引力波天文学笔记

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2023 年 12 月 5 日

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Printed in China

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第一章 引力波

1.1 Linearized Gravity

[8]. 流形 $\mathbb{R}^4$. 任意坐标系 $\{x^\mu\}$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} s + \mathrm{O}(s^2)$. 设 $g^{\mu\nu} = ?^{\mu\nu} + ??^{\mu\nu} s + \mathrm{O}(s^2)$, 则 $\delta^\mu{}_\lambda = ?^{\mu\nu}\eta_{\nu\lambda} + ?^{\mu\nu}\gamma_{\nu\lambda} s + ??^{\mu\nu}\eta_{\nu\lambda} s + \mathrm{O}(s^2)$, 所以 $?^{\mu\nu} = \eta^{\mu\nu}$, $??^{\mu\nu} = ??^{\mu\sigma}\delta_\sigma{}^\nu = ??^{\mu\sigma}\eta_{\sigma\lambda}\eta^{\lambda\nu} = -?^{\mu\sigma}\gamma_{\sigma\lambda}\eta^{\lambda\nu} = -\eta^{\mu\sigma}\gamma_{\sigma\lambda}\eta^{\lambda\nu} = -\gamma^{\mu\nu}$, 所以 $g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu} s + \mathrm{O}(s^2) = \eta^{\mu\nu} - h^{\mu\nu} + \mathrm{O}(s^2)$.

$$R_{\mu\nu\lambda\sigma} = \partial_\sigma \partial_{[\mu} h_{\lambda]\nu} - \partial_\nu \partial_{[\mu} h_{\lambda]\sigma} + \mathrm{O}(s^2). \tag{1.1}$$

$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\sigma}h_{\lambda\sigma} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$.

$$-\frac{1}{2}\partial^\lambda\partial_\lambda \bar{h}_{\mu\nu} + \partial^\lambda\partial_{(\mu}\bar{h}_{\nu)\lambda} - \frac{1}{2}\eta_{\mu\nu}\partial^\lambda\partial^\sigma\bar{h}_{\lambda\sigma} + \mathrm{O}(s^2) = 8\pi T_{\mu\nu}. \tag{1.2}$$

存在 $\{x^\mu\}$, 使得 $\partial^\nu\bar{h}_{\mu\nu} + \mathrm{O}(s^2) = 0$ (Lorentz gauge). [证: 设 $x'^\mu = x^\mu - \xi^\mu = x^\mu - \zeta^\mu s - \mathrm{O}(s^2)$, 则 $\frac{\partial ?}{\partial x'^\mu} = \frac{\partial ?}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial x'^\mu} = \frac{\partial ?}{\partial x^\lambda}(\delta^\lambda{}_\mu + \frac{\partial \xi^\lambda}{\partial x'^\mu}) = \frac{\partial ?}{\partial x^\nu} + \mathrm{O}(s^2)$, $g'_{\mu\nu} = g_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} = g_{\lambda\sigma}(\delta^\lambda{}_\mu + \frac{\partial \xi^\lambda}{\partial x'^\mu})(\delta^\sigma{}_\nu + \frac{\partial \xi^\sigma}{\partial x'^\nu}) = g_{\mu\nu} + g_{\mu\sigma}\frac{\partial \xi^\sigma}{\partial x'^\nu} + g_{\lambda\nu}\frac{\partial \xi^\lambda}{\partial x'^\mu} = g_{\mu\nu} + (\eta_{\mu\sigma} + \mathrm{O}(s))(\frac{\partial \xi^\sigma}{\partial x^\nu} + \mathrm{O}(s^2)) + (\eta_{\lambda\nu} + \mathrm{O}(s))(\frac{\partial \xi^\lambda}{\partial x^\mu} + \mathrm{O}(s^2)) = g_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \mathrm{O}(s^2)$, 所以 $h'_{\mu\nu} = g'_{\mu\nu} - \eta_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \mathrm{O}(s^2) = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \mathrm{O}(s^2)$, 因此存在 ξ^μ, 使得 $\partial'^\nu\bar{h}'_{\mu\nu} + \mathrm{O}(s^2) = 0$.] 令 $\{x^\mu\}$ 满足 $\partial^\nu\bar{h}_{\mu\nu} + \mathrm{O}(s^2) = 0$, 则

$$\partial^\lambda\partial_\lambda\bar{h}_{\mu\nu} + \mathrm{O}(s^2) = -16\pi T_{\mu\nu}. \tag{1.3}$$

略去 $\mathrm{O}(s^2)$ 条件: $h_{\mu\nu}$, $\partial_\lambda h_{\mu\nu}$...小. 下略 $\mathrm{O}(s^2)$.

Lorentz gauge 等价于协和坐标条件.

1.2 Radiation Gauge

[8]. 存在 $\{x^\mu\}$, 使得 “无源处” $h + \mathrm{O}(s^2) = 0$ (TT gauge [9]) 且 $h_{0\mu} + \mathrm{O}(s^2) = 0$. [4], 解 $\partial^\lambda \partial_\lambda \bar{h}_{\mu\nu} = 0$ 得 $h_{ij} = A_{ij}(\vec{k}) e^{ik^\mu x_\mu}$ (A_{ij} 称为 polarization tensor). $h_{(ij)} = 0$, $h = 0$, $\partial^j h_{ij} = 0 \Rightarrow A_{(ij)} = 0$, $A = 0$, $k^j A_{ij} = 0$. 令 $\vec{e}_z \parallel \vec{k}$,

$$h_{xy} = \begin{bmatrix} +h_+ & h_\times \\ h_\times & -h_+ \end{bmatrix} e^{i\omega(t-z)}. \tag{1.4}$$

[4]. Lorentz gauge $\to$ radiation gauge, $P_{ij} := \delta_{ij} - n_i n_j$, $\Lambda_{ijkl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$, $h^{\mathrm{r}}_{ij} = \Lambda_{ijkl} h^{\mathrm{L}}_{kl} = \Lambda_{ijkl} \bar{h}^{\mathrm{L}}_{kl}$. [7]. Step 1: 坐标系空间旋转, 使 $\vec{e}_z \parallel \vec{n}$. Step 2: 取 x, y 分量 h_{xy}. Step 3: 去迹. $[h_+ = \frac{1}{2}(h_{xx} - h_{yy}),\ h_\times = h_{xy} = h_{yx}.]$

1.3 Fourier Transformation

[4].

$$h_{ij} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\vec{k} \left[\mathcal{A}_{ij}(\vec{k}) e^{+ik_\mu x^\mu} + \mathcal{A}^*_{ij}(\vec{k}) e^{-ik_\mu x^\mu} \right] \tag{1.5}$$

$\mathrm{d}^2\vec{n} := \sin\theta \mathrm{d}\theta \mathrm{d}\phi$,

$$\begin{aligned} h_{ij} &= \int_0^\infty \mathrm{d}f\, f^2 \int \mathrm{d}^2\vec{n} \left[\mathcal{A}_{ij}(f, \vec{n}) e^{-2\pi i f(t - \vec{n}\cdot\vec{x})} + \text{c.c.} \right] && (1.6) \\ &= \int_0^\infty \mathrm{d}f \left[e^{-2\pi i f t} f^2 \int \mathrm{d}^2\vec{n}\, \mathcal{A}_{ij}(f, \vec{n}) e^{+2\pi i f \vec{n}\cdot\vec{x}} + \text{c.c.} \right] && (1.7) \\ &:= \int_0^\infty \mathrm{d}f \left[\tilde{h}_{ij}(f, \vec{x}) e^{-2\pi i f t} + \tilde{h}^*_{ij}(f, \vec{x}) e^{+2\pi i f t} \right] && (1.8) \\ &:= \int \mathrm{d}f\, \tilde{h}_{ij}(f, \vec{x}) e^{-2\pi i f t}. && (1.9) \end{aligned}$$

When we observe on Earth a GW emitted by a single astrophysical source, and the linear dimensions of the detector are much smaller than wavelength of the GW, choosing the origin of the coordinate system centered on the detector, $\tilde{h}_{ij}(f, \vec{x}) \approx \tilde{h}_{ij}(f) := \tilde{h}_{ij}(f, \vec{x} = \vec{0})$,

$$h_{ij} = \int \mathrm{d}f\, \tilde{h}_{ij}(f) e^{-2\pi i f t}. \tag{1.10}$$

The dependence on $\vec{x}$ must be kept in some cases (see [4]).

1.4 TT frame

TT gauge $\Rightarrow$ TT frame. free test body $x^\mu(\tau)$, $\frac{\mathrm{d}x^i}{\mathrm{d}t}|_{\tau=0} = 0 \Rightarrow \frac{\mathrm{d}x^0}{\mathrm{d}\tau} \equiv 1$ and $\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \equiv 0$.

设一测试体在 $(0,0,0)$，另一测试体在 $(\Delta x^1, \Delta x^2, \Delta x^3)$，定义 $\Delta x^2 = \delta_{ij}\Delta x^i \Delta x^j$，则 $\Delta s^2 = g_{ij}\Delta x^i \Delta x^j = \Delta x^2(1 + h_{ij}\frac{\Delta x^i}{\Delta x}\frac{\Delta x^j}{\Delta x})$，$\Delta s \approx \Delta x(1 + \frac{1}{2}h_{ij}\frac{\Delta x^i}{\Delta x}\frac{\Delta x^j}{\Delta x})$，$\Delta\ddot{s} \approx \frac{1}{2}\ddot{h}_{ij}\frac{\Delta x^i}{\Delta x}\frac{\Delta x^j}{\Delta x}\Delta x$. 定义 $n^i = \frac{\Delta x^i}{\Delta x}$，则 $\Delta\ddot{s} \approx n^i(\frac{1}{2}\ddot{h}_{ij}\Delta x^j)$. 定义 $\Delta s^i = \Delta s\, n^i$，则 $\Delta s = \Delta s\, n^i n_i = \Delta s^i n_i = n^i \Delta s_i$，则 $\Delta s_i \approx \frac{1}{2}\ddot{h}_{ij}\Delta x^j \approx \frac{1}{2}\ddot{h}_{ij}\Delta s^j$.

