

引力波天文学笔记

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第一章 引力波

1.1 Linearized Gravity

[8]. 流形 \mathbb{R}^4 . 任意坐标系 $\{x^\mu\}$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}s + O(s^2)$, 得

$$R_{\mu\nu\lambda\sigma} = \partial_\sigma \partial_{[\mu} h_{\lambda]\nu} - \partial_\nu \partial_{[\mu} h_{\lambda]\sigma} + O(s^2). \quad (1.1)$$

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\sigma}h_{\lambda\sigma} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h.$$

$$-\frac{1}{2}\partial^\lambda \partial_\lambda \bar{h}_{\mu\nu} + \partial^\lambda \partial_{(\mu} \bar{h}_{\nu)\lambda} - \frac{1}{2}\eta_{\mu\nu} \partial^\lambda \partial^\sigma \bar{h}_{\lambda\sigma} + O(s^2) = 8\pi T_{\mu\nu}. \quad (1.2)$$

存在 $\{x^\mu\}$, 使得 $\partial^\nu \bar{h}_{\mu\nu} + O(s^2) = 0$ (Lorentz gauge). 令 $\{x^\mu\}$ 满足 $\partial^\nu \bar{h}_{\mu\nu} + O(s^2) = 0$, 则

$$\partial^\lambda \partial_\lambda \bar{h}_{\mu\nu} + O(s^2) = -16\pi T_{\mu\nu}. \quad (1.3)$$

略去 $O(s^2)$ 条件: $h_{\mu\nu}, \partial_\lambda h_{\mu\nu} \dots$ 小.

1.2 Radiation Gauge

[8]. 存在 $\{x^\mu\}$, 使得 $h + O(s^2) = 0$ (TT gauge [9]) 且 $h_{0\mu} + O(s^2) = 0$.

1.3 Quadrupole Approximation

[8]. 下略 $O(s^2)$. 由(1.3)得

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4 \int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV'. \quad (1.4)$$

$$\hat{h}_{\mu\nu}(\omega, \vec{r}) := \frac{1}{\sqrt{2\pi}} \int \bar{h}_{\mu\nu}(t, \vec{r}) e^{i\omega t} dt \quad (1.5)$$

$$= 4 \int \frac{\hat{T}_{\mu\nu}(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega|\vec{r} - \vec{r}'|} dV'. \quad (1.6)$$

由 $\partial^\nu \bar{h}_{\mu\nu} = 0$,

$$-i\omega \hat{h}_{0\mu} = \sum_i \frac{\partial \hat{h}_{i\mu}}{\partial x^i}. \quad (1.7)$$

$|\vec{r}| \gg |\vec{r}'|$ 且 $\omega \ll 1/|\vec{r}'|$,

$$\hat{h}_{ij}(\omega, \vec{r}) = 4 \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{ij}(\omega, \vec{r}') dV'. \quad (1.8)$$

$$\int \hat{T}_{ij} dV' = \int \sum_k (\hat{T}_{kj} \frac{\partial x'^i}{\partial x'^k}) dV' \quad (1.9)$$

$$= \sum_k \left[\int \frac{\partial}{\partial x'^k} (\hat{T}_{kj} x'^i) dV' - \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \right] \quad (1.10)$$

$$= \sum_k \int \partial'_k (\hat{T}_{kj} x'^i) dV' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \quad (1.11)$$

$$= \int \hat{T}_{kj} x'^i dS' - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \quad (1.12)$$

$$= - \sum_k \int \frac{\partial \hat{T}_{kj}}{\partial x'^k} x'^i dV' \quad (1.13)$$

$$= - \int (\sum_k \partial'_k \hat{T}_{kj}) x'^i dV' \quad (1.14)$$

$$= - \int (\partial_0 \hat{T}_{0j}) x'^i dV' \quad (1.15)$$

$$= -i\omega \int \hat{T}_{0j} x'^i dV' \quad (1.16)$$

$$= \int \hat{T}_{(ij)} dV' \quad (1.17)$$

$$= -i\omega \int \hat{T}_{0(j} x'^i) dV' \quad (1.18)$$

$$= -\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) dV', \quad (1.19)$$

$$(1.20)$$

$$-\frac{i\omega}{2} \int (\hat{T}_{0j} x'^i + \hat{T}_{0i} x'^j) dV' = -\frac{i\omega}{2} \int \sum_k (\hat{T}_{0k} x'^i \frac{\partial x'^j}{\partial x'^k} + \hat{T}_{0k} \frac{\partial x'^i}{\partial x'^k} x'^j) dV' \quad (1.21)$$

