CS323 LECTURE NOTES - LECTURE 11

1 Interpolation and Approximation

1.1 Interpolation and Extrapolation

One of themain objectives of the search for mathematical models that describe the behavior of reality is to be able to predict future experimental results.

Suppose that we want to predict the population of the United States in the year 2030. Can we use data from past census to do our prediction? Suppos that we have census data stored on a table:

year	**population**
1950	
1960	
$\vdots$	$\vdots$
2010	

If we plot the data we might be able to find a tendency and make a prediction. This prediction of a value outside of the range of the given data is called *extrapolation*

If now we want to know the population in the year 2005, we can also use the same initial data, but now our "guess" will be within the initial range of data. This is called *interpolation*.

In the case of extrapolation, if the value that we want to predict is very far from the initial range, the prediction will be bad.

1.2 Polynomial interpolation

suppose that we have a set of pairs (x, y)

x_1	y_1
x_2	y_2
x_3	y_3
$\vdots$	$\vdots$
x_n	y_n

where there are no duplicate values of x, i.e., $x_i \neq x_j$ si $i \neq j$. We want to find a degree n polynomial $P_n(x)$ that goes exactly through each one of the points, i.e.

$$\begin{aligned}
P(x_1) &= y_1 \\
P(x_2) &= y_2 \\
P(x_3) &= y_3 \\
\vdots &\quad \vdots \\
P(x_n) &= y_n
\end{aligned}$$

The first question we can ask is: What is the degree of the polynomial that we are looking for? Notice that if we are given two points, the lowest degree polynomial that goes through the points is a line (degree 1 polynomial). If we are given 3 points, the polynomial will be a parabola (degree 2 polynomial), and so on. If we are given $n+1$ point, the polynomial will have degree=n.

We can easily prove the following theorem:

Theorem

the polynomial of degree n that goes exactly through $n+1$ points is unique.

Proof

We know that a given n-degree polynomial has exactly n roots. Suppose that there are two distinct n-degree polynomials $P_n(x)$ y $Q_n(x)$ that agree on the points $x_1, x_2, \ldots, x_{n+1}$, i.e.

$$P_{n-1}(x_i) = Q_{n-1}(x_i) \quad \forall i = 1, \ldots n+1$$

Let us define the following polynomial

$$R_n(x) = P_n(x) - Q_n(x)$$

This polynomial clearly satisfies

$$R_n(x_i) = 0 \quad \forall i = 1, \ldots, n+1$$

but we know from the **Fundamental Theorem of Algebra** that the only n-degree polynomial with $n+1$ roots is the 0 polynomial. Therefore,

$$R_n(x) = 0; \;\; \forall x \in \mathbb{R}$$

which implies that

$$P_n(x) = Q_n(x)$$

which completes the proof of the theorem.

1.3 Lagrange Polynomials

Suppose that we are able to find $n+1$ polynomials of degree n that satisfy the condition $\forall j = 1 \dots n+1$

$$\mathcal{L}_j(x_i) = \begin{cases} 1 & \text{si} \quad i = j \\ 0 & \text{si} \quad i \neq j \end{cases}$$

To see how we can use such polynomials suppose that we want to find a degree-2 polynomial that goes through the points:

$x_1 = 1$	$y_1 = 6$
$x_2 = 2$	$y_2 = 4$
$x_3 = 3$	$y_3 = 8$

$n = 2$

Suppose that we have three 2nd-degree polynomials $\mathcal{L}_1$, $\mathcal{L}_2$ y $\mathcal{L}_3$ such that:

$\mathcal{L}_1(x_1) = 1$	$\mathcal{L}_2(x_1) = 0$	$\mathcal{L}_3(x_1) = 0$
$\mathcal{L}_1(x_2) = 0$	$\mathcal{L}_2(x_2) = 1$	$\mathcal{L}_3(x_2) = 0$
$\mathcal{L}_1(x_3) = 0$	$\mathcal{L}_2(x_3) = 0$	$\mathcal{L}_3(x_3) = 1$