1.5 Proper detector frame

$$g_{\mu\nu} \approx (g_{\mu\nu})_{x=0} + (\partial_i g_{\mu\nu})_{x=0}\Delta x^i + \frac{1}{2}(\partial_i\partial_j g_{\mu\nu})_{x=0}\Delta x^i \Delta x^j \tag{1.11}$$

$$(g_{\mu\nu})_{x=0} = \eta_{\mu\nu} \tag{1.12}$$

$$(\Gamma^\sigma{}_{\mu\nu})_{x=0} = \frac{1}{2}\eta^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})_{x=0} = 0 \tag{1.13}$$

$$(\partial_\mu g_{\nu 0} + \partial_\nu g_{\mu 0} - \partial_0 g_{\mu\nu})_{x=0} = 0 \tag{1.14}$$

$$(\partial_\mu g_{\nu i} + \partial_\nu g_{\mu i} - \partial_i g_{\mu\nu})_{x=0} = 0 \tag{1.15}$$

$$(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})_{x=0} = 0 \tag{1.16}$$

$$t_{abc} + t_{bca} - t_{cab} = 0 \tag{1.17}$$

$$t_{cab} + t_{abc} - t_{bca} = 0 \tag{1.18}$$

$$t_{abc} = 0 \tag{1.19}$$

$$(R_{\mu\nu\sigma}{}^\rho)_{x=0} = (\partial_\nu \Gamma^\rho{}_{\mu\sigma} - \partial_\mu \Gamma^\rho{}_{\nu\sigma})_{x=0} \tag{1.20}$$

$$(\partial_\mu \Gamma^\rho{}_{\nu\sigma})_{x=0} = \frac{1}{2}[\partial_\mu g^{\rho\lambda}(\partial_\nu g_{\sigma\lambda} + \partial_\sigma g_{\nu\lambda} - \partial_\lambda g_{\nu\sigma}) + g^{\rho\lambda}\partial_\mu(\partial_\nu g_{\sigma\lambda} + \partial_\sigma g_{\nu\lambda} - \partial_\lambda g_{\nu\sigma})]_{x=0} \tag{1.21}$$

$$(\partial_\mu \Gamma^\rho{}_{\nu\sigma})_{x=0} = \frac{1}{2}[\eta^{\rho\lambda}\partial_\mu(\partial_\nu g_{\sigma\lambda} + \partial_\sigma g_{\nu\lambda} - \partial_\lambda g_{\nu\sigma})]_{x=0} \tag{1.22}$$

$$(\partial_\mu \Gamma_{\rho\nu\sigma})_{x=0} = \frac{1}{2}[\partial_\mu(\partial_\nu g_{\sigma\rho} + \partial_\sigma g_{\nu\rho} - \partial_\rho g_{\nu\sigma})]_{x=0} \tag{1.23}$$

$$(R_{\mu\nu\sigma\rho})_{x=0} = -\frac{1}{2}[(\partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\mu\partial_\rho g_{\nu\sigma}) - (\partial_\nu\partial_\sigma g_{\mu\rho} - \partial_\nu\partial_\rho g_{\mu\sigma})]_{x=0} \tag{1.24}$$

[5]

第二章　能量

[8],

$$G^{[1]}_{ab}(h^{[1]}_{cd}) + G^{[1]}_{ab}(h^{[2]}_{cd}) + G^{[2]}_{ab}(h^{[1]}_{cd}) = 8\pi T_{ab}, \tag{2.1}$$

$$G^{[1]}_{ab}(h^{[1]}_{cd} + h^{[2]}_{cd}) = 8\pi(T_{ab} + t_{ab}) := 8\pi(T_{ab} - \frac{G^{[2]}_{ab}(h^{[1]}_{cd})}{8\pi}), \tag{2.2}$$

Thus, in the 2nd order, $h^{[2]}_{ab}$ causes the same correction to g_{ab} as would be produced by ordinary matter with effect stress-energy tensor t_{ab}. If not $T_{ab} \gg t_{ab}$, derivations in 一 are not valid.

[4], $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$. $R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} \ldots$,

$$R^{(0)}_{\mu\nu} + [R^{(2)}_{\mu\nu}]^{\mathrm{low}} = 8\pi(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})^{\mathrm{low}}, \tag{2.3}$$

$$R^{(1)}_{\mu\nu} + [R^{(2)}_{\mu\nu}]^{\mathrm{high}} = 8\pi(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})^{\mathrm{high}}, \tag{2.4}$$

(2.3) $\Rightarrow$

$$\begin{aligned} R^{(0)}_{\mu\nu} &= 8\pi\langle(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})^{\mathrm{low}}\rangle - \langle[R^{(2)}_{\mu\nu}]^{\mathrm{low}}\rangle & (2.5)\\ &= 8\pi\langle(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})\rangle - \langle[R^{(2)}_{\mu\nu}]\rangle & (2.6)\\ &:= 8\pi(T^{(0)}_{\mu\nu} - \frac{1}{2}T^{(0)}g^{(0)}_{\mu\nu}) + 8\pi(t_{\mu\nu} - \frac{1}{2}tg^{(0)}_{\mu\nu}), & (2.7)\end{aligned}$$

$\Rightarrow$

$$G^{(0)}_{\mu\nu} = 8\pi(T^{(0)}_{\mu\nu} + t_{\mu\nu}). \tag{2.8}$$

In TT gauge,

$$t_{\mu\nu} = \frac{1}{32\pi}\langle\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\rangle. \tag{2.9}$$

第三章　多极矩

3.1 Quadrupole Approximation

[8]. 由(1.3)得

$$\bar{h}_{\mu\nu}(t,\vec{r}) = 4\int \frac{T_{\mu\nu}(t-|\vec{r}-\vec{r}'|,\vec{r}')}{|\vec{r}-\vec{r}'|}\,\mathrm{d}V'. \tag{3.1}$$

$$\begin{aligned}\hat{\bar{h}}_{\mu\nu}(\omega,\vec{r}) &:= \frac{1}{\sqrt{2\pi}}\int \bar{h}_{\mu\nu}(t,\vec{r})e^{i\omega t}\mathrm{d}t \qquad (3.2)\\ &= 4\int \frac{\hat{T}_{\mu\nu}(\omega,\vec{r}')}{|\vec{r}-\vec{r}'|}e^{i\omega|\vec{r}-\vec{r}'|}\,\mathrm{d}V'. \qquad (3.3)\end{aligned}$$

由 $\partial^\nu \bar{h}_{\mu\nu} = 0$,

$$-i\omega\hat{\bar{h}}_{0\mu} = \sum_i \frac{\partial\hat{\bar{h}}_{i\mu}}{\partial x^i}. \tag{3.4}$$

$|\vec{r}| \gg |\vec{r}'|$ 且 $\omega \ll 1/|\vec{r}'|$,

$$\hat{\bar{h}}_{ij}(\omega,\vec{r}) = 4\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{T}_{ij}(\omega,\vec{r}')\,\mathrm{d}V'. \tag{3.5}$$

$$\begin{aligned}\int \hat{T}_{ij}\,\mathrm{d}V' &= \int \sum_k (\hat{T}_{kj}\frac{\partial x'^i}{\partial x'^k})\,\mathrm{d}V' \qquad (3.6)\\ &= \sum_k \left[\int \frac{\partial}{\partial x'^k}(\hat{T}_{kj}x'^i)\,\mathrm{d}V' - \int \frac{\partial\hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V'\right] \qquad (3.7)\\ &= \sum_k \int \partial'_k\,(\hat{T}_{kj}x'^i)\,\mathrm{d}V' - \sum_k \int \frac{\partial\hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V' \qquad (3.8)\\ &= \int \hat{T}_{kj}x'^i\,\mathrm{d}S' - \sum_k \int \frac{\partial\hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V' \qquad (3.9)\end{aligned}$$

$$= -\sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i \, \mathrm{d}V' \tag{3.10}$$

$$= -\int (\sum_k \partial'_k \hat{T}_{kj}) x'^i \, \mathrm{d}V' \tag{3.11}$$

$$= -\int (\partial_0 \hat{T}_{0j}) x'^i \, \mathrm{d}V' \tag{3.12}$$

$$= -i\omega \int \hat{T}_{0j} x'^i \, \mathrm{d}V' \tag{3.13}$$

$$= \int \hat{T}_{(ij)} \, \mathrm{d}V' \tag{3.14}$$

$$= -i\omega \int \hat{T}_{0(j} x'^{i)} \, \mathrm{d}V' \tag{3.15}$$

$$= -\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) \, \mathrm{d}V', \tag{3.16}$$

$$-\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) \, \mathrm{d}V' = -\frac{i\omega}{2} \int \sum_k (\hat{T}_{0k} x'^i \frac{\partial x'^j}{\partial x'^k} + \hat{T}_{0k} \frac{\partial x'^i}{\partial x'^k} x'^j) \, \mathrm{d}V' \tag{3.17}$$

$$= -\frac{i\omega}{2} \sum_k \left[\int \frac{\partial}{\partial x'^k} (\hat{T}_{0k} x'^i x'^j) \, \mathrm{d}V' - \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, \mathrm{d}V' \right] \tag{3.18}$$

$$= -\frac{i\omega}{2} \sum_k \int \partial'_k \, (\hat{T}_{0k} x'^i x'^j) \, \mathrm{d}V' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, \mathrm{d}V' \tag{3.19}$$

$$= -\frac{i\omega}{2} \sum_k \int \hat{T}_{0k} x'^i x'^j \, \mathrm{d}S' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, \mathrm{d}V' \tag{3.20}$$

$$= \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, \mathrm{d}V' \tag{3.21}$$

$$= \frac{i\omega}{2} \int (\sum_k \partial'_k \hat{T}_{0k}) x'^i x'^j \, \mathrm{d}V' \tag{3.22}$$

$$= \frac{i\omega}{2} \int (\partial_0 \hat{T}_{00}) x'^i x'^j \, \mathrm{d}V' \tag{3.23}$$

$$= -\frac{\omega^2}{2} \int \hat{T}_{00} \, x'^i x'^j \, \mathrm{d}V'. \tag{3.24}$$

$$q_{ij}(t) := \int T_{00}\, x'^i x'^j \,\mathrm{d}V', \tag{3.25}$$

$$\hat{\bar{h}}_{ij}(\omega,\vec{r}) = -2\omega^2 \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\hat{q}_{ij}(\omega), \tag{3.26}$$

$$\bar{h}_{ij}(t,\vec{r}) = \frac{2}{|\vec{r}|}\frac{\mathrm{d}^2}{\mathrm{d}t^2} q_{ij}(t-|\vec{r}|). \tag{3.27}$$