$$= -\frac{i\omega}{2} \sum_k \left[\int \frac{\partial}{\partial x'^k} (\hat{T}_{0k} x'^i x'^j) dV' - \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \right] \quad (1.22)$$

$$= -\frac{i\omega}{2} \sum_k \int \partial'_k (\hat{T}_{0k} x'^i x'^j) dV' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \quad (1.23)$$

$$= -\frac{i\omega}{2} \sum_k \int \hat{T}_{0k} x'^i x'^j dS' + \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \quad (1.24)$$

$$= \frac{i\omega}{2} \sum_k \int \frac{\partial \hat{T}_{0k}}{\partial x'^k} x'^i x'^j dV' \quad (1.25)$$

$$= \frac{i\omega}{2} \int (\sum_k \partial'_k \hat{T}_{0k}) x'^i x'^j dV' \quad (1.26)$$

$$= \frac{i\omega}{2} \int (\partial_0 \hat{T}_{00}) x'^i x'^j dV' \quad (1.27)$$

$$= -\frac{\omega^2}{2} \int \hat{T}_{00} x'^i x'^j dV'. \quad (1.28)$$

$$q_{ij}(t) := \int T_{00} x'^i x'^j dV', \quad (1.29)$$

$$\hat{h}_{ij}(\omega, \vec{r}) = -2\omega^2 \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \hat{q}_{ij}(\omega), \quad (1.30)$$

$$\bar{h}_{ij}(t, \vec{r}) = \frac{2}{|\vec{r}|} \frac{d^2}{dt^2} q_{ij}(t - |\vec{r}|). \quad (1.31)$$

1.4 + Mode and \times Mode

寻新标架 $(e'^1)_a = (e^+)_a$, $(e'^2)_a = (e^\times)_a$, $(e'^3)_a = (e^r)_a$, $\bar{h}_{ij}(e^i)_a (e^j)_b = \bar{h}'_{ij}(e'^i)_a (e'^j)_b$, 取 x, y 分量后去迹, $h_+ = \frac{1}{2}(\bar{h}'_{11} - \bar{h}'_{22})$, $h_\times = \bar{h}'_{12} = \bar{h}'_{21}$? [7]
[2], $\vec{n} := \frac{\vec{r}}{|\vec{r}|}$,

$$h_{ij}^{\text{TT}} = \frac{2}{|\vec{r}|} \mathcal{P}_{ijkm} \frac{d^2}{dt^2} Q^{km}(t - |\vec{r}|), \quad (1.32)$$

$$\mathcal{P}_{ijkm} := (\delta_{ik} - \vec{n}_i \vec{n}_k) (\delta_{jm} - \vec{n}_j \vec{n}_m) - \frac{1}{2} (\delta_{ij} - \vec{n}_i \vec{n}_j) (\delta_{km} - \vec{n}_k \vec{n}_m), \quad (1.33)$$

$$Q^{km}(t) := \int T_{00} \left(x'^k x'^m - \frac{1}{3} \delta^{km} \sum_n x'^n x'^n \right) dV' \quad (1.34)$$

1.5 电磁—引力对比

$$A_\mu(t, \vec{r}) = \frac{\mu_0}{4\pi} \int \frac{J_\mu(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV' \quad (1.35)$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = 4G \int \frac{T_{\mu\nu}(t - |\vec{r} - \vec{r}'|, \vec{r}')}{|\vec{r} - \vec{r}'|} dV' \quad (1.36)$$

$$A_\mu(t, \vec{r}) = \frac{1}{\sqrt{2\pi}} \int \hat{A}_\mu(\omega, \vec{r}) e^{-i\omega t} d\omega \quad (1.37)$$

$$\bar{h}_{\mu\nu}(t, \vec{r}) = \frac{1}{\sqrt{2\pi}} \int \hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) e^{-i\omega t} d\omega \quad (1.38)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\hat{J}_\mu(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega|\vec{r} - \vec{r}'|} dV' \quad (1.39)$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G \int \frac{\hat{T}_{\mu\nu}(\omega, \vec{r}')}{|\vec{r} - \vec{r}'|} e^{i\omega|\vec{r} - \vec{r}'|} dV' \quad (1.40)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_\mu(\omega, \vec{r}') e^{-i\omega(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}')} dV' \quad (1.41)$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{\mu\nu}(\omega, \vec{r}') e^{-i\omega(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}')} dV' \quad (1.42)$$

$$\hat{A}_\mu(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_\mu(\omega, \vec{r}') \left[1 - i\omega \left(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}' \right) - \dots \right] dV' \quad (1.43)$$

$$\hat{\bar{h}}_{\mu\nu}(\omega, \vec{r}) = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{\mu\nu}(\omega, \vec{r}') \left[1 - i\omega \left(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}' \right) - \dots \right] dV' \quad (1.44)$$