(1)

Using these polynomials we can easily write an interpolating polynomial of degree 2 that goes exactly through each one of the given 3 points:

$$P_2(x) = \mathcal{L}_1(x)y_1 + \mathcal{L}_2(x)y_2 + \mathcal{L}_3(x)y_3$$

We can verify thta $P_2(x)$ goes through the 3 given points:

$$\begin{aligned} P_2(x_1) &= \mathcal{L}_1(x_1)y_1 + \mathcal{L}_2(x_1)y_2 + \mathcal{L}_3(x_1)y_3 \\ &= (1)y_1 + (0)y_2 + (0)y_3 \\ &= y_1 \end{aligned}$$

$$\begin{aligned} P_2(x_2) &= \mathcal{L}_1(x_2)y_1 + \mathcal{L}_2(x_2)y_2 + \mathcal{L}_3(x_2)y_3 \\ &= (0)y_1 + (1)y_2 + (0)y_3 \\ &= y_2 \end{aligned}$$

$$\begin{aligned}
P_2(x_2) &= \mathcal{L}_1(x_2)y_1 + \mathcal{L}_2(x_2)y_2 + \mathcal{L}_3(x_2)y_3 \\
&= (0)y_1 + (0)y_2 + (1)y_3 \\
&= y_3
\end{aligned}$$

We now only have to find $\mathcal{L}_1$, $\mathcal{L}_2$ y $\mathcal{L}_3$. Which is easy if we notice that

$$q(x) = (x - x2)(x - x3)$$

is a polynomial of degree 3 such that

$$\begin{aligned}
q(x_1) &= (x1 - x2)(x1 - x3) \\
q(x_2) &= 0 \\
q(x_3) &= 0
\end{aligned}$$

Since we know that $x_1 \neq x_2$ y $x_1 \neq x_3$, we can define

$$\mathcal{L}_1(x) = \frac{q(x)}{q(x_1)} = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

In the same way we have

$$\begin{aligned}
\mathcal{L}_2(x) &= \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} \\
\mathcal{L}_3(x) &= \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}
\end{aligned}$$

We can easily verify that these polynomials satisfy conditions **(1)**, therefore, the 2nd-degree polynomial that goes through the given points is:

$$P_2(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}y_1 + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}y_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}y_3$$

This polynomial is known as *The Lagrange Polynomial.* It can easily be generalized to a polynomial of degree n if we are given $n+1$ points:

$$\mathcal{L}_i(x) = \Pi_{j=1, j\neq i}^{n+1} \frac{x - x_j}{x_i - x_j}$$

(2)

and

$$P_n(x) = \sum_{i=1}^{n+1} \mathcal{L}_i(x) y_i$$

(3)

The number of factors that appear in each product that defines $\mathcal{L}_i$ is n, hence the degree of each polynomial is $n-1$.

Example

Find the Lagrange polynomial that goes through the points:

x	y
1	6
2	4
3	8

Notice that since we have 3 points, the degree of the polynomial that we are looking for must be 2. Using the formulas for $\mathcal{L}$ given above, we have that

$$\begin{aligned}
\mathcal{L}_1(x) &= \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}(x^2 - 5x + 6) \\
\mathcal{L}_2(x) &= \frac{(x-1)(x-3)}{(2-1)(2-3)} = -(x^2 - 4x + 3) \\
\mathcal{L}_3(x) &= \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{1}{2}(x^2 - 3x + 2)
\end{aligned}$$

If we substitute in $P_2(x)$ we get

$$P_2(x) = \frac{1}{2}(x^2 - 5x + 6)(6) - (x^2 - 4x + 3)(4) + \frac{1}{2}(x^2 - 3x + 2)(8)$$

So we have that the interpolating polynomial that goes through the given points is:

$$P_2(x) = 3x^2 - 11x + 14$$