3.2 电磁—引力对比

$$A_\mu(t,\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{J_\mu(t-|\vec{r}-\vec{r}'|\,,\vec{r}')}{|\vec{r}-\vec{r}'|}\mathrm{d}V' \tag{3.28}$$

$$\bar{h}_{\mu\nu}(t,\vec{r}) = 4G\int \frac{T_{\mu\nu}(t-|\vec{r}-\vec{r}'|\,,\vec{r}')}{|\vec{r}-\vec{r}'|}\,\mathrm{d}V' \tag{3.29}$$

$$A_\mu(t,\vec{r}) = \frac{1}{\sqrt{2\pi}}\int \hat{A}_\mu(\omega,\vec{r})e^{-i\omega t}\mathrm{d}t \tag{3.30}$$

$$\bar{h}_{\mu\nu}(t,\vec{r}) = \frac{1}{\sqrt{2\pi}}\int \hat{\bar{h}}_{\mu\nu}(\omega,\vec{r})e^{-i\omega t}\mathrm{d}t \tag{3.31}$$

$$\hat{A}_\mu(\omega,\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\hat{J}_\mu(\omega,\vec{r}')}{|\vec{r}-\vec{r}'|}e^{i\omega|\vec{r}-\vec{r}'|}\,\mathrm{d}V' \tag{3.32}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega,\vec{r}) = 4G\int \frac{\hat{T}_{\mu\nu}(\omega,\vec{r}')}{|\vec{r}-\vec{r}'|}e^{i\omega|\vec{r}-\vec{r}'|}\,\mathrm{d}V' \tag{3.33}$$

$$\hat{A}_\mu(\omega,\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{J}_\mu(\omega,\vec{r}')e^{-i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}')}\,\mathrm{d}V' \tag{3.34}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega,\vec{r}) = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{T}_{\mu\nu}(\omega,\vec{r}')e^{-i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}')}\,\mathrm{d}V' \tag{3.35}$$

$$\hat{A}_\mu(\omega,\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{J}_\mu(\omega,\vec{r}')\left[1-i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}')-\dots\right]\,\mathrm{d}V' \tag{3.36}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega,\vec{r}) = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{T}_{\mu\nu}(\omega,\vec{r}')\left[1-i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}')-\dots\right]\,\mathrm{d}V' \tag{3.37}$$

3.2.1 电偶极—引力对比

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_i \, \mathrm{d}V' \tag{3.38}$$

$$\hat{\bar{h}}_{ij} = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{ij} \, \mathrm{d}V' \tag{3.39}$$

$$\int \hat{J}_i \, \mathrm{d}V' = -i\omega \int \hat{J}_0 x'^i \, \mathrm{d}V' \tag{3.40}$$

$$\int \hat{T}_{ij} \, \mathrm{d}V' = -\frac{\omega^2}{2} \int \hat{T}_{00} \, x'^i x'^j \, \mathrm{d}V' \tag{3.41}$$

$$\hat{p}_i = \int \hat{J}_0 x'^i \, \mathrm{d}V' \tag{3.42}$$

$$\hat{q}_{ij} = \int \hat{T}_{00} \, x'^i x'^j \, \mathrm{d}V' \tag{3.43}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega \hat{p}_i) \tag{3.44}$$

$$\hat{\bar{h}}_{ij} = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-\frac{\omega^2}{2} \hat{q}_{ij}) \tag{3.45}$$

$$A_i = \frac{\mu_0}{4\pi} \frac{1}{|\vec{r}|} \frac{\mathrm{d}}{\mathrm{d}t} p_i(t - |\vec{r}|) \tag{3.46}$$

$$\bar{h}_{ij} = 4G \frac{1}{|\vec{r}|} \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} q_{ij}(t - |\vec{r}|) \tag{3.47}$$

3.2.2 电四极—引力对比

$$\hat{A}_i(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \ (-i\omega) \int \hat{J}_i(\omega, \vec{r}')(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}') \, \mathrm{d}V' \tag{3.48}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int \hat{J}_i' n^j x_j' \, \mathrm{d}V' \tag{3.49}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int n^j x_j' \hat{J}_i' \, \mathrm{d}V' \tag{3.50}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega)n^j\left[\int x'_{(j}\hat{J}'_{i)}\,\mathrm{d}V'\right] \tag{3.51}$$

$$\begin{aligned}
\int x'_{(j}\hat{J}'_{i)}\,\mathrm{d}V' &= \frac{1}{2}\int(\hat{J}'_j x'_i + \hat{J}'_i x'_j)\,\mathrm{d}V' && (3.52)\\
&= \frac{1}{2}\int\sum_k(\hat{J}'_k x'^i\frac{\partial x'^j}{\partial x'^k} + \hat{J}'_k\frac{\partial x'^i}{\partial x'^k}x'^j)\,\mathrm{d}V' && (3.53)\\
&= \frac{1}{2}\sum_k\left[\int\frac{\partial}{\partial x'^k}(\hat{J}'_k x'^i x'^j)\,\mathrm{d}V' - \int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V'\right] && (3.54)\\
&= \frac{1}{2}\sum_k\int\partial'_k\,(\hat{J}'_k x'^i x'^j)\,\mathrm{d}V' - \frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' && (3.55)\\
&= \frac{1}{2}\sum_k\int\hat{J}'_k x'^i x'^j\,\mathrm{d}S' - \frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' && (3.56)\\
&= -\frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' && (3.57)\\
&= -\frac{1}{2}\int(\sum_k\partial'_k\hat{J}'_k)x'^i x'^j\,\mathrm{d}V' && (3.58)\\
&= -\frac{1}{2}\int(\partial_0\hat{J}'_0)x'^i x'^j\,\mathrm{d}V' && (3.59)\\
&= -\frac{i\omega}{2}\int\hat{J}'_0 x'^i x'^j\,\mathrm{d}V' && (3.60)
\end{aligned}$$

$$\hat{D}_{ij} = \int\hat{J}'_0\,x'^i x'^j\,\mathrm{d}V' \tag{3.61}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-\frac{\omega^2}{2}n^j\hat{D}_{ij}) \tag{3.62}$$

$$A_i = \frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|}n^j\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}D_{ij}(t-|\vec{r}|) \tag{3.63}$$

第四章　双星系统

4.1　基本公式

$$\mathcal{M} := \mu^{3/5} M^{2/5} \tag{4.1}$$

$$h_+ = \frac{4\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}\frac{1+\cos^2\iota}{2}\cos\Phi(t) \tag{4.2}$$

$$h_\times = \frac{4\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}\cos\iota\sin\Phi(t) \tag{4.3}$$

$$h = F_+h_+ + F_\times h_\times \tag{4.4}$$

4.2　Post-Newtonian Approximation

4.3　Stationary Phase Approximation

[6], if $\zeta(t)$ varies slowly near $t = t_0$ where the phase has a stationary point: $\phi'(t_0) = 0$,

$$\int \zeta(t)e^{i\phi(t;f)}\,\mathrm{d}t = \int \zeta(t)e^{i[\phi(t_0)+\phi'(t_0)(t-t_0)+\frac{1}{2}\phi''(t_0)(t-t_0)^2+\dots]}\,\mathrm{d}t \tag{4.5}$$

$$\simeq e^{i\phi(t_0)}\int \zeta(t)e^{i[\frac{1}{2}\phi''(t_0)(t-t_0)^2]}\,\mathrm{d}t \tag{4.6}$$

$$\simeq e^{i\phi(t_0)}\int \zeta(t_0)e^{\frac{-\sqrt{-i\phi''(t_0)}^2(t-t_0)^2}{2}}\,\mathrm{d}t \tag{4.7}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{-i\phi''(t_0)}}\zeta(t_0)e^{i\phi(t_0)}. \tag{4.8}$$

$$h = \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\cos\Phi(t) \tag{4.9}$$

$$= \frac{\mathcal{M}}{D}[\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2}[e^{i\Phi(t)} + e^{-i\Phi(t)}] \tag{4.10}$$

$$\tilde{h}(f) = \int h(t) e^{i2\pi f t}\,\mathrm{d}t \tag{4.11}$$

$$= \int \frac{\mathcal{M}}{D}[\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2}[e^{i\Phi(t)} + e^{-i\Phi(t)}] e^{i2\pi f t}\,\mathrm{d}t \tag{4.12}$$

$$= \int \frac{\mathcal{M}}{D}[\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2}\{e^{i[2\pi f t + \Phi(t)]} + e^{i[2\pi f t - \Phi(t)]}\}\,\mathrm{d}t \tag{4.13}$$