1.5.1 电偶极—引力对比

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{J}_i dV' \quad (1.45)$$

$$\hat{\bar{h}}_{ij} = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \int \hat{T}_{ij} dV' \quad (1.46)$$

$$\int \hat{J}_i dV' = -i\omega \int \hat{J}_0 x'^i dV' \quad (1.47)$$

$$\int \hat{T}_{ij} dV' = -\frac{\omega^2}{2} \int \hat{T}_{00} x'^i x'^j dV' \quad (1.48)$$

$$p_i = \int \hat{J}_0 x'^i dV' \quad (1.49)$$

$$q_{ij} = \int \hat{T}_{00} x'^i x'^j dV' \quad (1.50)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega p_i) \quad (1.51)$$

$$\hat{\bar{h}}_{ij} = 4G \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \left(-\frac{\omega^2}{2} q_{ij}\right) \quad (1.52)$$

$$A_i = \frac{\mu_0}{4\pi} \frac{1}{|\vec{r}|} \frac{d}{dt} p_i(t - |\vec{r}|) \quad (1.53)$$

$$\bar{h}_{ij} = 4G \frac{1}{|\vec{r}|} \frac{1}{2} \frac{d^2}{dt^2} q_{ij}(t - |\vec{r}|) \quad (1.54)$$

1.5.2 电四极—引力对比

$$\hat{A}_i(\omega, \vec{r}) = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int \hat{J}_i(\omega, \vec{r}') \left(\frac{\vec{r}}{|\vec{r}|} \cdot \vec{r}'\right) dV' \quad (1.55)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int \hat{J}'_i n^j x'_j dV' \quad (1.56)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) \int n^j x'_j \hat{J}'_i dV' \quad (1.57)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} (-i\omega) n^j \left[\int x'_{(j)} \hat{J}'_i \, dV' \right] \quad (1.58)$$

$$\int x'_{(j)} \hat{J}'_i \, dV' = \frac{1}{2} \int (\hat{J}'_j x'_i + \hat{J}'_i x'_j) \, dV' \quad (1.59)$$

$$= \frac{1}{2} \int \sum_k (\hat{J}'_k x'^i \frac{\partial x'^j}{\partial x'^k} + \hat{J}'_k \frac{\partial x'^i}{\partial x'^k} x'^j) \, dV' \quad (1.60)$$

$$= \frac{1}{2} \sum_k \left[\int \frac{\partial}{\partial x'^k} (\hat{J}'_k x'^i x'^j) \, dV' - \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j \, dV' \right] \quad (1.61)$$

$$= \frac{1}{2} \sum_k \int \partial'_k (\hat{J}'_k x'^i x'^j) \, dV' - \frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j \, dV' \quad (1.62)$$

$$= \frac{1}{2} \sum_k \int \hat{J}'_k x'^i x'^j \, dS' - \frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j \, dV' \quad (1.63)$$

$$= -\frac{1}{2} \sum_k \int \frac{\partial \hat{J}'_k}{\partial x'^k} x'^i x'^j \, dV' \quad (1.64)$$

$$= -\frac{1}{2} \int (\sum_k \partial'_k \hat{J}'_k) x'^i x'^j \, dV' \quad (1.65)$$

$$= -\frac{1}{2} \int (\partial_0 \hat{J}'_0) x'^i x'^j \, dV' \quad (1.66)$$

$$= -\frac{i\omega}{2} \int \hat{J}'_0 x'^i x'^j \, dV' \quad (1.67)$$

$$D_{ij} = \int \hat{J}'_0 x'^i x'^j \, dV' \quad (1.68)$$

$$\hat{A}_i = \frac{\mu_0}{4\pi} \frac{e^{i\omega|\vec{r}|}}{|\vec{r}|} \left(-\frac{\omega^2}{2} n^j D_{ij} \right) \quad (1.69)$$

$$A_i = \frac{\mu_0}{4\pi} \frac{1}{|\vec{r}|} n^j \frac{1}{2} \frac{d^2}{dt^2} D_{ij}(t - |\vec{r}|) \quad (1.70)$$

