引力波天文学笔记

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目录

第一章　引力波

1.1 Linearized Gravity

[8]. 流形 $\mathbb{R}^4$. 任意坐标系 $\{x^\mu\}$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} s + \mathrm{O}(s^2)$, 得

$$R_{\mu\nu\lambda\sigma} = \partial_\sigma \partial_{[\mu} h_{\lambda]\nu} - \partial_\nu \partial_{[\mu} h_{\lambda]\sigma} + \mathrm{O}(s^2). \tag{1.1}$$

$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\sigma}h_{\lambda\sigma} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$.

$$-\frac{1}{2}\partial^\lambda \partial_\lambda \bar{h}_{\mu\nu} + \partial^\lambda \partial_{(\mu} \bar{h}_{\nu)\lambda} - \frac{1}{2}\eta_{\mu\nu}\partial^\lambda \partial^\sigma \bar{h}_{\lambda\sigma} + \mathrm{O}(s^2) = 8\pi T_{\mu\nu}. \tag{1.2}$$

存在 $\{x^\mu\}$, 使得 $\partial^\nu \bar{h}_{\mu\nu} + \mathrm{O}(s^2) = 0$ (Lorentz gauge). 令 $\{x^\mu\}$ 满足 $\partial^\nu \bar{h}_{\mu\nu} + \mathrm{O}(s^2) = 0$, 则

$$\partial^\lambda \partial_\lambda \bar{h}_{\mu\nu} + \mathrm{O}(s^2) = -16\pi T_{\mu\nu}. \tag{1.3}$$

略去 $\mathrm{O}(s^2)$ 条件: $h_{\mu\nu}$, $\partial_\lambda h_{\mu\nu}$...小.

1.2 Radiation Gauge

[8]. 存在 $\{x^\mu\}$, 使得 $h + \mathrm{O}(s^2) = 0$ (TT gauge [9]) 且 $h_{0\mu} + \mathrm{O}(s^2) = 0$.

1.3 Quadrupole Approximation

[8]. 下略 $\mathrm{O}(s^2)$. 由(1.3)得

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4 \int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} \, \mathrm{d}V'. \tag{1.4}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega,\vec{r}) := \frac{1}{\sqrt{2\pi}}\int \bar{h}_{\mu\nu}(t,\vec{r})e^{i\omega t}\mathrm{d}t \tag{1.5}$$

$$= 4\int \frac{\hat{T}_{\mu\nu}(\omega,\vec{r}')}{|\vec{r}-\vec{r}'|}e^{i\omega|\vec{r}-\vec{r}'|}\,\mathrm{d}V'. \tag{1.6}$$

由 $\partial^\nu \bar{h}_{\mu\nu} = 0$,

$$-i\omega\hat{\bar{h}}_{0\mu} = \sum_i \frac{\partial\hat{\bar{h}}_{i\mu}}{\partial x^i}. \tag{1.7}$$

$|\vec{r}| \gg |\vec{r}'|$ 且 $\omega \ll 1/|\vec{r}'|$,

$$\hat{\bar{h}}_{ij}(\omega,\vec{r}) = 4\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{T}_{ij}(\omega,\vec{r}')\,\mathrm{d}V'. \tag{1.8}$$

$$\int \hat{T}_{ij}\,\mathrm{d}V' = \int \sum_k (\hat{T}_{kj}\frac{\partial x'^i}{\partial x'^k})\,\mathrm{d}V' \tag{1.9}$$

$$= \sum_k \left[\int \frac{\partial}{\partial x'^k}(\hat{T}_{kj}x'^i)\,\mathrm{d}V' - \int \frac{\partial \hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V'\right] \tag{1.10}$$

$$= \sum_k \int \partial'_k\,(\hat{T}_{kj}x'^i)\,\mathrm{d}V' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V' \tag{1.11}$$

$$= \int \hat{T}_{kj}x'^i\,\mathrm{d}S' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V' \tag{1.12}$$

$$= -\sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V' \tag{1.13}$$

$$= -\int (\sum_k \partial'_k\hat{T}_{kj})x'^i\,\mathrm{d}V' \tag{1.14}$$

$$= -\int (\partial_0\hat{T}_{0j})x'^i\,\mathrm{d}V' \tag{1.15}$$

$$= -i\omega\int \hat{T}_{0j}x'^i\,\mathrm{d}V' \tag{1.16}$$

$$= \int \hat{T}_{(ij)}\,\mathrm{d}V' \tag{1.17}$$

$$= -i\omega\int \hat{T}_{0(j}x'^{i)}\,\mathrm{d}V' \tag{1.18}$$

$$= -\frac{i\omega}{2}\int (\hat{T}_{0j}x'^i + \hat{T}_{0i}x'^j)\,\mathrm{d}V', \tag{1.19}$$

$$\tag{1.20}$$

$$-\frac{i\omega}{2}\int(\hat{T}_{0j}x'^i+\hat{T}_{0i}x'^j)\,\mathrm{d}V' = -\frac{i\omega}{2}\int\sum_k(\hat{T}_{0k}x'^i\frac{\partial x'^j}{\partial x'^k}+\hat{T}_{0k}\frac{\partial x'^i}{\partial x'^k}x'^j)\,\mathrm{d}V' \tag{1.21}$$

