

# 引力波天文学笔记

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# 第一章 引力波

## 1.1 Linearized Gravity

[7]. 流形  $\mathbb{R}^4$ . 任意坐标系  $\{x^\mu\}$ ,  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}s + O(s^2)$ . 设  $g^{\mu\nu} = \eta^{\mu\nu} + \gamma^{\mu\nu}s + O(s^2)$ , 则  $\delta^\mu_\lambda = \eta^{\mu\nu}\eta_{\nu\lambda} + \gamma^{\mu\nu}\eta_{\nu\lambda}s + \gamma^{\mu\nu}\eta_{\nu\lambda}s + O(s^2)$ , 所以  $\gamma^{\mu\nu} = \eta^{\mu\nu}$ ,  $\gamma^{\mu\nu} = \eta^{\mu\sigma}\delta^\nu_\sigma = \eta^{\mu\sigma}\eta_{\sigma\lambda}\eta^{\lambda\nu} = -\eta^{\mu\sigma}\gamma_{\sigma\lambda}\eta^{\lambda\nu} = -\eta^{\mu\sigma}\gamma_{\sigma\lambda}\eta^{\lambda\nu} = -\gamma^{\mu\nu}$ , 所以  $g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu}s + O(s^2) = \eta^{\mu\nu} - h^{\mu\nu} + O(s^2)$ .

$$R_{\mu\nu\lambda\sigma} = \partial_\sigma\partial_{[\mu}h_{\lambda]\nu} - \partial_\nu\partial_{[\mu}h_{\lambda]\sigma} + O(s^2). \quad (1.1)$$

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\sigma}h_{\lambda\sigma} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h.$$

$$-\frac{1}{2}\partial^\lambda\partial_\lambda\bar{h}_{\mu\nu} + \partial^\lambda\partial_{(\mu}\bar{h}_{\nu)\lambda} - \frac{1}{2}\eta_{\mu\nu}\partial^\lambda\partial^\sigma\bar{h}_{\lambda\sigma} + O(s^2) = 8\pi T_{\mu\nu}. \quad (1.2)$$

存在  $\{x^\mu\}$ , 使得  $\partial^\nu\bar{h}_{\mu\nu} + O(s^2) = 0$  (Lorentz gauge). [证: 设  $x'^\mu = x^\mu - \xi^\mu = x^\mu - \zeta^\mu s - O(s^2)$ , 则  $\frac{\partial?}{\partial x'^\mu} = \frac{\partial?}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial x'^\mu} = \frac{\partial?}{\partial x^\lambda}(\delta^\lambda_\mu + \frac{\partial\xi^\lambda}{\partial x'^\mu}) = \frac{\partial?}{\partial x^\mu} + O(s^2)$ ,  $g'_{\mu\nu} = g_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} = g_{\lambda\sigma}(\delta^\lambda_\mu + \frac{\partial\xi^\lambda}{\partial x'^\mu})(\delta^\sigma_\nu + \frac{\partial\xi^\sigma}{\partial x'^\nu}) = g_{\mu\nu} + g_{\mu\sigma}\frac{\partial\xi^\sigma}{\partial x'^\nu} + g_{\lambda\nu}\frac{\partial\xi^\lambda}{\partial x'^\mu} = g_{\mu\nu} + (\eta_{\mu\sigma} + O(s))(\frac{\partial\xi^\sigma}{\partial x^\nu} + O(s^2)) + (\eta_{\lambda\nu} + O(s))(\frac{\partial\xi^\lambda}{\partial x^\mu} + O(s^2)) = g_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + O(s^2)$ , 所以  $h'_{\mu\nu} = g'_{\mu\nu} - \eta_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + O(s^2) = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + O(s^2)$ , 因此存在  $\xi^\mu$ , 使得  $\partial'^\nu h'_{\mu\nu} + O(s^2) = 0$ .] 令  $\{x^\mu\}$  满足  $\partial^\nu\bar{h}_{\mu\nu} + O(s^2) = 0$ , 则

$$\partial^\lambda\partial_\lambda\bar{h}_{\mu\nu} + O(s^2) = -16\pi T_{\mu\nu}. \quad (1.3)$$

略去  $O(s^2)$  条件:  $h_{\mu\nu}$ ,  $\partial_\lambda h_{\mu\nu}$ ...小. 下略  $O(s^2)$ .

## 1.2 Radiation Gauge

[7]. 存在  $\{x^\mu\}$ , 使得 “无源处”  $h + O(s^2) = 0$  (TT gauge [8]) 且  $h_{0\mu} + O(s^2) = 0$ . [4], 解  $\partial^\lambda\partial_\lambda\bar{h}_{\mu\nu} = 0$  得  $h_{ij} = A_{ij}(\vec{k})e^{ik^\mu x_\mu}$  ( $A_{ij}$  称为 polarization

tensor).  $h_{(ij)} = 0$ ,  $h = 0$ ,  $\partial^j h_{ij} = 0 \Rightarrow A_{(ij)} = 0$ ,  $A = 0$ ,  $k^j A_{ij} = 0$ . 令  $\vec{e}_z \parallel \vec{k}$ ,

$$h_{xy} = \begin{bmatrix} +h_+ & h_\times \\ h_\times & -h_+ \end{bmatrix} e^{i\omega(t-z)}. \quad (1.4)$$

[4]. Lorentz gauge  $\rightarrow$  radiation gauge,  $P_{ij} := \delta_{ij} - n_i n_j$ ,  $\Lambda_{ijkl} = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}$ ,  $h_{ij}^r = \Lambda_{ijkl} h_{kl}^L = \Lambda_{ijkl} \bar{h}_{kl}^L$ . [6]. Step 1: 坐标系空间旋转, 使  $\vec{e}_z \parallel \vec{n}$ . Step 2: 取  $x, y$  分量  $h_{xy}$ . Step 3: 去迹. [ $h_+ = \frac{1}{2}(h_{xx} - h_{yy})$ ,  $h_\times = h_{xy} = h_{yx}$ .]

### 1.3 Fourier Transformation

[4].

$$h_{ij} = \frac{1}{(2\pi)^3} \int d^3 \vec{k} \left[ \mathcal{A}_{ij}(\vec{k}) e^{+ik_\mu x^\mu} + \mathcal{A}_{ij}^*(\vec{k}) e^{-ik_\mu x^\mu} \right] \quad (1.5)$$

$$d^2 \vec{n} := \sin \theta d\theta d\phi,$$

$$h_{ij} = \int_0^\infty df f^2 \int d^2 \vec{n} \left[ \mathcal{A}_{ij}(f, \vec{n}) e^{-2\pi i f(t - \vec{n} \cdot \vec{x})} + \text{c.c.} \right] \quad (1.6)$$

$$= \int_0^\infty df \left[ e^{-2\pi i f t} f^2 \int d^2 \vec{n} \mathcal{A}_{ij}(f, \vec{n}) e^{+2\pi i f \vec{n} \cdot \vec{x}} + \text{c.c.} \right] \quad (1.7)$$

$$:= \int_0^\infty df \left[ \tilde{h}_{ij}(f, \vec{x}) e^{-2\pi i f t} + \tilde{h}_{ij}^*(f, \vec{x}) e^{+2\pi i f t} \right] \quad (1.8)$$

$$:= \int df \tilde{h}_{ij}(f, \vec{x}) e^{-2\pi i f t}. \quad (1.9)$$

When we observe on Earth a GW emitted by a single astrophysical source, and the linear dimensions of the detector are much smaller than wavelength of the GW, choosing the origin of the coordinate system centered on the detector,  $\tilde{h}_{ij}(f, \vec{x}) \approx \tilde{h}_{ij}(f) := \tilde{h}_{ij}(f, \vec{x} = \vec{0})$ ,

$$h_{ij} = \int df \tilde{h}_{ij}(f) e^{-2\pi i f t}. \quad (1.10)$$

The dependence on  $\vec{x}$  must be kept in some cases (see [4]).

