{h}_{\mu\nu}(\omega, \vec{r}) = 4G \int \frac{\hat{T}_{\mu\nu}(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega|\vec{r} - \vec{r}'|} dV' \quad (1.39)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_\mu(\omega, \vec{r}') e^{-i\omega(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}')} dV' \quad (1.40)$$

$$\hat{h}_{\mu\nu}(\omega, \vec{r}) = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{\mu\nu}(\omega, \vec{r}') e^{-i\omega(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}')} dV' \quad (1.41)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_\mu(\omega, \vec{r}') \left[1 - i\omega \left(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}' \right) - \dots \right] dV' \quad (1.42)$$

$$\hat{h}_{\mu\nu}(\omega, \vec{r}) = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{\mu\nu}(\omega, \vec{r}') \left[1 - i\omega \left(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}' \right) - \dots \right] dV' \quad (1.43)$$

1.5.1 电偶极—引力对比

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_i dV' \quad (1.44)$$

$$\hat{\bar{h}}_{ij} = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{ij} dV' \quad (1.45)$$

$$\int \hat{J}_i dV' = -i\omega \int \hat{J}_0 x'^i dV' \quad (1.46)$$

$$\int \hat{T}_{ij} dV' = -\frac{\omega^2}{2} \int \hat{T}_{00} x'^i x'^j dV' \quad (1.47)$$

$$\hat{p}_i = \int \hat{J}_0 x'^i dV' \quad (1.48)$$

$$\hat{q}_{ij} = \int \hat{T}_{00} x'^i x'^j dV' \quad (1.49)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega \hat{p}_i) \quad (1.50)$$

$$\hat{\bar{h}}_{ij} = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \left(-\frac{\omega^2}{2} \hat{q}_{ij}\right) \quad (1.51)$$

$$A_i = \frac{\mu_0}{4\pi} \frac{1}{|\vec{r}|} \frac{d}{dt} p_i(t - |\vec{r}|) \quad (1.52)$$

$$\bar{h}_{ij} = 4G \frac{1}{|\vec{r}|} \frac{1}{2} \frac{d^2}{dt^2} q_{ij}(t - |\vec{r}|) \quad (1.53)$$

1.5.2 电四极—引力对比

$$\hat{A}_i(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int \hat{J}_i(\omega, \vec{r}') \left(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}'\right) dV' \quad (1.54)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int \hat{J}'_i n^j x'_j dV' \quad (1.55)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int n^j x'_j \hat{J}'_i dV' \quad (1.56)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) n^j \left[\int x'_{(j} \hat{J}'_{i)} dV' \right] \quad (1.57)$$

$$\int x'_{(j} \hat{J}'_{i)} dV' = \frac{1}{2} \int (\hat{J}'_j x'_i + \hat{J}'_i x'_j) dV' \quad (1.58)$$

$$= \frac{1}{2} \int \sum_k (\hat{J}'_k x'^i \frac{\partial x'^j}{\partial x'^k} + \hat{J}'_k \frac{\partial x'^i}{\partial x'^k} x'^j) dV' \quad (1.59)$$

$$= \frac{1}{2} \sum_k \left[\int \frac{\partial}{\partial x'^k} (\hat{J}'_k x'^i x'^j) dV' - \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \right] \quad (1.60)$$

$$= \frac{1}{2} \sum_k \int \partial'_k (\hat{J}'_k x'^i x'^j) dV' - \frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \quad (1.61)$$

$$= \frac{1}{2} \sum_k \int \hat{J}'_k x'^i x'^j dS' - \frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \quad (1.62)$$

$$= -\frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \quad (1.63)$$

$$= -\frac{1}{2} \int (\sum_k \partial'_k \hat{J}'_k) x'^i x'^j dV' \quad (1.64)$$

$$= -\frac{1}{2} \int (\partial_0 \hat{J}'_0) x'^i x'^j dV' \quad (1.65)$$

$$= -\frac{i\omega}{2} \int \hat{J}'_0 x'^i x'^j dV' \quad (1.66)$$

$$\hat{D}_{ij} = \int \hat{J}'_0 x'^i x'^j dV' \quad (1.67)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \left(-\frac{\omega^2}{2} n^j \hat{D}_{ij} \right) \quad (1.68)$$

$$A_i = \frac{\mu_0}{4\pi} \frac{1}{|\vec{r}|} n^j \frac{1}{2} \frac{d^2}{dt^2} D_{ij}(t - |\vec{r}|) \quad (1.69)$$

第二章 能量

[7], [5]