$$= -\frac{i\omega}{2}\sum_k\left[\int\frac{\partial}{\partial x'^k}(\hat{T}_{0k}x'^ix'^j)\,\mathrm{d}V' - \int\frac{\partial\hat{T}_{0k}}{\partial x'^k}x'^ix'^j\,\mathrm{d}V'\right] \tag{1.22}$$

$$= -\frac{i\omega}{2}\sum_k\int\partial'_k\,(\hat{T}_{0k}x'^ix'^j)\,\mathrm{d}V' + \frac{i\omega}{2}\sum_k\int\frac{\partial\hat{T}_{0k}}{\partial x'^k}x'^ix'^j\,\mathrm{d}V' \tag{1.23}$$

$$= -\frac{i\omega}{2}\sum_k\int\hat{T}_{0k}x'^ix'^j\,\mathrm{d}S' + \frac{i\omega}{2}\sum_k\int\frac{\partial\hat{T}_{0k}}{\partial x'^k}x'^ix'^j\,\mathrm{d}V' \tag{1.24}$$

$$= \frac{i\omega}{2}\sum_k\int\frac{\partial\hat{T}_{0k}}{\partial x'^k}x'^ix'^j\,\mathrm{d}V' \tag{1.25}$$

$$= \frac{i\omega}{2}\int(\sum_k\partial'_k\hat{T}_{0k})x'^ix'^j\,\mathrm{d}V' \tag{1.26}$$

$$= \frac{i\omega}{2}\int(\partial_0\hat{T}_{00})x'^ix'^j\,\mathrm{d}V' \tag{1.27}$$

$$= -\frac{\omega^2}{2}\int\hat{T}_{00}\,x'^ix'^j\,\mathrm{d}V'. \tag{1.28}$$

$$q_{ij}(t) := \int T_{00}\,x'^ix'^j\,\mathrm{d}V', \tag{1.29}$$

$$\hat{\bar{h}}_{ij}(\omega,\vec{r}) = -2\omega^2\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\hat{q}_{ij}(\omega), \tag{1.30}$$

$$\bar{h}_{ij}(t,\vec{r}) = \frac{2}{|\vec{r}|}\frac{\mathrm{d}^2}{\mathrm{d}t^2}q_{ij}(t-|\vec{r}|). \tag{1.31}$$

1.4 $+$ Mode and $\times$ Mode

寻新标架 $(e'^1)_a = (e^+)_a$, $(e'^2)_a = (e^\times)_a$, $(e'^3)_a = (e^r)_a$, $\bar{h}_{ij}(e^i)_a(e^j)_b = \bar{h}'_{ij}(e'^i)_a(e'^j)_b$, 取 x, y 分量后去迹, $h_+ = \frac{1}{2}(\bar{h}'_{11}-\bar{h}'_{22})$, $h_\times = \bar{h}'_{12} = \bar{h}'_{21}$? [7] [2], $\vec{n} := \frac{\vec{r}}{|\vec{r}|}$,

$$h^{\mathrm{TT}}_{ij} = \frac{2}{|\vec{r}|}\mathcal{P}_{ijkm}\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q^{km}(t-|\vec{r}|), \tag{1.32}$$

$$\mathcal{P}_{ijkm} := (\delta_{ik} - \vec{n}_i \vec{n}_k)(\delta_{jm} - \vec{n}_j \vec{n}_m) - \frac{1}{2}(\delta_{ij} - \vec{n}_i \vec{n}_j)(\delta_{km} - \vec{n}_k \vec{n}_m), \tag{1.33}$$

$$Q^{km}(t) := \int T_{00} \left(x'^k x'^m - \frac{1}{3} \delta^{km} \sum_n x'^n x'^n \right) \mathrm{d}V' \tag{1.34}$$

1.5 电磁—引力对比

$$A_\mu(t, \vec{r}) = \frac{\mu_0}{4\pi} \int \frac{J_\mu(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} \mathrm{d}V' \tag{1.35}$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4G \int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} \mathrm{d}V' \tag{1.36}$$

$$A_\mu(t, \vec{r}) = \frac{1}{\sqrt{2\pi}} \int \hat{A}_\mu(\omega, \vec{r}) e^{-i\omega t} \mathrm{d}t \tag{1.37}$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = \frac{1}{\sqrt{2\pi}} \int \hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) e^{-i\omega t} \mathrm{d}t \tag{1.38}$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\hat{J}_\mu(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega|\vec{r} - \vec{r}'|} \mathrm{d}V' \tag{1.39}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G \int \frac{\hat{T}_{\mu\nu}(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega|\vec{r} - \vec{r}'|} \mathrm{d}V' \tag{1.40}$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_\mu(\omega, \vec{r}') e^{-i\omega(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}')} \mathrm{d}V' \tag{1.41}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{\mu\nu}(\omega, \vec{r}') e^{-i\omega(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}')} \mathrm{d}V' \tag{1.42}$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_\mu(\omega, \vec{r}') \left[1 - i\omega(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}') - \ldots \right] \mathrm{d}V' \tag{1.43}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{\mu\nu}(\omega, \vec{r}') \left[1 - i\omega(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}') - \ldots \right] \mathrm{d}V' \tag{1.44}$$

