引力波天文学笔记

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目录

第一章　引力波

1.1　Linearized Gravity

[8]. 流形 $\mathbb{R}^4$. 任意坐标系 $\{x^\mu\}$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}s + \mathrm{O}(s^2)$, $g^{\mu\nu} = ?^{\mu\nu} + ??^{\mu\nu}s + \mathrm{O}(s^2)$, 则 $\delta^\mu{}_\lambda = ?^{\mu\nu}\eta_{\nu\lambda} + ?^{\mu\nu}\gamma_{\nu\lambda}s + ??^{\mu\nu}\eta_{\nu\lambda}s + \mathrm{O}(s^2)$, 所以 $?^{\mu\nu} = \eta^{\mu\nu}$, $??^{\mu\nu} = ??^{\mu\sigma}\delta_\sigma{}^\nu = ??^{\mu\sigma}\eta_{\sigma\lambda}\eta^{\lambda\nu} = -?^{\mu\sigma}\gamma_{\sigma\lambda}\eta^{\lambda\nu} = -\eta^{\mu\sigma}\gamma_{\sigma\lambda}\eta^{\lambda\nu} = -\gamma^{\mu\nu}$, 所以 $g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu}s + \mathrm{O}(s^2) = \eta^{\mu\nu} - h^{\mu\nu} + \mathrm{O}(s^2)$.

$$R_{\mu\nu\lambda\sigma} = \partial_\sigma\partial_{[\mu}h_{\lambda]\nu} - \partial_\nu\partial_{[\mu}h_{\lambda]\sigma} + \mathrm{O}(s^2). \tag{1.1}$$

$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\sigma}h_{\lambda\sigma} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$.

$$-\frac{1}{2}\partial^\lambda\partial_\lambda\bar{h}_{\mu\nu} + \partial^\lambda\partial_{(\mu}\bar{h}_{\nu)\lambda} - \frac{1}{2}\eta_{\mu\nu}\partial^\lambda\partial^\sigma\bar{h}_{\lambda\sigma} + \mathrm{O}(s^2) = 8\pi T_{\mu\nu}. \tag{1.2}$$

存在 $\{x^\mu\}$, 使得 $\partial^\nu\bar{h}_{\mu\nu} + \mathrm{O}(s^2) = 0$ (Lorentz gauge). 设 $x'^\mu = x^\mu - \xi^\mu = x^\mu - \zeta^\mu s - \mathrm{O}(s^2)$, 则 $\frac{\partial ?}{\partial x'^\mu} = \frac{\partial ?}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial x'^\mu} = \frac{\partial ?}{\partial x^\lambda}(\delta^\lambda{}_\mu + \frac{\partial \xi^\lambda}{\partial x'^\mu}) = \frac{\partial ?}{\partial x^\nu} + \mathrm{O}(s^2)$, $g'_{\mu\nu} = g_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} = g_{\lambda\sigma}(\delta^\lambda{}_\mu + \frac{\partial \xi^\lambda}{\partial x'^\mu})(\delta^\sigma{}_\nu + \frac{\partial \xi^\sigma}{\partial x'^\nu}) = g_{\mu\nu} + g_{\mu\sigma}\frac{\partial \xi^\sigma}{\partial x'^\nu} + g_{\lambda\nu}\frac{\partial \xi^\lambda}{\partial x'^\mu} = g_{\mu\nu} + (\eta_{\mu\sigma} + \mathrm{O}(s))(\frac{\partial \xi^\sigma}{\partial x^\nu} + \mathrm{O}(s^2)) + (\eta_{\lambda\nu} + \mathrm{O}(s))(\frac{\partial \xi^\lambda}{\partial x^\mu} + \mathrm{O}(s^2)) = g_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \mathrm{O}(s^2)$, 所以 $h'_{\mu\nu} = g'_{\mu\nu} - \eta_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \mathrm{O}(s^2) = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + \mathrm{O}(s^2)$, 因此存在 ξ^μ, 使得 $\partial'^\nu\bar{h}'_{\mu\nu} + \mathrm{O}(s^2) = 0$ [8]. 令 $\{x^\mu\}$ 满足 $\partial^\nu\bar{h}_{\mu\nu} + \mathrm{O}(s^2) = 0$, 则

$$\partial^\lambda\partial_\lambda\bar{h}_{\mu\nu} + \mathrm{O}(s^2) = -16\pi T_{\mu\nu}. \tag{1.3}$$

略去 $\mathrm{O}(s^2)$ 条件: $h_{\mu\nu}$, $\partial_\lambda h_{\mu\nu}$...小.

1.2　Radiation Gauge

[8]. 存在 $\{x^\mu\}$, 使得 “无源处” $h + \mathrm{O}(s^2) = 0$ (TT gauge [9]) 且 $h_{0\mu} + \mathrm{O}(s^2) = 0$.

1.3 Quadrupole Approximation

[8]. 下略 $\mathrm{O}(s^2)$. 由(1.3)得

$$\bar{h}_{\mu\nu}(t,\vec{r}) = 4\int \frac{T_{\mu\nu}(t-|\vec{r}-\vec{r}'|,\vec{r}')}{|\vec{r}-\vec{r}'|}\,\mathrm{d}V'. \tag{1.4}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega,\vec{r}) := \frac{1}{\sqrt{2\pi}}\int \bar{h}_{\mu\nu}(t,\vec{r})e^{i\omega t}\mathrm{d}t \tag{1.5}$$

$$= 4\int \frac{\hat{T}_{\mu\nu}(\omega,\vec{r}')}{|\vec{r}-\vec{r}'|}e^{i\omega|\vec{r}-\vec{r}'|}\,\mathrm{d}V'. \tag{1.6}$$

由 $\partial^\nu \bar{h}_{\mu\nu}=0$,

$$-i\omega\hat{\bar{h}}_{0\mu} = \sum_i \frac{\partial\hat{\bar{h}}_{i\mu}}{\partial x^i}. \tag{1.7}$$

$|\vec{r}| \gg |\vec{r}'|$ 且 $\omega \ll 1/|\vec{r}'|$,

$$\hat{\bar{h}}_{ij}(\omega,\vec{r}) = 4\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{T}_{ij}(\omega,\vec{r}')\,\mathrm{d}V'. \tag{1.8}$$

$$\int \hat{T}_{ij}\,\mathrm{d}V' = \int \sum_k (\hat{T}_{kj}\frac{\partial x'^i}{\partial x'^k})\,\mathrm{d}V' \tag{1.9}$$

$$= \sum_k \left[\int \frac{\partial}{\partial x'^k}(\hat{T}_{kj}x'^i)\,\mathrm{d}V' - \int \frac{\partial \hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V'\right] \tag{1.10}$$

$$= \sum_k \int \partial'_k\,(\hat{T}_{kj}x'^i)\,\mathrm{d}V' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V' \tag{1.11}$$

$$= \int \hat{T}_{kj}x'^i\,\mathrm{d}S' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V' \tag{1.12}$$

$$= -\sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k}x'^i\,\mathrm{d}V' \tag{1.13}$$

$$= -\int (\sum_k \partial'_k\hat{T}_{kj})x'^i\,\mathrm{d}V' \tag{1.14}$$

$$= -\int (\partial_0\hat{T}_{0j})x'^i\,\mathrm{d}V' \tag{1.15}$$

$$= -i\omega\int \hat{T}_{0j}x'^i\,\mathrm{d}V' \tag{1.16}$$

$$= \int \hat{T}_{(ij)}\,\mathrm{d}V' \tag{1.17}$$

$$= -i\omega \int \hat{T}_{0(j} x'^{i)} \, \mathrm{d}V' \tag{1.18}$$

$$= -\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) \, \mathrm{d}V', \tag{1.19}$$