### 1.4 TT frame

TT gauge  $\Rightarrow$  TT frame. free test body  $x^\mu(\tau)$ ,  $\frac{dx^i}{d\tau}|_{\tau=0} = 0 \Rightarrow \frac{dx^0}{d\tau} \equiv 1$  and  $\frac{dx^i}{d\tau} \equiv 0$ .

## 第二章 能量

[7],

$$G_{ab}^{[1]}(h_{cd}^{[1]}) + G_{ab}^{[1]}(h_{cd}^{[2]}) + G_{ab}^{[2]}(h_{cd}^{[1]}) = 8\pi T_{ab}, \quad (2.1)$$

$$G_{ab}^{[1]}(h_{cd}^{[1]} + h_{cd}^{[2]}) = 8\pi(T_{ab} + t_{ab}) := 8\pi(T_{ab} - \frac{G_{ab}^{[2]}(h_{cd}^{[1]})}{8\pi}), \quad (2.2)$$

Thus, in the 2nd order,  $h_{ab}^{[2]}$  causes the same correction to  $g_{ab}$  as would be produced by ordinary matter with effect stress-energy tensor  $t_{ab}$ . If not  $T_{ab} \gg t_{ab}$ , derivations in — are not valid.

$$[4], g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}. \quad R_{\mu\nu} = R_{\mu\nu}^{(0)} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} \dots,$$

$$R_{\mu\nu}^{(0)} + [R_{\mu\nu}^{(2)}]^{\text{low}} = 8\pi(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})^{\text{low}}, \quad (2.3)$$

$$R_{\mu\nu}^{(1)} + [R_{\mu\nu}^{(2)}]^{\text{high}} = 8\pi(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})^{\text{high}}, \quad (2.4)$$

(2.3)  $\Rightarrow$

$$R_{\mu\nu}^{(0)} = 8\pi\langle(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})^{\text{low}}\rangle - \langle[R_{\mu\nu}^{(2)}]^{\text{low}}\rangle \quad (2.5)$$

$$= 8\pi\langle(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})\rangle - \langle[R_{\mu\nu}^{(2)}]\rangle \quad (2.6)$$

$$:= 8\pi(T_{\mu\nu}^{(0)} - \frac{1}{2}T^{(0)}g_{\mu\nu}^{(0)}) + 8\pi(t_{\mu\nu} - \frac{1}{2}tg_{\mu\nu}^{(0)}), \quad (2.7)$$

$\Rightarrow$

$$G_{\mu\nu}^{(0)} = 8\pi(T_{\mu\nu}^{(0)} + t_{\mu\nu}). \quad (2.8)$$

In TT gauge,

$$t_{\mu\nu} = \frac{1}{32\pi}\langle\partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\rangle. \quad (2.9)$$





## 第三章 多极矩

### 3.1 Quadrupole Approximation

[7]. 由(1.3)得

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4 \int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV'. \quad (3.1)$$

$$\hat{h}_{\mu\nu}(\omega, \vec{r}) := \frac{1}{\sqrt{2\pi}} \int \bar{h}_{\mu\nu}(t, \vec{r}) e^{i\omega t} dt \quad (3.2)$$

$$= 4 \int \frac{\hat{T}_{\mu\nu}(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega |\vec{r} - \vec{r}'|} dV'. \quad (3.3)$$

由  $\partial^\nu \bar{h}_{\mu\nu} = 0$ ,

$$-i\omega \hat{h}_{0\mu} = \sum_i \frac{\partial \hat{h}_{i\mu}}{\partial x^i}. \quad (3.4)$$

$|\vec{r}| \gg |\vec{r}'|$  且  $\omega \ll 1/|\vec{r}'|$ ,

$$\hat{h}_{ij}(\omega, \vec{r}) = 4 \frac{e^{i\omega |\vec{r}|}}{|\vec{r}|} \int \hat{T}_{ij}(\omega, \vec{r}') dV'. \quad (3.5)$$

$$\int \hat{T}_{ij} dV' = \int \sum_k (\hat{T}_{kj} \frac{\partial x'^i}{\partial x'^k}) dV' \quad (3.6)$$

$$= \sum_k \left[ \int \frac{\partial}{\partial x'^k} (\hat{T}_{kj} x'^i) dV' - \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \right] \quad (3.7)$$

$$= \sum_k \int \partial'_k (\hat{T}_{kj} x'^i) dV' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \quad (3.8)$$

$$= \int \hat{T}_{kj} x'^i dS' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \quad (3.9)$$

$$= - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i \, dV' \quad (3.10)$$

$$= - \int (\sum_k \partial'_k \hat{T}_{kj}) x'^i \, dV' \quad (3.11)$$

$$= - \int (\partial_0 \hat{T}_{0j}) x'^i \, dV' \quad (3.12)$$

$$= -i\omega \int \hat{T}_{0j} x'^i \, dV' \quad (3.13)$$

$$= \int \hat{T}_{(ij)} \, dV' \quad (3.14)$$

$$= -i\omega \int \hat{T}_{0(j} x'^{i)} \, dV' \quad (3.15)$$

$$= -\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) \, dV', \quad (3.16)$$

$$-\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) \, dV' = -\frac{i\omega}{2} \int \sum_k (\hat{T}_{0k} x'^i \frac{\partial x'^j}{\partial x'^k} + \hat{T}_{0k} \frac{\partial x'^i}{\partial x'^k} x'^j) \, dV' \quad (3.17)$$

$$= -\frac{i\omega}{2} \sum_k \left[ \int \frac{\partial}{\partial x'^k} (\hat{T}_{0k} x'^i x'^j) \, dV' - \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, dV' \right] \quad (3.18)$$

$$= -\frac{i\omega}{2} \sum_k \int \partial'_k (\hat{T}_{0k} x'^i x'^j) \, dV' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, dV' \quad (3.19)$$

$$= -\frac{i\omega}{2} \sum_k \int \hat{T}_{0k} x'^i x'^j \, dS' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, dV' \quad (3.20)$$

$$= \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, dV' \quad (3.21)$$

$$= \frac{i\omega}{2} \int (\sum_k \partial'_k \hat{T}_{0k}) x'^i x'^j \, dV' \quad (3.22)$$

$$= \frac{i\omega}{2} \int (\partial_0 \hat{T}_{00}) x'^i x'^j \, dV' \quad (3.23)$$

$$= -\frac{\omega^2}{2} \int \hat{T}_{00} x'^i x'^j \, dV'. \quad (3.24)$$

$$q_{ij}(t) := \int T_{00} x'^i x'^j dV', \quad (3.25)$$

$$\hat{h}_{ij}(\omega, \vec{r}) = -2\omega^2 \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \hat{q}_{ij}(\omega), \quad (3.26)$$

$$\bar{h}_{ij}(t, \vec{r}) = \frac{2}{|\vec{r}|} \frac{d^2}{dt^2} q_{ij}(t - |\vec{r}|). \quad (3.27)$$