1.5.1 电偶极—引力对比

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{J}_i \,\mathrm{d}V' \tag{1.45}$$

$$\hat{\bar{h}}_{ij} = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{T}_{ij}\,\mathrm{d}V' \tag{1.46}$$

$$\int \hat{J}_i \,\mathrm{d}V' = -i\omega\int \hat{J}_0 x'^i \,\mathrm{d}V' \tag{1.47}$$

$$\int \hat{T}_{ij}\,\mathrm{d}V' = -\frac{\omega^2}{2}\int \hat{T}_{00}\, x'^i x'^j \,\mathrm{d}V' \tag{1.48}$$

$$p_i = \int \hat{J}_0 x'^i \,\mathrm{d}V' \tag{1.49}$$

$$q_{ij} = \int \hat{T}_{00}\, x'^i x'^j \,\mathrm{d}V' \tag{1.50}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega p_i) \tag{1.51}$$

$$\hat{\bar{h}}_{ij} = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-\frac{\omega^2}{2}q_{ij}) \tag{1.52}$$

$$A_i = \frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|}\frac{\mathrm{d}}{\mathrm{d}t}p_i(t-|\vec{r}|) \tag{1.53}$$

$$\bar{h}_{ij} = 4G\frac{1}{|\vec{r}|}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}q_{ij}(t-|\vec{r}|) \tag{1.54}$$

1.5.2 电四极—引力对比

$$\hat{A}_i(\omega,\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\;\;(-i\omega)\int \hat{J}_i(\omega,\vec{r}')(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}')\,\mathrm{d}V' \tag{1.55}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega)\int \hat{J}'_i n^j x'_j \,\mathrm{d}V' \tag{1.56}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega)\int n^j x'_j \hat{J}'_i \,\mathrm{d}V' \tag{1.57}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega)n^j\left[\int x'_{(j}\hat{J}'_{i)}\,\mathrm{d}V'\right] \tag{1.58}$$

$$\begin{aligned}
\int x'_{(j}\hat{J}'_{i)}\,\mathrm{d}V' &= \frac{1}{2}\int(\hat{J}'_j x'_i + \hat{J}'_i x'_j)\,\mathrm{d}V' &(1.59)\\
&= \frac{1}{2}\int\sum_k(\hat{J}'_k x'^i\frac{\partial x'^j}{\partial x'^k} + \hat{J}'_k\frac{\partial x'^i}{\partial x'^k}x'^j)\,\mathrm{d}V' &(1.60)\\
&= \frac{1}{2}\sum_k\left[\int\frac{\partial}{\partial x'^k}(\hat{J}'_k x'^i x'^j)\,\mathrm{d}V' - \int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V'\right] &(1.61)\\
&= \frac{1}{2}\sum_k\int\partial'_k\,(\hat{J}'_k x'^i x'^j)\,\mathrm{d}V' - \frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' &(1.62)\\
&= \frac{1}{2}\sum_k\int\hat{J}'_k x'^i x'^j\,\mathrm{d}S' - \frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' &(1.63)\\
&= -\frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' &(1.64)\\
&= -\frac{1}{2}\int(\sum_k\partial'_k\hat{J}'_k)x'^i x'^j\,\mathrm{d}V' &(1.65)\\
&= -\frac{1}{2}\int(\partial_0\hat{J}'_0)x'^i x'^j\,\mathrm{d}V' &(1.66)\\
&= -\frac{i\omega}{2}\int\hat{J}'_0 x'^i x'^j\,\mathrm{d}V' &(1.67)
\end{aligned}$$

$$D_{ij} = \int\hat{J}'_0\,x'^i x'^j\,\mathrm{d}V' \tag{1.68}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-\frac{\omega^2}{2}n^j D_{ij}) \tag{1.69}$$

$$A_i = \frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|}n^j\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}D_{ij}(t-|\vec{r}|) \tag{1.70}$$

1.6 常数变易

1.6.1 formula

$$\bar{h}_{ij} = \frac{4G}{c^4}\frac{1}{|\vec{r}|}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}q_{ij}(t-\frac{|\vec{r}|}{c}) \tag{1.71}$$

$$A_{(\mathrm{E2})_i} = \frac{\mu}{4\pi}\frac{1}{|\vec{r}|}\frac{\mathrm{d}}{\mathrm{d}t}p_i(t - \frac{|\vec{r}|}{c}) \tag{1.72}$$

$$A_{(\mathrm{E4})_i} = \frac{\mu}{4\pi}\frac{1}{|\vec{r}|}n^j\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}D_{ij}(t - \frac{|\vec{r}|}{c}) \tag{1.73}$$

1.6.2 definition

$$\frac{4G}{c^4} := \frac{\mu_{\mathrm{G}}}{4\pi} \tag{1.74}$$

$$c := \frac{1}{\sqrt{\epsilon_{\mathrm{G}}\mu_{\mathrm{G}}}} \tag{1.75}$$

$$G := \frac{\mu_{\mathrm{G}}}{16\pi\epsilon_{\mathrm{G}}^2\mu_{\mathrm{G}}^2} \tag{1.76}$$

