

If we only take the lowest order effects of the motion of mass sources into account and neglect stresses, then we can define two vector fields, \vec{E} and \vec{B} , which are similar to the electric and magnetic field. We can define $A_\mu = -\frac{1}{4}c_0\bar{h}_{0\mu} = (\varphi/c_0, \vec{A})$, then A_μ , which are similar to the electromagnetic 4-potential, will satisfy

$$\partial^\nu \partial_\nu A_\mu = -\frac{4\pi G_0}{c_0^2} J_\mu, \quad \partial^\mu A_\mu = 0, \quad (1)$$

where $J_\mu = -c_0 T_{0\mu}/c_0^2 = (c_0\rho, \vec{j}) = \rho(c_0, \vec{v})$ is 4-momentum density. If we define $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial}{\partial t}\vec{A}$, $\vec{B} = \vec{\nabla} \times \vec{A}$, and $\varepsilon_{G0} = \frac{1}{4\pi G_0}$, $\mu_{G0} = \frac{4\pi G_0}{c_0^2}$, then there will be

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_{G0}\vec{E}) = \rho, \\ \vec{\nabla} \cdot \vec{B} = 0, \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}, \\ \vec{\nabla} \times (\mu_{G0}^{-1}\vec{B}) = \vec{j} + \frac{\partial}{\partial t}(\varepsilon_{G0}\vec{E}), \end{cases} \quad (2)$$

and the acceleration of mass particle \vec{a} will satisfy

$$\vec{a} = -\vec{E} - 4\vec{v} \times \vec{B}. \quad (3)$$

Since (2) are identical to Maxwell's equations, and (3) is identical to the Lorentz force equation except for an overall minus sign and a factor of 4 in the "magnetic force" term, we assume that the constants ε_{G0} and μ_{G0} can simply vary as constants ε_0 and μ_0 in electromagnetic theories, which means, (3) is still valid and (2) becomes

$$\begin{cases} \vec{\nabla} \cdot \vec{D} = \rho, \\ \vec{\nabla} \cdot \vec{B} = 0, \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}, \\ \vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial}{\partial t}\vec{D}, \end{cases} \quad (4)$$

where $\vec{D} = \varepsilon_G \vec{E}$, $\vec{B} = \mu_G \vec{H}$. Also, we define $c = \frac{1}{\sqrt{\varepsilon_G \mu_G}}$, $G = \frac{1}{4\pi \varepsilon_G}$, together with $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial}{\partial t}\vec{A}$, $\vec{B} = \vec{\nabla} \times \vec{A}$, and $\bar{h}_{0\mu} = -4A_\mu/c = -4(\varphi/c, \vec{A})/c$.

We assume that the source of GW is located in a spherical coordinate origin and $c = c(r)$, $G = G(r)$. We suppose that $\bar{h}_{0\mu}(\rho, \vec{j}; c_0, G_0)$ is the solution of $\bar{h}_{0\mu}$ when ρ and \vec{j} are assumed and $c \equiv c_0$, $G \equiv G_0$ everywhere, then if $c \equiv c_s$, $G \equiv G_s$, where c_s and G_s are two constants, within a region $r \leq R$, then $\bar{h}_{0\mu}(\rho, \vec{j}; c_s, G_s)$ can be a solution of $\bar{h}_{0\mu}$ within the region $r \leq R$, and now we can solve the equations

$$\begin{cases} \vec{\nabla} \cdot \vec{D} = 0, \\ \vec{\nabla} \cdot \vec{B} = 0, \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}, \\ \vec{\nabla} \times \vec{H} = +\frac{\partial}{\partial t}\vec{D}, \end{cases} \quad (5)$$

with the boundary conditions $\vec{E}|_{r=R} = \vec{E}(\rho, \vec{j}; c_s, G_s)$, $\vec{B}|_{r=R} = \vec{B}(\rho, \vec{j}; c_s, G_s)$, where $\vec{E}(\rho, \vec{j}; c_s, G_s)$ and $\vec{B}(\rho, \vec{j}; c_s, G_s)$ can be derived from $\bar{h}_{0\mu}(\rho, \vec{j}; c_s, G_s)$.