第三章 双星系统

3.1 基本公式

$$\mathcal{M} := \mu^{3/5} M^{2/5} \quad (3.1)$$

$$h_+ = \frac{4\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} \frac{1 + \cos^2 \iota}{2} \cos \Phi(t) \quad (3.2)$$

$$h_\times = \frac{4\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} \cos \iota \sin \Phi(t) \quad (3.3)$$

$$h = F_+ h_+ + F_\times h_\times \quad (3.4)$$

3.2 Post-Newtonian Approximation

3.3 Stationary Phase Approximation

[4], if $\zeta(t)$ varies slowly near $t = t_0$ where the phase has a stationary point: $\phi'(t_0) = 0$,

$$\int \zeta(t) e^{i\phi(t;f)} dt = \int \zeta(t) e^{i[\phi(t_0) + \phi'(t_0)(t-t_0) + \frac{1}{2}\phi''(t_0)(t-t_0)^2 + \dots]} dt \quad (3.5)$$

$$\simeq e^{i\phi(t_0)} \int \zeta(t) e^{i[\frac{1}{2}\phi''(t_0)(t-t_0)^2]} dt \quad (3.6)$$

$$\simeq e^{i\phi(t_0)} \int \zeta(t_0) e^{\frac{-\sqrt{-i\phi''(t_0)}^2 (t-t_0)^2}{2}} dt \quad (3.7)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{-i\phi''(t_0)}} \zeta(t_0) e^{i\phi(t_0)}. \quad (3.8)$$

$$h = \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \cos \Phi(t) \quad (3.9)$$

$$= \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} [e^{i\Phi(t)} + e^{-i\Phi(t)}] \quad (3.10)$$

$$\tilde{h}(f) = \int h(t) e^{i2\pi f t} dt \quad (3.11)$$

$$= \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} [e^{i\Phi(t)} + e^{-i\Phi(t)}] e^{i2\pi f t} dt \quad (3.12)$$

$$= \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} \{e^{i[2\pi f t + \Phi(t)]} + e^{i[2\pi f t - \Phi(t)]}\} dt \quad (3.13)$$

$$\simeq \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} e^{i[2\pi f t - \Phi(t)]} dt \quad (3.14)$$

$$= \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F]^{2/3} Q \frac{1}{2} e^{i[2\pi f t(F) - \Phi(F)]} \frac{dt}{dF} dF \quad (3.15)$$

$$\simeq \frac{\sqrt{2\pi}}{\sqrt{-i[2\pi f t(F) - \Phi(F)]''_{F=f}}} \quad (3.16)$$

$$\left[\frac{\mathcal{M}}{D} (\pi \mathcal{M} F)^{2/3} Q \frac{1}{2} \frac{dt}{dF} \right]_{F=f} e^{i[2\pi f t(f) - \Phi(f)]} \quad (3.17)$$

$$\simeq \frac{\sqrt{2\pi}}{\sqrt{-i \left\{ 2\pi f \left[-\frac{5}{256} \mathcal{M} (\pi \mathcal{M} F)^{-8/3} \right] - \left[\frac{1}{16} (\pi \mathcal{M} F)^{-5/3} \right] \right\}''_{F=f}}} \quad (3.18)$$

$$\left\{ \frac{\mathcal{M}}{D} (\pi \mathcal{M} F)^{2/3} Q \frac{1}{2} \left[\frac{5\pi \mathcal{M}^2}{96} (\pi \mathcal{M} F)^{-11/3} \right] \right\}_{F=f} e^{i[2\pi f t(f) - \Phi(f)]} \quad (3.19)$$

$$= \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{D} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4}]} \quad (\text{pnspace.py}) \quad (3.20)$$

另可考 [1]. 其中 $\frac{d\Phi}{dt} = 2\pi F$.

第四章 宇宙学效应

$$\frac{d\eta}{d(ct)} = \frac{1}{a} \quad (4.1)$$

$$ds^2 = -d(ct)^2 + a^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (4.2)$$

$$ds^2 = a^2 \left[-d\eta^2 + \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (4.3)$$

第五章 电磁引力

[3].

5.1 时空张量转化为空间张量

$$h_{ab} := g_{ab} + Z_a Z_b. \quad (5.1)$$

$$h_a{}^b = \delta_a{}^b + Z_a Z^b. \quad (5.2)$$

$$Z^a h_{ab} = 0. \quad (5.3)$$

$$V_{\langle a} := h_a{}^b V_b. \quad (5.4)$$

$$Z^a V_{\langle a} = 0. \quad (5.5)$$

$$T_{\langle ab \rangle} := h_{(a}{}^c h_{b)}{}^d T_{cd} - \frac{1}{3} h_{cd} T^{cd} h_{ab}. \quad (5.6)$$

$$Z^a (h_a{}^c h_b{}^d T_{cd}) = 0. \quad (5.7)$$

$$Z^a (h_b{}^c h_a{}^d T_{cd}) = 0. \quad (5.8)$$

$$Z^a (h_{(a}{}^c h_{b)}{}^d T_{cd}) = 0. \quad (5.9)$$

$$Z^a (h_{cd} T^{cd} h_{ab}) = 0. \quad (5.10)$$

$$Z^a T_{\langle ab \rangle} = 0. \quad (5.11)$$

$$T_{\langle (ab) \rangle} = T_{\langle ab \rangle}. \quad (5.12)$$

$$h^{ab} T_{\langle ab \rangle} = 0. \quad (5.13)$$

$$\varepsilon_{abc} := \varepsilon_{abcd} Z^d. \quad (5.14)$$

$$\varepsilon_{0123} := -\sqrt{|g|}. \quad (5.15)$$

$$T_a := \frac{1}{2} \varepsilon_{abc} T^{[bc]}. \quad (5.16)$$

$$[U, V]_a := \varepsilon_{abc} U^b V^c. \quad (5.17)$$

$$[S, T]_a := \varepsilon_{abc} g_{de} S^{bd} T^{ce}. \quad (5.18)$$

$$D_t T^{a\dots}_{b\dots} := Z^c \nabla_c T^{a\dots}_{b\dots}. \quad (5.19)$$

$${}^3\nabla_a T^{b\dots}_{c\dots} := h_a{}^p h^b{}_q \dots h_c{}^r \dots \nabla_p T^{q\dots}_{r\dots}. \quad (5.20)$$

$$(\operatorname{div} V) := {}^3\nabla^a V_a. \quad (5.21)$$

$$(\operatorname{curl} V)_a := \varepsilon_{bca} {}^3\nabla^b V^c. \quad (5.22)$$

$$(\operatorname{div} T)_a := {}^3\nabla^b T_{ab}. \quad (5.23)$$

$$(\operatorname{curl} T)_{ab} := \varepsilon_{cd(a} {}^3\nabla^c g_{b)e} T^{ed}. \quad (5.24)$$