1.6 常数变易

1.6.1 formula

$$\bar{h}_{ij} = \frac{4G}{c^4} \frac{1}{|\vec{r}|} \frac{1}{2} \frac{d^2}{dt^2} q_{ij}(t - \frac{|\vec{r}|}{c}) \quad (1.71)$$

$$A_{(\text{E2})i} = \frac{\mu}{4\pi} \frac{1}{|\vec{r}|} \frac{d}{dt} p_i(t - \frac{|\vec{r}|}{c}) \quad (1.72)$$

$$A_{(\text{E4})i} = \frac{\mu}{4\pi} \frac{1}{|\vec{r}|} n^j \frac{1}{2} \frac{d^2}{dt^2} D_{ij}(t - \frac{|\vec{r}|}{c}) \quad (1.73)$$

1.6.2 definition

$$\frac{4G}{c^4} := \frac{\mu_G}{4\pi} \quad (1.74)$$

$$c := \frac{1}{\sqrt{\epsilon_G \mu_G}} \quad (1.75)$$

$$G := \frac{\mu_G}{16\pi \epsilon_G^2 \mu_G^2} \quad (1.76)$$

1.6.3 energy

$$T_{ab} \propto F_{ac} F_b{}^c - \frac{1}{4} \eta_{ab} F_{cd} F^{cd} \quad (1.77)$$

$$T_{0i} \propto F_{0c} F_i{}^c - \frac{1}{4} \eta_{0i} F_{cd} F^{cd} = F_{0c} F_i{}^c \quad (1.78)$$

$$F_{ab} = \partial_a A_b - \partial_b A_a \quad (1.79)$$

$$F_a{}^b = \partial_a A^b - \partial^b A_a \quad (1.80)$$

$$F^{ab} = \partial^a A^b - \partial^b A^a \quad (1.81)$$

$$T_{0i} \propto (\partial_0 A_c - \partial_c A_0)(\partial_i A^c - \partial^c A_i) \quad (1.82)$$

$$A_0 = 0 \quad (1.83)$$

$$T_{0i} \propto \partial_0 A_j (\partial_i A^j - \partial^j A_i) \quad (1.84)$$

$$A_i(t, \vec{r}) = \Re[A_i e^{-i(\omega t - \vec{k} \cdot \vec{r})}] \quad (1.85)$$

$$\partial_0 A_j = \Re[-i\omega A_j e^{-i(\omega t - \vec{k} \cdot \vec{r})}] \quad (1.86)$$

$$\partial_i A^j = \Re[ik_i A^j e^{-i(\omega t - \vec{k} \cdot \vec{r})}] \quad (1.87)$$

$$\partial^j A_i = \Re[ik^j A_i e^{-i(\omega t - \vec{k} \cdot \vec{r})}] \quad (1.88)$$

$$\bar{T}_{0i} \propto (-i\omega A_j)[ik_i A^j - ik^j A_i] = \omega[|\vec{A}|^2 \vec{k} - (\vec{k} \cdot \vec{A}) \vec{A}] \quad (1.89)$$

$$\partial^a A_a = 0 \quad (1.90)$$

$$\partial^i A_i = 0 \quad (1.91)$$

$$\partial^i A_i = \Re[ik^i A_i e^{-i(\omega t - \vec{k} \cdot \vec{r})}] \quad (1.92)$$

$$k^i A_i = \vec{k} \cdot \vec{A} = 0 \quad (1.93)$$

$$S_{\text{EM}} \propto |\bar{T}_{0i}| \propto \omega k A^2 = \omega^2 A^2 \quad (1.94)$$

$$S_{\text{EM}} : [M][T]^{-3} \quad (1.95)$$

$$\omega : [T]^{-1} \quad (1.96)$$

$$A : [M][L][T]^{-2}[I]^{-1} \quad (1.97)$$

$$c : [L][T]^{-1} \quad (1.98)$$

$$\mu : [M][L][T]^{-2}[I]^{-2} \quad (1.99)$$

$$S_{\text{EM}} \propto \frac{\omega^2 A^2}{c\mu} \quad (1.100)$$

$$S_{\text{G}} \propto \dot{h}^2 \propto \omega^2 h^2 \quad (1.101)$$

$$S_{\text{G}} : [M][T]^{-3} \quad (1.102)$$

$$\omega : [T]^{-1} \quad (1.103)$$

$$c : [L][T]^{-1} \quad (1.104)$$

$$G : [L]^3[M]^{-1}[T]^{-2} \quad (1.105)$$

$$S_{\text{G}} \propto \frac{c^3 \omega^2 h^2}{G} \propto \frac{\omega^2 h^2}{c\mu_{\text{G}}} \quad (1.106)$$

1.6.4 experiment

双星系统引力辐射本为

$$h = \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{r} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F(t) dt\right] \quad (1.107)$$

设双星系统常量 c^* , μ_{G}^* , 则¹

$$h = \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{r/c^*} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F(t) dt\right] \quad (1.108)$$

¹ \mathcal{M} 和 c^* , μ_{G}^* 简并, 所以可以笼统地仍记作 \mathcal{M} .