$$-\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) \, \mathrm{d}V' = -\frac{i\omega}{2} \int \sum_k (\hat{T}_{0k} x'^i \frac{\partial x'^j}{\partial x'^k} + \hat{T}_{0k} \frac{\partial x'^i}{\partial x'^k} x'^j) \, \mathrm{d}V' \tag{1.20}$$

$$= -\frac{i\omega}{2} \sum_k \left[\int \frac{\partial}{\partial x'^k} (\hat{T}_{0k} x'^i x'^j) \, \mathrm{d}V' - \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, \mathrm{d}V' \right] \tag{1.21}$$

$$= -\frac{i\omega}{2} \sum_k \int \partial'_k \, (\hat{T}_{0k} x'^i x'^j) \, \mathrm{d}V' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, \mathrm{d}V' \tag{1.22}$$

$$= -\frac{i\omega}{2} \sum_k \int \hat{T}_{0k} x'^i x'^j \, \mathrm{d}S' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, \mathrm{d}V' \tag{1.23}$$

$$= \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j \, \mathrm{d}V' \tag{1.24}$$

$$= \frac{i\omega}{2} \int (\sum_k \partial'_k \hat{T}_{0k}) x'^i x'^j \, \mathrm{d}V' \tag{1.25}$$

$$= \frac{i\omega}{2} \int (\partial_0 \hat{T}_{00}) x'^i x'^j \, \mathrm{d}V' \tag{1.26}$$

$$= -\frac{\omega^2}{2} \int \hat{T}_{00} \, x'^i x'^j \, \mathrm{d}V'. \tag{1.27}$$

$$q_{ij}(t) := \int T_{00} \, x'^i x'^j \, \mathrm{d}V', \tag{1.28}$$

$$\hat{\bar{h}}_{ij}(\omega, \vec{r}) = -2\omega^2 \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \hat{q}_{ij}(\omega), \tag{1.29}$$

$$\bar{h}_{ij}(t, \vec{r}) = \frac{2}{|\vec{r}|} \frac{\mathrm{d}^2}{\mathrm{d}t^2} q_{ij}(t - |\vec{r}|). \tag{1.30}$$

1.4 + Mode and × Mode

Lorentz gauge → radiation gauge. 寻新标架 $(e'^1)_a = (e^+)_a$, $(e'^2)_a = (e^\times)_a$, $(e'^3)_a = (e^r)_a$, $\bar{h}_{ij}(e^i)_a(e^j)_b = \bar{h}'_{ij}(e'^i)_a(e'^j)_b$, 取 x, y 分量后去迹,

$h_+ = \frac{1}{2}(\bar{h}'_{11} - \bar{h}'_{22})$, $h_\times = \bar{h}'_{12} = \bar{h}'_{21}$. (?) [7], [4]

[2], $\vec{n} := \frac{\vec{r}}{|\vec{r}|}$,

$$h_{ij}^{\mathrm{TT}} = \frac{2}{|\vec{r}|}\mathcal{P}_{ijkm}\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q^{km}(t-|\vec{r}|), \tag{1.31}$$

$$\mathcal{P}_{ijkm} := (\delta_{ik} - \vec{n}_i\vec{n}_k)(\delta_{jm} - \vec{n}_j\vec{n}_m) - \frac{1}{2}(\delta_{ij} - \vec{n}_i\vec{n}_j)(\delta_{km} - \vec{n}_k\vec{n}_m), \tag{1.32}$$

$$Q^{km}(t) := \int T_{00}\left(x'^k x'^m - \frac{1}{3}\delta^{km}\sum_n x'^n x'^n\right)\mathrm{d}V' \tag{1.33}$$

1.5 电磁—引力对比

$$A_\mu(t,\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{J_\mu(t-|\vec{r}-\vec{r}'|,\vec{r}')}{|\vec{r}-\vec{r}'|}\mathrm{d}V' \tag{1.34}$$

$$\bar{h}_{\mu\nu}(t,\vec{r}) = 4G\int\frac{T_{\mu\nu}(t-|\vec{r}-\vec{r}'|,\vec{r}')}{|\vec{r}-\vec{r}'|}\mathrm{d}V' \tag{1.35}$$

$$A_\mu(t,\vec{r}) = \frac{1}{\sqrt{2\pi}}\int\hat{A}_\mu(\omega,\vec{r})e^{-i\omega t}\mathrm{d}t \tag{1.36}$$

$$\bar{h}_{\mu\nu}(t,\vec{r}) = \frac{1}{\sqrt{2\pi}}\int\hat{\bar{h}}_{\mu\nu}(\omega,\vec{r})e^{-i\omega t}\mathrm{d}t \tag{1.37}$$

$$\hat{A}_\mu(\omega,\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\hat{J}_\mu(\omega,\vec{r}')}{|\vec{r}-\vec{r}'|}e^{i\omega|\vec{r}-\vec{r}'|}\mathrm{d}V' \tag{1.38}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega,\vec{r}) = 4G\int\frac{\hat{T}_{\mu\nu}(\omega,\vec{r}')}{|\vec{r}-\vec{r}'|}e^{i\omega|\vec{r}-\vec{r}'|}\mathrm{d}V' \tag{1.39}$$

$$\hat{A}_\mu(\omega,\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int\hat{J}_\mu(\omega,\vec{r}')e^{-i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}')}\mathrm{d}V' \tag{1.40}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega,\vec{r}) = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int\hat{T}_{\mu\nu}(\omega,\vec{r}')e^{-i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}')}\mathrm{d}V' \tag{1.41}$$

$$\hat{A}_\mu(\omega,\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int\hat{J}_\mu(\omega,\vec{r}')\left[1 - i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}') - \ldots\right]\mathrm{d}V' \tag{1.42}$$

$$\hat{\bar{h}}_{\mu\nu}(\omega,\vec{r}) = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int\hat{T}_{\mu\nu}(\omega,\vec{r}')\left[1 - i\omega(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}') - \ldots\right]\mathrm{d}V' \tag{1.43}$$

1.5.1 电偶极—引力对比

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{J}_i \,\mathrm{d}V' \tag{1.44}$$

$$\hat{\bar{h}}_{ij} = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\int \hat{T}_{ij} \,\mathrm{d}V' \tag{1.45}$$

$$\int \hat{J}_i \,\mathrm{d}V' = -i\omega \int \hat{J}_0 x'^i \,\mathrm{d}V' \tag{1.46}$$

$$\int \hat{T}_{ij} \,\mathrm{d}V' = -\frac{\omega^2}{2}\int \hat{T}_{00}\, x'^i x'^j \,\mathrm{d}V' \tag{1.47}$$

$$\hat{p}_i = \int \hat{J}_0 x'^i \,\mathrm{d}V' \tag{1.48}$$

$$\hat{q}_{ij} = \int \hat{T}_{00}\, x'^i x'^j \,\mathrm{d}V' \tag{1.49}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega\hat{p}_i) \tag{1.50}$$

$$\hat{\bar{h}}_{ij} = 4G\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-\frac{\omega^2}{2}\hat{q}_{ij}) \tag{1.51}$$

$$A_i = \frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|}\frac{\mathrm{d}}{\mathrm{d}t}p_i(t-|\vec{r}|) \tag{1.52}$$

$$\bar{h}_{ij} = 4G\frac{1}{|\vec{r}|}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}q_{ij}(t-|\vec{r}|) \tag{1.53}$$