### 3.2 电磁—引力对比

$$A_\mu(t, \vec{r}) = \frac{\mu_0}{4\pi} \int \frac{J_\mu(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV' \quad (3.28)$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4G \int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV' \quad (3.29)$$

$$A_\mu(t, \vec{r}) = \frac{1}{\sqrt{2\pi}} \int \hat{A}_\mu(\omega, \vec{r}) e^{-i\omega t} d\omega \quad (3.30)$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = \frac{1}{\sqrt{2\pi}} \int \hat{h}_{\mu\nu}(\omega, \vec{r}) e^{-i\omega t} d\omega \quad (3.31)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\hat{J}_\mu(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega|\vec{r} - \vec{r}'|} dV' \quad (3.32)$$

$$\hat{h}_{\mu\nu}(\omega, \vec{r}) = 4G \int \frac{\hat{T}_{\mu\nu}(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega|\vec{r} - \vec{r}'|} dV' \quad (3.33)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_\mu(\omega, \vec{r}') e^{-i\omega(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}')} dV' \quad (3.34)$$

$$\hat{h}_{\mu\nu}(\omega, \vec{r}) = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{\mu\nu}(\omega, \vec{r}') e^{-i\omega(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}')} dV' \quad (3.35)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_\mu(\omega, \vec{r}') \left[ 1 - i\omega \left( \frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}' \right) - \dots \right] dV' \quad (3.36)$$

$$\hat{h}_{\mu\nu}(\omega, \vec{r}) = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{\mu\nu}(\omega, \vec{r}') \left[ 1 - i\omega \left( \frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}' \right) - \dots \right] dV' \quad (3.37)$$

### 3.2.1 电偶极—引力对比

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_i dV' \quad (3.38)$$

$$\hat{\bar{h}}_{ij} = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{ij} dV' \quad (3.39)$$

$$\int \hat{J}_i dV' = -i\omega \int \hat{J}_0 x'^i dV' \quad (3.40)$$

$$\int \hat{T}_{ij} dV' = -\frac{\omega^2}{2} \int \hat{T}_{00} x'^i x'^j dV' \quad (3.41)$$

$$\hat{p}_i = \int \hat{J}_0 x'^i dV' \quad (3.42)$$

$$\hat{q}_{ij} = \int \hat{T}_{00} x'^i x'^j dV' \quad (3.43)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega \hat{p}_i) \quad (3.44)$$

$$\hat{\bar{h}}_{ij} = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \left(-\frac{\omega^2}{2} \hat{q}_{ij}\right) \quad (3.45)$$

$$A_i = \frac{\mu_0}{4\pi} \frac{1}{|\vec{r}|} \frac{d}{dt} p_i(t - |\vec{r}|) \quad (3.46)$$

$$\bar{h}_{ij} = 4G \frac{1}{|\vec{r}|} \frac{1}{2} \frac{d^2}{dt^2} q_{ij}(t - |\vec{r}|) \quad (3.47)$$

### 3.2.2 电四极—引力对比

$$\hat{A}_i(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int \hat{J}_i(\omega, \vec{r}') \left(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}'\right) dV' \quad (3.48)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int \hat{J}'_i n^j x'_j dV' \quad (3.49)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int n^j x'_j \hat{J}'_i dV' \quad (3.50)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) n^j \left[ \int x'_{(j)} \hat{J}'_{i)} dV' \right] \quad (3.51)$$

$$\int x'_{(j)} \hat{J}'_{i)} dV' = \frac{1}{2} \int (\hat{J}'_j x'_i + \hat{J}'_i x'_j) dV' \quad (3.52)$$

$$= \frac{1}{2} \int \sum_k (\hat{J}'_k x'^i \frac{\partial x'^j}{\partial x'^k} + \hat{J}'_k \frac{\partial x'^i}{\partial x'^k} x'^j) dV' \quad (3.53)$$

$$= \frac{1}{2} \sum_k \left[ \int \frac{\partial}{\partial x'^k} (\hat{J}'_k x'^i x'^j) dV' - \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \right] \quad (3.54)$$

$$= \frac{1}{2} \sum_k \int \partial'_k (\hat{J}'_k x'^i x'^j) dV' - \frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \quad (3.55)$$

$$= \frac{1}{2} \sum_k \int \hat{J}'_k x'^i x'^j dS' - \frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \quad (3.56)$$

$$= -\frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j dV' \quad (3.57)$$

$$= -\frac{1}{2} \int (\sum_k \partial'_k \hat{J}'_k) x'^i x'^j dV' \quad (3.58)$$

$$= -\frac{1}{2} \int (\partial_0 \hat{J}'_0) x'^i x'^j dV' \quad (3.59)$$

$$= -\frac{i\omega}{2} \int \hat{J}'_0 x'^i x'^j dV' \quad (3.60)$$

$$\hat{D}_{ij} = \int \hat{J}'_0 x'^i x'^j dV' \quad (3.61)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \left( -\frac{\omega^2}{2} n^j \hat{D}_{ij} \right) \quad (3.62)$$

$$A_i = \frac{\mu_0}{4\pi} \frac{1}{|\vec{r}|} n^j \frac{1}{2} \frac{d^2}{dt^2} D_{ij}(t - |\vec{r}|) \quad (3.63)$$



## 第四章 双星系统

### 4.1 基本公式

$$\mathcal{M} := \mu^{3/5} M^{2/5} \quad (4.1)$$

$$h_+ = \frac{4\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} \frac{1 + \cos^2 \iota}{2} \cos \Phi(t) \quad (4.2)$$

$$h_\times = \frac{4\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} \cos \iota \sin \Phi(t) \quad (4.3)$$

$$h = F_+ h_+ + F_\times h_\times \quad (4.4)$$

### 4.2 Post-Newtonian Approximation

### 4.3 Stationary Phase Approximation

[5], if  $\zeta(t)$  varies slowly near  $t = t_0$  where the phase has a stationary point:  $\phi'(t_0) = 0$ ,

$$\int \zeta(t) e^{i\phi(t;f)} dt = \int \zeta(t) e^{i[\phi(t_0) + \phi'(t_0)(t-t_0) + \frac{1}{2}\phi''(t_0)(t-t_0)^2 + \dots]} dt \quad (4.5)$$

$$\simeq e^{i\phi(t_0)} \int \zeta(t) e^{i[\frac{1}{2}\phi''(t_0)(t-t_0)^2]} dt \quad (4.6)$$

$$\simeq e^{i\phi(t_0)} \int \zeta(t_0) e^{\frac{-\sqrt{-i\phi''(t_0)}^2 (t-t_0)^2}{2}} dt \quad (4.7)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{-i\phi''(t_0)}} \zeta(t_0) e^{i\phi(t_0)}. \quad (4.8)$$

$$h = \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \cos \Phi(t) \quad (4.9)$$