$$\simeq \int \frac{\mathcal{M}}{D}[\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} e^{i[2\pi f t - \Phi(t)]}\,\mathrm{d}t \tag{4.14}$$

$$= \int \frac{\mathcal{M}}{D}[\pi \mathcal{M} F]^{2/3} Q \frac{1}{2} e^{i[2\pi f t(F) - \Phi(F)]} \frac{\mathrm{d}t}{\mathrm{d}F}\,\mathrm{d}F \tag{4.15}$$

$$\simeq \frac{\sqrt{2\pi}}{\sqrt{-i[2\pi f t(F) - \Phi(F)]''_{F=f}}} \tag{4.16}$$

$$\left[\frac{\mathcal{M}}{D}(\pi \mathcal{M} F)^{2/3} Q \frac{1}{2} \frac{\mathrm{d}t}{\mathrm{d}F}\right]_{F=f} e^{i[2\pi f t(f) - \Phi(f)]} \tag{4.17}$$

$$\simeq \frac{\sqrt{2\pi}}{\sqrt{-i\left\{2\pi f\left[-\frac{5}{256}\mathcal{M}(\pi \mathcal{M} F)^{-8/3}\right] - \left[\frac{1}{16}(\pi \mathcal{M} F)^{-5/3}\right]\right\}''_{F=f}}} \tag{4.18}$$

$$\left\{\frac{\mathcal{M}}{D}(\pi \mathcal{M} F)^{2/3} Q \frac{1}{2}\left[\frac{5\pi \mathcal{M}^2}{96}(\pi \mathcal{M} F)^{-11/3}\right]\right\}_{F=f} e^{i[2\pi f t(f) - \Phi(f)]} \tag{4.19}$$

$$= \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{D} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4}]} \quad (\texttt{pnspa.py}) \tag{4.20}$$

另可考 [1]. 其中 $\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 2\pi F$.

第五章　宇宙学效应

[4],

$$\frac{\mathrm{d}\eta}{\mathrm{d}(ct)} = \frac{1}{a}, \tag{5.1}$$

$$\mathrm{d}s^2 = -\mathrm{d}(ct)^2 + a^2[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)], \tag{5.2}$$

$$\mathrm{d}s^2 = a^2[-\mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)]. \tag{5.3}$$

$\Box\Phi = 0$, $\Phi := f/r$, $f := g/a$,

$$\partial_r^2 g + (\partial_\eta^2 a/a)g - \partial_\eta^2 g = 0, \tag{5.4}$$

$\partial_\eta^2 a/a \sim \eta^2$, $\omega^2 \gg 1/\eta^2$,

$$g \simeq e^{i\omega(\eta - r)}. \tag{5.5}$$

第六章　干涉仪

[4]. 设入射电场 $\vec{E}_{\text{in}} = \vec{E}_0 e^{-i\omega_{\text{L}} t + i\vec{k}_{\text{L}}\cdot\vec{x}}$. 设 splitter 在 $\vec{x} = 0$ 处, 则 $\vec{E}_{\text{in}} = \vec{E}_0 e^{-i\omega_{\text{L}} t}$. $\vec{E}_{\text{out}} = \vec{E}_{\text{form x}} + \vec{E}_{\text{form y}}$, t 时的 $\vec{E}_{\text{form x}}$ 在 $t - \frac{2L_x}{c}$ 时入 splitter, t 时的 $\vec{E}_{\text{form y}}$ 在 $t - \frac{2L_y}{c}$ 时入 splitter, 考虑反相, $\vec{E}_{\text{form x}} = -\frac{1}{2}\vec{E}_0 e^{-i\omega_{\text{L}} t + 2ik_{\text{L}} L_x}$, $\vec{E}_{\text{form y}} = +\frac{1}{2}\vec{E}_0 e^{-i\omega_{\text{L}} t + 2ik_{\text{L}} L_y}$, $\vec{E}_{\text{out}} = \vec{E}_0 \sin(\phi_0) e^{-i\omega_{\text{L}}(t - \frac{2L}{c}) - i\frac{\pi}{2}}$, where $\phi_0 = k_{\text{L}}(L_y - L_x)$ and $L = (L_x + L_y)/2$.

6.1　简单简单解释

reflector 接收, 相移 $2\pi[(1 \pm h)L/\lambda'_{\text{L}}]$, splitter 再接收, 相移 $2\pi[(1 \pm h)L/\lambda''_{\text{L}}]$, 其中 $\lambda''_{\text{L}}/\lambda'_{\text{L}} = \lambda'_{\text{L}}/\lambda_{\text{L}} = 1 \pm (\mathrm{d}h/\mathrm{d}t)L/c$, $(1/\lambda''_{\text{L}})/(1/\lambda'_{\text{L}}) = (1/\lambda'_{\text{L}})/(1/\lambda_{\text{L}}) = 1 \mp (\mathrm{d}h/\mathrm{d}t)L/c$, 总相移 $2\pi[2(1 \pm h)(1 \mp (\mathrm{d}h/\mathrm{d}t)L/c)L/\lambda_{\text{L}}] \approx 2\pi[2(1 \pm h \mp (\mathrm{d}h/\mathrm{d}t)L/c)L/\lambda_{\text{L}}]$. 若 $\omega_{\text{gw}} L/c \ll 1$ $(v/c \ll h)$, 则 $h \gg (\mathrm{d}h/\mathrm{d}t)L/c \approx h(\omega_{\text{gw}} L/c)$. 注: $(\mathrm{d}h/\mathrm{d}t)|_{t_2} - (\mathrm{d}h/\mathrm{d}t)|_{t_1} \approx (\mathrm{d}^2 h/\mathrm{d}t^2)(L/c) \approx (\mathrm{d}h/\mathrm{d}t)(\omega_{\text{gw}} L/c) \ll (\mathrm{d}h/\mathrm{d}t)$.

6.2　简单解释

$h := h_0 \sin(\omega_{\text{gw}} t)$. splitter 在 $(0, 0)$, reflector x 在 $(L(1+h), 0)$, reflector y 在 $(0, L(1-h))$. 设 photon $t = t_0$ 到 splitter, $t = t_1$ 到 x reflector, $t = t_2$ 到 splitter, $c(t_1 - t_0) = L(1 \pm h(t_1))$, $c(t_2 - t_1) = L(1 \pm h(t_1))$.

解: $t_0 = t_1 - (L/c)(1 \pm h(t_1))$, $t_2 = t_1 + (L/c)(1 \pm h(t_1))$, $\omega_{\text{gw}}(t_2 - (L/c)) = (\omega_{\text{gw}} t_1) \pm \omega_{\text{gw}}(L/c)h_0 \sin(\omega_{\text{gw}} t_1)$, $\omega_{\text{gw}} t_1 \approx \omega_{\text{gw}}(t_2 - (L/c)) \mp \omega_{\text{gw}}(L/c)h_0 \sin(\omega_{\text{gw}}(t_2 - (L/c)))$, $t_1 = t_2 - (L/c)(1 \pm h(t_2 - (L/c)))$, $h(t_1) \approx h(t_2 - (L/c))$, $t_0 = t_2 - 2(L/c)(1 \pm h(t_2 - (L/c)))$.

$E_{\text{in}}(t) := e^{-i\omega_{\text{L}} t}$, $E_{\text{out}}(t_2) = E_{\text{in}}(t_0)$, $E_{\text{out}}(t) = e^{-i\omega_{\text{L}}(t - 2(L/c)(1 \pm h(t - (L/c))))}$.

6.3 TT frame 解释

设 splitter 在 $(0,0)$, reflector x 在 $(L_x,0)$, reflector y 在 $(0,L_y)$, 显然无 GW 时如上.

设 GW 只有 +mode 且方向为 z_+, $h_+ = h_0\cos[\omega_{\rm gw}(t-z/c)]$,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + (1+h_+)\mathrm{d}x^2 + (1-h_+)\mathrm{d}y^2 + \mathrm{d}z^2. \tag{6.1}$$

$h_+(t) := h_+|_{z=0}$. 光 $\mathrm{d}s^2=0$, 保留一阶项, x 方向光轨迹

$$\mathrm{d}x = \pm c\mathrm{d}t[1-\frac{1}{2}h_+(t)], \tag{6.2}$$

y 方向光轨迹

$$\mathrm{d}y = \pm c\mathrm{d}t[1+\frac{1}{2}h_+(t)], \tag{6.3}$$

$+$ 号是 splitter 到 reflector, $-$ 号是 reflector 到 splitter.

设 photon $t=t_0$ 到 splitter, $t=t_1$ 到 x reflector, $t=t_2$ 到 splitter, 则

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{1}{2}\int_{t_0}^{t_2}\mathrm{d}t' h_+(t') \tag{6.4}$$

$$\approx \frac{2L_x}{c} + \frac{1}{2}\int_{t_0}^{t_0+\frac{2L_x}{c}}\mathrm{d}t' h_+(t') \tag{6.5}$$

$$= \frac{2L_x}{c} + \frac{L_x}{c}h_+(t_0+\frac{L_x}{c})\,\mathrm{sinc}(\omega_{\rm gw}\frac{L_x}{c}). \tag{6.6}$$

$\omega_{\rm gw}\frac{L_x}{c} \ll 1$, $t_2-t_0 \approx \frac{2L_x}{c} + \frac{L_x}{c}h_+(t_1)$. $\omega_{\rm gw}\frac{L_x}{c} \gg 1$, $t_2-t_0 \approx \frac{2L_x}{c}$.

y 方向, x 改成 y, $+h_+$ 改成 $-h_+$.