1.5.2 电四极—引力对比

$$\hat{A}_i(\omega,\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}\ \ (-i\omega)\int \hat{J}_i(\omega,\vec{r}')(\frac{\vec{r}}{|\vec{r}|}\cdot\vec{r}')\,\mathrm{d}V' \tag{1.54}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega)\int \hat{J}'_i n^j x'_j \,\mathrm{d}V' \tag{1.55}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega)\int n^j x'_j \hat{J}'_i \,\mathrm{d}V' \tag{1.56}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-i\omega)n^j\left[\int x'_{(j}\hat{J}'_{i)}\,\mathrm{d}V'\right] \tag{1.57}$$

$$\begin{aligned}
\int x'_{(j}\hat{J}'_{i)}\,\mathrm{d}V' &= \frac{1}{2}\int(\hat{J}'_j x'_i + \hat{J}'_i x'_j)\,\mathrm{d}V' && (1.58)\\
&= \frac{1}{2}\int\sum_k(\hat{J}'_k x'^i\frac{\partial x'^j}{\partial x'^k} + \hat{J}'_k\frac{\partial x'^i}{\partial x'^k}x'^j)\,\mathrm{d}V' && (1.59)\\
&= \frac{1}{2}\sum_k\left[\int\frac{\partial}{\partial x'^k}(\hat{J}'_k x'^i x'^j)\,\mathrm{d}V' - \int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V'\right] && (1.60)\\
&= \frac{1}{2}\sum_k\int\partial'_k\,(\hat{J}'_k x'^i x'^j)\,\mathrm{d}V' - \frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' && (1.61)\\
&= \frac{1}{2}\sum_k\int\hat{J}'_k x'^i x'^j\,\mathrm{d}S' - \frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' && (1.62)\\
&= -\frac{1}{2}\sum_k\int\frac{\partial\hat{J}'_k}{\partial x'^k}x'^i x'^j\,\mathrm{d}V' && (1.63)\\
&= -\frac{1}{2}\int(\sum_k\partial'_k\hat{J}'_k)x'^i x'^j\,\mathrm{d}V' && (1.64)\\
&= -\frac{1}{2}\int(\partial_0\hat{J}'_0)x'^i x'^j\,\mathrm{d}V' && (1.65)\\
&= -\frac{i\omega}{2}\int\hat{J}'_0 x'^i x'^j\,\mathrm{d}V' && (1.66)
\end{aligned}$$

$$\hat{D}_{ij} = \int\hat{J}'_0\,x'^i x'^j\,\mathrm{d}V' \tag{1.67}$$

$$\hat{A}_i = \frac{\mu_0}{4\pi}\frac{e^{i\omega|\vec{r}|}}{|\vec{r}|}(-\frac{\omega^2}{2}n^j\hat{D}_{ij}) \tag{1.68}$$

$$A_i = \frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|}n^j\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}D_{ij}(t-|\vec{r}|) \tag{1.69}$$

第二章　探测

2.1　TT frame

TT gauge $\Rightarrow$ TT frame.

If a test mass is at rest at $\tau = 0$ ($\Rightarrow \mathrm{d}x^i/\mathrm{d}\tau = 0$), $\mathrm{d}x^\mu/\mathrm{d}\tau = 0$, $\mathrm{d}^2x^i/\mathrm{d}\tau^2 = 0$. In the TT frame, particles which were at rest before the arrival of the wave remain at rest even after the arrival ofthe wave.

two test masses, proper position s_i, $\ddot{s}_i = (1/2)\ddot{h}_{ij}s_j$.

第三章　能量

[8], [4], [6],

$$G_{ab}^{[1]}(h_{cd}^{[1]}) + G_{ab}^{[1]}(h_{cd}^{[2]}) + G_{ab}^{[2]}(h_{cd}^{[1]}) = 8\pi T_{ab}, \tag{3.1}$$

$$G_{ab}^{[1]}(h_{cd}^{[1]} + h_{cd}^{[2]}) = 8\pi(T_{ab} + t_{ab}) := 8\pi(T_{ab} - \frac{G_{ab}^{[2]}(h_{cd}^{[1]})}{8\pi}), \tag{3.2}$$

Thus, in the 2nd order, $h_{ab}^{[2]}$ causes the same correction to g_{ab} as would be produced by ordinary matter with effect stress-energy tensor t_{ab}.

If not $T_{ab} \gg t_{ab}$, derivations in 一 are not valid.

第四章　双星系统

4.1　基本公式

$$\mathcal{M} := \mu^{3/5} M^{2/5} \tag{4.1}$$

$$h_+ = \frac{4\mathcal{M}}{D}[\pi \mathcal{M} F(t)]^{2/3} \frac{1+\cos^2 \iota}{2} \cos\Phi(t) \tag{4.2}$$

$$h_\times = \frac{4\mathcal{M}}{D}[\pi \mathcal{M} F(t)]^{2/3} \cos\iota \sin\Phi(t) \tag{4.3}$$

$$h = F_+ h_+ + F_\times h_\times \tag{4.4}$$

4.2　Post-Newtonian Approximation

4.3　Stationary Phase Approximation

[5], if $\zeta(t)$ varies slowly near $t = t_0$ where the phase has a stationary point: $\phi'(t_0) = 0$,

$$\int \zeta(t) e^{i\phi(t;f)}\,\mathrm{d}t = \int \zeta(t) e^{i[\phi(t_0)+\phi'(t_0)(t-t_0)+\frac{1}{2}\phi''(t_0)(t-t_0)^2+\dots]}\,\mathrm{d}t \tag{4.5}$$

$$\simeq e^{i\phi(t_0)} \int \zeta(t) e^{i[\frac{1}{2}\phi''(t_0)(t-t_0)^2]}\,\mathrm{d}t \tag{4.6}$$

$$\simeq e^{i\phi(t_0)} \int \zeta(t_0) e^{\frac{-\sqrt{-i\phi''(t_0)}^2 (t-t_0)^2}{2}}\,\mathrm{d}t \tag{4.7}$$

$$= \frac{\sqrt{2\pi}}{\sqrt{-i\phi''(t_0)}} \zeta(t_0) e^{i\phi(t_0)}. \tag{4.8}$$

$$h = \frac{\mathcal{M}}{D}[\pi \mathcal{M} F(t)]^{2/3} Q \cos\Phi(t) \tag{4.9}$$

$$= \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}[e^{i\Phi(t)} + e^{-i\Phi(t)}] \tag{4.10}$$

$$\begin{aligned}
\tilde{h}(f) &= \int h(t)e^{i2\pi ft}\,\mathrm{d}t && (4.11)\\
&= \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}[e^{i\Phi(t)} + e^{-i\Phi(t)}]e^{i2\pi ft}\,\mathrm{d}t && (4.12)\\
&= \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}\{e^{i[2\pi ft+\Phi(t)]} + e^{i[2\pi ft-\Phi(t)]}\}\,\mathrm{d}t && (4.13)\\
&\simeq \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F(t)]^{2/3}Q\frac{1}{2}e^{i[2\pi ft-\Phi(t)]}\,\mathrm{d}t && (4.14)\\
&= \int \frac{\mathcal{M}}{D}[\pi\mathcal{M}F]^{2/3}Q\frac{1}{2}e^{i[2\pi ft(F)-\Phi(F)]}\frac{\mathrm{d}t}{\mathrm{d}F}\,\mathrm{d}F && (4.15)\\
&\simeq \frac{\sqrt{2\pi}}{\sqrt{-i[2\pi ft(F)-\Phi(F)]''_{F=f}}} && (4.16)\\
&\left[\frac{\mathcal{M}}{D}(\pi\mathcal{M}F)^{2/3}Q\frac{1}{2}\frac{\mathrm{d}t}{\mathrm{d}F}\right]_{F=f} e^{i[2\pi ft(f)-\Phi(f)]} && (4.17)\\
&\simeq \frac{\sqrt{2\pi}}{\sqrt{-i\left\{2\pi f\left[-\frac{5}{256}\mathcal{M}(\pi\mathcal{M}F)^{-8/3}\right]-\left[\frac{1}{16}(\pi\mathcal{M}F)^{-5/3}\right]\right\}''_{F=f}}} && (4.18)\\
&\left\{\frac{\mathcal{M}}{D}(\pi\mathcal{M}F)^{2/3}Q\frac{1}{2}\left[\frac{5\pi\mathcal{M}^2}{96}(\pi\mathcal{M}F)^{-11/3}\right]\right\}_{F=f} e^{i[2\pi ft(f)-\Phi(f)]} && (4.19)\\
&= \frac{\sqrt{30}}{48\pi^{2/3}}\frac{\mathcal{M}^{5/6}Q}{D}f^{-7/6}e^{i[2\pi ft(f)-\Phi(f)-\frac{\pi}{4}]} \quad (\texttt{pnspa.py}) && (4.20)
\end{aligned}$$

另可考 [1]. 其中 $\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 2\pi F$.