$$= \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} [e^{i\Phi(t)} + e^{-i\Phi(t)}] \quad (4.10)$$

$$\tilde{h}(f) = \int h(t) e^{i2\pi f t} dt \quad (4.11)$$

$$= \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} [e^{i\Phi(t)} + e^{-i\Phi(t)}] e^{i2\pi f t} dt \quad (4.12)$$

$$= \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} \{e^{i[2\pi f t + \Phi(t)]} + e^{i[2\pi f t - \Phi(t)]}\} dt \quad (4.13)$$

$$\simeq \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} e^{i[2\pi f t - \Phi(t)]} dt \quad (4.14)$$

$$= \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F]^{2/3} Q \frac{1}{2} e^{i[2\pi f t(F) - \Phi(F)]} \frac{dt}{dF} dF \quad (4.15)$$

$$\simeq \frac{\sqrt{2\pi}}{\sqrt{-i[2\pi f t(F) - \Phi(F)]''_{F=f}}} \quad (4.16)$$

$$\left[ \frac{\mathcal{M}}{D} (\pi \mathcal{M} F)^{2/3} Q \frac{1}{2} \frac{dt}{dF} \right]_{F=f} e^{i[2\pi f t(f) - \Phi(f)]} \quad (4.17)$$

$$\simeq \frac{\sqrt{2\pi}}{\sqrt{-i \left\{ 2\pi f \left[ -\frac{5}{256} \mathcal{M} (\pi \mathcal{M} F)^{-8/3} \right] - \left[ \frac{1}{16} (\pi \mathcal{M} F)^{-5/3} \right] \right\}''_{F=f}}} \quad (4.18)$$

$$\left\{ \frac{\mathcal{M}}{D} (\pi \mathcal{M} F)^{2/3} Q \frac{1}{2} \left[ \frac{5\pi \mathcal{M}^2}{96} (\pi \mathcal{M} F)^{-11/3} \right] \right\}_{F=f} e^{i[2\pi f t(f) - \Phi(f)]} \quad (4.19)$$

$$= \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{D} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4}]} \quad (\text{pnspace.py}) \quad (4.20)$$

另可考 [1]. 其中  $\frac{d\Phi}{dt} = 2\pi F$ .



## 第五章 宇宙学效应

[4],

$$\frac{d\eta}{d(ct)} = \frac{1}{a}, \quad (5.1)$$

$$ds^2 = -d(ct)^2 + a^2 \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (5.2)$$

$$ds^2 = a^2 \left[ -d\eta^2 + \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (5.3)$$

$$\square\Phi = 0, \quad \Phi := f/r, \quad f := g/a,$$

$$\partial_r^2 g + (\partial_\eta^2 a/a)g - \partial_\eta^2 g = 0, \quad (5.4)$$

$$\partial_\eta^2 a/a \sim \eta^2, \quad \omega^2 \gg 1/\eta^2,$$

$$g \simeq e^{i\omega(\eta-r)}. \quad (5.5)$$



## 第六章 数据分析

[2], [4].

$$R(\tau) := E(N_t N_{t+\tau}), \quad (6.1)$$

$$\frac{1}{2}S_N(f) := \tilde{R}(f) := \int R(\tau) e^{i2\pi f\tau} d\tau. \quad (6.2)$$

$$\langle p|q \rangle := 4\text{Re} \int_0^\infty \frac{\tilde{p}^*(f)\tilde{q}(f)}{S_N(f)} df. \quad (6.3)$$

### 6.1 parameter estimation

$$p(\mu|d) \propto p(\mu) \exp \left[ -\frac{1}{2} \sum_{m,n} C_{mn}^{-1} (d_m - h_m)(d_n - h_n) \right], \quad (6.4)$$

$$p(\mu|d) \propto p(\mu) \exp \left[ -\frac{1}{2} \langle d - h | d - h \rangle \right]. \quad (6.5)$$

### 6.2 sensitivity

$$\Gamma_{mn} = E(\langle d - h | \partial_m h \rangle \langle d - h | \partial_n h \rangle) = \langle \partial_m h | \partial_n h \rangle. \quad (6.6)$$



## 第七章 电磁引力

[3].

### 7.1 时空张量转化为空间张量

$$h_{ab} := g_{ab} + Z_a Z_b. \quad (7.1)$$

$$h_a{}^b = \delta_a{}^b + Z_a Z^b. \quad (7.2)$$

$$Z^a h_{ab} = 0. \quad (7.3)$$

$$V_{\langle a} := h_a{}^b V_b. \quad (7.4)$$

$$Z^a V_{\langle a} = 0. \quad (7.5)$$

$$T_{\langle ab \rangle} := h_{(a}{}^c h_{b)}{}^d T_{cd} - \frac{1}{3} h_{cd} T^{cd} h_{ab}. \quad (7.6)$$

$$Z^a (h_a{}^c h_b{}^d T_{cd}) = 0. \quad (7.7)$$

$$Z^a (h_b{}^c h_a{}^d T_{cd}) = 0. \quad (7.8)$$

$$Z^a (h_{(a}{}^c h_{b)}{}^d T_{cd}) = 0. \quad (7.9)$$

$$Z^a (h_{cd} T^{cd} h_{ab}) = 0. \quad (7.10)$$

$$Z^a T_{\langle ab \rangle} = 0. \quad (7.11)$$

$$T_{\langle (ab) \rangle} = T_{\langle ab \rangle}. \quad (7.12)$$

$$h^{ab} T_{\langle ab \rangle} = 0. \quad (7.13)$$

$$\varepsilon_{abc} := \varepsilon_{abcd} Z^d. \quad (7.14)$$

$$\varepsilon_{0123} := -\sqrt{|g|}. \quad (7.15)$$

$$T_a := \frac{1}{2} \varepsilon_{abc} T^{[bc]}. \quad (7.16)$$

$$[U, V]_a := \varepsilon_{abc} U^b V^c. \quad (7.17)$$

$$[S, T]_a := \varepsilon_{abc} g_{de} S^{bd} T^{ce}. \quad (7.18)$$

$$D_t T^{a\dots}_{b\dots} := Z^c \nabla_c T^{a\dots}_{b\dots}. \quad (7.19)$$

$${}^3\nabla_a T^{b\dots}_{c\dots} := h_a{}^p h^b{}_q \dots h_c{}^r \dots \nabla_p T^{q\dots}_{r\dots}. \quad (7.20)$$

$$(\operatorname{div} V) := {}^3\nabla^a V_a. \quad (7.21)$$

$$(\operatorname{curl} V)_a := \varepsilon_{bca} {}^3\nabla^b V^c. \quad (7.22)$$

$$(\operatorname{div} T)_a := {}^3\nabla^b T_{ab}. \quad (7.23)$$

$$(\operatorname{curl} T)_{ab} := \varepsilon_{cd(a} {}^3\nabla^c g_{b)e} T^{ed}. \quad (7.24)$$

