

If we only take the lowest order effects of the motion of mass sources into account and neglect stresses, then we can define two vector fields,  $\vec{E}$  and  $\vec{B}$ , which are similar to the electric and magnetic field. We can define  $A_\mu = -\frac{1}{4}c_0\bar{h}_{0\mu} = (\varphi/c_0, \vec{A})$ , then  $A_\mu$ , which are similar to the electromagnetic 4-potential, will satisfy

$$\partial^\nu \partial_\nu A_\mu = -\frac{4\pi G_0}{c_0^2} J_\mu, \quad \partial^\mu A_\mu = 0, \quad (1)$$

where  $J_\mu = -c_0 T_{0\mu}/c_0^2 = (c_0\rho, \vec{j}) = \rho(c_0, \vec{v})$  is 4-momentum density. If we define  $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial}{\partial t}\vec{A}$ ,  $\vec{B} = \vec{\nabla} \times \vec{A}$ , and  $\varepsilon_{G0} = \frac{1}{4\pi G_0}$ ,  $\mu_{G0} = \frac{4\pi G_0}{c_0^2}$ , then there will be

$$\begin{cases} \vec{\nabla} \cdot (\varepsilon_{G0}\vec{E}) = \rho, \\ \vec{\nabla} \cdot \vec{B} = 0, \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}, \\ \vec{\nabla} \times (\mu_{G0}^{-1}\vec{B}) = \vec{j} + \frac{\partial}{\partial t}(\varepsilon_{G0}\vec{E}), \end{cases} \quad (2)$$

and the acceleration of mass particle  $\vec{a}$  will satisfy

$$\vec{a} = -\vec{E} - 4\vec{v} \times \vec{B}. \quad (3)$$

Since (2) are identical to Maxwell's equations, and (3) is identical to the Lorentz force equation except for an overall minus sign and a factor of 4 in the "magnetic force" term, we assume that the constants  $\varepsilon_{G0}$  and  $\mu_{G0}$  can simply vary as constants  $\varepsilon_0$  and  $\mu_0$  in electromagnetic theories, which means, (3) is still valid and (2) becomes

$$\begin{cases} \vec{\nabla} \cdot \vec{D} = \rho, \\ \vec{\nabla} \cdot \vec{B} = 0, \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}, \\ \vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial}{\partial t}\vec{D}, \end{cases} \quad (4)$$

where  $\vec{D} = \varepsilon_G \vec{E}$ ,  $\vec{B} = \mu_G \vec{H}$ . Also, we define  $c = \frac{1}{\sqrt{\varepsilon_G \mu_G}}$ ,  $G = \frac{1}{4\pi \varepsilon_G}$ , together with  $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial}{\partial t}\vec{A}$ ,  $\vec{B} = \vec{\nabla} \times \vec{A}$ , and  $\bar{h}_{0\mu} = -4A_\mu/c = -4(\varphi/c, \vec{A})/c$ .

We assume that the source of GW is located in a spherical coordinate origin and  $c = c(r)$ ,  $G = G(r)$ . We suppose that  $\bar{h}_{0\mu}(\rho, \vec{j}; c_0, G_0)$  is the solution of  $\bar{h}_{0\mu}$  when  $\rho$  and  $\vec{j}$  are assumed and  $c \equiv c_0$ ,  $G \equiv G_0$  everywhere, then if  $c \equiv c_s$ ,  $G \equiv G_s$ , where  $c_s$  and  $G_s$  are two constants, within a region  $r \leq R$ , then  $\bar{h}_{0\mu}(\rho, \vec{j}; c_s, G_s)$  can be a solution of  $\bar{h}_{0\mu}$  within the region  $r \leq R$ , and now we can solve the equations

$$\begin{cases} \vec{\nabla} \cdot \vec{D} = 0, \\ \vec{\nabla} \cdot \vec{B} = 0, \\ \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}, \\ \vec{\nabla} \times \vec{H} = +\frac{\partial}{\partial t}\vec{D}, \end{cases} \quad (5)$$

with the boundary conditions  $\vec{E}|_{r=R} = \vec{E}(\rho, \vec{j}; c_s, G_s)$ ,  $\vec{B}|_{r=R} = \vec{B}(\rho, \vec{j}; c_s, G_s)$ , where  $\vec{E}(\rho, \vec{j}; c_s, G_s)$  and  $\vec{B}(\rho, \vec{j}; c_s, G_s)$  can be derived from  $\bar{h}_{0\mu}(\rho, \vec{j}; c_s, G_s)$ .