第五章　宇宙学效应

$$\frac{\mathrm{d}\eta}{\mathrm{d}(ct)} = \frac{1}{a} \tag{5.1}$$

$$\mathrm{d}s^2 = -\mathrm{d}(ct)^2 + a^2[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)] \tag{5.2}$$

$$\mathrm{d}s^2 = a^2[-\mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)] \tag{5.3}$$

第六章　电磁引力

[3].

6.1　时空张量转化为空间张量

$$h_{ab} := g_{ab} + Z_a Z_b. \tag{6.1}$$

$$h_a{}^b = \delta_a{}^b + Z_a Z^b. \tag{6.2}$$

$$Z^a h_{ab} = 0. \tag{6.3}$$

$$V_{\langle a\rangle} := h_a{}^b V_b. \tag{6.4}$$

$$Z^a V_{\langle a\rangle} = 0. \tag{6.5}$$

$$T_{\langle ab\rangle} := h_{(a}{}^c h_{b)}{}^d T_{cd} - \frac{1}{3} h_{cd} T^{cd} h_{ab}. \tag{6.6}$$

$$Z^a (h_a{}^c h_b{}^d T_{cd}) = 0. \tag{6.7}$$

$$Z^a (h_b{}^c h_a{}^d T_{cd}) = 0. \tag{6.8}$$

$$Z^a (h_{(a}{}^c h_{b)}{}^d T_{cd}) = 0. \tag{6.9}$$

$$Z^a (h_{cd} T^{cd} h_{ab}) = 0. \tag{6.10}$$

$$Z^a T_{\langle ab\rangle} = 0. \tag{6.11}$$

$$T_{(\langle ab\rangle)} = T_{\langle ab\rangle}. \tag{6.12}$$

$$h^{ab} T_{\langle ab\rangle} = 0. \tag{6.13}$$

$$\varepsilon_{abc} := \varepsilon_{abcd} Z^d. \tag{6.14}$$

$$\varepsilon_{0123} := -\sqrt{|g|}. \tag{6.15}$$

$$T_a := \frac{1}{2} \varepsilon_{abc} T^{[bc]}. \tag{6.16}$$

$$[U, V]_a := \varepsilon_{abc} U^b V^c. \tag{6.17}$$

$$[S, T]_a := \varepsilon_{abc} g_{de} S^{bd} T^{ce}. \tag{6.18}$$

$$\mathrm{D}_t T^{a\ldots}{}_{b\ldots} := Z^c \nabla_c T^{a\ldots}{}_{b\ldots}. \tag{6.19}$$

$${}^3\nabla_a T^{b\ldots}{}_{c\ldots} := h_a{}^p h^b{}_q \ldots h_c{}^r \ldots \nabla_p T^{q\ldots}{}_{r\ldots}. \tag{6.20}$$

$$(\operatorname{div} V) := {}^3\nabla^a V_a. \tag{6.21}$$

$$(\operatorname{curl} V)_a := \varepsilon_{bca} {}^3\nabla^b V^c. \tag{6.22}$$

$$(\operatorname{div} T)_a := {}^3\nabla^b T_{ab}. \tag{6.23}$$

$$(\operatorname{curl} T)_{ab} := \varepsilon_{cd(a} {}^3\nabla^c g_{b)e} T^{ed}. \tag{6.24}$$

6.2 电磁空间矢量

$${}^*F_{ab} := \frac{1}{2} \varepsilon_{abcd} F^{cd} \tag{6.25}$$

$$E_a := F_{ab} Z^b = E_{\langle a \rangle}. \tag{6.26}$$

$$B_a := {}^*F_{ab} Z^b = B_{\langle a \rangle}. \tag{6.27}$$

$$\rho = -Z^a J_a. \tag{6.28}$$

$$j_a = h_a{}^b J_b. \tag{6.29}$$

$$\nabla_{[a} F_{bc]} = 0. \tag{6.30}$$

$$\nabla^a F_{ab} = \mu J_b. \tag{6.31}$$

$$(\operatorname{div} E) = \mu\rho - \dots. \tag{6.32}$$

$$(\operatorname{div} B) = + \dots. \tag{6.33}$$

$$(\operatorname{curl} E)_a + \dots = -\mathrm{D}_t B_{\langle a \rangle} - \dots. \tag{6.34}$$

$$(\operatorname{curl} B)_a + \dots = \mu j_a + \mathrm{D}_t E_{\langle a \rangle} + \dots. \tag{6.35}$$

6.3 引力空间张量

$$^*C_{abcd} := \frac{1}{2}\varepsilon_{abef} C^{ef}{}_{cd}. \tag{6.36}$$

$$E_{ab} := C_{acbd} Z^c Z^d = E_{\langle ab \rangle}. \tag{6.37}$$

$$B_{ab} := {}^*C_{acbd} Z^c Z^d = B_{\langle ab \rangle}. \tag{6.38}$$

$$(\operatorname{div} E)_a = \kappa \frac{1}{3} {}^3\nabla_a \rho - \dots. \tag{6.39}$$

$$(\operatorname{div} B)_a = \kappa(\rho + p)\omega_a + \dots. \tag{6.40}$$

$$(\operatorname{curl} E)_{ab} + \dots = -\mathrm{D}_t B_{\langle ab \rangle} - \dots. \tag{6.41}$$

$$(\operatorname{curl} B)_{ab} + \dots = \kappa \frac{1}{2}(\rho + p)\sigma_{ab} + \mathrm{D}_t E_{\langle ab \rangle} + \dots. \tag{6.42}$$

第七章　Varying G

7.1 Modification of Amplitude

$$\partial^c\partial_c\bar{h}_{ab} = -16\pi\frac{G_0}{c_0^4}T_{ab}, \quad \partial^a\bar{h}_{ab} = 0 \tag{7.1}$$

$$\Gamma^c{}_{ab} = \frac{1}{2}\eta^{cd}(2\partial_{(a}h_{b)d} - \partial_d h_{ab}) \tag{7.2}$$

$$U^a\partial_a U^c + \Gamma^c{}_{ab}U^aU^b = 0 \tag{7.3}$$

$$U^a\partial_a U^c = -\frac{1}{2}\eta^{cd}(2\partial_{(a}h_{b)d} - \partial_d h_{ab})U^aU^b \tag{7.4}$$

$$T_{ab} = c_0^2(2U_{(a}J_{b)} + U^cJ_cU_aU_b) \tag{7.5}$$

$$J_b c_0^2 = -U^aT_{ab} \tag{7.6}$$

$$A_b = -\frac{1}{4}U^a\bar{h}_{ab} \tag{7.7}$$

$$A_0 = -\frac{1}{4}c_0\bar{h}_{00} = -\frac{1}{2}c_0(\bar{h}_{00} - \frac{1}{2}\eta_{00}\eta^{00}\bar{h}_{00}) = -\frac{1}{2}c_0h_{00} \tag{7.8}$$