## 7.2 电磁空间矢量

$${}^*F_{ab} := \frac{1}{2} \varepsilon_{abcd} F^{cd} \quad (7.25)$$

$$E_a := F_{ab} Z^b = E_{\langle a \rangle}. \quad (7.26)$$

$$B_a := {}^*F_{ab} Z^b = B_{\langle a \rangle}. \quad (7.27)$$

$$\rho = -Z^a J_a. \quad (7.28)$$

$$j_a = h_a{}^b J_b. \quad (7.29)$$

$$\nabla_{[a} F_{bc]} = 0. \quad (7.30)$$

$$\nabla^a F_{ab} = \mu J_b. \quad (7.31)$$

$$(\operatorname{div} E) = \mu\rho - \dots \quad (7.32)$$

$$(\operatorname{div} B) = + \dots \quad (7.33)$$

$$(\operatorname{curl} E)_a + \dots = -D_t B_{\langle a} - \dots \quad (7.34)$$

$$(\operatorname{curl} B)_a + \dots = \mu j_a + D_t E_{\langle a} + \dots \quad (7.35)$$

### 7.3 引力空间张量

$${}^*C_{abcd} := \frac{1}{2}\varepsilon_{abef}C^ef_{cd}. \quad (7.36)$$

$$E_{ab} := C_{acbd}Z^cZ^d = E_{\langle ab \rangle}. \quad (7.37)$$

$$B_{ab} := {}^*C_{acbd}Z^cZ^d = B_{\langle ab \rangle}. \quad (7.38)$$

$$(\operatorname{div} E)_a = \kappa \frac{1}{3} \nabla_a \rho - \dots \quad (7.39)$$

$$(\operatorname{div} B)_a = \kappa(\rho + p)\omega_a + \dots \quad (7.40)$$

$$(\operatorname{curl} E)_{ab} + \dots = -D_t B_{\langle ab \rangle} - \dots \quad (7.41)$$

$$(\operatorname{curl} B)_{ab} + \dots = \kappa \frac{1}{2}(\rho + p)\sigma_{ab} + D_t E_{\langle ab \rangle} + \dots \quad (7.42)$$





## 第八章 Varying $G$

### 8.1 Modification of Amplitude

$$\partial^c \partial_c \bar{h}_{ab} = -16\pi \frac{G_0}{c_0^4} T_{ab}, \quad \partial^a \bar{h}_{ab} = 0 \quad (8.1)$$

$$\Gamma^c_{ab} = \frac{1}{2} \eta^{cd} (2\partial_{(a} h_{b)d} - \partial_d h_{ab}) \quad (8.2)$$

$$U^a \partial_a U^c + \Gamma^c_{ab} U^a U^b = 0 \quad (8.3)$$

$$U^a \partial_a U^c = -\frac{1}{2} \eta^{cd} (2\partial_{(a} h_{b)d} - \partial_d h_{ab}) U^a U^b \quad (8.4)$$

$$T_{ab} = c_0^2 (2U_{(a} J_{b)} + U^c J_c U_a U_b) \quad (8.5)$$

$$J_b c_0^2 = -U^a T_{ab} \quad (8.6)$$

$$A_b = -\frac{1}{4} U^a \bar{h}_{ab} \quad (8.7)$$

$$A_0 = -\frac{1}{4} c_0 \bar{h}_{00} = -\frac{1}{2} c_0 (\bar{h}_{00} - \frac{1}{2} \eta_{00} \eta^{00} \bar{h}_{00}) = -\frac{1}{2} c_0 h_{00} \quad (8.8)$$

$$A_i = -\frac{1}{4} c_0 \bar{h}_{0i} = -\frac{1}{4} c_0 h_{0i} \quad (8.9)$$

$$U^\mu \partial_\mu U^i = -\frac{1}{2} \eta^{i\sigma} (\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) U^\mu U^\nu \quad (8.10)$$

$$-\frac{1}{2} \eta^{i\sigma} (\partial_0 h_{0\sigma} + \partial_0 h_{0\sigma} - \partial_\sigma h_{00}) U^0 U^0 = \frac{1}{2} c_0^2 \eta^{i\sigma} \partial_\sigma h_{00} \quad (8.11)$$

$$= \frac{1}{2} c_0^2 \partial^i h_{00} \quad (8.12)$$

$$= -c_0 \partial^i A_0 \quad (8.13)$$

$$= -E^i \quad (8.14)$$

$$-\frac{1}{2} \eta^{i\sigma} (\partial_0 h_{j\sigma} + \partial_j h_{0\sigma} - \partial_\sigma h_{0j}) U^0 U^j = -\frac{1}{2} c_0 \eta^{i\sigma} (\partial_j h_{0\sigma} - \partial_\sigma h_{0j}) v^j \quad (8.15)$$

$$= -\frac{1}{2} c_0 \eta^{ik} (\partial_j h_{0k} - \partial_k h_{0j}) v^j \quad (8.16)$$

$$= 2\eta^{ik} (\partial_j A_k - \partial_k A_j) v^j \quad (8.17)$$

$$= -2\eta^{ik} (\partial_k A_j - \partial_j A_k) v^j \quad (8.18)$$

$$= -2(\partial^i A_j - \partial_j A^i) v^j \quad (8.19)$$

$$= -2\varepsilon^i{}_{jk} v^j B^k \quad (8.20)$$

$$-\frac{1}{2} \eta^{i\sigma} (\partial_j h_{k\sigma} + \partial_k h_{j\sigma} - \partial_\sigma h_{jk}) U^j U^k = 0 \quad (8.21)$$

$$a^i = -E^i - 4\varepsilon^i{}_{jk} v^j B^k \quad (8.22)$$

$$\partial^i \left( \frac{1}{4\pi G_0} E_i \right) = \rho \quad (8.23)$$

$$\partial^i B_i = 0 \quad (8.24)$$

$$\varepsilon^i{}_{jk} \partial^j E^k = -\partial_t B^i \quad (8.25)$$

$$\varepsilon^i{}_{jk} \partial^j \left( \frac{c_0^2}{4\pi G_0} B^k \right) = j^i + \partial_t \left( \frac{1}{4\pi G_0} E^i \right) \quad (8.26)$$

$$\varepsilon_{G0} := \frac{1}{4\pi G_0}, \quad \mu_{G0} := \frac{4\pi G_0}{c_0^2} \quad (8.27)$$

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_{G0} \vec{E}) = \rho \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \\ \vec{\nabla} \times (\mu_{G0}^{-1} \vec{B}) = \vec{j} + \frac{\partial}{\partial t} (\varepsilon_{G0} \vec{E}) \end{cases} \quad (8.28)$$

$$\vec{a} = -\vec{E} - 4\vec{v} \times \vec{B} \quad (8.29)$$

$$\varepsilon_G = \frac{1}{4\pi G}, \quad \mu_G = \frac{4\pi G}{c^2} \quad (8.30)$$

$$x^\mu = (ct, x, y, z) \quad (8.31)$$

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_G \vec{E}) = \rho \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \\ \vec{\nabla} \times (\mu_G^{-1} \vec{B}) = \vec{j} + \frac{\partial}{\partial t} (\varepsilon_G \vec{E}) \end{cases} \quad (8.32)$$

$$\vec{a} = -\vec{E} - 4\vec{v} \times \vec{B} \quad (8.33)$$

$$A_\mu = -\frac{1}{4} c \bar{h}_{0\mu} \quad (8.34)$$

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \varepsilon_G^{-1} \rho \\ \vec{\nabla} \times \vec{B} = \mu_G \vec{j} + \varepsilon_G \mu_G \frac{\partial}{\partial t} \vec{E} \end{cases} \quad (8.35)$$

$$\frac{1}{c^2} \frac{\partial}{\partial t} \varphi + \vec{\nabla} \cdot \vec{A} = 0 \quad (8.36)$$

$$\begin{cases} -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi + \vec{\nabla}^2 \varphi = \varepsilon_G^{-1} \rho \\ -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} + \vec{\nabla}^2 \vec{A} = \mu_G \vec{j} \end{cases} \quad (8.37)$$

$$\begin{cases} -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} c^{-1} \varphi + \vec{\nabla}^2 c^{-1} \varphi = \mu_G c \rho \\ -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} + \vec{\nabla}^2 \vec{A} = \mu_G \vec{j} \end{cases} \quad (8.38)$$

