$\dot{h}_{20}$ of Inspiral Binary

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If $\dot{h}_{l0} \approx 0$,

$$h_{l0}|_{t_1}^{t_2} = \ldots \Re\left[\frac{4}{D}\int \Psi_2^\circ|_{t_1}^{t_2} \ldots \, d\theta d\phi - D\cdots \times \left(\int_{t_1}^{t_2} \dot{h}\dot{\bar{h}}\, dt - (\dot{h}\bar{h})|_{t_1}^{t_2}\right)\right]. \tag{1}$$

$$\left(\int_{t_1}^{t_2} \dot{h}\dot{\bar{h}}\, dt - (\dot{h}\bar{h})|_{t_1}^{t_2}\right) = \left(\int_{t_1}^{t_2} \dot{h}\dot{\bar{h}}\, dt - \int_{t_1}^{t_2} (\ddot{h}\bar{h} + \dot{h}\dot{\bar{h}})\, dt\right) \tag{2}$$

$$= -\left(\int_{t_1}^{t_2} \ddot{h}\bar{h}\, dt\right), \tag{3}$$

$$h_{l0}|_{t_1}^{t_2} = \ldots \Re\left[\frac{4}{D}\int \Psi_2^\circ|_{t_1}^{t_2} \ldots \, d\theta d\phi + D\cdots \times \left(\int_{t_1}^{t_2} \ddot{h}\bar{h}\, dt\right)\right] \tag{4}$$

$$= \ldots \Re\left[\frac{4}{D}\int \left(\int_{t_1}^{t_2} \frac{\partial \Psi_2^\circ}{\partial t}\, dt\right) \ldots \, d\theta d\phi + D\cdots \times \left(\int_{t_1}^{t_2} \ddot{h}\bar{h}\, dt\right)\right] \tag{5}$$

$$= \ldots \Re\left[\frac{4}{D}\int_{t_1}^{t_2} \left(\int \frac{\partial \Psi_2^\circ}{\partial t} \ldots \, d\theta d\phi\right) dt + D\cdots \times \int_{t_1}^{t_2} \ddot{h}\bar{h}\, dt\right]. \tag{6}$$

$$
\begin{aligned}
\Psi_2^\circ &= -\frac{M}{\gamma^4}(1 - v_x \sin\theta\cos\phi - v_y \sin\theta\sin\phi - v_z\cos\theta)^{-3} && (7)\\
&= -\frac{M(1 - v_x^2 - v_y^2 - v_z^2)^2}{(1 - v_x \sin\theta\cos\phi - v_y \sin\theta\sin\phi - v_z\cos\theta)^3} && (8)\\
&= -\frac{M(\frac{M^2 - p_x^2 - p_y^2 - p_z^2}{M^2})^2}{(\frac{M - p_x\sin\theta\cos\phi - p_y\sin\theta\sin\phi - p_z\cos\theta}{M})^3} \quad (\vec{p} := M\vec{v}) && (9)\\
&= -\frac{(M^2 - p_x^2 - p_y^2 - p_z^2)^2}{(M - p_x\sin\theta\cos\phi - p_y\sin\theta\sin\phi - p_z\cos\theta)^3}, && (10)
\end{aligned}
$$

$$\Psi_2^\circ = -\frac{(M^2 - p_x^2 - p_y^2 - p_z^2)^2}{(M - p_x \sin\theta\cos\phi - p_y \sin\theta\sin\phi - p_z\cos\theta)^3},$$

$$\begin{aligned}\frac{\partial \Psi_2^\circ}{\partial t} = &-\frac{2(M^2 - p_x^2 - p_y^2 - p_z^2)(2M\frac{dM}{dt} - 2p_x\frac{dp_x}{dt} - 2p_y\frac{dp_y}{dt} - 2p_z\frac{dp_z}{dt})}{(M - p_x \sin\theta\cos\phi - p_y \sin\theta\sin\phi - p_z\cos\theta)^3} \\ &- \frac{-3(M^2 - p_x^2 - p_y^2 - p_z^2)^2(\frac{dM}{dt} - \frac{dp_x}{dt}\sin\theta\cos\phi - \frac{dp_y}{dt}\sin\theta\sin\phi - \frac{dp_z}{dt}\cos\theta)}{(M - p_x \sin\theta\cos\phi - p_y \sin\theta\sin\phi - p_z\cos\theta)^4}.\end{aligned} \tag{11}$$

