## 11.3 Weak interpolation

$\Omega = [-1, 1]$ $\qquad f(x) = \cos(4\pi x)$

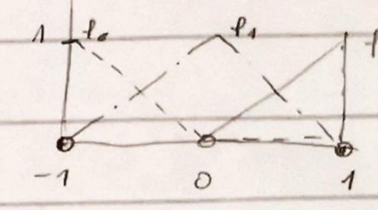
### 11.3.1 Global basis

$\{x_i\} = \{-1, 0, 1\}$

$$\varphi_0 = \begin{cases} -x \; ; & x \in [-1,0] \\ 0 \; ; & \text{else} \end{cases}$$

$$\varphi_1 = \begin{cases} x+1 \; ; & x \in [-1,0] \\ -x+2 \; ; & x \in [0,1] \\ 0 \; ; & \text{else} \end{cases}$$

$$\varphi_2 = \begin{cases} x-1 \; ; & [0,1] \\ 0 \; ; & \text{else} \end{cases}$$




- $\varphi_0(-1) = 1$, $\varphi_0(0) = 0$ $\qquad \varphi_0 = ax + n = -x$

  $1 = -a + n \Rightarrow a = -1$

  $0 = -a \cdot 0 + n \Rightarrow n = 0$

- $\varphi_1(0) = 1$, $\varphi_1(-1) = 0$ $\qquad \varphi_1 = ax + n = x + 1$

  $1 = n \Rightarrow n = 1$

  $0 = -a + 1 \Rightarrow a = 1$

### 11.3.2 Nodal interpolation

First we compute the coefficients $\{F_j\}$

$$F_0 = f(-1) = \cos(-4\pi) = 1$$

$$F_1 = f(0) = \cos(0) = 1$$

$$F_2 = f(1) = \cos(4\pi) = 1$$

Combining coeficients and basis to form $f_s$.

$$f_s = \begin{cases} \varphi_0 F_0 + \varphi_1 F_1 \; ; & x \in [-1,0] \\ \varphi_1 F_1 + \varphi_2 F_2 \; ; & x \in [0,1] \\ 0 \; ; & \text{else} \end{cases} = \begin{cases} (-x) + (x+1) \; ; & x \in [-1,0] \\ (-x+2) + (x-1) \; ; & x \in [0,1] \\ 0 \; ; & \text{else} \end{cases}$$

$$f_s = \begin{cases} 1 \; ; & x \in [-1,0] \\ 1 \; ; & x \in [0,1] \\ 0 \; ; & \text{else} \end{cases}$$

11.3.3 Weak interpolation

We set up a linear system to compute coefficients $\{F_j\}$.

$$M_{ij} = \int_{-1}^{1} \varphi_j \cdot \varphi_i \quad ; \quad M = \begin{bmatrix} m_{00} & m_{01} & 0 \\ m_{01} & m_{11} & m_{12} \\ 0 & m_{12} & m_{22} \end{bmatrix} = \begin{bmatrix} 1/3 & -1/6 & 0 \\ -1/6 & 2/3 & -5/6 \\ 0 & -5/6 & 1/3 \end{bmatrix}$$

$$m_{00} = \int_{-1}^{0} (-x)^2 \, dx = \frac{1}{3} x^3 \Big|_{-1}^{0} = \frac{1}{3}$$

$$m_{11} = \int_{-1}^{0} (x+1)^2 \, dx + \int_{0}^{1} (-x+2)^2 \, dx = \frac{2}{3}$$

$$m_{22} = \int_{0}^{1} (x-1)^2 \, dx = \frac{1}{3}$$

$$m_{01} = \int_{-1}^{0} -x(x+1) \, dx = -\frac{x^3}{3} - \frac{x^2}{2} \Big|_{-1}^{0} = -\left(\frac{1}{3} - \frac{1}{2}\right) = +\frac{1}{6}$$

$$m_{12} = \int_{0}^{1} (-x+2)(x-1) \, dx = \int_{0}^{1} (-x^2 + 3x - 2) \, dx = -\frac{x^3}{3} + \frac{3x^2}{2} - 2x \Big|_{0}^{1} = -\frac{2}{6} + \frac{9}{6} - \frac{12}{6}$$

$$= -\cancel{5/6}$$

$$1/6$$

$f_1$

$$l = \begin{bmatrix} l_0 \\ l_1 \\ l_2 \end{bmatrix} \quad ; \quad f(x) = \cos(4\pi x) \quad ; \quad l_j = \int_{-1}^{1} f \varphi_j$$

$$l_0 = \int_{-1}^{0} \varphi_0 \cdot f(x) dx = \int_{-1}^{0} (-x) \cdot \cos(4\pi x) \, dx = \left[ -\frac{\cos(4\pi x)}{16\pi^2} - \frac{x \sin[4\pi x]}{4\pi} \right]_{-1}^{0} = 0$$

$$l_1 = \int_{-1}^{1} \varphi_1 f(x) \, dx = \int_{-1}^{0} (x+1) \cos(4\pi x) \, dx + \int_{0}^{1} (-x+2) \cos(4\pi x) \, dx =$$

$$= \left[ \frac{\cos(4\pi x)}{16\pi^2} + \frac{\sin(4\pi x)}{4\pi} + \frac{x \sin(4\pi x)}{4\pi} \right]_{-1}^{0} + \left[ -\frac{\cos(4\pi x)}{16\pi^2} + \frac{\sin(4\pi x)}{2\pi} - \frac{x \sin[4\pi x]}{4\pi} \right]_{0}^{1} =$$

$$= 0 + 0 = 0$$

$$l_2 = \int_{0}^{1} \varphi_2 \cdot f(x) \, dx = \int_{0}^{1} (x-1) \cos(4\pi x) \, dx = [\ldots = 0$$

We now solve $MF = l$ where $F$ are coefficients

$$1/3 \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 2 & -5/2 \\ 0 & -5/2 & 1 \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow F = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Combining coefficients and basis to form $f_w$:

$$f_w = F_0 \varphi_0 + F_1 \varphi_1 + F_2 \varphi_2 = \begin{cases} 0 & ; \; x \in [-1, 1] \\ 0 & ; \; x \in [1, 2) \\ 0 & ; \; \text{else} \end{cases}$$

11.3.4 Comparison

The strong interpolation relies only on the function values in the nodal points and is therefore interpolated correctly in those points, but as a function it is a constant of 1 (since it is based only on nodal values).
On the other hand, the weak interpolation is based on the average value of the whole function. On average the function is indeed a constant 0.