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_G \vec{E}) = 0 \\ \vec{\nabla} \cdot (\mu_G \vec{H}) = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} (\mu_G \vec{H}) \\ \vec{\nabla} \times \vec{H} = +\frac{\partial}{\partial t} (\varepsilon_G \vec{E}) \end{cases} \quad (8.39)$$

$$E_r = 0, \quad H_r = 0 \quad (8.40)$$

$$\begin{cases} \frac{\varepsilon_G}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta E_\theta) + \frac{\varepsilon_G}{r \sin \theta} \frac{\partial}{\partial \phi} (E_\phi) = 0 \\ \frac{\mu_G}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta H_\theta) + \frac{\mu_G}{r \sin \theta} \frac{\partial}{\partial \phi} (H_\phi) = 0 \\ \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta E_\phi) - \frac{\partial}{\partial \phi} (E_\theta) \right] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r} (r E_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \vec{e}_\phi = -\mu_G \frac{\partial}{\partial t} (H_\theta \vec{e}_\theta + H_\phi \vec{e}_\phi) \\ \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta H_\phi) - \frac{\partial}{\partial \phi} (H_\theta) \right] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r} (r H_\theta) \vec{e}_\phi = +\varepsilon_G \frac{\partial}{\partial t} (E_\theta \vec{e}_\theta + E_\phi \vec{e}_\phi) \end{cases} \quad (8.41)$$

$$\vec{E} = E_\theta \vec{e}_\theta, \quad \vec{H} = H_\phi \vec{e}_\phi \quad (8.42)$$

$$\begin{cases} \frac{\varepsilon_G}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta E_\theta) = 0 \\ \frac{\mu_G}{r \sin \theta} \frac{\partial}{\partial \phi} (H_\phi) = 0 \\ -\frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (E_\theta) \vec{e}_r + \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \vec{e}_\phi = -\mu_G \frac{\partial}{\partial t} (H_\phi) \vec{e}_\phi \\ +\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta H_\phi) \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) \vec{e}_\theta = +\varepsilon_G \frac{\partial}{\partial t} (E_\theta) \vec{e}_\theta \end{cases} \quad (8.43)$$

$$\begin{cases} \frac{\partial}{\partial r} (r E_\theta) + \mu_G \frac{\partial}{\partial t} (r H_\phi) = 0 \\ \frac{\partial}{\partial r} (r H_\phi) + \varepsilon_G \frac{\partial}{\partial t} (r E_\theta) = 0 \end{cases} \quad (8.44)$$

$$\begin{cases} \mu_G \frac{\partial}{\partial r} \mu_G^{-1} \frac{\partial}{\partial r} (r E_\theta) - \varepsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t} (r E_\theta) = 0 \\ \varepsilon_G \frac{\partial}{\partial r} \varepsilon_G^{-1} \frac{\partial}{\partial r} (r H_\phi) - \varepsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t} (r H_\phi) = 0 \end{cases} \quad (8.45)$$

$$\begin{cases} \mu_G \frac{\partial}{\partial r} \mu_G^{-1} \frac{\partial}{\partial r} (r E_\theta) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (r E_\theta) = 0 \\ \varepsilon_G \frac{\partial}{\partial r} \varepsilon_G^{-1} \frac{\partial}{\partial r} (r H_\phi) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (r H_\phi) = 0 \end{cases} \quad (8.46)$$

$$\begin{cases} \frac{\partial}{\partial r} \frac{\partial}{\partial r} (r E_\theta) - \frac{\partial}{\partial r} (\ln \mu_G) \frac{\partial}{\partial r} (r E_\theta) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (r E_\theta) = 0 \\ \frac{\partial}{\partial r} \frac{\partial}{\partial r} (r H_\phi) - \frac{\partial}{\partial r} (\ln \varepsilon_G) \frac{\partial}{\partial r} (r H_\phi) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (r H_\phi) = 0 \end{cases} \quad (8.47)$$

$$\frac{\partial^2}{\partial r^2} f(r, t) - p(r) \frac{\partial}{\partial r} f(r, t) - \frac{\partial^2}{\partial(ct)^2} f(r, t) = 0 \quad (8.48)$$

$$f(r, t) = f(r) e^{-ikct} \quad (8.49)$$

$$\frac{d^2}{dr^2} f(r) - p(r) \frac{d}{dr} f(r) + k^2 f(r) = 0 \quad (8.50)$$

$$\frac{d^2}{dr^2} f(r) - p \frac{d}{dr} f(r) + k^2 f(r) = 0 \quad (8.51)$$

$$f(r) = e^{(p/2)r} [C_+ e^{i\sqrt{k^2 - (p/2)^2}r} + C_- e^{-i\sqrt{k^2 - (p/2)^2}r}] \quad (8.52)$$

$$f(r, t) = e^{(p/2)r} [C_+ e^{i(+\sqrt{k^2 - (p/2)^2}r - kct)} + C_- e^{i(-\sqrt{k^2 - (p/2)^2}r - kct)}] \quad (8.53)$$

$$f(r, t) = e^{(p/2)r} [C_+ e^{i(+\sqrt{(\omega/c)^2 - (p/2)^2}r - \omega t)} + C_- e^{i(-\sqrt{(\omega/c)^2 - (p/2)^2}r - \omega t)}] \quad (8.54)$$

$$f(r, t) = e^{\int (p/2) dr} [C_+ e^{i(+\int \sqrt{(\omega/c)^2 - (p/2)^2} dr - \omega t)} + C_- e^{i(-\int \sqrt{(\omega/c)^2 - (p/2)^2} dr - \omega t)}] \quad (8.55)$$

$$\begin{cases} r_2 |E_\theta|_{r=r_2} = r_1 |E_\theta|_{r=r_1} e^{\int_{r_1}^{r_2} \frac{1}{2} \frac{\partial}{\partial r} (\ln \mu_G) dr} \\ r_2 |H_\phi|_{r=r_2} = r_1 |H_\phi|_{r=r_1} e^{\int_{r_1}^{r_2} \frac{1}{2} \frac{\partial}{\partial r} (\ln \varepsilon_G) dr} \end{cases} \quad (8.56)$$

$$\begin{cases} E_2 = \sqrt{\frac{\mu_{G2}}{\mu_{G1}} \frac{r_1}{r_2}} E_1 \\ H_2 = \sqrt{\frac{\varepsilon_{G2}}{\varepsilon_{G1}} \frac{r_1}{r_2}} H_1 \end{cases} \quad (8.57)$$