$\vec{E}_{\rm in} = \vec{E}_0 e^{-i\omega_{\rm L}t}$, t 时的 $\vec{E}_{\text{form x}}$ 在 $t-\frac{2L_x}{c}-\frac{L_x}{c}h_+(t-\frac{L_x}{c})\,\mathrm{sinc}(\omega_{\rm gw}\frac{L_x}{c})$ 时入 splitter, t 时的 $\vec{E}_{\text{form y}}$ 在 $t-\frac{2L_y}{c}+\frac{L_y}{c}h_+(t-\frac{L_y}{c})\,\mathrm{sinc}(\omega_{\rm gw}\frac{L_y}{c})$ 时入 splitter, $\vec{E}_{\text{form x}} = -\frac{1}{2}\vec{E}_0 e^{-i\omega_{\rm L}(t-\frac{2L}{c})+i\phi_0+i\Delta\phi(t)}$, $\vec{E}_{\text{form y}} = +\frac{1}{2}\vec{E}_0 e^{-i\omega_{\rm L}(t-\frac{2L}{c})-i\phi_0-i\Delta\phi(t)}$, where $\phi_0 = k_{\rm L}(L_y-L_x)$, $\Delta\phi(t) = h_+(t-\frac{L}{c})k_{\rm L}L\,\mathrm{sinc}(\omega_{\rm gw}\frac{L}{c})$, and $L = (L_x+L_y)/2$. Finally, $\vec{E}_{\rm out} = \vec{E}_0\sin[\phi_0+\Delta\phi(t)]e^{-i\omega_{\rm L}(t-\frac{2L}{c})-i\frac{\pi}{2}}$.

第七章　数据分析

[2], [4].

$$R(\tau) := \mathrm{E}(N_t N_{t+\tau}), \tag{7.1}$$

$$\frac{1}{2} S_N(f) := \tilde{R}(f) := \int R(\tau) e^{i2\pi f\tau} \, \mathrm{d}\tau. \tag{7.2}$$

$$\langle p|q\rangle := 4\mathrm{Re} \int_0^\infty \frac{\tilde{p}^*(f)\tilde{q}(f)}{S_N(f)} \, \mathrm{d}f. \tag{7.3}$$

7.1 matched filtering

$$\hat{\mathcal{S}} := \int S_t K(t) \, \mathrm{d}t \tag{7.4}$$

$$\frac{\mathcal{S}}{\mathcal{N}} := \frac{\mathrm{E}(\int (h(t) + N_t) K(t) \, \mathrm{d}t)}{\sqrt{\mathrm{D}(\int N_t K(t) \, \mathrm{d}t)}} \tag{7.5}$$

$$= \frac{\int h(t) K(t) \, \mathrm{d}t}{\sqrt{\int \mathrm{E}(N_{t_1} N_{t_2}) K(t_1) K(t_2) \, \mathrm{d}t_1 \mathrm{d}t_2}} \tag{7.6}$$

$$= \frac{\int h(t) K(t) \, \mathrm{d}t}{\sqrt{\int R(t_2 - t_1) K(t_1) K(t_2) \, \mathrm{d}t_1 \mathrm{d}t_2}} \tag{7.7}$$

$$= \frac{\int \tilde{h}(f) \tilde{K}^*(f) \, \mathrm{d}f}{\sqrt{\int \frac{1}{2} S_N(f) \tilde{K}(f) \tilde{K}^*(f) \, \mathrm{d}f}} \tag{7.8}$$

$$= \frac{\langle \frac{1}{2} S_N \tilde{K} | h \rangle}{\langle \frac{1}{2} S_N \tilde{K} | \frac{1}{2} S_N \tilde{K} \rangle^{1/2}} \tag{7.9}$$

$$\max \left(\frac{\mathcal{S}}{\mathcal{N}}\right) = \langle h|h\rangle^{1/2} \tag{7.10}$$

7.2 parameter estimation

$$p(\mu|d) \propto p(\mu) \exp\left[-\frac{1}{2}\sum_{m,n} C^{-1}_{mn}(d_m - h_m)(d_n - h_n)\right], \tag{7.11}$$

$$p(\mu|d) \propto p(\mu) \exp\left[-\frac{1}{2}\langle d - h|d - h\rangle\right]. \tag{7.12}$$

7.3 sensitivity

$$\Gamma_{mn} = \mathrm{E}(\langle d - h|\partial_m h\rangle\langle d - h|\partial_n h\rangle) = \langle\partial_m h|\partial_n h\rangle. \tag{7.13}$$

第八章　电磁引力

[3].

8.1　时空张量转化为空间张量

$$h_{ab} := g_{ab} + Z_a Z_b. \tag{8.1}$$

$$h_a{}^b = \delta_a{}^b + Z_a Z^b. \tag{8.2}$$

$$Z^a h_{ab} = 0. \tag{8.3}$$

$$V_{\langle a\rangle} := h_a{}^b V_b. \tag{8.4}$$

$$Z^a V_{\langle a\rangle} = 0. \tag{8.5}$$

$$T_{\langle ab\rangle} := h_{(a}{}^c h_{b)}{}^d T_{cd} - \frac{1}{3} h_{cd} T^{cd} h_{ab}. \tag{8.6}$$

$$Z^a (h_a{}^c h_b{}^d T_{cd}) = 0. \tag{8.7}$$

$$Z^a (h_b{}^c h_a{}^d T_{cd}) = 0. \tag{8.8}$$

$$Z^a (h_{(a}{}^c h_{b)}{}^d T_{cd}) = 0. \tag{8.9}$$

$$Z^a (h_{cd} T^{cd} h_{ab}) = 0. \tag{8.10}$$

$$Z^a T_{\langle ab\rangle} = 0. \tag{8.11}$$

$$T_{(\langle ab\rangle)} = T_{\langle ab\rangle}. \tag{8.12}$$

$$h^{ab} T_{\langle ab\rangle} = 0. \tag{8.13}$$

$$\varepsilon_{abc} := \varepsilon_{abcd} Z^d. \tag{8.14}$$

$$\varepsilon_{0123} := -\sqrt{|g|}. \tag{8.15}$$

$$T_a := \frac{1}{2}\varepsilon_{abc} T^{[bc]}. \tag{8.16}$$

$$[U, V]_a := \varepsilon_{abc} U^b V^c. \tag{8.17}$$

$$[S, T]_a := \varepsilon_{abc} g_{de} S^{bd} T^{ce}. \tag{8.18}$$

$$\mathrm{D}_t T^{a\dots}{}_{b\dots} := Z^c \nabla_c T^{a\dots}{}_{b\dots}. \tag{8.19}$$

$${}^3\nabla_a T^{b\dots}{}_{c\dots} := h_a{}^p h^b{}_q \dots h_c{}^r \dots \nabla_p T^{q\dots}{}_{r\dots}. \tag{8.20}$$

$$(\operatorname{div} V) := {}^3\nabla^a V_a. \tag{8.21}$$

$$(\operatorname{curl} V)_a := \varepsilon_{bca} {}^3\nabla^b V^c. \tag{8.22}$$

$$(\operatorname{div} T)_a := {}^3\nabla^b T_{ab}. \tag{8.23}$$

$$(\operatorname{curl} T)_{ab} := \varepsilon_{cd(a} {}^3\nabla^c g_{b)e} T^{ed}. \tag{8.24}$$

8.2 电磁空间矢量

$${}^*F_{ab} := \frac{1}{2}\varepsilon_{abcd} F^{cd} \tag{8.25}$$

$$E_a := F_{ab} Z^b = E_{\langle a\rangle}. \tag{8.26}$$

$$B_a := {}^*F_{ab} Z^b = B_{\langle a\rangle}. \tag{8.27}$$

$$\rho = -Z^a J_a. \tag{8.28}$$

$$j_a = h_a{}^b J_b. \tag{8.29}$$

$$\nabla_{[a} F_{bc]} = 0. \tag{8.30}$$

$$\nabla^a F_{ab} = \mu J_b. \tag{8.31}$$

$$(\operatorname{div} E) = \mu\rho - \dots. \tag{8.32}$$

$$(\operatorname{div} B) = + \dots. \tag{8.33}$$

$$(\operatorname{curl} E)_a + \dots = -\mathrm{D}_t B_{\langle a\rangle} - \dots. \tag{8.34}$$

$$(\operatorname{curl} B)_a + \dots = \mu j_a + \mathrm{D}_t E_{\langle a\rangle} + \dots. \tag{8.35}$$

8.3 引力空间张量

$$^{*}C_{abcd} := \frac{1}{2}\varepsilon_{abef}C^{ef}{}_{cd}. \tag{8.36}$$

$$E_{ab} := C_{acbd}Z^c Z^d = E_{\langle ab\rangle}. \tag{8.37}$$

$$B_{ab} := {}^{*}C_{acbd}Z^c Z^d = B_{\langle ab\rangle}. \tag{8.38}$$

$$(\operatorname{div} E)_a = \kappa\frac{1}{3}{}^3\nabla_a\rho - \dots. \tag{8.39}$$

$$(\operatorname{div} B)_a = \kappa(\rho + p)\omega_a + \dots. \tag{8.40}$$

$$(\operatorname{curl} E)_{ab} + \dots = -\mathrm{D}_t B_{\langle ab\rangle} - \dots. \tag{8.41}$$

$$(\operatorname{curl} B)_{ab} + \dots = \kappa\frac{1}{2}(\rho + p)\sigma_{ab} + \mathrm{D}_t E_{\langle ab\rangle} + \dots. \tag{8.42}$$