5.2 电磁空间矢量

$${}^*F_{ab} := \frac{1}{2} \varepsilon_{abcd} F^{cd} \quad (5.25)$$

$$E_a := F_{ab} Z^b = E_{\langle a \rangle}. \quad (5.26)$$

$$B_a := {}^*F_{ab} Z^b = B_{\langle a \rangle}. \quad (5.27)$$

$$\rho = -Z^a J_a. \quad (5.28)$$

$$j_a = h_a{}^b J_b. \quad (5.29)$$

$$\nabla_{[a} F_{bc]} = 0. \quad (5.30)$$

$$\nabla^a F_{ab} = \mu J_b. \quad (5.31)$$

$$(\operatorname{div} E) = \mu\rho - \dots \quad (5.32)$$

$$(\operatorname{div} B) = + \dots \quad (5.33)$$

$$(\operatorname{curl} E)_a + \dots = -D_t B_{\langle a} - \dots \quad (5.34)$$

$$(\operatorname{curl} B)_a + \dots = \mu j_a + D_t E_{\langle a} + \dots \quad (5.35)$$

5.3 引力空间张量

$${}^*C_{abcd} := \frac{1}{2}\varepsilon_{abef}C^ef_{cd}. \quad (5.36)$$

$$E_{ab} := C_{acbd}Z^cZ^d = E_{\langle ab \rangle}. \quad (5.37)$$

$$B_{ab} := {}^*C_{acbd}Z^cZ^d = B_{\langle ab \rangle}. \quad (5.38)$$

$$(\operatorname{div} E)_a = \kappa \frac{1}{3} \nabla_a \rho - \dots \quad (5.39)$$

$$(\operatorname{div} B)_a = \kappa(\rho + p)\omega_a + \dots \quad (5.40)$$

$$(\operatorname{curl} E)_{ab} + \dots = -D_t B_{\langle ab \rangle} - \dots \quad (5.41)$$

$$(\operatorname{curl} B)_{ab} + \dots = \kappa \frac{1}{2}(\rho + p)\sigma_{ab} + D_t E_{\langle ab \rangle} + \dots \quad (5.42)$$

第六章 Varying G

6.1 Modification of Amplitude

$$\partial^c \partial_c \bar{h}_{ab} = -16\pi \frac{G_0}{c_0^4} T_{ab}, \quad \partial^a \bar{h}_{ab} = 0 \quad (6.1)$$

$$\Gamma^c_{ab} = \frac{1}{2} \eta^{cd} (2\partial_{(a} h_{b)d} - \partial_d h_{ab}) \quad (6.2)$$

$$U^a \partial_a U^c + \Gamma^c_{ab} U^a U^b = 0 \quad (6.3)$$

$$U^a \partial_a U^c = -\frac{1}{2} \eta^{cd} (2\partial_{(a} h_{b)d} - \partial_d h_{ab}) U^a U^b \quad (6.4)$$

$$T_{ab} = c_0^2 (2U_{(a} J_{b)} + U^c J_c U_a U_b) \quad (6.5)$$

$$J_b c_0^2 = -U^a T_{ab} \quad (6.6)$$

$$A_b = -\frac{1}{4} U^a \bar{h}_{ab} \quad (6.7)$$

$$A_0 = -\frac{1}{4} c_0 \bar{h}_{00} = -\frac{1}{2} c_0 (\bar{h}_{00} - \frac{1}{2} \eta_{00} \eta^{00} \bar{h}_{00}) = -\frac{1}{2} c_0 h_{00} \quad (6.8)$$

$$A_i = -\frac{1}{4} c_0 \bar{h}_{0i} = -\frac{1}{4} c_0 h_{0i} \quad (6.9)$$

$$U^\mu \partial_\mu U^i = -\frac{1}{2} \eta^{i\sigma} (\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) U^\mu U^\nu \quad (6.10)$$

$$-\frac{1}{2} \eta^{i\sigma} (\partial_0 h_{0\sigma} + \partial_0 h_{0\sigma} - \partial_\sigma h_{00}) U^0 U^0 = \frac{1}{2} c_0^2 \eta^{i\sigma} \partial_\sigma h_{00} \quad (6.11)$$

$$= \frac{1}{2} c_0^2 \partial^i h_{00} \quad (6.12)$$

$$= -c_0 \partial^i A_0 \quad (6.13)$$

$$= -E^i \quad (6.14)$$

$$-\frac{1}{2} \eta^{i\sigma} (\partial_0 h_{j\sigma} + \partial_j h_{0\sigma} - \partial_\sigma h_{0j}) U^0 U^j = -\frac{1}{2} c_0 \eta^{i\sigma} (\partial_j h_{0\sigma} - \partial_\sigma h_{0j}) v^j \quad (6.15)$$