1.6.3 energy

$$T_{ab} \propto F_{ac}F_b{}^c - \frac{1}{4}\eta_{ab}F_{cd}F^{cd} \tag{1.77}$$

$$T_{0i} \propto F_{0c}F_i{}^c - \frac{1}{4}\eta_{0i}F_{cd}F^{cd} = F_{0c}F_i{}^c \tag{1.78}$$

$$F_{ab} = \partial_a A_b - \partial_b A_a \tag{1.79}$$

$$F_a{}^b = \partial_a A^b - \partial^b A_a \tag{1.80}$$

$$F^{ab} = \partial^a A^b - \partial^b A^a \tag{1.81}$$

$$T_{0i} \propto (\partial_0 A_c - \partial_c A_0)(\partial_i A^c - \partial^c A_i) \tag{1.82}$$

$$A_0 = 0 \tag{1.83}$$

$$T_{0i} \propto \partial_0 A_j(\partial_i A^j - \partial^j A_i) \tag{1.84}$$

$$A_i(t, \vec{r}) = \Re[A_i e^{-i(\omega t - \vec{k}\cdot\vec{r})}] \tag{1.85}$$

$$\partial_0 A_j = \Re[-i\omega A_i e^{-i(\omega t - \vec{k}\cdot\vec{r})}] \tag{1.86}$$

$$\partial_i A^j = \Re[ik_i A^j e^{-i(\omega t - \vec{k}\cdot\vec{r})}] \tag{1.87}$$

$$\partial^j A_i = \Re[ik^j A_i e^{-i(\omega t - \vec{k}\cdot\vec{r})}] \tag{1.88}$$

$$\bar{T}_{0i} \propto (-i\omega A_j)[ik_i A^j - ik^j A_i] = \omega[|\vec{A}|^2\vec{k} - (\vec{k}\cdot\vec{A})\vec{A}] \tag{1.89}$$

$$\partial^a A_a = 0 \tag{1.90}$$

$$\partial^i A_i = 0 \tag{1.91}$$

$$\partial^i A_i = \Re[ik^i A_i e^{-i(\omega t - \vec{k}\cdot\vec{r})}] \tag{1.92}$$

$$k^i A_i = \vec{k}\cdot\vec{A} = 0 \tag{1.93}$$

$$S_{\mathrm{EM}} \propto \left|\bar{T}_{0i}\right| \propto \omega k A^2 = \omega^2 A^2 \tag{1.94}$$

$$S_{\mathrm{EM}} : [M][T]^{-3} \tag{1.95}$$

$$\omega : [T]^{-1} \tag{1.96}$$

$$A : [M][L][T]^{-2}[I]^{-1} \tag{1.97}$$

$$c : [L][T]^{-1} \tag{1.98}$$

$$\mu : [M][L][T]^{-2}[I]^{-2} \tag{1.99}$$

$$S_{\mathrm{EM}} \propto \frac{\omega^2 A^2}{c\mu} \tag{1.100}$$

$$S_{\mathrm{G}} \propto \dot{h}^2 \propto \omega^2 h^2 \tag{1.101}$$

$$S_{\mathrm{G}} : [M][T]^{-3} \tag{1.102}$$

$$\omega : [T]^{-1} \tag{1.103}$$

$$c : [L][T]^{-1} \tag{1.104}$$

$$G : [L]^3[M]^{-1}[T]^{-2} \tag{1.105}$$

$$S_{\mathrm{G}} \propto \frac{c^3\omega^2 h^2}{G} \propto \frac{\omega^2 h^2}{c\mu_{\mathrm{G}}} \tag{1.106}$$

1.6.4 experiment

双星系统引力辐射本为

$$h = \frac{\mathcal{M}[\pi\mathcal{M}F(t)]^{2/3}}{r} Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi F(t)\,\mathrm{d}t] \tag{1.107}$$

设双星系统常量 c^*, μ_{G}^*, 则[1]

$$h = \frac{\mathcal{M}[\pi\mathcal{M}F(t)]^{2/3}}{r/c^*} Q(\theta,\phi,\psi,\iota)\cos[\int 2\pi F(t)\,\mathrm{d}t] \tag{1.108}$$

[1] $\mathcal{M}$ 和 c^*, μ_{G}^* 简并, 所以可以笼统地仍记作 $\mathcal{M}$.