第九章　Varying G

9.1　Modification of Amplitude

$$\partial^c\partial_c\bar{h}_{ab} = -16\pi\frac{G_0}{c_0^4}T_{ab}, \quad \partial^a\bar{h}_{ab} = 0 \tag{9.1}$$

$$\Gamma^c{}_{ab} = \frac{1}{2}\eta^{cd}(2\partial_{(a}h_{b)d} - \partial_d h_{ab}) \tag{9.2}$$

$$U^a\partial_a U^c + \Gamma^c{}_{ab}U^aU^b = 0 \tag{9.3}$$

$$U^a\partial_a U^c = -\frac{1}{2}\eta^{cd}(2\partial_{(a}h_{b)d} - \partial_d h_{ab})U^aU^b \tag{9.4}$$

$$T_{ab} = c_0^2(2U_{(a}J_{b)} + U^cJ_cU_aU_b) \tag{9.5}$$

$$J_bc_0^2 = -U^aT_{ab} \tag{9.6}$$

$$A_b = -\frac{1}{4}U^a\bar{h}_{ab} \tag{9.7}$$

$$A_0 = -\frac{1}{4}c_0\bar{h}_{00} = -\frac{1}{2}c_0(\bar{h}_{00} - \frac{1}{2}\eta_{00}\eta^{00}\bar{h}_{00}) = -\frac{1}{2}c_0h_{00} \tag{9.8}$$

$$A_i = -\frac{1}{4}c_0\bar{h}_{0i} = -\frac{1}{4}c_0h_{0i} \tag{9.9}$$

$$U^\mu\partial_\mu U^i = -\frac{1}{2}\eta^{i\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu})U^\mu U^\nu \tag{9.10}$$

$$\begin{aligned} -\frac{1}{2}\eta^{i\sigma}(\partial_0h_{0\sigma} + \partial_0h_{0\sigma} - \partial_\sigma h_{00})U^0U^0 &= \frac{1}{2}c_0^2\eta^{i\sigma}\partial_\sigma h_{00} && (9.11)\\ &= \frac{1}{2}c_0^2\partial^i h_{00} && (9.12)\end{aligned}$$

$$= -c_0 \partial^i A_0 \tag{9.13}$$

$$= -E^i \tag{9.14}$$

$$-\frac{1}{2}\eta^{i\sigma}(\partial_0 h_{j\sigma} + \partial_j h_{0\sigma} - \partial_\sigma h_{0j})U^0 U^j = -\frac{1}{2}c_0 \eta^{i\sigma}(\partial_j h_{0\sigma} - \partial_\sigma h_{0j})v^j \tag{9.15}$$

$$= -\frac{1}{2}c_0 \eta^{ik}(\partial_j h_{0k} - \partial_k h_{0j})v^j \tag{9.16}$$

$$= 2\eta^{ik}(\partial_j A_k - \partial_k A_j)v^j \tag{9.17}$$

$$= -2\eta^{ik}(\partial_k A_j - \partial_j A_k)v^j \tag{9.18}$$

$$= -2(\partial^i A_j - \partial_j A^i)v^j \tag{9.19}$$

$$= -2\varepsilon^i{}_{jk} v^j B^k \tag{9.20}$$

$$-\frac{1}{2}\eta^{i\sigma}(\partial_j h_{k\sigma} + \partial_k h_{j\sigma} - \partial_\sigma h_{jk})U^j U^k = 0 \tag{9.21}$$

$$a^i = -E^i - 4\varepsilon^i{}_{jk} v^j B^k \tag{9.22}$$

$$\partial^i (\frac{1}{4\pi G_0} E_i) = \rho \tag{9.23}$$

$$\partial^i B_i = 0 \tag{9.24}$$

$$\varepsilon^i{}_{jk} \partial^j E^k = -\partial_t B^i \tag{9.25}$$

$$\varepsilon^i{}_{jk} \partial^j (\frac{c_0^2}{4\pi G_0} B^k) = j^i + \partial_t (\frac{1}{4\pi G_0} E^i) \tag{9.26}$$

$$\varepsilon_{\mathrm{G}0} := \frac{1}{4\pi G_0}, \quad \mu_{\mathrm{G}0} := \frac{4\pi G_0}{c_0^2} \tag{9.27}$$

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_{\mathrm{G}0} \vec{E}) = \rho \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \\ \vec{\nabla} \times (\mu_{\mathrm{G}0}^{-1} \vec{B}) = \vec{j} + \frac{\partial}{\partial t} (\varepsilon_{\mathrm{G}0} \vec{E}) \end{cases} \tag{9.28}$$

$$\vec{a} = -\vec{E} - 4\vec{v} \times \vec{B} \tag{9.29}$$

$$\varepsilon_{\mathrm{G}} = \frac{1}{4\pi G}, \quad \mu_{\mathrm{G}} = \frac{4\pi G}{c^2} \tag{9.30}$$

$$x^\mu = (ct, x, y, z) \tag{9.31}$$

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_{\mathrm{G}} \vec{E}) = \rho \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \\ \vec{\nabla} \times (\mu_{\mathrm{G}}^{-1} \vec{B}) = \vec{j} + \frac{\partial}{\partial t} (\varepsilon_{\mathrm{G}} \vec{E}) \end{cases} \tag{9.32}$$

$$\vec{a} = -\vec{E} - 4\vec{v} \times \vec{B} \tag{9.33}$$

$$A_\mu = -\frac{1}{4} c \bar{h}_{0\mu} \tag{9.34}$$

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \varepsilon_{\mathrm{G}}^{-1} \rho \\ \vec{\nabla} \times \vec{B} = \mu_{\mathrm{G}} \vec{j} + \varepsilon_{\mathrm{G}} \mu_{\mathrm{G}} \frac{\partial}{\partial t} \vec{E} \end{cases} \tag{9.35}$$

$$\frac{1}{c^2} \frac{\partial}{\partial t} \varphi + \vec{\nabla} \cdot \vec{A} = 0 \tag{9.36}$$

$$\begin{cases} -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi + \vec{\nabla}^2 \varphi = \varepsilon_{\mathrm{G}}^{-1} \rho \\ -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} + \vec{\nabla}^2 \vec{A} = \mu_{\mathrm{G}} \vec{j} \end{cases} \tag{9.37}$$

$$\begin{cases} -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} c^{-1} \varphi + \vec{\nabla}^2 c^{-1} \varphi = \mu_{\mathrm{G}} c \rho \\ -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} + \vec{\nabla}^2 \vec{A} = \mu_{\mathrm{G}} \vec{j} \end{cases} \tag{9.38}$$

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_{\mathrm{G}} \vec{E}) = 0 \\ \vec{\nabla} \cdot (\mu_{\mathrm{G}} \vec{H}) = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} (\mu_{\mathrm{G}} \vec{H}) \\ \vec{\nabla} \times \vec{H} = +\frac{\partial}{\partial t} (\varepsilon_{\mathrm{G}} \vec{E}) \end{cases} \tag{9.39}$$

$$E_r = 0, \quad H_r = 0 \tag{9.40}$$

$$\begin{cases} \frac{\varepsilon_{\mathrm{G}}}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta E_\theta) + \frac{\varepsilon_{\mathrm{G}}}{r \sin\theta} \frac{\partial}{\partial \phi} (E_\phi) = 0 \\ \frac{\mu_{\mathrm{G}}}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta H_\theta) + \frac{\mu_{\mathrm{G}}}{r \sin\theta} \frac{\partial}{\partial \phi} (H_\phi) = 0 \\ \frac{1}{r \sin\theta} [\frac{\partial}{\partial \theta} (\sin\theta E_\phi) - \frac{\partial}{\partial \phi} (E_\theta)] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r} (r E_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \vec{e}_\phi = -\mu_{\mathrm{G}} \frac{\partial}{\partial t} (H_\theta \vec{e}_\theta + H_\phi \vec{e}_\phi) \\ \frac{1}{r \sin\theta} [\frac{\partial}{\partial \theta} (\sin\theta H_\phi) - \frac{\partial}{\partial \phi} (H_\theta)] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r} (r H_\theta) \vec{e}_\phi = +\varepsilon_{\mathrm{G}} \frac{\partial}{\partial t} (E_\theta \vec{e}_\theta + E_\phi \vec{e}_\phi) \end{cases} \tag{9.41}$$

$$\vec{E} = E_\theta \vec{e}_\theta, \quad \vec{H} = H_\phi \vec{e}_\phi \tag{9.42}$$

$$\begin{cases} \frac{\varepsilon_{\mathrm{G}}}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) = 0 \\ \frac{\mu_{\mathrm{G}}}{r\sin\theta}\frac{\partial}{\partial\phi}(H_\phi) = 0 \\ -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(E_\theta)\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta)\vec{e}_\phi = -\mu_{\mathrm{G}}\frac{\partial}{\partial t}(H_\phi)\vec{e}_\phi \\ +\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta H_\phi)\vec{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\vec{e}_\theta = +\varepsilon_{\mathrm{G}}\frac{\partial}{\partial t}(E_\theta)\vec{e}_\theta \end{cases} \tag{9.43}$$

$$\begin{cases} \frac{\partial}{\partial r}(rE_\theta) + \mu_{\mathrm{G}}\frac{\partial}{\partial t}(rH_\phi) = 0 \\ \frac{\partial}{\partial r}(rH_\phi) + \varepsilon_{\mathrm{G}}\frac{\partial}{\partial t}(rE_\theta) = 0 \end{cases} \tag{9.44}$$

$$\begin{cases} \mu_{\mathrm{G}}\frac{\partial}{\partial r}\mu_{\mathrm{G}}^{-1}\frac{\partial}{\partial r}(rE_\theta) - \varepsilon_{\mathrm{G}}\mu_{\mathrm{G}}\frac{\partial}{\partial t}\frac{\partial}{\partial t}(rE_\theta) = 0 \\ \varepsilon_{\mathrm{G}}\frac{\partial}{\partial r}\varepsilon_{\mathrm{G}}^{-1}\frac{\partial}{\partial r}(rH_\phi) - \varepsilon_{\mathrm{G}}\mu_{\mathrm{G}}\frac{\partial}{\partial t}\frac{\partial}{\partial t}(rH_\phi) = 0 \end{cases} \tag{9.45}$$

