Discrete Systems (Part II) . Theory

1. Preparation

First we derive the creation matrix $M_{ij} = \int_\Omega \varphi_j \varphi_i$

We go from integrating aer the whole domain to integrating on the reference triangle and summing aer all triangles:

$$M_{ij} = \int_\Omega \varphi_j \varphi_i =$$

$$= \sum_{T \in \mathcal{T}_h} \int_T \varphi_j \varphi_i =$$

$$= \sum_{T \in \mathcal{T}_h} \int_T \phi^T_{D(j,T)}(x,y)\, \phi^T_{D(i,T)}(x,y)\, dx\,dy =$$ ... going from global to local test and trial functions

$$= \sum_{\substack{T \in supp(\varphi_i) \\ T \in supp(\varphi_j)}} \int_T \hat{\phi}_{D(j,T)}(F_T^{-1}(x,y))\, \hat{\phi}_{D(i,T)}(F_T^{-1}(x,y))\, dx\,dy =$$ ... defining local basis

$$= \sum_{\substack{T \in supp\, \varphi_i \\ T \in supp\, \varphi_j}} |\det(DF_T)| \int_{\hat{T}} \hat{\phi}_{D(j,T)}(\hat{x},\hat{y})\, \hat{\phi}_{D(i,T)}(\hat{x},\hat{y})\, d\hat{x}\,d\hat{y} \approx$$ ... integral transformation

$$\approx \sum_{\substack{T \in supp\, \varphi_i \\ T \in supp\, \varphi_j}} |\det(DF_T)| \sum_{\hat{x},\hat{y},\hat{\omega}} \hat{\omega} \cdot \hat{\phi}_{D(j,T)}(\hat{x},\hat{y})\, \hat{\phi}_{D(i,T)}(\hat{x},\hat{y})$$ ... numerical integration by quadrature rule.

Now we derive diffusion part of the stiffness matrix D_{ij}.

$$D_{ij} = \int_\Omega \nabla \varphi_j \cdot \nabla \varphi_i = \sum_{T \in \mathcal{T}_h} \int_T \nabla \phi^T_{D(j,T)}(x,y) \cdot \nabla \phi^T_{D(i,T)}(x,y)\, dx\,dy$$ ... splitting the integral and going from local to global.

$$= \sum_{\substack{T \in supp(\varphi_i) \\ T \in supp(\varphi_j)}} \int_T \nabla \hat{\phi}_{D(j,T)}(F_T^{-1}(x,y)) \cdot \nabla \hat{\phi}_{D(i,T)}(F_T^{-1}(x,y))\, dx\,dy$$ ... defining local basis

$$= \sum_{T \in \dots} \int_T DF_T^{-T} \cdot \nabla \hat{\phi}_{D(j,T)}(F_T^{-1}(x,y)) \cdot DF_T^{-T} \nabla \hat{\phi}_{D(i,T)}(F_T^{-1}(x,y))\, dx\,dy$$ ... chain rule for 2D

$$= \sum_{T \in \dots} |\det(DF_T)| \int_{\hat{T}} DF_T^{-T} \nabla \hat{\phi}_{D(j,T)}(\hat{x},\hat{y}) \cdot DF_T^{-T} \nabla \hat{\phi}_{D(i,T)}(\hat{x},\hat{y})\, d\hat{x}\,d\hat{y}$$ ... integral transform

$$\approx \sum_{\substack{T \in supp\, \varphi_i \\ T \in supp\, \varphi_j}} |\det(DF_T)| \sum_{\hat{x},\hat{y},\hat{\omega}} \hat{\omega} \cdot DF_T^{-T} \nabla \hat{\phi}_{D(j,T)}(\hat{x},\hat{y}) \cdot DF_T^{-T} \nabla \hat{\phi}_{D(i,T)}(\hat{x},\hat{y})\, d\hat{x}\,d\hat{y}$$ ... numerical integration

2. Optimal Approximation

The approximation is the result of the quadrature rule:
We can exactly calculate M and D using the quadrature rule if the underlying function is a polynomial.
We know $\hat{\Phi} \in P_1$, so $\hat{x}, \hat{y}$ and $\hat{\omega}$ have to be of order $k+1$.