$$A_i = -\frac{1}{4}c_0\bar{h}_{0i} = -\frac{1}{4}c_0h_{0i} \tag{7.9}$$

$$U^\mu\partial_\mu U^i = -\frac{1}{2}\eta^{i\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu})U^\mu U^\nu \tag{7.10}$$

$$\begin{aligned} -\frac{1}{2}\eta^{i\sigma}(\partial_0 h_{0\sigma} + \partial_0 h_{0\sigma} - \partial_\sigma h_{00})U^0U^0 &= \frac{1}{2}c_0^2\eta^{i\sigma}\partial_\sigma h_{00} && (7.11)\\ &= \frac{1}{2}c_0^2\partial^i h_{00} && (7.12)\end{aligned}$$

$$= -c_0 \partial^i A_0 \tag{7.13}$$

$$= -E^i \tag{7.14}$$

$$-\frac{1}{2}\eta^{i\sigma}(\partial_0 h_{j\sigma} + \partial_j h_{0\sigma} - \partial_\sigma h_{0j})U^0 U^j = -\frac{1}{2}c_0 \eta^{i\sigma}(\partial_j h_{0\sigma} - \partial_\sigma h_{0j})v^j \tag{7.15}$$

$$= -\frac{1}{2}c_0 \eta^{ik}(\partial_j h_{0k} - \partial_k h_{0j})v^j \tag{7.16}$$

$$= 2\eta^{ik}(\partial_j A_k - \partial_k A_j)v^j \tag{7.17}$$

$$= -2\eta^{ik}(\partial_k A_j - \partial_j A_k)v^j \tag{7.18}$$

$$= -2(\partial^i A_j - \partial_j A^i)v^j \tag{7.19}$$

$$= -2\varepsilon^i{}_{jk} v^j B^k \tag{7.20}$$

$$-\frac{1}{2}\eta^{i\sigma}(\partial_j h_{k\sigma} + \partial_k h_{j\sigma} - \partial_\sigma h_{jk})U^j U^k = 0 \tag{7.21}$$

$$a^i = -E^i - 4\varepsilon^i{}_{jk} v^j B^k \tag{7.22}$$

$$\partial^i(\frac{1}{4\pi G_0} E_i) = \rho \tag{7.23}$$

$$\partial^i B_i = 0 \tag{7.24}$$

$$\varepsilon^i{}_{jk}\partial^j E^k = -\partial_t B^i \tag{7.25}$$

$$\varepsilon^i{}_{jk}\partial^j(\frac{c_0^2}{4\pi G_0} B^k) = j^i + \partial_t(\frac{1}{4\pi G_0} E^i) \tag{7.26}$$

$$\varepsilon_{\mathrm{G0}} := \frac{1}{4\pi G_0}, \quad \mu_{\mathrm{G0}} := \frac{4\pi G_0}{c_0^2} \tag{7.27}$$

$$\begin{cases} \vec{\nabla}\cdot(\varepsilon_{\mathrm{G0}}\vec{E}) = \rho \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B} \\ \vec{\nabla}\times(\mu_{\mathrm{G0}}^{-1}\vec{B}) = \vec{j} + \frac{\partial}{\partial t}(\varepsilon_{\mathrm{G0}}\vec{E}) \end{cases} \tag{7.28}$$

$$\vec{a} = -\vec{E} - 4\vec{v}\times\vec{B} \tag{7.29}$$

$$\varepsilon_{\mathrm{G}} = \frac{1}{4\pi G}, \quad \mu_{\mathrm{G}} = \frac{4\pi G}{c^2} \tag{7.30}$$

$$x^\mu = (ct, x, y, z) \tag{7.31}$$

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_{\mathrm{G}} \vec{E}) = \rho \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \\ \vec{\nabla} \times (\mu_{\mathrm{G}}^{-1} \vec{B}) = \vec{j} + \frac{\partial}{\partial t} (\varepsilon_{\mathrm{G}} \vec{E}) \end{cases} \tag{7.32}$$

$$\vec{a} = -\vec{E} - 4\vec{v} \times \vec{B} \tag{7.33}$$

$$A_\mu = -\frac{1}{4} c \bar{h}_{0\mu} \tag{7.34}$$

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \varepsilon_{\mathrm{G}}^{-1} \rho \\ \vec{\nabla} \times \vec{B} = \mu_{\mathrm{G}} \vec{j} + \varepsilon_{\mathrm{G}} \mu_{\mathrm{G}} \frac{\partial}{\partial t} \vec{E} \end{cases} \tag{7.35}$$

$$\frac{1}{c^2} \frac{\partial}{\partial t} \varphi + \vec{\nabla} \cdot \vec{A} = 0 \tag{7.36}$$

$$\begin{cases} -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi + \vec{\nabla}^2 \varphi = \varepsilon_{\mathrm{G}}^{-1} \rho \\ -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} + \vec{\nabla}^2 \vec{A} = \mu_{\mathrm{G}} \vec{j} \end{cases} \tag{7.37}$$

$$\begin{cases} -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} c^{-1} \varphi + \vec{\nabla}^2 c^{-1} \varphi = \mu_{\mathrm{G}} c \rho \\ -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} + \vec{\nabla}^2 \vec{A} = \mu_{\mathrm{G}} \vec{j} \end{cases} \tag{7.38}$$

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_{\mathrm{G}} \vec{E}) = 0 \\ \vec{\nabla} \cdot (\mu_{\mathrm{G}} \vec{H}) = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} (\mu_{\mathrm{G}} \vec{H}) \\ \vec{\nabla} \times \vec{H} = +\frac{\partial}{\partial t} (\varepsilon_{\mathrm{G}} \vec{E}) \end{cases} \tag{7.39}$$

$$E_r = 0, \quad H_r = 0 \tag{7.40}$$

$$\begin{cases} \frac{\varepsilon_{\mathrm{G}}}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta E_\theta) + \frac{\varepsilon_{\mathrm{G}}}{r \sin\theta} \frac{\partial}{\partial \phi} (E_\phi) = 0 \\ \frac{\mu_{\mathrm{G}}}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta H_\theta) + \frac{\mu_{\mathrm{G}}}{r \sin\theta} \frac{\partial}{\partial \phi} (H_\phi) = 0 \\ \frac{1}{r \sin\theta} [\frac{\partial}{\partial \theta} (\sin\theta E_\phi) - \frac{\partial}{\partial \phi} (E_\theta)] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r} (r E_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r} (r E_\theta) \vec{e}_\phi = -\mu_{\mathrm{G}} \frac{\partial}{\partial t} (H_\theta \vec{e}_\theta + H_\phi \vec{e}_\phi) \\ \frac{1}{r \sin\theta} [\frac{\partial}{\partial \theta} (\sin\theta H_\phi) - \frac{\partial}{\partial \phi} (H_\theta)] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r} (r H_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r} (r H_\theta) \vec{e}_\phi = +\varepsilon_{\mathrm{G}} \frac{\partial}{\partial t} (E_\theta \vec{e}_\theta + E_\phi \vec{e}_\phi) \end{cases} \tag{7.41}$$

$$\vec{E} = E_\theta \vec{e}_\theta, \quad \vec{H} = H_\phi \vec{e}_\phi \tag{7.42}$$

$$\begin{cases} \frac{\varepsilon_{\mathrm{G}}}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) = 0 \\ \frac{\mu_{\mathrm{G}}}{r\sin\theta}\frac{\partial}{\partial\phi}(H_\phi) = 0 \\ -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(E_\theta)\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta)\vec{e}_\phi = -\mu_{\mathrm{G}}\frac{\partial}{\partial t}(H_\phi)\vec{e}_\phi \\ +\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta H_\phi)\vec{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\vec{e}_\theta = +\varepsilon_{\mathrm{G}}\frac{\partial}{\partial t}(E_\theta)\vec{e}_\theta \end{cases} \tag{7.43}$$