在双星系统参考系中, $S_G r^2 \propto \omega^2 h^2 r^2 / c \mu_G$ 守恒, ω 不变, 所以实验室处

$$h = \sqrt{\frac{c^* \mu_G^*}{c \mu_G}} \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{d/c^*} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F(t) dt\right] \quad (1.109)$$

实验室因红移, 实际测得

$$h = \sqrt{\frac{c^* \mu_G^*}{c \mu_G}} \frac{\mathcal{M}[\pi \mathcal{M} F(t)]^{2/3}}{d/c^*} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi \frac{F(t)}{1+z} dt\right] \quad (1.110)$$

记 $F_{\text{obs}}(t) = F(t)/(1+z)$, $\mathcal{M}_{\text{obs}} = \mathcal{M}(1+z)$, 光度距离 $d_L = d_L(1+z)$, 则

$$h = \sqrt{\frac{c^* \mu_G^*}{c \mu_G}} \frac{\mathcal{M}_{\text{obs}}[\pi \mathcal{M}_{\text{obs}} F_{\text{obs}}(t)]^{2/3}}{d_L/c^*} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F_{\text{obs}}(t) dt\right] \quad (1.111)$$

则

$$h = \sqrt{\frac{c^{*3} \mu_G^*}{c^3 \mu_G}} \frac{\mathcal{M}_{\text{obs}}[\pi \mathcal{M}_{\text{obs}} F_{\text{obs}}(t)]^{2/3}}{d_L/c} Q(\theta, \phi, \psi, \iota) \cos\left[\int 2\pi F_{\text{obs}}(t) dt\right] \quad (1.112)$$

用独立手段测得 d_L , 引力波测距测得 d_L^* , 则

$$\frac{1}{d_L^*} = \sqrt{\frac{c^{*3} \mu_G^*}{c^3 \mu_G}} \frac{1}{d_L} \quad (1.113)$$

第二章 双星系统

2.1 基本公式

$$\mathcal{M} := \mu^{3/5} M^{2/5} \quad (2.1)$$

$$h_+ = \frac{4\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} \frac{1 + \cos^2 \iota}{2} \cos \Phi(t) \quad (2.2)$$

$$h_\times = \frac{4\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} \cos \iota \sin \Phi(t) \quad (2.3)$$

$$h = F_+ h_+ + F_\times h_\times \quad (2.4)$$

2.2 Post-Newtonian Approximation

2PN: [3, 6]

2.3 Stationary Phase Approximation

[6], if $\zeta(t)$ varies slowly near $t = t_0$ where the phase has a stationary point: $\phi'(t_0) = 0$,

$$\int \zeta(t) e^{i\phi(t;f)} dt = \int \zeta(t) e^{i[\phi(t_0) + \phi'(t_0)(t-t_0) + \frac{1}{2}\phi''(t_0)(t-t_0)^2 + \dots]} dt \quad (2.5)$$

$$\simeq e^{i\phi(t_0)} \int \zeta(t) e^{i[\frac{1}{2}\phi''(t_0)(t-t_0)^2]} dt \quad (2.6)$$

$$\simeq e^{i\phi(t_0)} \int \zeta(t_0) e^{\frac{-\sqrt{-i\phi''(t_0)}^2 (t-t_0)^2}{2}} dt \quad (2.7)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{-i\phi''(t_0)}} \zeta(t_0) e^{i\phi(t_0)}. \quad (2.8)$$

$$h = \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \cos \Phi(t) \quad (2.9)$$

$$= \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} [e^{i\Phi(t)} + e^{-i\Phi(t)}] \quad (2.10)$$

$$\tilde{h}(f) = \int h(t) e^{i2\pi f t} dt \quad (2.11)$$

$$= \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} [e^{i\Phi(t)} + e^{-i\Phi(t)}] e^{i2\pi f t} dt \quad (2.12)$$

$$= \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} \{e^{i[2\pi f t + \Phi(t)]} + e^{i[2\pi f t - \Phi(t)]}\} dt \quad (2.13)$$

$$\simeq \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F(t)]^{2/3} Q \frac{1}{2} e^{i[2\pi f t - \Phi(t)]} dt \quad (2.14)$$

$$= \int \frac{\mathcal{M}}{D} [\pi \mathcal{M} F]^{2/3} Q \frac{1}{2} e^{i[2\pi f t(F) - \Phi(F)]} \frac{dt}{dF} dF \quad (2.15)$$