在双星系统参考系中, $S_{\mathrm{G}} r^2 \propto \omega^2 h^2 r^2 / c\mu_{\mathrm{G}}$ 守恒, ω 不变, 所以实验室处

$$h = \sqrt{\frac{c^* \mu_{\mathrm{G}}^*}{c\mu_{\mathrm{G}}}} \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{d/c^*} Q(\theta, \phi, \psi, \iota) \cos[\int 2\pi F(t)\, \mathrm{d}t] \tag{1.109}$$

实验室因红移, 实际测得

$$h = \sqrt{\frac{c^* \mu_{\mathrm{G}}^*}{c\mu_{\mathrm{G}}}} \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{d/c^*} Q(\theta, \phi, \psi, \iota) \cos[\int 2\pi \frac{F(t)}{1+z}\, \mathrm{d}t] \tag{1.110}$$

记 $F_{\mathrm{obs}}(t) = F(t)/(1+z)$, $\mathcal{M}_{\mathrm{obs}} = \mathcal{M}(1+z)$, 光度距离 $d_{\mathrm{L}} = d_{\mathrm{L}}(1+z)$, 则

$$h = \sqrt{\frac{c^* \mu_{\mathrm{G}}^*}{c\mu_{\mathrm{G}}}} \frac{\mathcal{M}_{\mathrm{obs}}[\pi \mathcal{M}_{\mathrm{obs}} F_{\mathrm{obs}}(t)]^{2/3}}{d_{\mathrm{L}}/c^*} Q(\theta, \phi, \psi, \iota) \cos[\int 2\pi F_{\mathrm{obs}}(t)\, \mathrm{d}t] \tag{1.111}$$

则

$$h = \sqrt{\frac{c^{*3} \mu_{\mathrm{G}}^*}{c^3\mu_{\mathrm{G}}}} \frac{\mathcal{M}_{\mathrm{obs}}[\pi \mathcal{M}_{\mathrm{obs}} F_{\mathrm{obs}}(t)]^{2/3}}{d_{\mathrm{L}}/c} Q(\theta, \phi, \psi, \iota) \cos[\int 2\pi F_{\mathrm{obs}}(t)\, \mathrm{d}t] \tag{1.112}$$

用独立手段测得 d_{L}, 引力波测距测得 d_{L}^*, 则

$$\frac{1}{d_{\mathrm{L}}^*} = \sqrt{\frac{c^{*3} \mu_{\mathrm{G}}^*}{c^3\mu_{\mathrm{G}}}} \frac{1}{d_{\mathrm{L}}} \tag{1.113}$$

第二章　双星系统

2.1　基本公式

$$\mathcal{M} := \mu^{3/5} M^{2/5} \tag{2.1}$$

$$h_+ = \frac{4\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}\frac{1+\cos^2\iota}{2}\cos\Phi(t) \tag{2.2}$$

$$h_\times = \frac{4\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}\cos\iota\sin\Phi(t) \tag{2.3}$$

$$h = F_+h_+ + F_\times h_\times \tag{2.4}$$

2.2　Post-Newtonian Approximation

2PN: [3, 6]

2.3　Stationary Phase Approximation

[6], if $\zeta(t)$ varies slowly near $t = t_0$ where the phase has a stationary point: $\phi'(t_0) = 0$

$$\begin{aligned}
\int \zeta(t)e^{i\phi(t;f)}\,\mathrm{d}t &= \int \zeta(t)e^{i[\phi(t_0)+\phi'(t_0)(t-t_0)+\frac{1}{2}\phi''(t_0)(t-t_0)^2+\ldots]}\,\mathrm{d}t && (2.5)\\
&\simeq e^{i\phi(t_0)}\int \zeta(t)e^{i[\frac{1}{2}\phi''(t_0)(t-t_0)^2]}\,\mathrm{d}t && (2.6)\\
&\simeq e^{i\phi(t_0)}\int \zeta(t_0)e^{\frac{-\sqrt{-i\phi''(t_0)}^2(t-t_0)^2}{2}}\,\mathrm{d}t && (2.7)\\
&= \frac{\sqrt{2\pi}}{\sqrt{-i\phi''(t_0)}}\zeta(t_0)e^{i\phi(t_0)}. && (2.8)
\end{aligned}$$

$$
\begin{aligned}
h &= \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\cos\Phi(t) && (2.9)\\
&= \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}[e^{i\Phi(t)}+e^{-i\Phi(t)}] && (2.10)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h}(f) &= \int h(t)e^{i2\pi ft}\,\mathrm{d}t && (2.11)\\
&= \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}[e^{i\Phi(t)}+e^{-i\Phi(t)}]e^{i2\pi ft}\,\mathrm{d}t && (2.12)\\
&= \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}\{e^{i[2\pi ft+\Phi(t)]}+e^{i[2\pi ft-\Phi(t)]}\}\,\mathrm{d}t && (2.13)\\
&\simeq \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}e^{i[2\pi ft-\Phi(t)]}\,\mathrm{d}t && (2.14)\\
&= \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F]^{2/3}Q\frac{1}{2}e^{i[2\pi ft(F)-\Phi(F)]}\frac{\mathrm{d}t}{\mathrm{d}F}\,\mathrm{d}F && (2.15)\\
&\simeq \frac{\sqrt{2\pi}}{\sqrt{-i[2\pi ft(F)-\Phi(F)]''_{F=f}}} && (2.16)\\
&\left[\frac{\mathcal{M}}{D}(\pi\mathcal{M}F)^{2/3}Q\frac{1}{2}\frac{\mathrm{d}t}{\mathrm{d}F}\right]_{F=f} e^{i[2\pi ft(f)-\Phi(f)]} && (2.17)\\
&\simeq \frac{\sqrt{2\pi}}{\sqrt{-i\left\{2\pi f\left[-\frac{5}{256}\mathcal{M}(\pi\mathcal{M}F)^{-8/3}\right]-\left[\frac{1}{16}(\pi\mathcal{M}F)^{-5/3}\right]\right\}''_{F=f}}} && (2.18)\\
&\left\{\frac{\mathcal{M}}{D}(\pi\mathcal{M}F)^{2/3}Q\frac{1}{2}\left[\frac{5\pi\mathcal{M}^2}{96}(\pi\mathcal{M}F)^{-11/3}\right]\right\}_{F=f} e^{i[2\pi ft(f)-\Phi(f)]} && (2.19)\\
&= \frac{\sqrt{30}}{48\pi^{2/3}}\frac{\mathcal{M}^{5/6}Q}{D}f^{-7/6}e^{i[2\pi ft(f)-\Phi(f)-\frac{\pi}{4}]} \quad (\texttt{pnspa.py}) && (2.20)
\end{aligned}
$$