$$\begin{cases} \mu_{\mathrm{G}}\frac{\partial}{\partial r}\mu_{\mathrm{G}}^{-1}\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial}{\partial(ct)}\frac{\partial}{\partial(ct)}(rE_\theta) = 0 \\ \varepsilon_{\mathrm{G}}\frac{\partial}{\partial r}\varepsilon_{\mathrm{G}}^{-1}\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial}{\partial(ct)}\frac{\partial}{\partial(ct)}(rH_\phi) = 0 \end{cases} \tag{9.46}$$

$$\begin{cases} \frac{\partial}{\partial r}\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial}{\partial r}(\ln\mu_{\mathrm{G}})\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial}{\partial(ct)}\frac{\partial}{\partial(ct)}(rE_\theta) = 0 \\ \frac{\partial}{\partial r}\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial}{\partial r}(\ln\varepsilon_{\mathrm{G}})\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial}{\partial(ct)}\frac{\partial}{\partial(ct)}(rH_\phi) = 0 \end{cases} \tag{9.47}$$

$$\frac{\partial^2}{\partial r^2}f(r,t) - p(r)\frac{\partial}{\partial r}f(r,t) - \frac{\partial^2}{\partial(ct)^2}f(r,t) = 0 \tag{9.48}$$

$$f(r,t) = f(r)e^{-ikct} \tag{9.49}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}f(r) - p(r)\frac{\mathrm{d}}{\mathrm{d}r}f(r) + k^2 f(r) = 0 \tag{9.50}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}f(r) - p\frac{\mathrm{d}}{\mathrm{d}r}f(r) + k^2 f(r) = 0 \tag{9.51}$$

$$f(r) = e^{(p/2)r}[C_+ e^{i\sqrt{k^2-(p/2)^2}r} + C_- e^{-i\sqrt{k^2-(p/2)^2}r}] \tag{9.52}$$

$$f(r,t) = e^{(p/2)r}[C_+ e^{i(+\sqrt{k^2-(p/2)^2}r-kct)} + C_- e^{i(-\sqrt{k^2-(p/2)^2}r-kct)}] \tag{9.53}$$

$$f(r,t) = e^{(p/2)r}[C_+ e^{i(+\sqrt{(\omega/c)^2-(p/2)^2}r-\omega t)} + C_- e^{i(-\sqrt{(\omega/c)^2-(p/2)^2}r-\omega t)}] \tag{9.54}$$

$$f(r,t) = e^{\int(p/2)\mathrm{d}r}[C_+ e^{i(+\int\sqrt{(\omega/c)^2-(p/2)^2}\mathrm{d}r-\omega t)} + C_- e^{i(-\int\sqrt{(\omega/c)^2-(p/2)^2}\mathrm{d}r-\omega t)}] \tag{9.55}$$

$$\begin{cases} r_2\left|E_\theta\right|_{r=r_2} = r_1\left|E_\theta\right|_{r=r_1} e^{\int_{r_1}^{r_2}\frac{1}{2}\frac{\partial}{\partial r}(\ln\mu_{\mathrm{G}})\,\mathrm{d}r} \\ r_2\left|H_\phi\right|_{r=r_2} = r_1\left|H_\phi\right|_{r=r_1} e^{\int_{r_1}^{r_2}\frac{1}{2}\frac{\partial}{\partial r}(\ln\varepsilon_{\mathrm{G}})\,\mathrm{d}r} \end{cases} \tag{9.56}$$

$$\begin{cases} E_2 = \sqrt{\frac{\mu_{\mathrm{G}2}}{\mu_{\mathrm{G}1}}}\frac{r_1}{r_2}E_1 \\ H_2 = \sqrt{\frac{\varepsilon_{\mathrm{G}2}}{\varepsilon_{\mathrm{G}1}}}\frac{r_1}{r_2}H_1 \end{cases} \tag{9.57}$$

$$\begin{cases} E_2/c_2 = \sqrt{\frac{\mu_{\mathrm{G}2}}{\mu_{\mathrm{G}1}}}\frac{c_1}{c_2}\frac{r_1}{r_2}E_1/c_1 \\ B_2 = \sqrt{\frac{\mu_{\mathrm{G}2}}{\mu_{\mathrm{G}1}}}\frac{c_1}{c_2}\frac{r_1}{r_2}B_1 \end{cases} \tag{9.58}$$

$$\begin{cases} (\omega/c_2)c_2(\bar{h}_{00})_2 = \sqrt{\frac{\mu_{\mathrm{G}2}}{\mu_{\mathrm{G}1}}}\frac{c_1}{c_2}\frac{r_1}{r_2}(\omega/c_1)c_1(\bar{h}_{00})_1 \\ (\omega/c_2)c_2(\bar{h}_{0i})_2 = \sqrt{\frac{\mu_{\mathrm{G}2}}{\mu_{\mathrm{G}1}}}\frac{c_1}{c_2}\frac{r_1}{r_2}(\omega/c_1)c_1(\bar{h}_{0i})_1 \end{cases} \tag{9.59}$$

$$h_2 = \sqrt{\frac{c_1^4/G_1}{c_2^4/G_2}}\frac{r_1}{r_2}h_1 \tag{9.60}$$

双星系统引力辐射本为

$$h = \frac{\mathcal{M}[\pi\mathcal{M}F(t)]^{2/3}}{r}Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi F(t)\,\mathrm{d}t] \tag{9.61}$$

设双星系统常量 c^*, G^*, 一观者临近双星系统且与双星系统相对静止, 其与双星系统距离为 r, 测得强度 h_r, 频率 F_r, 则[1]

$$h_r = \frac{\mathcal{M}[\pi\mathcal{M}F_r(t)]^{2/3}}{r/c^*}Q(\theta,\phi,\psi,\iota) \tag{9.62}$$

设地球观者与双星系统距离为 d, 双星系统红移为 z, 测得强度 h_d, 频率 $F_d = F_r/(1+z)$, 则

$$h_d = \sqrt{\frac{c^{*4}/G^*}{c^4/G}}\frac{r}{d}h_r \tag{9.63}$$

$$= \sqrt{\frac{c^{*4}/G^*}{c^4/G}}\frac{\mathcal{M}[\pi\mathcal{M}F_r(t)]^{2/3}}{d/c^*}Q(\theta,\phi,\psi,\iota) \tag{9.64}$$

所以地球观者测得

$$h = \sqrt{\frac{c^{*4}/G^*}{c^4/G}}\frac{\mathcal{M}[\pi\mathcal{M}F_r(t)]^{2/3}}{d/c^*}Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi\frac{F_r(t)}{1+z}\,\mathrm{d}t] \tag{9.65}$$

记 $F_{\mathrm{obs}}(t) = F_r(t)/(1+z)$, $\mathcal{M}_{\mathrm{obs}} = \mathcal{M}(1+z)$, 光度距离 $d_{\mathrm{L}} = d(1+z)$, 则

$$h = \sqrt{\frac{c^{*4}/G^*}{c^4/G}}\frac{\mathcal{M}[\pi\mathcal{M}F_r(t)]^{2/3}}{d(1+z)/c^*}Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi F_{\mathrm{obs}}(t)\,\mathrm{d}t] \tag{9.66}$$

[1] $\mathcal{M}$ 和 c^*, G^* 简并, 所以可以笼统地仍记作 $\mathcal{M}$.

$$= \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}_{\rm obs}[\pi\mathcal{M}_{\rm obs}F_{\rm obs}(t)]^{2/3}}{d_{\rm L}/c^*} Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi F_{\rm obs}(t)\,{\rm d}t] \tag{9.67}$$

$$= \sqrt{\frac{c^{*6}/G^*}{c^6/G}} \frac{\mathcal{M}_{\rm obs}[\pi\mathcal{M}_{\rm obs}F_{\rm obs}(t)]^{2/3}}{d_{\rm L}/c} Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi F_{\rm obs}(t)\,{\rm d}t] \tag{9.68}$$

用引力波测距测得 $d_{\rm L,G}$, 则

$$d_{\rm L,G} = d_{\rm L}\sqrt{\frac{c^6/G}{c^{*6}/G^*}} \tag{9.69}$$

[6]

$$h(t) = \frac{\mathcal{M}[\pi\mathcal{M}F(t)]^{2/3}}{\xi\, d_{\rm L}} Q(\text{angles})\cos\Phi(t) \tag{9.70}$$

$$\tilde{h}(f) = \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6}Q}{\xi\, d_{\rm L}} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4}]} \tag{9.71}$$

问题转化为估计 ξ

$$p(\mu) \propto p^{(0)}(\mu)\exp[-\frac{1}{2}\Gamma_{ab}(\mu^a - \hat{\mu}^a)(\mu^b - \hat{\mu}^b)] \tag{9.72}$$

$$p^{(0)}(\mu) \propto \exp[-\frac{1}{2}\Gamma^{(0)}_{ab}(\mu^a - \bar{\mu}^a)(\mu^b - \bar{\mu}^b)] \tag{9.73}$$

设待估参数为 $\mu = (\ln\xi, \ln(d_{\rm L}/d_{\rm L0}), \ln Q, \dots)$, $\dots$ 为其他参数 (如 $\mathcal{M}$), 则 $\tilde{h}_{,\ln\xi} = \tilde{h}_{,\ln(d_{\rm L}/d_{\rm L0})} = -\tilde{h}_{,\ln Q} = -\tilde{h}$, $\tilde{h}$ 对其他参数求偏导皆为纯虚数, 则由 $\Gamma_{ab} = \langle h_{,a}|h_{,b}\rangle$ 和 $\mathrm{SNR} := \rho = \sqrt{\langle h|h\rangle}$ 得

$$\Gamma_{ab} = \begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 & 0 & \dots \\ \rho^2 & \rho^2 & -\rho^2 & 0 & \dots \\ -\rho^2 & -\rho^2 & \rho^2 & 0 & \dots \\ 0 & 0 & 0 & ? & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{9.74}$$