$$\begin{cases} E_2/c_2 = \sqrt{\frac{\mu_{G2}}{\mu_{G1}} \frac{c_1}{c_2} \frac{r_1}{r_2}} E_1/c_1 \\ B_2 = \sqrt{\frac{\mu_{G2}}{\mu_{G1}} \frac{c_1}{c_2} \frac{r_1}{r_2}} B_1 \end{cases} \quad (8.58)$$

$$\begin{cases} (\omega/c_2)c_2(\bar{h}_{00})_2 = \sqrt{\frac{\mu_{G2}}{\mu_{G1}} \frac{c_1}{c_2} \frac{r_1}{r_2}} (\omega/c_1)c_1(\bar{h}_{00})_1 \\ (\omega/c_2)c_2(\bar{h}_{0i})_2 = \sqrt{\frac{\mu_{G2}}{\mu_{G1}} \frac{c_1}{c_2} \frac{r_1}{r_2}} (\omega/c_1)c_1(\bar{h}_{0i})_1 \end{cases} \quad (8.59)$$

$$h_2 = \sqrt{\frac{c_1^4/G_1}{c_2^4/G_2} \frac{r_1}{r_2}} h_1 \quad (8.60)$$

双星系统引力辐射本为

$$h = \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{r} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F(t) dt\right] \quad (8.61)$$

设双星系统常量  $c^*$ ,  $G^*$ , 一观者临近双星系统且与双星系统相对静止, 其与双星系统距离为  $r$ , 测得强度  $h_r$ , 频率  $F_r$ , 则<sup>1</sup>

$$h_r = \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{r/c^*} Q(\theta, \phi, \psi, \iota) \quad (8.62)$$

设地球观者与双星系统距离为  $d$ , 双星系统红移为  $z$ , 测得强度  $h_d$ , 频率  $F_d = F_r/(1+z)$ , 则

$$h_d = \sqrt{\frac{c^{*4}/G^*}{c^4/G} \frac{r}{d}} h_r \quad (8.63)$$

$$= \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{d/c^*} Q(\theta, \phi, \psi, \iota) \quad (8.64)$$

所以地球观者测得

$$h = \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{d/c^*} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi \frac{F_r(t)}{1+z} dt\right] \quad (8.65)$$

记  $F_{\text{obs}}(t) = F_r(t)/(1+z)$ ,  $\mathcal{M}_{\text{obs}} = \mathcal{M}(1+z)$ , 光度距离  $d_L = d(1+z)$ , 则

$$h = \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{d(1+z)/c^*} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F_{\text{obs}}(t) dt\right] \quad (8.66)$$

<sup>1</sup> $\mathcal{M}$  和  $c^*$ ,  $G^*$  简并, 所以可以笼统地仍记作  $\mathcal{M}$ .

$$= \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}_{\text{obs}}[\pi \mathcal{M}_{\text{obs}} F_{\text{obs}}(t)]^{2/3}}{d_{\text{L}}/c^*} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F_{\text{obs}}(t) dt\right] \quad (8.67)$$

$$= \sqrt{\frac{c^{*6}/G^*}{c^6/G}} \frac{\mathcal{M}_{\text{obs}}[\pi \mathcal{M}_{\text{obs}} F_{\text{obs}}(t)]^{2/3}}{d_{\text{L}}/c} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F_{\text{obs}}(t) dt\right] \quad (8.68)$$

用引力波测距测得  $d_{\text{L,G}}$ , 则

$$d_{\text{L,G}} = d_{\text{L}} \sqrt{\frac{c^6/G}{c^{*6}/G^*}} \quad (8.69)$$

[5]

$$h(t) = \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{\xi d_{\text{L}}} Q(\text{angles}) \cos \Phi(t) \quad (8.70)$$

$$\tilde{h}(f) = \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{\xi d_{\text{L}}} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4}]} \quad (8.71)$$

问题转化为估计  $\xi$

$$p(\mu) \propto p^{(0)}(\mu) \exp\left[-\frac{1}{2} \Gamma_{ab}(\mu^a - \hat{\mu}^a)(\mu^b - \hat{\mu}^b)\right] \quad (8.72)$$

$$p^{(0)}(\mu) \propto \exp\left[-\frac{1}{2} \Gamma_{ab}^{(0)}(\mu^a - \bar{\mu}^a)(\mu^b - \bar{\mu}^b)\right] \quad (8.73)$$

设待估参数为  $\mu = (\ln \xi, \ln(d_{\text{L}}/d_{\text{L}0}), \ln Q, \dots)$ ,  $\dots$  为其他参数 (如  $\mathcal{M}$ ), 则  $\tilde{h}_{,\ln \xi} = \tilde{h}_{,\ln(d_{\text{L}}/d_{\text{L}0})} = -\tilde{h}_{,\ln Q} = -\tilde{h}$ ,  $\tilde{h}$  对其他参数求偏导皆为纯虚数, 则由  $\Gamma_{ab} = \langle h_{,a} | h_{,b} \rangle$  和  $\text{SNR} := \rho = \sqrt{\langle h | h \rangle}$  得

$$\Gamma_{ab} = \begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 & 0 & \dots \\ \rho^2 & \rho^2 & -\rho^2 & 0 & \dots \\ -\rho^2 & -\rho^2 & \rho^2 & 0 & \dots \\ 0 & 0 & 0 & ? & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (8.74)$$

又设

$$\Gamma_{ab}^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1/\sigma_{\ln d_{\text{L}}}^2 & 0 & 0 & \dots \\ 0 & 0 & 1/\sigma_{\ln Q}^2 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (8.75)$$

则由  $\Sigma_{ab} = (\Gamma_{ab}^{(0)} + \Gamma_{ab})^{-1}$  得

$$\Sigma_{ab} = \begin{bmatrix} \begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 \\ \rho^2 & \rho^2 + 1/\sigma_{\ln(d_L/d_{L0})}^2 & -\rho^2 \\ -\rho^2 & -\rho^2 & \rho^2 + 1/\sigma_{\ln Q}^2 \end{bmatrix}^{-1} & 0 \\ 0 & [?]^{-1} \end{bmatrix} \quad (8.76)$$

而

$$\begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 \\ \rho^2 & \rho^2 + 1/\sigma_{\ln(d_L/d_{L0})}^2 & -\rho^2 \\ -\rho^2 & -\rho^2 & \rho^2 + 1/\sigma_{\ln Q}^2 \end{bmatrix}^{-1} \quad (8.77)$$

$$= \begin{bmatrix} 1/\rho^2 + \sigma_{\ln(d_L/d_{L0})}^2 + \sigma_{\ln Q}^2 & -\sigma_{\ln(d_L/d_{L0})}^2 & \sigma_{\ln Q}^2 \\ -\sigma_{\ln(d_L/d_{L0})}^2 & \sigma_{\ln(d_L/d_{L0})}^2 & 0 \\ \sigma_{\ln Q}^2 & 0 & \sigma_{\ln Q}^2 \end{bmatrix} \quad (8.78)$$