$$= -\frac{1}{2} c_0 \eta^{ik} (\partial_j h_{0k} - \partial_k h_{0j}) v^j \quad (6.16)$$

$$= 2\eta^{ik} (\partial_j A_k - \partial_k A_j) v^j \quad (6.17)$$

$$= -2\eta^{ik} (\partial_k A_j - \partial_j A_k) v^j \quad (6.18)$$

$$= -2(\partial^i A_j - \partial_j A^i) v^j \quad (6.19)$$

$$= -2\varepsilon^i{}_{jk} v^j B^k \quad (6.20)$$

$$-\frac{1}{2} \eta^{i\sigma} (\partial_j h_{k\sigma} + \partial_k h_{j\sigma} - \partial_\sigma h_{jk}) U^j U^k = 0 \quad (6.21)$$

$$a^i = -E^i - 4\varepsilon^i{}_{jk} v^j B^k \quad (6.22)$$

$$\partial^i \left(\frac{1}{4\pi G_0} E_i \right) = \rho \quad (6.23)$$

$$\partial^i B_i = 0 \quad (6.24)$$

$$\varepsilon^i{}_{jk} \partial^j E^k = -\partial_t B^i \quad (6.25)$$

$$\varepsilon^i{}_{jk} \partial^j \left(\frac{c_0^2}{4\pi G_0} B^k \right) = j^i + \partial_t \left(\frac{1}{4\pi G_0} E^i \right) \quad (6.26)$$

$$\varepsilon_{G0} := \frac{1}{4\pi G_0}, \quad \mu_{G0} := \frac{4\pi G_0}{c_0^2} \quad (6.27)$$

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_{G0} \vec{E}) = \rho \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \\ \vec{\nabla} \times (\mu_{G0}^{-1} \vec{B}) = \vec{j} + \frac{\partial}{\partial t} (\varepsilon_{G0} \vec{E}) \end{cases} \quad (6.28)$$

$$\vec{a} = -\vec{E} - 4\vec{v} \times \vec{B} \quad (6.29)$$

$$\varepsilon_G = \frac{1}{4\pi G}, \quad \mu_G = \frac{4\pi G}{c^2} \quad (6.30)$$

$$x^\mu = (ct, x, y, z) \quad (6.31)$$

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_G \vec{E}) = \rho \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \\ \vec{\nabla} \times (\mu_G^{-1} \vec{B}) = \vec{j} + \frac{\partial}{\partial t} (\varepsilon_G \vec{E}) \end{cases} \quad (6.32)$$

$$\vec{a} = -\vec{E} - 4\vec{v} \times \vec{B} \quad (6.33)$$

$$A_\mu = -\frac{1}{4} c \bar{h}_{0\mu} \quad (6.34)$$

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \varepsilon_G^{-1} \rho \\ \vec{\nabla} \times \vec{B} = \mu_G \vec{j} + \varepsilon_G \mu_G \frac{\partial}{\partial t} \vec{E} \end{cases} \quad (6.35)$$

$$\frac{1}{c^2} \frac{\partial}{\partial t} \varphi + \vec{\nabla} \cdot \vec{A} = 0 \quad (6.36)$$

$$\begin{cases} -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi + \vec{\nabla}^2 \varphi = \varepsilon_G^{-1} \rho \\ -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} + \vec{\nabla}^2 \vec{A} = \mu_G \vec{j} \end{cases} \quad (6.37)$$

$$\begin{cases} -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} c^{-1} \varphi + \vec{\nabla}^2 c^{-1} \varphi = \mu_G c \rho \\ -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} + \vec{\nabla}^2 \vec{A} = \mu_G \vec{j} \end{cases} \quad (6.38)$$

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_G \vec{E}) = 0 \\ \vec{\nabla} \cdot (\mu_G \vec{H}) = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} (\mu_G \vec{H}) \\ \vec{\nabla} \times \vec{H} = +\frac{\partial}{\partial t} (\varepsilon_G \vec{E}) \end{cases} \quad (6.39)$$

$$E_r = 0, \quad H_r = 0 \quad (6.40)$$

$$\begin{cases} \frac{\varepsilon_G}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta E_\theta) + \frac{\varepsilon_G}{r \sin \theta} \frac{\partial}{\partial \phi} (E_\phi) = 0 \\ \frac{\mu_G}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta H_\theta) + \frac{\mu_G}{r \sin \theta} \frac{\partial}{\partial \phi} (H_\phi) = 0 \\ \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta E_\phi) - \frac{\partial}{\partial \phi} (E_\theta) \right] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r} (r E_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \vec{e}_\phi = -\mu_G \frac{\partial}{\partial t} (H_\theta \vec{e}_\theta + H_\phi \vec{e}_\phi) \\ \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta H_\phi) - \frac{\partial}{\partial \phi} (H_\theta) \right] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r} (r H_\theta) \vec{e}_\phi = +\varepsilon_G \frac{\partial}{\partial t} (E_\theta \vec{e}_\theta + E_\phi \vec{e}_\phi) \end{cases} \quad (6.41)$$