又设

$$\Gamma^{(0)}_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1/\sigma^2_{\ln d_{\rm L}} & 0 & 0 & \dots \\ 0 & 0 & 1/\sigma^2_{\ln Q} & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{9.75}$$

则由 $\Sigma_{ab} = (\Gamma^{(0)}_{ab} + \Gamma_{ab})^{-1}$ 得

$$\Sigma_{ab} = \begin{bmatrix} \begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 \\ \rho^2 & \rho^2 + 1/\sigma^2_{\ln(d_{\rm L}/d_{\rm L0})} & -\rho^2 \\ -\rho^2 & -\rho^2 & \rho^2 + 1/\sigma^2_{\ln Q} \end{bmatrix}^{-1} & 0 \\ 0 & [?]^{-1} \end{bmatrix} \tag{9.76}$$

而

$$\begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 \\ \rho^2 & \rho^2 + 1/\sigma^2_{\ln(d_{\rm L}/d_{\rm L0})} & -\rho^2 \\ -\rho^2 & -\rho^2 & \rho^2 + 1/\sigma^2_{\ln Q} \end{bmatrix}^{-1} \tag{9.77}$$

$$= \begin{bmatrix} 1/\rho^2 + \sigma^2_{\ln(d_{\rm L}/d_{\rm L0})} + \sigma^2_{\ln Q} & -\sigma^2_{\ln(d_{\rm L}/d_{\rm L0})} & \sigma^2_{\ln Q} \\ -\sigma^2_{\ln(d_{\rm L}/d_{\rm L0})} & \sigma^2_{\ln(d_{\rm L}/d_{\rm L0})} & 0 \\ \sigma^2_{\ln Q} & 0 & \sigma^2_{\ln Q} \end{bmatrix} \tag{9.78}$$

9.2 Modification of Phase

$$\frac{d^2}{dz^2}H(z) + 2p(z)\frac{d}{dz}H(z) + \left[\omega^2 + q(z)\right]H(z) = 0. \tag{9.79}$$

$$H = Ae^{i\Phi}. \tag{9.80}$$

$k = \frac{d\Phi}{dz}$,

$$\frac{d^2A}{dz^2} + 2p\frac{dA}{dz} + \left[\omega^2\left(1 - \frac{k^2}{\omega^2}\right) + q\right]A = 0, \tag{9.81}$$

$$2\frac{dA}{dz}k + A\frac{dk}{dz} + 2pAk = 0, \tag{9.82}$$

$$2\frac{1}{A}\frac{dA}{dz} + \frac{1}{k}\frac{dk}{dz} + 2p = 0, \tag{9.83}$$

$$A \propto e^{-\int p\,dz}k^{-1/2}. \tag{9.84}$$

$\Gamma = e^{\int p\,dz}$ and $K = (k/\omega)^{-1/2}$,

$$\frac{d^2K}{dz^2} - \left(\frac{1}{\Gamma}\frac{d^2\Gamma}{dz^2} - q\right)K + \omega^2K(1 - K^{-4}) = 0, \tag{9.85}$$

$\Xi = \frac{1}{\Gamma}\frac{d^2\Gamma}{dz^2} - q$ and make $\omega = 1$,

$$\frac{d^2K}{dz^2} + K[(1 - \Xi) - K^{-4}] = 0. \tag{9.86}$$

$\Xi = \text{const}$,

$$K = (1-\Xi)^{-1/4} = 1 + \frac{1}{4}\Xi + \frac{5}{32}\Xi^2 + O(\Xi^3), \tag{9.87}$$

$$k = (1-\Xi)^{1/2} = 1 - \frac{1}{2}\Xi - \frac{1}{8}\Xi^2 + O(\Xi^3), \tag{9.88}$$

$\Xi \neq \text{const}$, $\Xi(z) = \kappa^2\tilde{\Xi}(\tilde{z})$, where $\tilde{z} = \kappa z$.

$$K^3\frac{d^2K}{d\tilde{z}^2}\kappa^2 - K^4\tilde{\Xi}(\tilde{z})\kappa^2 + K^4 - 1 = 0. \tag{9.89}$$

$$K = \sum_{n=0}^{\infty} K_n(\tilde{z})\kappa^{2n}, \tag{9.90}$$

$$K_0^4 - 1 = 0, \tag{9.91}$$

$$K_0^3K_0'' - K_0^4\tilde{\Xi} + 4K_0^3K_1 = 0, \tag{9.92}$$

$$(K_0^3K_1'' + 3K_0^2K_1K_0'') - 4K_0^3K_1\tilde{\Xi} + (4K_0^3K_2 + 6K_0^2K_1^2) = 0. \tag{9.93}$$

$$K_0 = 1, \tag{9.94}$$

$$K_1 = \frac{1}{4}\tilde{\Xi}, \tag{9.95}$$

$$K_2 = \frac{5}{32}\tilde{\Xi}^2 - \frac{1}{16}\frac{d^2\tilde{\Xi}}{d\tilde{z}^2}, \tag{9.96}$$

$$h(z,t) = h(z)e^{-i\omega t} \tag{9.97}$$

$$h(z) = \Gamma^{-1}(z)K(z)(C_+e^{+i\omega\int K^{-2}(z)\,\mathrm{d}z} + C_-e^{-i\omega\int K^{-2}(z)\,\mathrm{d}z}) \tag{9.98}$$

$$h(z,t) = \int_{-\infty}^{+\infty}\tilde{h}(z;f)e^{-i2\pi ft}\,\mathrm{d}f \tag{9.99}$$

$$\tilde{h}(z;f) = \Gamma^{-1}(z)K(z;f)[C_+(f)e^{+i2\pi f\int K^{-2}(z;f)\,\mathrm{d}z} + C_-(f)e^{-i2\pi f\int K^{-2}(z;f)\,\mathrm{d}z}] \tag{9.100}$$

$$\tilde{h}_0(z;f) = \Gamma^{-1}(0)K(0;f)[C_+(f)e^{+i2\pi f\int\,\mathrm{d}z} + C_-(f)e^{-i2\pi f\int\,\mathrm{d}z}] \tag{9.101}$$

$$\tilde{h}(z;f) = \frac{\Gamma^{-1}(z)K(z;f)}{\Gamma^{-1}(0)K(0;f)}[C_+(f)e^{+i2\pi f\int K^{-2}(z)\,\mathrm{d}z} + C_-(f)e^{-i2\pi f\int K^{-2}(z)\,\mathrm{d}z}] \tag{9.102}$$

$$\tilde{h}_0(z;f) = [C_+(f)e^{+i2\pi f\int\,\mathrm{d}z} + C_-(f)e^{-i2\pi f\int\,\mathrm{d}z}] \tag{9.103}$$

$$K(z) = 1 + \frac{1}{4\omega^2}\Xi(z) \tag{9.104}$$

$$\tilde{h}(z;f) = \frac{\Gamma^{-1}(z)}{\Gamma^{-1}(0)}[1+\frac{\Xi(z)-\Xi(0)}{4}(2\pi f)^{-2}][C_+(f)e^{+i2\pi fz}e^{-i2\pi f\int\frac{\Xi(z)}{2}(2\pi f)^{-2}\,\mathrm{d}z}+C_-(f)e^{-i2\pi fz}e^{+i2\pi f\int\frac{\Xi(z)}{2}(2\pi f)^{-2}\,\mathrm{d}z}] \tag{9.105}$$

$$\tilde{h}(z;f) = \gamma(z)[1+\xi(z)(2\pi f)^{-2}][C_+(f)e^{+i2\pi fz}e^{+i\Omega(z)(2\pi f)^{-1}}+C_-(f)e^{-i2\pi fz}e^{-i\Omega(z)(2\pi f)^{-1}}] \tag{9.106}$$

$$\tilde{h}_0(z;f) = [C_+(f)e^{+i2\pi fz} + C_-(f)e^{-i2\pi fz}] \tag{9.107}$$

$$\tilde{h}_0(z;f) = C_+(f)e^{+i2\pi fz} + C_-(f)e^{-i2\pi fz} \tag{9.108}$$

$$\partial_z\tilde{h}_0(z;f) = C_+(f)(i2\pi f)e^{+i2\pi fz} - C_-(f)(i2\pi f)e^{-i2\pi fz} \tag{9.109}$$

$$C_+(f)e^{+i2\pi fz} = \frac{1}{2}[\tilde{h}_0(z;f) + \partial_z\tilde{h}_0(z;f)(i2\pi f)^{-1}] \tag{9.110}$$

$$C_-(f)e^{-i2\pi fz} = \frac{1}{2}[\tilde{h}_0(z;f) - \partial_z\tilde{h}_0(z;f)(i2\pi f)^{-1}] \tag{9.111}$$

$$\tilde{h}_0(f) = C_+(f)e^{+i2\pi fz} \tag{9.112}$$

$$\tilde{h}(f) = \gamma(1+\xi f^{-2})e^{i\Omega f^{-1}}\tilde{h}_0(f) \tag{9.113}$$

$$\tilde{h}_0(f) = \mathcal{A}f^{-7/6}e^{i\{2\pi ft_c-\varphi_c-\frac{\pi}{4}+\frac{3}{128}(\pi\mathcal{M}f)^{-5/3}[1+\frac{20}{9}(\frac{743}{366}+\frac{11}{4}\eta)(\pi Mf)^{2/3}]\}} \tag{9.114}$$

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