$$\begin{cases} \frac{\partial}{\partial r}(rE_\theta) + \mu_{\mathrm{G}}\frac{\partial}{\partial t}(rH_\phi) = 0 \\ \frac{\partial}{\partial r}(rH_\phi) + \varepsilon_{\mathrm{G}}\frac{\partial}{\partial t}(rE_\theta) = 0 \end{cases} \tag{7.44}$$

$$\begin{cases} \mu_{\mathrm{G}}\frac{\partial}{\partial r}\mu_{\mathrm{G}}^{-1}\frac{\partial}{\partial r}(rE_\theta) - \varepsilon_{\mathrm{G}}\mu_{\mathrm{G}}\frac{\partial}{\partial t}\frac{\partial}{\partial t}(rE_\theta) = 0 \\ \varepsilon_{\mathrm{G}}\frac{\partial}{\partial r}\varepsilon_{\mathrm{G}}^{-1}\frac{\partial}{\partial r}(rH_\phi) - \varepsilon_{\mathrm{G}}\mu_{\mathrm{G}}\frac{\partial}{\partial t}\frac{\partial}{\partial t}(rH_\phi) = 0 \end{cases} \tag{7.45}$$

$$\begin{cases} \mu_{\mathrm{G}}\frac{\partial}{\partial r}\mu_{\mathrm{G}}^{-1}\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial}{\partial(ct)}\frac{\partial}{\partial(ct)}(rE_\theta) = 0 \\ \varepsilon_{\mathrm{G}}\frac{\partial}{\partial r}\varepsilon_{\mathrm{G}}^{-1}\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial}{\partial(ct)}\frac{\partial}{\partial(ct)}(rH_\phi) = 0 \end{cases} \tag{7.46}$$

$$\begin{cases} \frac{\partial}{\partial r}\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial}{\partial r}(\ln\mu_{\mathrm{G}})\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial}{\partial(ct)}\frac{\partial}{\partial(ct)}(rE_\theta) = 0 \\ \frac{\partial}{\partial r}\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial}{\partial r}(\ln\varepsilon_{\mathrm{G}})\frac{\partial}{\partial r}(rH_\phi) - \frac{\partial}{\partial(ct)}\frac{\partial}{\partial(ct)}(rH_\phi) = 0 \end{cases} \tag{7.47}$$

$$\frac{\partial^2}{\partial r^2}f(r,t) - p(r)\frac{\partial}{\partial r}f(r,t) - \frac{\partial^2}{\partial(ct)^2}f(r,t) = 0 \tag{7.48}$$

$$f(r,t) = f(r)e^{-ikct} \tag{7.49}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}f(r) - p(r)\frac{\mathrm{d}}{\mathrm{d}r}f(r) + k^2 f(r) = 0 \tag{7.50}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}f(r) - p\frac{\mathrm{d}}{\mathrm{d}r}f(r) + k^2 f(r) = 0 \tag{7.51}$$

$$f(r) = e^{(p/2)r}[C_+e^{i\sqrt{k^2-(p/2)^2}r} + C_-e^{-i\sqrt{k^2-(p/2)^2}r}] \tag{7.52}$$

$$f(r,t) = e^{(p/2)r}[C_+e^{i(+\sqrt{k^2-(p/2)^2}r-kct)} + C_-e^{i(-\sqrt{k^2-(p/2)^2}r-kct)}] \tag{7.53}$$

$$f(r,t) = e^{(p/2)r}[C_+e^{i(+\sqrt{(\omega/c)^2-(p/2)^2}r-\omega t)} + C_-e^{i(-\sqrt{(\omega/c)^2-(p/2)^2}r-\omega t)}] \tag{7.54}$$

$$f(r,t) = e^{\int(p/2)\mathrm{d}r}[C_+e^{i(+\int\sqrt{(\omega/c)^2-(p/2)^2}\mathrm{d}r-\omega t)} + C_-e^{i(-\int\sqrt{(\omega/c)^2-(p/2)^2}\mathrm{d}r-\omega t)}] \tag{7.55}$$

$$\begin{cases} r_2\left|E_\theta\right|_{r=r_2} = r_1\left|E_\theta\right|_{r=r_1} e^{\int_{r_1}^{r_2}\frac{1}{2}\frac{\partial}{\partial r}(\ln\mu_{\mathrm{G}})\,\mathrm{d}r} \\ r_2\left|H_\phi\right|_{r=r_2} = r_1\left|H_\phi\right|_{r=r_1} e^{\int_{r_1}^{r_2}\frac{1}{2}\frac{\partial}{\partial r}(\ln\varepsilon_{\mathrm{G}})\,\mathrm{d}r} \end{cases} \tag{7.56}$$

$$\begin{cases} E_2 = \sqrt{\frac{\mu_{\mathrm{G}2}}{\mu_{\mathrm{G}1}}} \frac{r_1}{r_2} E_1 \\ H_2 = \sqrt{\frac{\varepsilon_{\mathrm{G}2}}{\varepsilon_{\mathrm{G}1}}} \frac{r_1}{r_2} H_1 \end{cases} \tag{7.57}$$

$$\begin{cases} E_2/c_2 = \sqrt{\frac{\mu_{\mathrm{G}2}}{\mu_{\mathrm{G}1}}} \frac{c_1}{c_2} \frac{r_1}{r_2} E_1/c_1 \\ B_2 = \sqrt{\frac{\mu_{\mathrm{G}2}}{\mu_{\mathrm{G}1}}} \frac{c_1}{c_2} \frac{r_1}{r_2} B_1 \end{cases} \tag{7.58}$$

$$\begin{cases} (\omega/c_2) c_2 (\bar{h}_{00})_2 = \sqrt{\frac{\mu_{\mathrm{G}2}}{\mu_{\mathrm{G}1}}} \frac{c_1}{c_2} \frac{r_1}{r_2} (\omega/c_1) c_1 (\bar{h}_{00})_1 \\ (\omega/c_2) c_2 (\bar{h}_{0i})_2 = \sqrt{\frac{\mu_{\mathrm{G}2}}{\mu_{\mathrm{G}1}}} \frac{c_1}{c_2} \frac{r_1}{r_2} (\omega/c_1) c_1 (\bar{h}_{0i})_1 \end{cases} \tag{7.59}$$

$$h_2 = \sqrt{\frac{c_1^4/G_1}{c_2^4/G_2}} \frac{r_1}{r_2} h_1 \tag{7.60}$$

双星系统引力辐射本为

$$h = \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{r} Q(\theta, \phi, \psi, \iota) \cos[\int 2\pi F(t)\, \mathrm{d}t] \tag{7.61}$$

设双星系统常量 c^*, G^*, 一观者临近双星系统且与双星系统相对静止, 其与双星系统距离为 r, 测得强度 h_r, 频率 F_r, 则[1]

$$h_r = \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{r/c^*} Q(\theta, \phi, \psi, \iota) \tag{7.62}$$

设地球观者与双星系统距离为 d, 双星系统红移为 z, 测得强度 h_d, 频率 $F_d = F_r/(1+z)$, 则

$$h_d = \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{r}{d} h_r \tag{7.63}$$

$$= \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{d/c^*} Q(\theta, \phi, \psi, \iota) \tag{7.64}$$

所以地球观者测得

$$h = \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{d/c^*} Q(\theta, \phi, \psi, \iota) \cos[\int 2\pi \frac{F_r(t)}{1+z}\, \mathrm{d}t] \tag{7.65}$$

