Instationary PDEs - Theory

Assuming a domain $\Omega \subset \mathbb{R}^d$, and $f:[0,T] \times \Omega \to \mathbb{R}$ and $g:[0,T] \times \Gamma \subset \partial\Omega \to \mathbb{R}$.
We want to find a $u \in [0,T] \times V$ for which the following hold

$$\int_\Omega \partial_t u \phi + cu\phi + a \nabla u \nabla \phi = \int_\Omega f \phi \quad ; \quad \forall \phi \in V \quad \forall t \in [0,T] \qquad (1)$$

where $u(0,x) = u_0(x)$ for $\forall x \in \Omega$. a and c = const., For all $t \in [0,T]$
$V = \{u \in H^1 | u = g$ on Γ, $\partial_n u = 0$ on $\partial\Omega \setminus \Gamma\}$.

By using a time and space discretization, we get a linear system $AU^{n+1} = b$ for $n = 0, 1, \dots, N$.

Time discretization

We discretize a time derivative as an implicit Euler method:

$$\frac{du}{dt}(t^{n+1}) \approx \frac{u^{n+1} - u^n}{\tau}$$

By approximating the time derivative in eq. (1) with the implicit Euler method and evaluating all other terms at t^{n+1} we get

$$\int_\Omega \frac{1}{\tau}(u^{n+1} - u^n)\phi + cu^{n+1}\phi + a\nabla u^{n+1}\nabla\phi = \int_\Omega f(t^n)\phi \quad ; \quad \forall \phi \in V, \forall n = 0,1,\dots,N$$

$$\int_\Omega \frac{1}{\tau}u^{n+1}\phi + cu^{n+1}\phi + a\nabla u^{n+1}\nabla\phi = \int_\Omega \frac{1}{\tau}u^n\phi + f(t^n)\phi \quad ; \quad \forall \phi \in V \quad (2) \quad n = 0,1,\dots,N$$

Space disretization

By a full finite element disvetization we have a disorete function space V_h given by a lagrangian basis $\{\varphi_i\}_{i=0}^{N_p}$. We want to find the discrete solution $u_h^n = \sum_{j=0}^{N_p-1} U_j^n \varphi_j$ such that

$$\sum_{j=0}^{N_p-1} U_j^{n+1} \int_\Omega \left(\frac{1}{\tau} + c\right)\varphi_i\varphi_j + a\nabla\varphi_i\nabla\varphi_j = \sum_{j=0}^{N_p-1} U_j^n \int_\Omega \frac{1}{\tau}\varphi_i\varphi_j + f(t^n)\varphi_i \quad ; \forall i \in \text{free DOFs}$$

This is now the linear system $AU^{n+1} = b$ where for free DOFs we have:

$$A_{ij} = \left(\frac{1}{\tau} + c\right)\int_\Omega \varphi_i\varphi_j + a\int_\Omega \nabla\varphi_i\nabla\varphi_j = \left(\frac{1}{\tau} + c\right)M_{ij} + aD_{ij} \quad ; \quad i \in \text{free DOFs}$$

$$b_j = \frac{1}{\tau}\sum_{i=0}^{N_p-1} U_j^n \int_\Omega \frac{1}{\tau}\varphi_i\varphi_j + f(t^n)\varphi_i$$

$\forall i = 0, \dots, N_p - 1$
$\forall n = 0, 1, \dots, N$

Approximation of integrals

↳ Mass matrix:

$$M_{ij} = \int_\Omega \ell_j \ell_i =$$
$$= \sum_T \int_T \ell_j \ell_i =$$
$$= \sum_{T \in \text{supp}(\ell_i) \cap \text{supp}(\ell_j)} \int_T \ell_j \ell_i =$$
$$= \sum_{T \in \dots} \int_T \phi_{D(j,T)} \phi_{D(i,T)} =$$
$$= \sum_{T \in \dots} \int_{\hat{T}} \phi_{D(j,T)} (F_T(\hat{x})) \phi_{D(i,T)} (F_T(\hat{x})) \, |\det DF^{-1}| \, d\hat{x} =$$
$$= \sum_{T \in \dots} \int_{\hat{T}} \hat{\phi}_{D(j,T)} (F_T^{-1}(F_T(\hat{x})) \phi_{D(i,T)} (F_T^{-1}(F_T(x)) \, |\det DF^{-1}| \, d\hat{x} \approx$$
$$\approx \sum_{T \in \dots} |\det DF^{-1}| \sum_{\hat{x},\hat{\omega}} \hat{\omega} \hat{\phi}_{D(j,T)} (\hat{x}) \; \phi_{D(i,T)}$$

↳ diffusion matrix:

$$D_{ij} = \int_\Omega \nabla \ell_j \nabla \ell_i =$$
$$= \sum_T \int_T \nabla \ell_j \nabla \ell_i =$$
$$= \sum_{T \in \text{supp}(j) \cap \text{supp}(i)} \int_T \nabla \ell_j \nabla \ell_i =$$
$$= \sum_{T \in \dots} \int_T \nabla \phi_{D(j,T)} \nabla \phi_{D(i,T)} =$$
$$= \sum_{T \in \dots} \int_{\hat{T}} \nabla \phi_{D(j,T)} (F_T(\hat{x})) \nabla \phi_{D(i,T)} (F_T(\hat{x})) \, |\det DF^{-1}| \, d\hat{x}$$
$$= \sum_{T \in \dots} \int_{\hat{T}} \nabla \left(\hat{\phi}_{D(j,T)} (F_T^{-1}(F_T(\hat{x})) \right) \nabla \left(\hat{\phi}_{D(i,T)} (F_T^{-1}(F_T(\hat{x})) \right) |\det DF^{-1}| \, d\hat{x} =$$
$$= \sum_{T \in \dots} \int_{\hat{T}} DF_T^{-T} \nabla \hat{\phi}_{D(j,T)} (\hat{x}) \cdot DF_T^{-T} \nabla \hat{\phi}_{D(i,T)} (\hat{x}) \, d\hat{x} \approx$$
$$\approx \sum_{T \in \dots} |\det DF^{-1}| \sum_{\hat{x},\hat{\omega}} \hat{\omega} \, DF_T^{-T} \nabla \hat{\phi}_{D(j,T)} (\hat{x}) \cdot DF_T^{-T} \nabla \phi_{D(i,T)} (\hat{x})$$

↳ Load vector:

$$b_j = \frac{1}{\tau} \sum_{i=0}^{n_p - 1} M_{ij} U_i^n + \int_\Omega f(t^n) \ell_i =$$
$$= \quad \dots \quad + \sum_T \int_T f(t^n) \phi_{D(i,T)} =$$
$$= \quad \dots \quad + \sum_T \int_T f \hat{\phi}_{D(i,T)} (F_T^{-1}(x)) \approx$$
$$\approx \quad \dots \quad + \sum_T |\det (DF^{-1})| \sum_{x,\hat{\omega}} f(t^n) \hat{\phi}_{D(i,T)}$$