Again, since (2) are identical to Maxwell's equations, we can use the Liénard–Wiechert potentials formula to calculate the \vec{E} and \vec{B} field produced by a moving particle, whose position is \vec{r}' , in vacuum, and the results are

$$\vec{E} = \frac{m}{4\pi\epsilon_{G0}} \left[\frac{(\vec{n}' - \vec{\beta})}{(1 - \vec{\beta} \cdot \vec{n}')^3 \gamma^2 r'^2} + \frac{\vec{n}' \times \{(\vec{n}' - \vec{\beta}) \times \dot{\vec{\beta}}\}}{(1 - \vec{\beta} \cdot \vec{n}')^3 c_0 r'} \right]_{\text{ret}}, \vec{B} = \left[\frac{\vec{n}' \times \vec{E}}{c_0} \right]_{\text{ret}}, \quad (6)$$

where $r' = |\vec{r}'|$, $\vec{n}' = \vec{r}'/r'$, $\vec{\beta} = \dot{\vec{r}}'/c_0$, $\gamma = 1/\sqrt{1 - |\vec{\beta}|^2}$, and “ret” denotes that the fields are “retarded fields”. Both \vec{E} and \vec{B} in (6) can be divided into two parts: one, called “inherent field part”, is proportional to $1/r'^2$ and the other one, called “radiation part”, is proportional to $1/r'$. We only take the lowest order effects of the motion of mass sources into account, therefore we can assume that $|\vec{\beta}| \ll 1$. The “inherent field part” of \vec{E} will be parallel to \vec{r}' and the “inherent field part” of \vec{B} will be nearly zero vector. The “radiation part” of \vec{E} and \vec{B} will be perpendicular to \vec{r}' . Since (5) are linear equations, we can discuss two parts of \vec{E} and \vec{B} separately for a binary GW source. If we assume that the distance between two components of binary is much less than R mentioned above, then the “inherent field part” of \vec{E} will have only r component, and the “radiation part” of \vec{E} and \vec{B} will have no r component.

For the “inherent field part”, (5) will become $\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 D_r) = 0$, then $D_r \propto 1/r^2$, and $E_r \propto \epsilon_G^{-1}(1/r^2)$. As mentioned below, for the “radiation part”, $E \propto c^{-1} \epsilon_G^{-1/2}(1/r)$, therefore the “inherent field part” will vanish in the region far away from source.

For the “radiation part”, from the third and fourth equation of (5), we can derive

$$\begin{cases} [\vec{e}_r \cdot (\vec{\nabla} \times \vec{E})] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r}(r E_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r}(r E_\theta) \vec{e}_\phi = -\mu_G \frac{\partial}{\partial t}(H_\theta \vec{e}_\theta + H_\phi \vec{e}_\phi), \\ [\vec{e}_r \cdot (\vec{\nabla} \times \vec{H})] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r}(r H_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r}(r H_\theta) \vec{e}_\phi = +\epsilon_G \frac{\partial}{\partial t}(E_\theta \vec{e}_\theta + E_\phi \vec{e}_\phi), \end{cases} \quad (7)$$

and then

$$\begin{cases} \frac{1}{r} \frac{\partial}{\partial r}(r E_\theta) = -\mu_G \frac{\partial}{\partial t} H_\phi, \\ \frac{1}{r} \frac{\partial}{\partial r}(r E_\phi) = +\mu_G \frac{\partial}{\partial t} H_\theta, \\ \frac{1}{r} \frac{\partial}{\partial r}(r H_\theta) = +\epsilon_G \frac{\partial}{\partial t} E_\phi, \\ \frac{1}{r} \frac{\partial}{\partial r}(r H_\phi) = -\epsilon_G \frac{\partial}{\partial t} E_\theta, \end{cases} \quad (8)$$