记 $F_{\mathrm{obs}}(t) = F_r(t)/(1+z)$, $\mathcal{M}_{\mathrm{obs}} = \mathcal{M}(1+z)$, 光度距离 $d_{\mathrm{L}} = d(1+z)$, 则

$$h = \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}[\pi \mathcal{M} F_r(t)]^{2/3}}{d(1+z)/c^*} Q(\theta, \phi, \psi, \iota) \cos[\int 2\pi F_{\mathrm{obs}}(t)\, \mathrm{d}t] \tag{7.66}$$

[1] $\mathcal{M}$ 和 c^*, G^* 简并, 所以可以笼统地仍记作 $\mathcal{M}$.

$$= \sqrt{\frac{c^{*4}/G^*}{c^4/G}} \frac{\mathcal{M}_{\text{obs}}[\pi \mathcal{M}_{\text{obs}} F_{\text{obs}}(t)]^{2/3}}{d_{\text{L}}/c^*} Q(\theta, \phi, \psi, \iota) \cos[\int 2\pi F_{\text{obs}}(t)\, \mathrm{d}t] \tag{7.67}$$

$$= \sqrt{\frac{c^{*6}/G^*}{c^6/G}} \frac{\mathcal{M}_{\text{obs}}[\pi \mathcal{M}_{\text{obs}} F_{\text{obs}}(t)]^{2/3}}{d_{\text{L}}/c} Q(\theta, \phi, \psi, \iota) \cos[\int 2\pi F_{\text{obs}}(t)\, \mathrm{d}t] \tag{7.68}$$

用引力波测距测得 $d_{\text{L,G}}$, 则

$$d_{\text{L,G}} = d_{\text{L}} \sqrt{\frac{c^6/G}{c^{*6}/G^*}} \tag{7.69}$$

[5]

$$h(t) = \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{\xi\, d_{\text{L}}} Q(\text{angles}) \cos \Phi(t) \tag{7.70}$$

$$\tilde{h}(f) = \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{\xi\, d_{\text{L}}} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4}]} \tag{7.71}$$

问题转化为估计 ξ

$$p(\mu) \propto p^{(0)}(\mu) \exp[-\frac{1}{2}\Gamma_{ab}(\mu^a - \hat{\mu}^a)(\mu^b - \hat{\mu}^b)] \tag{7.72}$$

$$p^{(0)}(\mu) \propto \exp[-\frac{1}{2}\Gamma^{(0)}_{ab}(\mu^a - \bar{\mu}^a)(\mu^b - \bar{\mu}^b)] \tag{7.73}$$

设待估参数为 $\mu = (\ln \xi, \ln(d_{\text{L}}/d_{\text{L0}}), \ln Q, \dots), \dots$ 为其他参数 (如 $\mathcal{M}$), 则 $\tilde{h}_{,\ln \xi} = \tilde{h}_{,\ln(d_{\text{L}}/d_{\text{L0}})} = -\tilde{h}_{,\ln Q} = -\tilde{h}$, $\tilde{h}$ 对其他参数求偏导皆为纯虚数, 则由 $\Gamma_{ab} = \langle h_{,a} | h_{,b} \rangle$ 和 SNR $:= \rho = \sqrt{\langle h | h \rangle}$ 得

$$\Gamma_{ab} = \begin{bmatrix} \rho^2 & \rho^2 & -\rho^2 & 0 & \dots \\ \rho^2 & \rho^2 & -\rho^2 & 0 & \dots \\ -\rho^2 & -\rho^2 & \rho^2 & 0 & \dots \\ 0 & 0 & 0 & ? & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{7.74}$$

又设

$$\Gamma^{(0)}_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1/\sigma^2_{\ln d_{\text{L}}} & 0 & 0 & \dots \\ 0 & 0 & 1/\sigma^2_{\ln Q} & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{7.75}$$

则由 $\Sigma_{ab}=(\Gamma^{(0)}_{ab}+\Gamma_{ab})^{-1}$ 得

$$\Sigma_{ab}=\begin{bmatrix}\begin{bmatrix}\rho^2 & \rho^2 & -\rho^2\\ \rho^2 & \rho^2+1/\sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L}0})} & -\rho^2\\ -\rho^2 & -\rho^2 & \rho^2+1/\sigma^2_{\ln Q}\end{bmatrix}^{-1} & 0\\ 0 & [?]^{-1}\end{bmatrix} \tag{7.76}$$

而

$$\begin{bmatrix}\rho^2 & \rho^2 & -\rho^2\\ \rho^2 & \rho^2+1/\sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L}0})} & -\rho^2\\ -\rho^2 & -\rho^2 & \rho^2+1/\sigma^2_{\ln Q}\end{bmatrix}^{-1} \tag{7.77}$$

$$=\begin{bmatrix}1/\rho^2+\sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L}0})}+\sigma^2_{\ln Q} & -\sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L}0})} & \sigma^2_{\ln Q}\\ -\sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L}0})} & \sigma^2_{\ln(d_{\mathrm{L}}/d_{\mathrm{L}0})} & 0\\ \sigma^2_{\ln Q} & 0 & \sigma^2_{\ln Q}\end{bmatrix} \tag{7.78}$$

7.2 Modification of Phase

$$\frac{d^2}{dz^2}H(z)+2p(z)\frac{d}{dz}H(z)+\left[\omega^2+q(z)\right]H(z)=0. \tag{7.79}$$

$$H=Ae^{i\Phi}. \tag{7.80}$$

$k=\frac{d\Phi}{dz}$,

$$\frac{d^2A}{dz^2}+2p\frac{dA}{dz}+\left[\omega^2\left(1-\frac{k^2}{\omega^2}\right)+q\right]A=0, \tag{7.81}$$

$$2\frac{dA}{dz}k+A\frac{dk}{dz}+2pAk=0, \tag{7.82}$$

$$2\frac{1}{A}\frac{dA}{dz}+\frac{1}{k}\frac{dk}{dz}+2p=0, \tag{7.83}$$

$$A\propto e^{-\int p\,dz}k^{-1/2}. \tag{7.84}$$

$\Gamma=e^{\int p\,dz}$ and $K=(k/\omega)^{-1/2}$,

$$\frac{d^2K}{dz^2}-\left(\frac{1}{\Gamma}\frac{d^2\Gamma}{dz^2}-q\right)K+\omega^2K(1-K^{-4})=0, \tag{7.85}$$

$\Xi=\frac{1}{\Gamma}\frac{d^2\Gamma}{dz^2}-q$ and make $\omega=1$,

$$\frac{d^2K}{dz^2}+K[(1-\Xi)-K^{-4}]=0. \tag{7.86}$$

$\Xi = \text{const}$,

$$K = (1-\Xi)^{-1/4} = 1 + \frac{1}{4}\Xi + \frac{5}{32}\Xi^2 + O(\Xi^3), \tag{7.87}$$

$$k = (1-\Xi)^{1/2} = 1 - \frac{1}{2}\Xi - \frac{1}{8}\Xi^2 + O(\Xi^3), \tag{7.88}$$

$\Xi \neq \text{const}$, $\Xi(z) = \kappa^2\tilde{\Xi}(\tilde{z})$, where $\tilde{z} = \kappa z$.