或 [1], $h(t)=2A(t)\cos\phi(t)$, $\mathrm{d}\ln A/\mathrm{d}t \ll \mathrm{d}\phi/\mathrm{d}t$ 且 $|\mathrm{d}^2\phi/\mathrm{d}t^2| \ll (\mathrm{d}\phi/\mathrm{d}t)^2$.

第三章　电磁引力

[5].

3.1　时空张量转化为空间张量

$$h_{ab} := g_{ab} + Z_a Z_b. \tag{3.1}$$

$$h_a{}^b = \delta_a{}^b + Z_a Z^b. \tag{3.2}$$

$$Z^a h_{ab} = 0. \tag{3.3}$$

$$V_{\langle a \rangle} := h_a{}^b V_b. \tag{3.4}$$

$$Z^a V_{\langle a \rangle} = 0. \tag{3.5}$$

$$T_{\langle ab \rangle} := h_{(a}{}^c h_{b)}{}^d T_{cd} - \frac{1}{3} h_{cd} T^{cd} h_{ab}. \tag{3.6}$$

$$Z^a (h_a{}^c h_b{}^d T_{cd}) = 0. \tag{3.7}$$

$$Z^a (h_b{}^c h_a{}^d T_{cd}) = 0. \tag{3.8}$$

$$Z^a (h_{(a}{}^c h_{b)}{}^d T_{cd}) = 0. \tag{3.9}$$

$$Z^a (h_{cd} T^{cd} h_{ab}) = 0. \tag{3.10}$$

$$Z^a T_{\langle ab \rangle} = 0. \tag{3.11}$$

$$T_{(\langle ab \rangle)} = T_{\langle ab \rangle}. \tag{3.12}$$

$$h^{ab} T_{\langle ab \rangle} = 0. \tag{3.13}$$

$$\varepsilon_{abc} := \varepsilon_{abcd} Z^d. \tag{3.14}$$

$$\varepsilon_{0123} := -\sqrt{|g|}. \tag{3.15}$$

$$T_a := \frac{1}{2} \varepsilon_{abc} T^{[bc]}. \tag{3.16}$$

$$[U, V]_a := \varepsilon_{abc} U^b V^c. \tag{3.17}$$

$$[S, T]_a := \varepsilon_{abc} g_{de} S^{bd} T^{ce}. \tag{3.18}$$

$$\mathrm{D}_t T^{a\dots}{}_{b\dots} := Z^c \nabla_c T^{a\dots}{}_{b\dots}. \tag{3.19}$$

$${}^3\nabla_a T^{b\dots}{}_{c\dots} := h_a{}^p h^b{}_q \dots h_c{}^r \dots \nabla_p T^{q\dots}{}_{r\dots}. \tag{3.20}$$

$$(\operatorname{div} V) := {}^3\nabla^a V_a. \tag{3.21}$$

$$(\operatorname{curl} V)_a := \varepsilon_{bca} {}^3\nabla^b V^c. \tag{3.22}$$

$$(\operatorname{div} T)_a := {}^3\nabla^b T_{ab}. \tag{3.23}$$

$$(\operatorname{curl} T)_{ab} := \varepsilon_{cd(a} {}^3\nabla^c g_{b)e} T^{ed}. \tag{3.24}$$

3.2 电磁空间矢量

$${}^*F_{ab} := \frac{1}{2} \varepsilon_{abcd} F^{cd} \tag{3.25}$$

$$E_a := F_{ab} Z^b = E_{\langle a \rangle}. \tag{3.26}$$

$$B_a := {}^*F_{ab} Z^b = B_{\langle a \rangle}. \tag{3.27}$$

$$\rho = -Z^a J_a. \tag{3.28}$$

$$j_a = h_a{}^b J_b. \tag{3.29}$$

$$\nabla_{[a} F_{bc]} = 0. \tag{3.30}$$

$$\nabla^a F_{ab} = \mu J_b. \tag{3.31}$$

$$(\operatorname{div} E) = \mu\rho - \ldots. \tag{3.32}$$

$$(\operatorname{div} B) = + \ldots. \tag{3.33}$$

$$(\operatorname{curl} E)_a + \cdots = -\mathrm{D}_t B_{\langle a\rangle} - \ldots. \tag{3.34}$$

$$(\operatorname{curl} B)_a + \cdots = \mu j_a + \mathrm{D}_t E_{\langle a\rangle} + \ldots. \tag{3.35}$$