Again, since (2) are identical to Maxwell's equations, we can use the Liénard–Wiechert potentials formula to calculate the  $\vec{E}$  and  $\vec{B}$  field produced by a moving particle, whose position is  $\vec{r}'$ , in vacuum, and the results are

$$\vec{E} = \frac{m}{4\pi\epsilon_{G0}} \left[ \frac{(\vec{n}' - \vec{\beta})}{(1 - \vec{\beta} \cdot \vec{n}')^3 \gamma^2 r'^2} + \frac{\vec{n}' \times \{(\vec{n}' - \vec{\beta}) \times \dot{\vec{\beta}}\}}{(1 - \vec{\beta} \cdot \vec{n}')^3 c_0 r'} \right]_{\text{ret}}, \vec{B} = \left[ \frac{\vec{n}' \times \vec{E}}{c_0} \right]_{\text{ret}}, \quad (6)$$

where  $r' = |\vec{r}'|$ ,  $\vec{n}' = \vec{r}'/r'$ ,  $\vec{\beta} = \dot{\vec{r}}'/c_0$ ,  $\gamma = 1/\sqrt{1 - |\vec{\beta}|^2}$ , and “ret” denotes that the fields are “retarded fields”. Both  $\vec{E}$  and  $\vec{B}$  in (6) can be divided into two parts: one, called “inherent field part”, is proportional to  $1/r'^2$  and the other one, called “radiation part”, is proportional to  $1/r'$ . We only take the lowest order effects of the motion of mass sources into account, therefore we can assume that  $|\vec{\beta}| \ll 1$ . The “inherent field part” of  $\vec{E}$  will be parallel to  $\vec{r}'$  and the “inherent field part” of  $\vec{B}$  will be nearly zero vector. The “radiation part” of  $\vec{E}$  and  $\vec{B}$  will be perpendicular to  $\vec{r}'$ . Since (5) are linear equations, we can discuss two parts of  $\vec{E}$  and  $\vec{B}$  separately for a binary GW source. If we assume that the distance between two components of binary is much less than  $R$  mentioned above, then the “inherent field part” of  $\vec{E}$  will have only  $r$  component, and the “radiation part” of  $\vec{E}$  and  $\vec{B}$  will have no  $r$  component.

For the “inherent field part”, (5) will become  $\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 D_r) = 0$ , then  $D_r \propto 1/r^2$ , and  $E_r \propto \epsilon_G^{-1}(1/r^2)$ . As mentioned below, for the “radiation part”,  $E \propto c^{-1} \epsilon_G^{-1/2}(1/r)$ , therefore the “inherent field part” will vanish in the region far away from source.

For the “radiation part”, from the third and fourth equation of (5), we can derive

$$\begin{cases} [\vec{e}_r \cdot (\vec{\nabla} \times \vec{E})] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r}(r E_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r}(r E_\theta) \vec{e}_\phi = -\mu_G \frac{\partial}{\partial t}(H_\theta \vec{e}_\theta + H_\phi \vec{e}_\phi), \\ [\vec{e}_r \cdot (\vec{\nabla} \times \vec{H})] \vec{e}_r - \frac{1}{r} \frac{\partial}{\partial r}(r H_\phi) \vec{e}_\theta + \frac{1}{r} \frac{\partial}{\partial r}(r H_\theta) \vec{e}_\phi = +\epsilon_G \frac{\partial}{\partial t}(E_\theta \vec{e}_\theta + E_\phi \vec{e}_\phi), \end{cases} \quad (7)$$

and then

$$\begin{cases} \frac{1}{r} \frac{\partial}{\partial r}(r E_\theta) = -\mu_G \frac{\partial}{\partial t} H_\phi, \\ \frac{1}{r} \frac{\partial}{\partial r}(r E_\phi) = +\mu_G \frac{\partial}{\partial t} H_\theta, \\ \frac{1}{r} \frac{\partial}{\partial r}(r H_\theta) = +\epsilon_G \frac{\partial}{\partial t} E_\phi, \\ \frac{1}{r} \frac{\partial}{\partial r}(r H_\phi) = -\epsilon_G \frac{\partial}{\partial t} E_\theta, \end{cases} \quad (8)$$

$$\begin{cases} \frac{\partial}{\partial r}(r E_\theta) = -\mu_G \frac{\partial}{\partial t}(r H_\phi), \\ \frac{\partial}{\partial r}(r E_\phi) = +\mu_G \frac{\partial}{\partial t}(r H_\theta), \\ \frac{\partial}{\partial r}(r H_\theta) = +\epsilon_G \frac{\partial}{\partial t}(r E_\phi), \\ \frac{\partial}{\partial r}(r H_\phi) = -\epsilon_G \frac{\partial}{\partial t}(r E_\theta), \end{cases} \quad (9)$$