$$\simeq \frac{\sqrt{2\pi}}{\sqrt{-i[2\pi f t(F) - \Phi(F)]''_{F=f}}} \quad (2.16)$$

$$\left[\frac{\mathcal{M}}{D} (\pi \mathcal{M} F)^{2/3} Q \frac{1}{2} \frac{dt}{dF} \right]_{F=f} e^{i[2\pi f t(f) - \Phi(f)]} \quad (2.17)$$

$$\simeq \frac{\sqrt{2\pi}}{\sqrt{-i \left\{ 2\pi f \left[-\frac{5}{256} \mathcal{M} (\pi \mathcal{M} F)^{-8/3} \right] - \left[\frac{1}{16} (\pi \mathcal{M} F)^{-5/3} \right] \right\}''_{F=f}}} \quad (2.18)$$

$$\left\{ \frac{\mathcal{M}}{D} (\pi \mathcal{M} F)^{2/3} Q \frac{1}{2} \left[\frac{5\pi \mathcal{M}^2}{96} (\pi \mathcal{M} F)^{-11/3} \right] \right\}_{F=f} e^{i[2\pi f t(f) - \Phi(f)]} \quad (2.19)$$

$$= \frac{\sqrt{30}}{48\pi^{2/3}} \frac{\mathcal{M}^{5/6} Q}{D} f^{-7/6} e^{i[2\pi f t(f) - \Phi(f) - \frac{\pi}{4}]} \quad (\text{pnspace.py}) \quad (2.20)$$

或 [1], $h(t) = 2A(t) \cos \phi(t)$, $d \ln A/dt \ll d\phi/dt$ 且 $|d^2\phi/dt^2| \ll (d\phi/dt)^2$.

第三章 电磁引力

[5].

3.1 时空张量转化为空间张量

$$h_{ab} := g_{ab} + Z_a Z_b. \quad (3.1)$$

$$h_a{}^b = \delta_a{}^b + Z_a Z^b. \quad (3.2)$$

$$Z^a h_{ab} = 0. \quad (3.3)$$

$$V_{\langle a} := h_a{}^b V_b. \quad (3.4)$$

$$Z^a V_{\langle a} = 0. \quad (3.5)$$

$$T_{\langle ab \rangle} := h_{(a}{}^c h_{b)}{}^d T_{cd} - \frac{1}{3} h_{cd} T^{cd} h_{ab}. \quad (3.6)$$

$$Z^a (h_a{}^c h_b{}^d T_{cd}) = 0. \quad (3.7)$$

$$Z^a (h_b{}^c h_a{}^d T_{cd}) = 0. \quad (3.8)$$

$$Z^a (h_{(a}{}^c h_{b)}{}^d T_{cd}) = 0. \quad (3.9)$$

$$Z^a (h_{cd} T^{cd} h_{ab}) = 0. \quad (3.10)$$

$$Z^a T_{\langle ab \rangle} = 0. \quad (3.11)$$

$$T_{\langle (ab) \rangle} = T_{\langle ab \rangle}. \quad (3.12)$$

$$h^{ab} T_{\langle ab \rangle} = 0. \quad (3.13)$$

$$\varepsilon_{abc} := \varepsilon_{abcd} Z^d. \quad (3.14)$$

$$\varepsilon_{0123} := -\sqrt{|g|}. \quad (3.15)$$

$$T_a := \frac{1}{2} \varepsilon_{abc} T^{[bc]}. \quad (3.16)$$

$$[U, V]_a := \varepsilon_{abc} U^b V^c. \quad (3.17)$$

$$[S, T]_a := \varepsilon_{abc} g_{de} S^{bd} T^{ce}. \quad (3.18)$$

$$D_t T^{a\dots}_{b\dots} := Z^c \nabla_c T^{a\dots}_{b\dots}. \quad (3.19)$$

$${}^3\nabla_a T^{b\dots}_{c\dots} := h_a{}^p h^b{}_q \dots h_c{}^r \dots \nabla_p T^{q\dots}_{r\dots}. \quad (3.20)$$

$$(\operatorname{div} V) := {}^3\nabla^a V_a. \quad (3.21)$$

$$(\operatorname{curl} V)_a := \varepsilon_{bca} {}^3\nabla^b V^c. \quad (3.22)$$

$$(\operatorname{div} T)_a := {}^3\nabla^b T_{ab}. \quad (3.23)$$

$$(\operatorname{curl} T)_{ab} := \varepsilon_{cd(a} {}^3\nabla^c g_{b)e} T^{ed}. \quad (3.24)$$