3.3 引力空间张量

$$^{*}C_{abcd} := \frac{1}{2}\varepsilon_{abef}C^{ef}{}_{cd}. \tag{3.36}$$

$$E_{ab} := C_{acbd}Z^cZ^d = E_{\langle ab\rangle}. \tag{3.37}$$

$$B_{ab} := {}^{*}C_{acbd}Z^cZ^d = B_{\langle ab\rangle}. \tag{3.38}$$

$$(\operatorname{div} E)_a = \kappa\frac{1}{3}{}^3\nabla_a\rho - \ldots. \tag{3.39}$$

$$(\operatorname{div} B)_a = \kappa(\rho + p)\omega_a + \ldots. \tag{3.40}$$

$$(\operatorname{curl} E)_{ab} + \cdots = -\mathrm{D}_t B_{\langle ab\rangle} - \ldots. \tag{3.41}$$

$$(\operatorname{curl} B)_{ab} + \cdots = \kappa\frac{1}{2}(\rho + p)\sigma_{ab} + \mathrm{D}_t E_{\langle ab\rangle} + \ldots. \tag{3.42}$$

第四章　Fisher 矩阵法

[4], 论证见FinnNotes.

4.1　判断观测数据中有无信号

$\Omega = A_0 \cup A_m$, A_0 为事件 "无信号", A_m 为事件 "有信号", 测量结果为 $G_t(\omega)$, 噪声 $N_t(\omega)$, 信号 $M_t(\omega)$,

$$G_t(\omega) = \begin{cases} N_t(\omega) & \omega \in A_0, \\ N_t(\omega) + M_t(\omega) & \omega \in A_m, \end{cases} \tag{4.1}$$

实测得 g_t, $A_g := \{\omega : G_t(\omega) = g_t\}$, A_g 为事件为 "测得 g_t", 求 $\mathbf{P}(A_m|A_g)$. 另认为信号依赖于参数 $\vec{\mu}$, $A_m = \cup A_{\vec{\mu}}$, $A_{\vec{\mu}}$ 为事件 "有信号且参数为 μ", $p(\vec{\mu}) := p(A_{\vec{\mu}}|A_m)$

$$\mathbf{P}(A_m|A_g) = \frac{\Lambda}{\Lambda + \mathbf{P}(A_0)/\mathbf{P}(A_m)}, \tag{4.2}$$

$$\Lambda := \int \mathrm{d}\vec{\mu}\, \lambda(\vec{\mu}), \tag{4.3}$$

$$\lambda(\vec{\mu}) := p(\vec{\mu}) \exp[2\left\langle g(t)|m_{\vec{\mu}}(t)\right\rangle - \left\langle m_{\vec{\mu}}(t), m_{\vec{\mu}}(t)\right\rangle] \tag{4.4}$$

$$\langle \xi(t), \zeta(t)\rangle := \int \mathrm{d}f\, \frac{\tilde{\xi}(f)\tilde{\zeta}(f)^*}{S_n(|f|)}, \tag{4.5}$$

$$\tilde{q}(f) := \int \mathrm{d}t\, q(t) \exp[2\pi i f t]. \tag{4.6}$$

4.2 认定有信号后参数估计 (MLE)

实测得 g_t 且认定有信号, 事件 $A_g \cap A_m$, 求使 $p(A_{\vec{\mu}}|A_g \cap A_m)$ 最大的 $\vec{\mu}$, 记作 $\hat{\vec{\mu}}$.

$$p(A_{\vec{\mu}}|A_g) = \frac{\lambda(\vec{\mu})}{\Lambda + \mathbf{P}(A_0)/\mathbf{P}(A_m)}, \tag{4.7}$$

$$p(A_{\vec{\mu}}|A_g \cap A_m) = \frac{\lambda(\vec{\mu})}{\Lambda}, \tag{4.8}$$

$$\frac{\partial \ln p(\vec{\mu})}{\partial \vec{\mu}}|_{\vec{\mu}=\hat{\vec{\mu}}} + 2\left\langle \frac{\partial m_{\vec{\mu}}}{\partial \vec{\mu}}|_{\vec{\mu}=\hat{\vec{\mu}}}(t), g(t) - m_{\vec{\mu}}|_{\vec{\mu}=\hat{\vec{\mu}}}(t) \right\rangle = 0. \tag{4.9}$$