$$K^3 \frac{d^2K}{d\tilde{z}^2}\kappa^2 - K^4\tilde{\Xi}(\tilde{z})\kappa^2 + K^4 - 1 = 0. \tag{7.89}$$

$$K = \sum_{n=0}^{\infty} K_n(\tilde{z})\kappa^{2n}, \tag{7.90}$$

$$K_0^4 - 1 = 0, \tag{7.91}$$

$$K_0^3K_0'' - K_0^4\tilde{\Xi} + 4K_0^3K_1 = 0, \tag{7.92}$$

$$(K_0^3K_1'' + 3K_0^2K_1K_0'') - 4K_0^3K_1\tilde{\Xi} + (4K_0^3K_2 + 6K_0^2K_1^2) = 0. \tag{7.93}$$

$$K_0 = 1, \tag{7.94}$$

$$K_1 = \frac{1}{4}\tilde{\Xi}, \tag{7.95}$$

$$K_2 = \frac{5}{32}\tilde{\Xi}^2 - \frac{1}{16}\frac{d^2\tilde{\Xi}}{d\tilde{z}^2}, \tag{7.96}$$

[5]

$$h(t) = \frac{\mathcal{M}[\pi\mathcal{M}F(t)]^{2/3}}{d_{\mathrm{L}}} Q(\text{angles}) \cos\Phi(t) \tag{7.97}$$

$$\tilde{h}(f) = \frac{\sqrt{30}}{48\pi^{2/3}}\frac{\mathcal{M}^{5/6}Q}{d_{\mathrm{L}}} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4}]} \tag{7.98}$$

$$\tilde{h}(f) = \int h(t)e^{2\pi i f t}\mathrm{d}t \tag{7.99}$$

$$h(t) = \int \tilde{h}(f)e^{-2\pi i f t}\mathrm{d}f \tag{7.100}$$

$$A = \frac{\sqrt{30}}{48\pi^{2/3}}\frac{\mathcal{M}^{5/6}Q}{d_{\mathrm{L}}} f^{-7/6}\,\mathrm{d}f \tag{7.101}$$

$$A \propto \Gamma^{-1}K \tag{7.102}$$

$$K|_{z=d_{\mathrm{L}}} = K|_{z=0} = 1 \tag{7.103}$$

$$\frac{A|_{z=d_{\mathrm{L}}}}{A|_{z=d_0}} = e^{-\int_0^{d_{\mathrm{L}}} p\,dz} \tag{7.104}$$

$$A=\frac{\sqrt{30}}{48\pi^{2/3}}\frac{\mathcal{M}^{5/6}Q}{d_{\mathrm{L}}}f^{-7/6}e^{-\int_0^{d_{\mathrm{L}}}p\,dz}\,\mathrm{d}f \tag{7.105}$$

$$A=\frac{\sqrt{30}}{48\pi^{2/3}}\frac{\mathcal{M}^{5/6}Q}{e^{\int_0^{d_{\mathrm{L}}}p\,dz}d_{\mathrm{L}}}f^{-7/6}\,\mathrm{d}f \tag{7.106}$$

$$A=\frac{\sqrt{30}}{48\pi^{2/3}}\frac{\mathcal{M}^{5/6}Q}{\xi d_{\mathrm{L}}}f^{-7/6}\,\mathrm{d}f \tag{7.107}$$

$$A=\mathcal{A}f^{-7/6}\,\mathrm{d}f \tag{7.108}$$

$$k=\omega[1-\frac{1}{2}\frac{\Xi}{\omega^2}] \tag{7.109}$$

$$\psi=\int k\,dz=2\pi ft(f)-\Phi(f)-\frac{\pi}{4} \tag{7.110}$$

$$\psi=\int k\,dz=2\pi ft(f)-\Phi(f)-\frac{\pi}{4}-\int_0^{d_{\mathrm{L}}}\frac{1}{2}\frac{\Xi}{\omega^2}\omega\,dz \tag{7.111}$$

$$\psi=\int k\,dz=2\pi ft(f)-\Phi(f)-\frac{\pi}{4}-\frac{1}{2}\frac{\int_0^{d_{\mathrm{L}}}\Xi\,dz}{(2\pi f)^2}(2\pi f) \tag{7.112}$$

$$\psi=\int k\,dz=2\pi ft(f)-\Phi(f)-\frac{\pi}{4}-\Omega(2\pi f)^{-1} \tag{7.113}$$

$$\tilde{h}(f)=\mathcal{A}f^{-7/6}e^{i[2\pi ft(f)-\Phi(f)-\frac{\pi}{4}-\Omega(2\pi f)^{-1}]} \tag{7.114}$$

$$2\pi f\Delta t(f)-\Delta\Phi(f)=(2\pi f)^{-1} \tag{7.115}$$

$$\frac{\mathrm{d}\Delta\Phi/\mathrm{d}f}{\mathrm{d}\Delta t/\mathrm{d}f}=2\pi f \tag{7.116}$$

$$\Delta\Phi(f)=-2(2\pi f)^{-1} \tag{7.117}$$

$$\Delta t(f)=-(2\pi f)^{-2} \tag{7.118}$$

$$\Phi_{\mathrm{1PN}}(f)=-\frac{1}{16}\frac{5}{3}(\frac{743}{336}+\frac{11}{4}\eta)(\pi\mathcal{M}f)^{-5/3}(\pi Mf)^{2/3} \tag{7.119}$$

$$t_{\mathrm{1PN}}(f)=-\frac{5}{256}\frac{4}{3}(\frac{743}{336}+\frac{11}{4}\eta)\mathcal{M}(\pi\mathcal{M}f)^{-8/3}(\pi Mf)^{2/3} \tag{7.120}$$

$$\Delta\Phi(f)=2\Omega(2\pi f)^{-1} \tag{7.121}$$

$$\Delta t(f)=\Omega(2\pi f)^{-2} \tag{7.122}$$

$$\frac{1}{16}\frac{5}{3}(\frac{743}{336}+\frac{11}{4}\eta)\mathcal{M}^{-5/3}M^{2/3}-\Omega \tag{7.123}$$

$$h_0(t)=\sum_k[C_+(k)e^{+ik\int\,dt}+C_-(k)e^{-ik\int\,dt}]e^{ikd_L} \tag{7.124}$$

$$h(t) = \sum_k \Gamma^{-1} K[C_+(k) e^{+ik\int K^{-2}\,dt} + C_-(k) e^{-ik\int K^{-2}\,dt}] e^{ikd_L} \tag{7.125}$$

$$h_0(t) = \int [C_+(k) e^{+ik\int\,dt} + C_-(k) e^{-ik\int\,dt}] e^{ikd_L} dk \tag{7.126}$$

$$h(t) = \int \Gamma^{-1} K[C_+(k) e^{+ik\int K^{-2}\,dt} + C_-(k) e^{-ik\int K^{-2}\,dt}] e^{ikd_L} dk \tag{7.127}$$

$$h_0(t) = \int [C_+(-\omega) e^{-i\omega d_L} + C_-(+\omega) e^{+i\omega d_L}] e^{-i\omega t} d\omega \tag{7.128}$$

$$= \int \tilde{h}_0(\omega) e^{-i\omega t} d\omega \tag{7.129}$$

$$h(t) = \int [C_+(-\omega) e^{-i\omega d_L} + C_-(+\omega) e^{+i\omega d_L}] e^{-i\omega\int K^{-2}\,dt} d\omega \tag{7.130}$$

$$= \int \tilde{h}_0(\omega) e^{-i\omega t} e^{+i\omega\int \frac{\Xi(t)}{2\omega^2}} d\omega \tag{7.131}$$

如果直接认为 $\dot{G}/G$ 是常数, 那 $\int \Xi(t)/2\omega^2 = \eta(\dot{G}/G)^2 t/2\omega^2$, 这里暂用 $\sum_\omega$ 表示积分.

$$h(t) = \sum_\omega \tilde{h}_0(\omega) e^{-i\omega t} e^{+i\omega\eta(\dot{G}/G)^2 t/2\omega^2} \tag{7.132}$$

$$= \sum_\omega \tilde{h}_0(\omega) e^{-i\omega(1-\eta(\dot{G}/G)^2/2\omega^2)t} \tag{7.133}$$

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