$$\vec{E} = E_\theta \vec{e}_\theta, \quad \vec{H} = H_\phi \vec{e}_\phi \quad (6.42)$$

$$\begin{cases} \frac{\varepsilon_G}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta E_\theta) = 0 \\ \frac{\mu_G}{r \sin \theta} \frac{\partial}{\partial \phi} (H_\phi) = 0 \\ -\frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (E_\theta) \vec{e}_r + \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \vec{e}_\phi = -\mu_G \frac{\partial}{\partial t} (H_\phi) \vec{e}_\phi \\ +\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta H_\phi) \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) \vec{e}_\theta = +\varepsilon_G \frac{\partial}{\partial t} (E_\theta) \vec{e}_\theta \end{cases} \quad (6.43)$$

$$\begin{cases} \frac{\partial}{\partial r} (r E_\theta) + \mu_G \frac{\partial}{\partial t} (r H_\phi) = 0 \\ \frac{\partial}{\partial r} (r H_\phi) + \varepsilon_G \frac{\partial}{\partial t} (r E_\theta) = 0 \end{cases} \quad (6.44)$$

$$\begin{cases} \mu_G \frac{\partial}{\partial r} \mu_G^{-1} \frac{\partial}{\partial r} (r E_\theta) - \varepsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t} (r E_\theta) = 0 \\ \varepsilon_G \frac{\partial}{\partial r} \varepsilon_G^{-1} \frac{\partial}{\partial r} (r H_\phi) - \varepsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t} (r H_\phi) = 0 \end{cases} \quad (6.45)$$

$$\begin{cases} \mu_G \frac{\partial}{\partial r} \mu_G^{-1} \frac{\partial}{\partial r} (r E_\theta) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (r E_\theta) = 0 \\ \varepsilon_G \frac{\partial}{\partial r} \varepsilon_G^{-1} \frac{\partial}{\partial r} (r H_\phi) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (r H_\phi) = 0 \end{cases} \quad (6.46)$$

$$\begin{cases} \frac{\partial}{\partial r} \frac{\partial}{\partial r} (r E_\theta) - \frac{\partial}{\partial r} (\ln \mu_G) \frac{\partial}{\partial r} (r E_\theta) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (r E_\theta) = 0 \\ \frac{\partial}{\partial r} \frac{\partial}{\partial r} (r H_\phi) - \frac{\partial}{\partial r} (\ln \varepsilon_G) \frac{\partial}{\partial r} (r H_\phi) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (r H_\phi) = 0 \end{cases} \quad (6.47)$$

$$\frac{\partial^2}{\partial r^2} f(r, t) - p(r) \frac{\partial}{\partial r} f(r, t) - \frac{\partial^2}{\partial(ct)^2} f(r, t) = 0 \quad (6.48)$$

$$f(r, t) = f(r) e^{-ikct} \quad (6.49)$$

$$\frac{d^2}{dr^2} f(r) - p(r) \frac{d}{dr} f(r) + k^2 f(r) = 0 \quad (6.50)$$

$$\frac{d^2}{dr^2} f(r) - p \frac{d}{dr} f(r) + k^2 f(r) = 0 \quad (6.51)$$

$$f(r) = e^{(p/2)r} [C_+ e^{i\sqrt{k^2 - (p/2)^2}r} + C_- e^{-i\sqrt{k^2 - (p/2)^2}r}] \quad (6.52)$$

$$f(r, t) = e^{(p/2)r} [C_+ e^{i(+\sqrt{k^2 - (p/2)^2}r - kct)} + C_- e^{i(-\sqrt{k^2 - (p/2)^2}r - kct)}] \quad (6.53)$$

$$f(r, t) = e^{(p/2)r} [C_+ e^{i(+\sqrt{(\omega/c)^2 - (p/2)^2}r - \omega t)} + C_- e^{i(-\sqrt{(\omega/c)^2 - (p/2)^2}r - \omega t)}] \quad (6.54)$$

$$f(r, t) = e^{\int (p/2) dr} [C_+ e^{i(+\int \sqrt{(\omega/c)^2 - (p/2)^2} dr - \omega t)} + C_- e^{i(-\int \sqrt{(\omega/c)^2 - (p/2)^2} dr - \omega t)}] \quad (6.55)$$

$$\begin{cases} r_2 |E_\theta|_{r=r_2} = r_1 |E_\theta|_{r=r_1} e^{\int_{r_1}^{r_2} \frac{1}{2} \frac{\partial}{\partial r} (\ln \mu_G) dr} \\ r_2 |H_\phi|_{r=r_2} = r_1 |H_\phi|_{r=r_1} e^{\int_{r_1}^{r_2} \frac{1}{2} \frac{\partial}{\partial r} (\ln \varepsilon_G) dr} \end{cases} \quad (6.56)$$

$$\begin{cases} E_2 = \sqrt{\frac{\mu_{G2}}{\mu_{G1}} \frac{r_1}{r_2}} E_1 \\ H_2 = \sqrt{\frac{\varepsilon_{G2}}{\varepsilon_{G1}} \frac{r_1}{r_2}} H_1 \end{cases} \quad (6.57)$$