4.3 灵敏度

若由 g_t 求得 MLE 为 $\hat{\vec{\mu}}$, 则记 $g \Rightarrow \hat{\vec{\mu}}$, $A_{\hat{\vec{\mu}}} := \cup_{g\Rightarrow\hat{\vec{\mu}}} A_g$, $A_{\hat{\vec{\mu}}}$ 为事件 "测得 MLE 为 $\hat{\vec{\mu}}$", $A_{\tilde{\vec{\mu}}}$ 为事件 "有信号且参数为 $\tilde{\vec{\mu}}$", 求 $p(A_{\tilde{\vec{\mu}}}|A_{\hat{\vec{\mu}}})$. 高 SNR, $\tilde{\vec{\mu}} := \hat{\vec{\mu}} + \delta\vec{\mu}$,

$$p(A_{\hat{\vec{\mu}}+\delta\vec{\mu}}|A_{\hat{\vec{\mu}}}) = \frac{\exp[-\frac{1}{2}\sum \mathcal{C}_{ij}^{-1}(\delta\mu_i - \overline{\delta\mu_i})(\delta\mu_j - \overline{\delta\mu_j})]}{[(2\pi)^N \det(\mathcal{C}_{ij})^{1/2}]}, \tag{4.10}$$

$$\mathcal{C}_{ij}^{-1} = 2\left\langle \frac{\partial m_{\vec{\mu}}}{\partial \mu_i}|_{\vec{\mu}=\hat{\vec{\mu}}}(t), \frac{\partial m_{\vec{\mu}}}{\partial \mu_j}|_{\vec{\mu}=\hat{\vec{\mu}}}(t) \right\rangle, \tag{4.11}$$

$$\overline{\delta\mu_i} = -\sum \mathcal{C}_{ij} \frac{\partial \ln p(\vec{\mu})}{\partial \mu_j}|_{\vec{\mu}=\hat{\vec{\mu}}}. \tag{4.12}$$

4.4 认定有信号后参数估计 (分布)

[6],

$$p(A_{\vec{\mu}}|A_g \cap A_m) \propto p^{(0)}(\vec{\mu}) \exp[-\frac{1}{2}\langle m_{\vec{\mu}}(t) - g(t)|m_{\vec{\mu}}(t) - g(t)\rangle], \tag{4.13}$$

$$\langle \xi(t)|\zeta(t)\rangle := 2\int_0^\infty \frac{\tilde{\xi}(f)^*\tilde{\zeta}(f) + \tilde{\xi}(f)\tilde{\zeta}(f)^*}{S_n(f)}\,\mathrm{d}f, \tag{4.14}$$

$$\tilde{q}(f) := \int_{-\infty}^{\infty} q(t)e^{2\pi i f t}\,\mathrm{d}t, \tag{4.15}$$

$$\frac{\partial}{\partial \vec{\mu}}\langle m_{\vec{\mu}}(t) - g(t)|m_{\vec{\mu}}(t) - g(t)\rangle = \left\langle \frac{\partial}{\partial \vec{\mu}} m_{\vec{\mu}}(t)|m_{\vec{\mu}}(t) - g(t) \right\rangle, \tag{4.16}$$

μ^a 估计为 $\hat{\mu}^a$, 高 SNR,

$$\left\langle m_{,a}(t;\mu^b)|g(t) - m(t;\mu^b)\right\rangle|_{\mu^b=\hat{\mu}^b} = 0, \tag{4.17}$$

$$\Gamma_{ab} := \langle m_{,a}(t)|m_{,b}(t)\rangle\,, \tag{4.18}$$

$$p(A_{\mu^a}|A_g \cap A_m) \propto p^{(0)}(\mu^a)\exp[-\frac{1}{2}\Gamma_{ab}(\mu^a - \hat{\mu}^a)(\mu^b - \hat{\mu}^b)], \tag{4.19}$$

$$p^{(0)}(\mu^a) :\propto \exp[-\frac{1}{2}\Gamma^{(0)}_{ab}(\mu^a - \bar{\mu}^a)(\mu^b - \bar{\mu}^b)]. \tag{4.20}$$

参考文献

[1] K. G. Arun, B. R. Iyer, B. S. Sathyaprakash, and P. A. Sundararajan. Parameter estimation of inspiralling compact binaries using 3.5 post-newtonian gravitational wave phasing: The nonspinning case. *Phys. Rev. D*, 71:084008, Apr 2005.

[2] L. Blanchet. Gravitational radiation from relativistic sources. In J.-A. Marck and J.-P. Lasota, editors, *Relativistic Gravitation and Gravitational Radiation*, page 33, Jan. 1997.

[3] L. Blanchet, T. Damour, B. R. Iyer, C. M. Will, and A. G. Wiseman. Gravitational-radiation damping of compact binary systems to second post-newtonian order. *Phys. Rev. Lett.*, 74:3515–3518, May 1995.

[4] L. S. Finn. Detection, measurement, and gravitational radiation. *Phys. Rev. D*, 46:5236–5249, Dec 1992.

[5] R. Maartens. Nonlinear gravito-electromagnetism. *General Relativity and Gravitation*, 40(6):1203–1217, June 2008.

[6] E. Poisson and C. M. Will. Gravitational waves from inspiraling compact binaries: Parameter estimation using second-post-newtonian waveforms. *Phys. Rev. D*, 52:848–855, Jul 1995.

[7] B. S. Sathyaprakash and B. F. Schutz. Physics, astrophysics and cosmology with gravitational waves. *Living Reviews in Relativity*, 12(1):2, Dec. 2009.

[8] R. M. Wald. *General Relativity*. University of Chicago Pr., 1984.

[9] 王运永. 引力波探测. 科学出版社, 2020.