$$\begin{cases} \mu_G \frac{\partial}{\partial r} [\mu_G^{-1} \frac{\partial}{\partial r}(r E_\theta)] = -\mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial r}(r H_\phi) = \epsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t}(r E_\theta), \\ \mu_G \frac{\partial}{\partial r} [\mu_G^{-1} \frac{\partial}{\partial r}(r E_\phi)] = +\mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial r}(r H_\theta) = \epsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t}(r E_\phi), \\ \epsilon_G \frac{\partial}{\partial r} [\epsilon_G^{-1} \frac{\partial}{\partial r}(r H_\theta)] = +\mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial r}(r E_\phi) = \epsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t}(r H_\theta), \\ \epsilon_G \frac{\partial}{\partial r} [\epsilon_G^{-1} \frac{\partial}{\partial r}(r H_\phi)] = -\mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial r}(r E_\theta) = \epsilon_G \mu_G \frac{\partial}{\partial t} \frac{\partial}{\partial t}(r H_\phi), \end{cases} \quad (10)$$

therefore

$$\begin{cases} \frac{\partial}{\partial r} \frac{\partial}{\partial r} (rE_\theta) - \frac{\partial}{\partial r} (\ln \mu_G) \frac{\partial}{\partial r} (rE_\theta) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (rE_\theta) = 0, \\ \frac{\partial}{\partial r} \frac{\partial}{\partial r} (rE_\phi) - \frac{\partial}{\partial r} (\ln \mu_G) \frac{\partial}{\partial r} (rE_\phi) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (rE_\phi) = 0, \\ \frac{\partial}{\partial r} \frac{\partial}{\partial r} (rH_\theta) - \frac{\partial}{\partial r} (\ln \varepsilon_G) \frac{\partial}{\partial r} (rH_\theta) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (rH_\theta) = 0, \\ \frac{\partial}{\partial r} \frac{\partial}{\partial r} (rH_\phi) - \frac{\partial}{\partial r} (\ln \varepsilon_G) \frac{\partial}{\partial r} (rH_\phi) - \frac{\partial}{\partial(ct)} \frac{\partial}{\partial(ct)} (rH_\phi) = 0. \end{cases} \quad (11)$$

All equations in (11) have the form like

$$\frac{\partial^2}{\partial r^2} f(r, t) - p(r) \frac{\partial}{\partial r} f(r, t) - \frac{\partial^2}{\partial(ct)^2} f(r, t) = 0. \quad (12)$$

If we solve (12) by the method of separation of variables, we can suppose that  $f(r, t) = f(r)e^{-ikct}$ , and then

$$\frac{d^2}{dr^2} f(r) - p(r) \frac{d}{dr} f(r) + k^2 f(r) = 0. \quad (13)$$

If  $p$  is constant, then

$$f(r) = e^{(p/2)r} [C_+ e^{i\sqrt{k^2 - (p/2)^2}r} + C_- e^{-i\sqrt{k^2 - (p/2)^2}r}], \quad (14)$$

where  $C_+$  and  $C_-$  are two constants, and

$$f(r, t) = e^{(p/2)r} [C_+ e^{i(\sqrt{(\omega/c)^2 - (p/2)^2}r - \omega t)} + C_- e^{i(-\sqrt{(\omega/c)^2 - (p/2)^2}r - \omega t)}], \quad (15)$$

where  $\omega = kc$ . If  $dp/dr$  is small, we can divide the region  $r > R$  into many spherical shells, and in each of them  $p$  is nearly constant. To make  $f(r, t)$  continuous anytime, we suppose that the “angular frequency”  $\omega$  in each shell is same, and then there will be a approximate solution

$$f(r, t) = e^{\int (p/2)dr} [C_+ e^{i(\int \sqrt{(\omega/c)^2 - (p/2)^2}dr - \omega t)} + C_- e^{i(-\int \sqrt{(\omega/c)^2 - (p/2)^2}dr - \omega t)}], \quad (16)$$

therefore

$$f(r, t) = e^{\int O(p)dr} [C_+ e^{i(\int [(\omega/c)^2 + O(p^2)]dr - \omega t)} + C_- e^{i(\int [(\omega/c)^2 + O(p^2)]dr - \omega t)}]. \quad (17)$$

If  $p$  is small, (16) can be approximated to

$$f(r, t) = e^{\int (p/2)dr} [C_+ e^{i(\int (\omega/c)dr - \omega t)} + C_- e^{i(-\int (\omega/c)dr - \omega t)}], \quad (18)$$

then wave of  $\vec{E}$  and  $\vec{H}$  will approximately arrive Earth at the same time, and also  $rE \propto e^{\int (\frac{d}{dr} (\ln \mu_G)/2)dr} \propto \mu_G^{1/2}$ ,  $rH \propto e^{\int (\frac{d}{dr} (\ln \varepsilon_G)/2)dr} \propto \varepsilon_G^{1/2}$ .