$$\begin{cases} E_2/c_2 = \sqrt{\frac{\mu_{G2}}{\mu_{G1}} \frac{c_1}{c_2} \frac{r_1}{r_2}} E_1/c_1 \\ B_2 = \sqrt{\frac{\mu_{G2}}{\mu_{G1}} \frac{c_1}{c_2} \frac{r_1}{r_2}} B_1 \end{cases} \quad (6.58)$$

$$\begin{cases} (\omega/c_2)c_2(\bar{h}_{00})_2 = \sqrt{\frac{\mu_{G2}}{\mu_{G1}} \frac{c_1}{c_2} \frac{r_1}{r_2}} (\omega/c_1)c_1(\bar{h}_{00})_1 \\ (\omega/c_2)c_2(\bar{h}_{0i})_2 = \sqrt{\frac{\mu_{G2}}{\mu_{G1}} \frac{c_1}{c_2} \frac{r_1}{r_2}} (\omega/c_1)c_1(\bar{h}_{0i})_1 \end{cases} \quad (6.59)$$

$$h_2 = \sqrt{\frac{c_1^4/G_1}{c_2^4/G_2} \frac{r_1}{r_2}} h_1 \quad (6.60)$$

双星系统引力辐射本为

$$h = \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{r} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F(t) dt\right] \quad (6.61)$$

设双星系统常量 c^* , G^* , 一观者临近双星系统且与双星系统相对静止, 其与双星系统距离为 r , 测得强度 h_r , 频率 F_r , 则¹

$$h_r = \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{r/c^*} Q(\theta, \phi, \psi, \iota) \quad (6.62)$$

设地球观者与双星系统距离为 d , 双星系统红移为 z , 测得强度 h_d , 频率 $F_d = F_r/(1+z)$, 则

$$h_d = \sqrt{\frac{c^{*4}/G^*}{c^4/G} \frac{r}{d}} h_r \quad (6.63)$$

$$= \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{d/c^*} Q(\theta, \phi, \psi, \iota) \quad (6.64)$$

所以地球观者测得

$$h = \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{d/c^*} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi \frac{F_r(t)}{1+z} dt\right] \quad (6.65)$$

记 $F_{\text{obs}}(t) = F_r(t)/(1+z)$, $\mathcal{M}_{\text{obs}} = \mathcal{M}(1+z)$, 光度距离 $d_L = d(1+z)$, 则

$$h = \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{d(1+z)/c^*} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F_{\text{obs}}(t) dt\right] \quad (6.66)$$

¹ \mathcal{M} 和 c^* , G^* 简并, 所以可以笼统地仍记作 \mathcal{M} .

$$= \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}_{\text{obs}}[\pi \mathcal{M}_{\text{obs}} F_{\text{obs}}(t)]^{2/3}}{d_{\text{L}}/c^*} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F_{\text{obs}}(t) dt\right] \quad (6.67)$$

$$= \sqrt{\frac{c^{*6}/G^*}{c^6/G}} \frac{\mathcal{M}_{\text{obs}}[\pi \mathcal{M}_{\text{obs}} F_{\text{obs}}(t)]^{2/3}}{d_{\text{L}}/c} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F_{\text{obs}}(t) dt\right] \quad (6.68)$$

用引力波测距测得 $d_{\text{L,G}}$, 则

$$d_{\text{L,G}} = d_{\text{L}} \sqrt{\frac{c^6/G}{c^{*6}/G^*}} \quad (6.69)$$

[4]

$$h(t) = \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{\xi d_{\text{L}}} Q(\text{angles}) \cos \Phi(t) \quad (6.70)$$

$$\tilde{h}(f) = \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{\xi d_{\text{L}}} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4}]} \quad (6.71)$$

问题转化为估计 ξ

$$p(\mu) \propto p^{(0)}(\mu) \exp\left[-\frac{1}{2} \Gamma_{ab}(\mu^a - \hat{\mu}^a)(\mu^b - \hat{\mu}^b)\right] \quad (6.72)$$

$$p^{(0)}(\mu) \propto \exp\left[-\frac{1}{2} \Gamma_{ab}^{(0)}(\mu^a - \bar{\mu}^a)(\mu^b - \bar{\mu}^b)\right] \quad (6.73)$$

设待估参数为 $\mu = (\ln \xi, \ln(d_{\text{L}}/d_{\text{L}0}), \ln Q, \dots)$, \dots 为其他参数 (如 \mathcal{M}), 则 $\tilde{h}_{,\ln \xi} = \tilde{h}_{,\ln(d_{\text{L}}/d_{\text{L}0})} = -\tilde{h}_{,\ln Q} = -\tilde{h}$, \tilde{h} 对其他参数求偏导皆为纯虚数, 则由 $\Gamma_{ab} = \langle h_{,a} | h_{,b} \rangle$ 和 $\text{SNR} := \rho = \sqrt{\langle h | h \rangle}$ 得

$$\Gamma_{ab} = \begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 & 0 & \dots \\ \rho^2 & \rho^2 & -\rho^2 & 0 & \dots \\ -\rho^2 & -\rho^2 & \rho^2 & 0 & \dots \\ 0 & 0 & 0 & ? & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (6.74)$$

又设

$$\Gamma_{ab}^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1/\sigma_{\ln d_{\text{L}}}^2 & 0 & 0 & \dots \\ 0 & 0 & 1/\sigma_{\ln Q}^2 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (6.75)$$