If $|\vec{p}| \approx E_{\mathrm{GW}} \ll M$, $\left|\frac{d\vec{p}}{dt}\right| \approx \frac{dE_{\mathrm{GW}}}{dt}$,

$$\begin{aligned}\frac{\partial \Psi_2^\circ}{\partial t} = &-\frac{2(M^2)(2M\frac{dM}{dt})}{(M)^3} \\ &- \frac{-3(M^2)^2(\frac{dM}{dt} - \frac{dp_x}{dt}\sin\theta\cos\phi - \frac{dp_y}{dt}\sin\theta\sin\phi - \frac{dp_z}{dt}\cos\theta)}{(M)^4}. &(12) \\ = &-4\frac{dM}{dt} + 3(\frac{dM}{dt} - \frac{dp_x}{dt}\sin\theta\cos\phi - \frac{dp_y}{dt}\sin\theta\sin\phi - \frac{dp_z}{dt}\cos\theta), &(13) \\ = &-\frac{dM}{dt} - 3(\frac{dp_x}{dt}\sin\theta\cos\phi - \frac{dp_y}{dt}\sin\theta\sin\phi - \frac{dp_z}{dt}\cos\theta). &(14)\end{aligned}$$

$$\frac{\partial \Psi_2^\circ}{\partial t} = -\frac{dM}{dt} - 3\left(\frac{dp_x}{dt}\sin\theta\cos\phi - \frac{dp_y}{dt}\sin\theta\sin\phi - \frac{dp_z}{dt}\cos\theta\right).$$

$$\frac{4}{D}\int_{t_1}^{t_2}\left(\int \frac{\partial \Psi_2^\circ}{\partial t}\left[{}^0Y_{l0}\right]\sin\theta\, d\theta d\phi\right)\, dt = 0. \tag{15}$$

$$\int_0^\pi \left[{}^0Y_{l0}\right]\sin\theta\, d\theta \propto \int_{-1}^{1} P_l(x)\, dx \propto \int_{-1}^{1} P_l(x)P_0(x)\, dx = 0,$$

$$\int_0^{2\pi}\cos\phi\, d\phi = 0, \quad \int_0^{2\pi}\sin\phi\, d\phi = 0,$$

$$\int_0^\pi \left[{}^0Y_{l0}\right]\cos\theta\sin\theta\, d\theta \propto \int_{-1}^{1} P_l(x)x\, dx \propto \int_{-1}^{1} P_l(x)P_1(x)\, dx = 0.$$

$$h_{l0}|_{t_1}^{t_2} = \sqrt{\frac{(l-2)!}{(l+2)!}} D\,\Re\left[\sum_{\substack{l'\geqslant 2\\ l''\geqslant 2\\ 0<|m'|\leqslant l'\\ 0<|m''|\leqslant l''}} \Gamma^{2-20}_{l'l''lm'-m''0}\left(-\int_{t_1}^{t_2} \ddot{h}_{l'm'}\bar{h}_{l''m''}\,dt\right)\right], \tag{16}$$

$$-\int_{t_1}^{t_2} \ddot{h}_{l'm'}\bar{h}_{l''m''}\,dt = \int_{t_1}^{t_2} \dot{h}_{l'm'}\dot{\bar{h}}_{l''m''}\,dt - (\dot{h}_{l'm'}\bar{h}_{l''m''})|_{t_1}^{t_2}. \tag{17}$$

If (2,2) mode dominants,

$$h_{l0}|_{t_1}^{t_2} = -2\sqrt{\frac{(l-2)!}{(l+2)!}}\Gamma^{2-20}_{22l2-20} D\,\Re\left[\int_{t_1}^{t_2} \ddot{h}_{22}\bar{h}_{22}\,dt\right] \tag{18}$$

$$= 2\sqrt{\frac{(l-2)!}{(l+2)!}}\Gamma^{2-20}_{22l2-20} D\,\Re\left[\int_{t_1}^{t_2} \dot{h}_{22}\dot{\bar{h}}_{22}\,dt - (\dot{h}_{22}\bar{h}_{22})|_{t_1}^{t_2}\right]. \tag{19}$$

This formula is used in code.