3.2 电磁空间矢量

$${}^*F_{ab} := \frac{1}{2} \varepsilon_{abcd} F^{cd} \quad (3.25)$$

$$E_a := F_{ab} Z^b = E_{\langle a \rangle}. \quad (3.26)$$

$$B_a := {}^*F_{ab} Z^b = B_{\langle a \rangle}. \quad (3.27)$$

$$\rho = -Z^a J_a. \quad (3.28)$$

$$j_a = h_a{}^b J_b. \quad (3.29)$$

$$\nabla_{[a} F_{bc]} = 0. \quad (3.30)$$

$$\nabla^a F_{ab} = \mu J_b. \quad (3.31)$$

$$(\operatorname{div} E) = \mu\rho - \dots \quad (3.32)$$

$$(\operatorname{div} B) = + \dots \quad (3.33)$$

$$(\operatorname{curl} E)_a + \dots = -D_t B_{\langle a} - \dots \quad (3.34)$$

$$(\operatorname{curl} B)_a + \dots = \mu j_a + D_t E_{\langle a} + \dots \quad (3.35)$$

3.3 引力空间张量

$${}^*C_{abcd} := \frac{1}{2}\varepsilon_{abef}C^ef_{cd}. \quad (3.36)$$

$$E_{ab} := C_{acbd}Z^cZ^d = E_{\langle ab \rangle}. \quad (3.37)$$

$$B_{ab} := {}^*C_{acbd}Z^cZ^d = B_{\langle ab \rangle}. \quad (3.38)$$

$$(\operatorname{div} E)_a = \kappa \frac{1}{3} {}^3\nabla_a \rho - \dots \quad (3.39)$$

$$(\operatorname{div} B)_a = \kappa(\rho + p)\omega_a + \dots \quad (3.40)$$

$$(\operatorname{curl} E)_{ab} + \dots = -D_t B_{\langle ab \rangle} - \dots \quad (3.41)$$

$$(\operatorname{curl} B)_{ab} + \dots = \kappa \frac{1}{2}(\rho + p)\sigma_{ab} + D_t E_{\langle ab \rangle} + \dots \quad (3.42)$$

第四章 Fisher 矩阵法

[4], 论证见[FinnNotes](#).

4.1 判断观测数据中是否有信号

$\Omega = A_0 \cup A_m$, A_0 为事件 “无信号”, A_m 为事件 “有信号”, 测量结果为 $G_t(\omega)$, 噪声 $N_t(\omega)$, 信号 $M_t(\omega)$,

$$G_t(\omega) = \begin{cases} N_t(\omega) & \omega \in A_0, \\ N_t(\omega) + M_t(\omega) & \omega \in A_m, \end{cases} \quad (4.1)$$

实测得 g_t , $A_g := \{\omega : G_t(\omega) = g_t\}$, A_g 为事件为 “测得 g_t ”, 求 $\mathbf{P}(A_m|A_g)$.
另认为信号依赖于参数 $\vec{\mu}$, $A_m = \cup A_{\vec{\mu}}$, $A_{\vec{\mu}}$ 为事件 “有信号且参数为 μ ”,
 $p(\vec{\mu}) := p(A_{\vec{\mu}}|A_m)$

$$\mathbf{P}(A_m|A_g) = \frac{\Lambda}{\Lambda + \mathbf{P}(A_0)/\mathbf{P}(A_m)}, \quad (4.2)$$

$$\Lambda := \int d\vec{\mu} \lambda(\vec{\mu}), \quad (4.3)$$

$$\lambda(\vec{\mu}) := p(\vec{\mu}) \exp[2 \langle g(t) | m_{\vec{\mu}}(t) \rangle - \langle m_{\vec{\mu}}(t), m_{\vec{\mu}}(t) \rangle] \quad (4.4)$$

$$\langle \xi(t), \zeta(t) \rangle := \int df \frac{\tilde{\xi}(f) \tilde{\zeta}(f)^*}{S_n(|f|)}, \quad (4.5)$$

$$\tilde{q}(f) := \int dt q(t) \exp[2\pi i f t]. \quad (4.6)$$

4.2 认定有信号后参数估计 (MLE)

实测得 g_t 且认定有信号, 事件 $A_g \cap A_m$, 求使 $p(A_{\vec{\mu}}|A_g \cap A_m)$ 最大的 $\vec{\mu}$, 记作 $\hat{\vec{\mu}}$.