则由 $\Sigma_{ab} = (\Gamma_{ab}^{(0)} + \Gamma_{ab})^{-1}$ 得

$$\Sigma_{ab} = \begin{bmatrix} \begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 \\ \rho^2 & \rho^2 + 1/\sigma_{\ln(d_L/d_{L0})}^2 & -\rho^2 \\ -\rho^2 & -\rho^2 & \rho^2 + 1/\sigma_{\ln Q}^2 \end{bmatrix}^{-1} & 0 \\ 0 & [?]^{-1} \end{bmatrix} \quad (6.76)$$

而

$$\begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 \\ \rho^2 & \rho^2 + 1/\sigma_{\ln(d_L/d_{L0})}^2 & -\rho^2 \\ -\rho^2 & -\rho^2 & \rho^2 + 1/\sigma_{\ln Q}^2 \end{bmatrix}^{-1} \quad (6.77)$$

$$= \begin{bmatrix} 1/\rho^2 + \sigma_{\ln(d_L/d_{L0})}^2 + \sigma_{\ln Q}^2 & -\sigma_{\ln(d_L/d_{L0})}^2 & \sigma_{\ln Q}^2 \\ -\sigma_{\ln(d_L/d_{L0})}^2 & \sigma_{\ln(d_L/d_{L0})}^2 & 0 \\ \sigma_{\ln Q}^2 & 0 & \sigma_{\ln Q}^2 \end{bmatrix} \quad (6.78)$$

6.2 Modification of Phase

$$\frac{d^2}{dz^2} H(z) + 2p(z) \frac{d}{dz} H(z) + [\omega^2 + q(z)] H(z) = 0. \quad (6.79)$$

$$H = A e^{i\Phi}. \quad (6.80)$$

$$k = \frac{d\Phi}{dz},$$

$$\frac{d^2 A}{dz^2} + 2p \frac{dA}{dz} + \left[\omega^2 \left(1 - \frac{k^2}{\omega^2} \right) + q \right] A = 0, \quad (6.81)$$

$$2 \frac{dA}{dz} k + A \frac{dk}{dz} + 2p A k = 0, \quad (6.82)$$

$$2 \frac{1}{A} \frac{dA}{dz} + \frac{1}{k} \frac{dk}{dz} + 2p = 0, \quad (6.83)$$

$$A \propto e^{-\int p dz} k^{-1/2}. \quad (6.84)$$

$$\Gamma = e^{\int p dz} \text{ and } K = (k/\omega)^{-1/2},$$

$$\frac{d^2 K}{dz^2} - \left(\frac{1}{\Gamma} \frac{d^2 \Gamma}{dz^2} - q \right) K + \omega^2 K (1 - K^{-4}) = 0, \quad (6.85)$$

$$\Xi = \frac{1}{\Gamma} \frac{d^2 \Gamma}{dz^2} - q \text{ and make } \omega = 1,$$

$$\frac{d^2 K}{dz^2} + K[(1 - \Xi) - K^{-4}] = 0. \quad (6.86)$$

$$\Xi = \text{const},$$

$$K = (1 - \Xi)^{-1/4} = 1 + \frac{1}{4}\Xi + \frac{5}{32}\Xi^2 + O(\Xi^3), \quad (6.87)$$

$$k = (1 - \Xi)^{1/2} = 1 - \frac{1}{2}\Xi - \frac{1}{8}\Xi^2 + O(\Xi^3), \quad (6.88)$$

$$\Xi \neq \text{const}, \Xi(z) = \kappa^2 \tilde{\Xi}(\tilde{z}), \text{ where } \tilde{z} = \kappa z.$$

$$K^3 \frac{d^2 K}{d\tilde{z}^2} \kappa^2 - K^4 \tilde{\Xi}(\tilde{z}) \kappa^2 + K^4 - 1 = 0. \quad (6.89)$$

$$K = \sum_{n=0}^{\infty} K_n(\tilde{z}) \kappa^{2n}, \quad (6.90)$$

$$K_0^4 - 1 = 0, \quad (6.91)$$

$$K_0^3 K_0'' - K_0^4 \tilde{\Xi} + 4K_0^3 K_1 = 0, \quad (6.92)$$

$$(K_0^3 K_1'' + 3K_0^2 K_1 K_0'') - 4K_0^3 K_1 \tilde{\Xi} + (4K_0^3 K_2 + 6K_0^2 K_1^2) = 0. \quad (6.93)$$

$$K_0 = 1, \quad (6.94)$$

$$K_1 = \frac{1}{4} \tilde{\Xi}, \quad (6.95)$$

$$K_2 = \frac{5}{32} \tilde{\Xi}^2 - \frac{1}{16} \frac{d^2 \tilde{\Xi}}{d\tilde{z}^2}, \quad (6.96)$$