If $f_{\mathrm{GW}} \approx$ const, $\int_{t_1}^{t_2} \ddot{h}_{22}\bar{h}_{22}\, dt = -\int_{t_1}^{t_2} \dot{h}_{22}\dot{\bar{h}}_{22}\, dt = -\int_{t_1}^{t_2} \left|\dot{h}_{22}\right|^2\, dt$,
$\left(\left|\int_{t_1}^{t_2} \ddot{h}_{22}\bar{h}_{22}\, dt + \int_{t_1}^{t_2} \dot{h}_{22}\dot{\bar{h}}_{22}\, dt\right| = \left|(\dot{h}_{22}\bar{h}_{22})|_{t_1}^{t_2}\right| \ll \int_{t_1}^{t_2} \left|\dot{h}_{22}\right|^2\, dt\right)$,

$$h_{l0}|_{t_1}^{t_2} = 2\sqrt{\frac{(l-2)!}{(l+2)!}}\Gamma^{2-20}_{22l2-20} D \int_{t_1}^{t_2} \left|\dot{h}_{22}\right|^2\, dt. \qquad (20)$$

By the way,

$$h = \sum h_{lm} \left[{}^0Y_{lm}\right], \tag{21}$$

$$\dot{h} = \sum \dot{h}_{lm} \left[{}^0Y_{lm}\right], \tag{22}$$

$$\dot{\bar{h}}\dot{h} = \sum \overline{\left[{}^0Y_{\tilde{l}\tilde{m}}\right]} \dot{\bar{h}}_{\tilde{l}\tilde{m}} \dot{h}_{lm} \left[{}^0Y_{lm}\right], \tag{23}$$

$$\int \frac{\left|\dot{h}\right|^2}{16\pi} D^2 \sin\theta \, d\theta d\phi = \int \sum \overline{\left[{}^0Y_{\tilde{l}\tilde{m}}\right]} \frac{\dot{\bar{h}}_{\tilde{l}\tilde{m}} \dot{h}_{lm}}{16\pi} \left[{}^0Y_{lm}\right] D^2 \sin\theta \, d\theta d\phi, \tag{24}$$

$$\frac{dE_{\mathrm{GW}}}{dt} = D^2 \sum \left(\frac{\dot{\bar{h}}_{\tilde{l}\tilde{m}} \dot{h}_{lm}}{16\pi} \int \overline{\left[{}^0Y_{\tilde{l}\tilde{m}}\right]} \left[{}^0Y_{lm}\right] \sin\theta \, d\theta d\phi \right), \tag{25}$$

$$\frac{dE_{\mathrm{GW}}}{dt} = \frac{D^2}{16\pi} \sum \left|\dot{h}_{lm}\right|^2. \tag{26}$$

$$L_{\mathrm{GW}} = \frac{dE_{\mathrm{GW}}}{dt} = \frac{D^2}{16\pi} \sum \left| \dot{h}_{lm} \right|^2, \tag{27}$$

$$\bar{F}_{\mathrm{GW}} := \frac{L_{\mathrm{GW}}}{4\pi D^2} = \frac{1}{64\pi^2} \sum \left| \dot{h}_{lm} \right|^2. \tag{28}$$

If (2,2) mode dominants,

$$L_{\mathrm{GW}} = \frac{D^2}{8\pi} \left| \dot{h}_{22} \right|^2, \tag{29}$$

$$\bar{F}_{\mathrm{GW}} = \frac{1}{32\pi^2} \left| \dot{h}_{22} \right|^2. \tag{30}$$

$$h_{l0}\Big|_{t_1}^{t_2} = 2\sqrt{\frac{(l-2)!}{(l+2)!}}\Gamma^{2-20}_{22l2-20} D \int_{t_1}^{t_2} \left|\dot{h}_{22}\right|^2 \, dt,$$

$$\dot{h}_{l0} = 2\sqrt{\frac{(l-2)!}{(l+2)!}}\Gamma^{2-20}_{22l2-20} D \left|\dot{h}_{22}\right|^2 \tag{31}$$

$$= 64\pi^2\sqrt{\frac{(l-2)!}{(l+2)!}}\Gamma^{2-20}_{22l2-20} \bar{F}_{\mathrm{GW}} D \tag{32}$$

$$= 16\pi\sqrt{\frac{(l-2)!}{(l+2)!}}\Gamma^{2-20}_{22l2-20} \frac{L_{\mathrm{GW}}}{D} \tag{33}$$

If $l = 2$, $\Gamma^{2-20}_{2222-20} = \frac{1}{7}\sqrt{\frac{5}{\pi}}$,

$$\dot{h}_{20} = \frac{8}{7}\sqrt{\frac{5\pi}{6}}\frac{L_{\mathsf{GW}}}{D} = \frac{32\pi}{7}\sqrt{\frac{5\pi}{6}}\bar{F}_{\mathsf{GW}}D = \frac{1}{7}\sqrt{\frac{5}{6\pi}}D\left|\dot{h}_{22}\right|^2 . \quad (34)$$