Discrete Systems (Part I) - Theory

Assume a domain Ω and on which we formulate a weak formulation of the diffusion-reaction equation

$$\int_\Omega c u \phi + a \nabla u \nabla \phi = \int_\Omega f \phi \qquad (1)$$

where f is an external force $f: \Omega \to \mathbb{R}$, $u \in V$ is the response we wish to find. We have $a, c > 0 \in V$ as chosen constants and ϕ as the test function $\forall \phi \in V$. For this part we assume boundary conditions $|\partial\Omega$ as $u=0$. Therefore $V = \{ u \in H_0^1 \mid u = 0 \text{ on } \partial\Omega \}$.

By a full finite element discretization we have a discrete function space V_h given by a lagrangian basis $\{\ell_i\}_{i=0}^{N_p-1}$, where N_p is the number of points. We want to find a discrete solution $u_h = \sum_{i=0}^{N_p-1} U_i \ell_i$ such that

$$\sum_{i=0}^{N_p-1} U_i \int_\Omega c \ell_j \ell_i + a \nabla \ell_i \nabla \ell_j = \int_\Omega f \ell_j \qquad \forall_j \text{ free DOF} \qquad (2)$$

1. Lemma of Lax-Milgram

We assume that the domain Ω can be divided into a mesh τ_h of triangles T_i and the Dirichlet boundary condition. We define

$$V_h = \{ v \in V_0 : \phi|_{T_i} \text{ is linear } \forall T_i \in \tau_h, \phi \text{ is continous on } \Omega \}$$

We need to find a unique discrete solution $u_h \in V_h$ for eq. (2), such that

$$A(u_h, \phi) = L(\phi) \quad \forall v \in V_h$$

We chose a basis for V_h : $\ell_j \in V_h$ such that

$$\phi_j(x_i) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad ; \; j = 1, 2, \dots, N_{p-1}$$

where x_i are the vertices of the triangles.

For the unique solution we solve the system

$$u_h(x) = \sum_{j=1}^{N_p-1} U_j \ell_j(x) \quad \text{where} \quad U_j - u_h(x_j).$$

We can write in matrix form:

$$AU = L \quad \text{where} \quad A_{ji} = A(\ell_i, \ell_j) \quad \text{and} \quad L_j = L(\phi_j).$$

Where A is a bounded, coerive bilinear form and L a linear bounded form. Following Lax-Milgrams Lemma, the problem has a unique solution.

2. Approximation of the integral

↳ optional task:

- The approximation property can be characterized by Galerkin-Orthogonality. We search for the approximation to $u \approx u_h \in V_h \subset V$, where V_h is a finite dimensional subset of V. In the Galerkin method we search for the best approximation u_h with respect to the bilinear form $A = A(u,v)$ as defined previously.

The Galerkin-orthogonality states that

$$A(u - u_h, v_h) = 0$$

which implies ortogonality of the error with respect to A.
The proof is given as:

$$A(u-u_h, v_h) = A(u, v_h) - A(u_h, v_h) = L(v_h) - L(v_h) = 0$$

↳ derive step by step the approximation:

$$\int_\Omega f(x,y) \cdot \varphi_j(x,y)\, dx\, dy \approx$$

$$= \sum_{T \in \tau_h} \int_T f(x,y) \cdot \phi^T_{D(j,T)}(x,y)\, dx\, dy =$$

$$= \sum_{T \in \tau_h} \int_T f(x,y) \cdot \hat{\phi}_{D(j,T)}(F_T^{-1}(x,y))\, dx\, dy =$$

$$= \sum_{T \in \tau_h} |\det(DF_T)| \int_{\hat{T}} f(F_T(\hat{x},\hat{y})) \cdot \hat{\phi}_{D(j,T)}(\hat{x},\hat{y})\, d\hat{x}\, d\hat{y} =$$

$$= \sum_{T \in \tau_h} |\det(DF_T)| \sum_{\hat{x},\hat{y},\hat{\omega}} \hat{\omega} \cdot f(F_T(\hat{x},\hat{y})) \cdot \hat{\phi}_{D(j,T)}(\hat{x},\hat{y}) \qquad \forall j \in DOF$$

3. Optimal Approximation

The approximation above is exact when the function f is linear on domain Ω, such that the selected test function $\varphi_j(x,y)$ which is a Lagrangian hat function with values $\hat{\omega} \cdot f$ evaluate to an exact function.