[4]

$$h(t) = \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{d_L} Q(\text{angles}) \cos \Phi(t) \quad (6.97)$$

$$\tilde{h}(f) = \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{d_L} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4}]} \quad (6.98)$$

$$\tilde{h}(f) = \int h(t) e^{2\pi i f t} dt \quad (6.99)$$

$$h(t) = \int \tilde{h}(f) e^{-2\pi i f t} df \quad (6.100)$$

$$A = \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{d_L} f^{-7/6} df \quad (6.101)$$

$$A \propto \Gamma^{-1} K \quad (6.102)$$

$$K|_{z=d_L} = K|_{z=0} = 1 \quad (6.103)$$

$$\frac{A|_{z=d_L}}{A|_{z=d_0}} = e^{-\int_0^{d_L} p \, dz} \quad (6.104)$$

$$A = \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{d_L} f^{-7/6} e^{-\int_0^{d_L} p \, dz} \, df \quad (6.105)$$

$$A = \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{e^{\int_0^{d_L} p \, dz} d_L} f^{-7/6} \, df \quad (6.106)$$

$$A = \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{\xi d_L} f^{-7/6} \, df \quad (6.107)$$

$$A = \mathcal{A} f^{-7/6} \, df \quad (6.108)$$

$$k = \omega \left[1 - \frac{1}{2} \frac{\Xi}{\omega^2} \right] \quad (6.109)$$

$$\psi = \int k \, dz = 2\pi f t(f) - \Phi(f) - \frac{\pi}{4} \quad (6.110)$$

$$\psi = \int k \, dz = 2\pi f t(f) - \Phi(f) - \frac{\pi}{4} - \int_0^{d_L} \frac{1}{2} \frac{\Xi}{\omega^2} \omega \, dz \quad (6.111)$$

$$\psi = \int k \, dz = 2\pi f t(f) - \Phi(f) - \frac{\pi}{4} - \frac{1}{2} \frac{\int_0^{d_L} \Xi \, dz}{(2\pi f)^2} (2\pi f) \quad (6.112)$$

$$\psi = \int k \, dz = 2\pi f t(f) - \Phi(f) - \frac{\pi}{4} - \Omega(2\pi f)^{-1} \quad (6.113)$$

$$\tilde{h}(f) = \mathcal{A} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4} - \Omega(2\pi f)^{-1}]} \quad (6.114)$$

$$2\pi f \Delta t(f) - \Delta \Phi(f) = (2\pi f)^{-1} \quad (6.115)$$

$$\frac{d\Delta \Phi / df}{d\Delta t / df} = 2\pi f \quad (6.116)$$

$$\Delta \Phi(f) = -2(2\pi f)^{-1} \quad (6.117)$$

$$\Delta t(f) = -(2\pi f)^{-2} \quad (6.118)$$

$$\Phi_{\text{1PN}}(f) = -\frac{1}{16} \frac{5}{3} \left(\frac{743}{336} + \frac{11}{4} \eta \right) (\pi \mathcal{M} f)^{-5/3} (\pi M f)^{2/3} \quad (6.119)$$

$$t_{\text{1PN}}(f) = -\frac{5}{256} \frac{4}{3} \left(\frac{743}{336} + \frac{11}{4} \eta \right) \mathcal{M} (\pi \mathcal{M} f)^{-8/3} (\pi M f)^{2/3} \quad (6.120)$$

$$\Delta \Phi(f) = 2\Omega(2\pi f)^{-1} \quad (6.121)$$

$$\Delta t(f) = \Omega(2\pi f)^{-2} \quad (6.122)$$

$$\frac{1}{16} \frac{5}{3} \left(\frac{743}{336} + \frac{11}{4} \eta \right) \mathcal{M}^{-5/3} M^{2/3} - \Omega \quad (6.123)$$

$$h_0(t) = \sum_k [C_+(k) e^{+ik \int dt} + C_-(k) e^{-ik \int dt}] e^{ikd_L} \quad (6.124)$$

$$h(t) = \sum_k \Gamma^{-1} K[C_+(k)e^{+ik \int K^{-2} dt} + C_-(k)e^{-ik \int K^{-2} dt}]e^{ikd_L} \quad (6.125)$$

$$h_0(t) = \int [C_+(k)e^{+ik \int dt} + C_-(k)e^{-ik \int dt}]e^{ikd_L} dk \quad (6.126)$$

$$h(t) = \int \Gamma^{-1} K[C_+(k)e^{+ik \int K^{-2} dt} + C_-(k)e^{-ik \int K^{-2} dt}]e^{ikd_L} dk \quad (6.127)$$

$$h_0(t) = \int [C_+(-\omega)e^{-i\omega d_L} + C_-(+\omega)e^{+i\omega d_L}]e^{-i\omega t} d\omega \quad (6.128)$$

$$= \int \tilde{h}_0(\omega)e^{-i\omega t} d\omega \quad (6.129)$$

$$h(t) = \int [C_+(-\omega)e^{-i\omega d_L} + C_-(+\omega)e^{+i\omega d_L}]e^{-i\omega \int K^{-2} dt} d\omega \quad (6.130)$$

$$= \int \tilde{h}_0(\omega)e^{-i\omega t} e^{+i\omega \int \frac{\Xi(t)}{2\omega^2} d\omega} d\omega \quad (6.131)$$

如果直接认为 \dot{G}/G 是常数, 那 $\int \Xi(t)/2\omega^2 = \eta(\dot{G}/G)^2 t/2\omega^2$, 这里暂用 \sum_ω 表示积分.

$$h(t) = \sum_\omega \tilde{h}_0(\omega)e^{-i\omega t} e^{+i\omega \eta(\dot{G}/G)^2 t/2\omega^2} \quad (6.132)$$

$$= \sum_\omega \tilde{h}_0(\omega)e^{-i\omega(1-\eta(\dot{G}/G)^2/2\omega^2)t} \quad (6.133)$$

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