$$p(A_{\vec{\mu}}|A_g) = \frac{\lambda(\vec{\mu})}{\Lambda + \mathbf{P}(A_0)/\mathbf{P}(A_m)}, \quad (4.7)$$

$$p(A_{\vec{\mu}}|A_g \cap A_m) = \frac{\lambda(\vec{\mu})}{\Lambda}, \quad (4.8)$$

$$\frac{\partial \ln p(\vec{\mu})}{\partial \vec{\mu}}|_{\vec{\mu}=\hat{\vec{\mu}}} + 2 \left\langle \frac{\partial m_{\vec{\mu}}}{\partial \vec{\mu}}|_{\vec{\mu}=\hat{\vec{\mu}}}(t), g(t) - m_{\vec{\mu}}|_{\vec{\mu}=\hat{\vec{\mu}}}(t) \right\rangle = 0. \quad (4.9)$$

4.3 灵敏度

若由 g_t 求得 MLE 为 $\hat{\vec{\mu}}$, 则记 $g \Rightarrow \hat{\vec{\mu}}$, $A_{\hat{\vec{\mu}}} := \cup_{g \Rightarrow \hat{\vec{\mu}}} A_g$, $A_{\hat{\vec{\mu}}}$ 为事件 “测得 MLE 为 $\hat{\vec{\mu}}$ ”, $A_{\tilde{\vec{\mu}}}$ 为事件 “有信号且参数为 $\tilde{\vec{\mu}}$ ”, 求 $p(A_{\tilde{\vec{\mu}}}|A_{\hat{\vec{\mu}}})$. 高 SNR, $\tilde{\vec{\mu}} := \hat{\vec{\mu}} + \delta \vec{\mu}$,

$$p(A_{\tilde{\vec{\mu}}+\delta \vec{\mu}}|A_{\hat{\vec{\mu}}}) = \frac{\exp[-\frac{1}{2} \sum \mathcal{C}_{ij}^{-1}(\delta \mu_i - \overline{\delta \mu_i})(\delta \mu_j - \overline{\delta \mu_j})]}{[(2\pi)^N \det(\mathcal{C}_{ij})^{1/2}]}, \quad (4.10)$$

$$\mathcal{C}_{ij}^{-1} = 2 \left\langle \frac{\partial m_{\vec{\mu}}}{\partial \mu_i}|_{\vec{\mu}=\hat{\vec{\mu}}}(t), \frac{\partial m_{\vec{\mu}}}{\partial \mu_j}|_{\vec{\mu}=\hat{\vec{\mu}}}(t) \right\rangle, \quad (4.11)$$

$$\overline{\delta \mu_i} = - \sum \mathcal{C}_{ij} \frac{\partial \ln p(\vec{\mu})}{\partial \mu_j}|_{\vec{\mu}=\hat{\vec{\mu}}}. \quad (4.12)$$

4.4 认定有信号后参数估计 (分布)

[6],

$$p(A_{\vec{\mu}}|A_g \cap A_m) \propto p^{(0)}(\vec{\mu}) \exp[-\frac{1}{2} \langle m_{\vec{\mu}}(t) - g(t) | m_{\vec{\mu}}(t) - g(t) \rangle], \quad (4.13)$$

$$\langle \xi(t) | \zeta(t) \rangle := 2 \int_0^\infty \frac{\tilde{\xi}(f)^* \tilde{\zeta}(f) + \tilde{\xi}(f) \tilde{\zeta}(f)^*}{S_n(f)} df, \quad (4.14)$$

$$\tilde{q}(f) := \int_{-\infty}^\infty q(t) e^{2\pi i f t} dt, \quad (4.15)$$

$$\frac{\partial}{\partial \vec{\mu}} \langle m_{\vec{\mu}}(t) - g(t) | m_{\vec{\mu}}(t) - g(t) \rangle = \left\langle \frac{\partial}{\partial \vec{\mu}} m_{\vec{\mu}}(t) | m_{\vec{\mu}}(t) - g(t) \right\rangle, \quad (4.16)$$

μ^a 估计为 $\hat{\mu}^a$, 高 SNR,

$$\langle m_{,a}(t; \mu^b) | g(t) - m(t; \mu^b) \rangle |_{\mu^b = \hat{\mu}^b} = 0, \quad (4.17)$$

$$\Gamma_{ab} := \langle m_{,a}(t) | m_{,b}(t) \rangle, \quad (4.18)$$

$$p(A_{\mu^a} | A_g \cap A_m) \propto p^{(0)}(\mu^a) \exp[-\frac{1}{2} \Gamma_{ab} (\mu^a - \hat{\mu}^a)(\mu^b - \hat{\mu}^b)], \quad (4.19)$$

$$p^{(0)}(\mu^a) : \propto \exp[-\frac{1}{2} \Gamma_{ab}^{(0)} (\mu^a - \bar{\mu}^a)(\mu^b - \bar{\mu}^b)]. \quad (4.20)$$

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