## 8.2 Modification of Phase

$$\frac{d^2}{dz^2} H(z) + 2p(z) \frac{d}{dz} H(z) + [\omega^2 + q(z)] H(z) = 0. \quad (8.79)$$

$$H = A e^{i\Phi}. \quad (8.80)$$

$$k = \frac{d\Phi}{dz},$$

$$\frac{d^2 A}{dz^2} + 2p \frac{dA}{dz} + \left[ \omega^2 \left( 1 - \frac{k^2}{\omega^2} \right) + q \right] A = 0, \quad (8.81)$$

$$2 \frac{dA}{dz} k + A \frac{dk}{dz} + 2p A k = 0, \quad (8.82)$$

$$2 \frac{1}{A} \frac{dA}{dz} + \frac{1}{k} \frac{dk}{dz} + 2p = 0, \quad (8.83)$$

$$A \propto e^{-\int p dz} k^{-1/2}. \quad (8.84)$$

$$\Gamma = e^{\int p dz} \text{ and } K = (k/\omega)^{-1/2},$$

$$\frac{d^2 K}{dz^2} - \left( \frac{1}{\Gamma} \frac{d^2 \Gamma}{dz^2} - q \right) K + \omega^2 K (1 - K^{-4}) = 0, \quad (8.85)$$

$$\Xi = \frac{1}{\Gamma} \frac{d^2 \Gamma}{dz^2} - q \text{ and make } \omega = 1,$$

$$\frac{d^2 K}{dz^2} + K[(1 - \Xi) - K^{-4}] = 0. \quad (8.86)$$

$$\Xi = \text{const},$$

$$K = (1 - \Xi)^{-1/4} = 1 + \frac{1}{4}\Xi + \frac{5}{32}\Xi^2 + O(\Xi^3), \quad (8.87)$$

$$k = (1 - \Xi)^{1/2} = 1 - \frac{1}{2}\Xi - \frac{1}{8}\Xi^2 + O(\Xi^3), \quad (8.88)$$

$$\Xi \neq \text{const}, \Xi(z) = \kappa^2 \tilde{\Xi}(\tilde{z}), \text{ where } \tilde{z} = \kappa z.$$

$$K^3 \frac{d^2 K}{d\tilde{z}^2} \kappa^2 - K^4 \tilde{\Xi}(\tilde{z}) \kappa^2 + K^4 - 1 = 0. \quad (8.89)$$

$$K = \sum_{n=0}^{\infty} K_n(\tilde{z}) \kappa^{2n}, \quad (8.90)$$

$$K_0^4 - 1 = 0, \quad (8.91)$$

$$K_0^3 K_0'' - K_0^4 \tilde{\Xi} + 4K_0^3 K_1 = 0, \quad (8.92)$$

$$(K_0^3 K_1'' + 3K_0^2 K_1 K_0'') - 4K_0^3 K_1 \tilde{\Xi} + (4K_0^3 K_2 + 6K_0^2 K_1^2) = 0. \quad (8.93)$$

$$K_0 = 1, \quad (8.94)$$

$$K_1 = \frac{1}{4} \tilde{\Xi}, \quad (8.95)$$

$$K_2 = \frac{5}{32} \tilde{\Xi}^2 - \frac{1}{16} \frac{d^2 \tilde{\Xi}}{d\tilde{z}^2}, \quad (8.96)$$

$$h(z, t) = h(z) e^{-i\omega t} \quad (8.97)$$

$$h(z) = \Gamma^{-1}(z) K(z) (C_+ e^{+i\omega \int K^{-2}(z) dz} + C_- e^{-i\omega \int K^{-2}(z) dz}) \quad (8.98)$$

$$h(z, t) = \int_{-\infty}^{+\infty} \tilde{h}(z; f) e^{-i2\pi f t} df \quad (8.99)$$

$$\tilde{h}(z; f) = \Gamma^{-1}(z) K(z; f) [C_+(f) e^{+i2\pi f \int K^{-2}(z; f) dz} + C_-(f) e^{-i2\pi f \int K^{-2}(z; f) dz}] \quad (8.100)$$

$$\tilde{h}_0(z; f) = \Gamma^{-1}(0) K(0; f) [C_+(f) e^{+i2\pi f \int dz} + C_-(f) e^{-i2\pi f \int dz}] \quad (8.101)$$

$$\tilde{h}(z; f) = \frac{\Gamma^{-1}(z) K(z; f)}{\Gamma^{-1}(0) K(0; f)} [C_+(f) e^{+i2\pi f \int K^{-2}(z) dz} + C_-(f) e^{-i2\pi f \int K^{-2}(z) dz}] \quad (8.102)$$

$$\tilde{h}_0(z; f) = [C_+(f) e^{+i2\pi f \int dz} + C_-(f) e^{-i2\pi f \int dz}] \quad (8.103)$$

$$K(z) = 1 + \frac{1}{4\omega^2} \Xi(z) \quad (8.104)$$



$$\tilde{h}(z; f) = \frac{\Gamma^{-1}(z)}{\Gamma^{-1}(0)} \left[ 1 + \frac{\Xi(z) - \Xi(0)}{4} (2\pi f)^{-2} \right] [C_+(f) e^{+i2\pi f z} e^{-i2\pi f \int \frac{\Xi(z)}{2} (2\pi f)^{-2} dz} + C_-(f) e^{-i2\pi f z} e^{+i2\pi f \int \frac{\Xi(z)}{2} (2\pi f)^{-2} dz}] \quad (8.105)$$

$$\tilde{h}(z; f) = \gamma(z) [1 + \xi(z) (2\pi f)^{-2}] [C_+(f) e^{+i2\pi f z} e^{+i\Omega(z)(2\pi f)^{-1}} + C_-(f) e^{-i2\pi f z} e^{-i\Omega(z)(2\pi f)^{-1}}] \quad (8.106)$$

$$\tilde{h}_0(z; f) = [C_+(f) e^{+i2\pi f z} + C_-(f) e^{-i2\pi f z}] \quad (8.107)$$

$$\tilde{h}_0(z; f) = C_+(f) e^{+i2\pi f z} + C_-(f) e^{-i2\pi f z} \quad (8.108)$$

$$\partial_z \tilde{h}_0(z; f) = C_+(f) (i2\pi f) e^{+i2\pi f z} - C_-(f) (i2\pi f) e^{-i2\pi f z} \quad (8.109)$$

$$C_+(f) e^{+i2\pi f z} = \frac{1}{2} [\tilde{h}_0(z; f) + \partial_z \tilde{h}_0(z; f) (i2\pi f)^{-1}] \quad (8.110)$$

$$C_-(f) e^{-i2\pi f z} = \frac{1}{2} [\tilde{h}_0(z; f) - \partial_z \tilde{h}_0(z; f) (i2\pi f)^{-1}] \quad (8.111)$$

$$\tilde{h}_0(f) = C_+(f) e^{+i2\pi f z} \quad (8.112)$$

$$\tilde{h}(f) = \gamma(1 + \xi f^{-2}) e^{i\Omega f^{-1}} \tilde{h}_0(f) \quad (8.113)$$

$$\tilde{h}_0(f) = \mathcal{A} f^{-7/6} e^{i\{2\pi f t_c - \varphi_c - \frac{\pi}{4} + \frac{3}{128} (\pi \mathcal{M} f)^{-5/3} [1 + \frac{20}{9} (\frac{743}{366} + \frac{11}{4} \eta) (\pi M f)^{2/3}]\}} \quad (8.114)$$



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