$$\begin{cases} \frac{\partial}{\partial r}(r E_\theta) = -\mu_G \frac{\partial}{\partial t}(r H_\phi), \\ \frac{\partial}{\partial r}(r E_\phi) = +\mu_G \frac{\partial}{\partial t}(r H_\theta), \\ \frac{\partial}{\partial r}(r H_\theta) = +\epsilon_G \frac{\partial}{\partial t}(r E_\phi), \\ \frac{\partial}{\partial r}(r H_\phi) = -\epsilon_G \frac{\partial}{\partial t}(r E_\theta), \end{cases} \quad (9)$$

$$\begin{cases} \mu_G \frac{\partial}{\partial r} [\mu_G^{-1} \frac{\partial}{\partial r}(r E_\theta)] = -\mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial r}(r H_\phi) = \epsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t}(r E_\theta), \\ \mu_G \frac{\partial}{\partial r} [\mu_G^{-1} \frac{\partial}{\partial r}(r E_\phi)] = +\mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial r}(r H_\theta) = \epsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t}(r E_\phi), \\ \epsilon_G \frac{\partial}{\partial r} [\epsilon_G^{-1} \frac{\partial}{\partial r}(r H_\theta)] = +\mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial r}(r E_\phi) = \epsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t}(r H_\theta), \\ \epsilon_G \frac{\partial}{\partial r} [\epsilon_G^{-1} \frac{\partial}{\partial r}(r H_\phi)] = -\mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial r}(r E_\theta) = \epsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t}(r H_\phi), \end{cases} \quad (10)$$

therefore

$$\begin{cases} \frac{\partial}{\partial r} \frac{\partial}{\partial r} (rE_\theta) - \frac{\partial}{\partial r} (\ln \mu_G) \frac{\partial}{\partial r} (rE_\theta) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (rE_\theta) = 0, \\ \frac{\partial}{\partial r} \frac{\partial}{\partial r} (rE_\phi) - \frac{\partial}{\partial r} (\ln \mu_G) \frac{\partial}{\partial r} (rE_\phi) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (rE_\phi) = 0, \\ \frac{\partial}{\partial r} \frac{\partial}{\partial r} (rH_\theta) - \frac{\partial}{\partial r} (\ln \varepsilon_G) \frac{\partial}{\partial r} (rH_\theta) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (rH_\theta) = 0, \\ \frac{\partial}{\partial r} \frac{\partial}{\partial r} (rH_\phi) - \frac{\partial}{\partial r} (\ln \varepsilon_G) \frac{\partial}{\partial r} (rH_\phi) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (rH_\phi) = 0. \end{cases} \quad (11)$$

All equations in (11) have the form like

$$\frac{\partial^2}{\partial r^2} f(r, t) - p(r) \frac{\partial}{\partial r} f(r, t) - \frac{\partial^2}{\partial(ct)^2} f(r, t) = 0. \quad (12)$$

If we solve (12) by the method of separation of variables, we can suppose that $f(r, t) = f(r)e^{-ikct}$, and then

$$\frac{d^2}{dr^2} f(r) - p(r) \frac{d}{dr} f(r) + k^2 f(r) = 0. \quad (13)$$

If p is constant, then

$$f(r) = e^{(p/2)r} [C_+ e^{i\sqrt{k-(p/2)^2}r} + C_- e^{-i\sqrt{k-(p/2)^2}r}], \quad (14)$$

where C_+ and C_- are two constants, and

$$f(r, t) = e^{(p/2)r} [C_+ e^{i(\int \sqrt{(\omega/c)-(p/2)^2} dr - \omega t)} + C_- e^{i(-\int \sqrt{(\omega/c)-(p/2)^2} dr - \omega t)}], \quad (15)$$

where $\omega = kc$. If dp/dr is small, we can divide the region $r > R$ into many spherical shells, and in each of them p is nearly constant. To make $f(r, t)$ continuous anytime, we suppose that the “angular frequency” ω in each shell is same, and then there will be a approximate solution

$$f(r, t) = e^{\int (p/2) dr} [C_+ e^{i(\int \sqrt{(\omega/c)-(p/2)^2} dr - \omega t)} + C_- e^{i(-\int \sqrt{(\omega/c)-(p/2)^2} dr - \omega t)}]. \quad (16)$$