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Brief paper

Spacecraft attitude control using magnetic actuators☆

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Abstract

The problem of inertial pointing for a spacecraft with magnetic actuators is addressed and an almost global solution to the problem is obtained by means of static attitude and rate feedback. A local solution based on dynamic attitude feedback is also presented. Simulation results demonstrate the practical applicability of the proposed approach.

Keywords: Attitude control; Spacecraft magnetic control; Nonlinear control; Averaging theory

1. Introduction

The problem of attitude regulation of rigid spacecraft, i.e., spacecraft modelled by the Euler's equations and by a suitable parameterisation of the attitude, has been widely studied in recent years. If the spacecraft is equipped with three independent actuators, a complete solution to the set point and tracking control problems is available. In Wen and Kreutz-Delgado (1991) and Fjellstad and Fossen (1994) these problems have been solved by means of PD-like control laws, i.e., control laws which make use of the angular velocity and of the attitude, whereas in Akella (2001) and Caccavale and Villani (1999), building on the general results developed in Battilotti (1996) and Lizarralde and Wen (1996) the same problems have been solved using dynamic output feedback control laws. It is worth noting that if only two independent actuators are available, i.e., if the spacecraft is underactuated, as discussed in detail in Byrnes and Isidori (1991), the problem of attitude regulation is not solvable by means of continuous (static or dynamic) time-invariant control laws, whereas a time-varying control law, achieving local asymptotic (nonexponential) stability, has been proposed in Morin, Samson, Pomet, and Jiang (1995).

The above results, however, are not directly applicable if the spacecraft is equipped with magnetic coils as actuators. Indeed, magnetic actuators operate on the basis of the interaction between a set of three orthogonal current-driven coils with the geomagnetic field and this has a number of implications which make the magnetic spacecraft control problem significantly different from the conventional attitude regulation one. First of all, it is not possible by means of magnetic actuators to provide three independent control torques at each time instant. In addition, the behaviour of these actuators is intrinsically time-varying, as the control mechanism hinges on the variations of the Earth magnetic field along the spacecraft orbit. Nevertheless, attitude stabilisation is possible because *on average* the system possesses strong controllability properties for a wide range of orbit inclinations.

A considerable amount of work has been dedicated in recent years to the problems of analysis and design of magnetic control laws in the linear case, i.e., control laws for *nominal* operation of a satellite near its equilibrium attitude. In particular, nominal and robust stability and performance have been studied, using either tools from periodic control theory exploiting the (quasi) periodic behaviour of the system near an equilibrium (see, e.g., Pittelkau (1993), Wisniewski and Markley (1999), Lovera, De Marchi, and Bittanti (2002)

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and Psiaki (2001)) or other techniques aiming at developing suitable time-varying controllers Steyn (1994) and Curti and Diani (1999).

On the other hand, very little attention has been dedicated to global formulations of the magnetic attitude control problem. In Wisniewski and Blanke (1999), Damaren (2002) and Arduini and Baiocco (1997) the attitude regulation problem for Earth pointing spacecraft has been addressed exploiting periodicity assumptions on the system, hence resorting to standard passivity arguments to prove local asymptotic stabilisability of stable open loop equilibria. In Wang and Shtessel (1998) similar arguments have been used to analyse a state feedback control law for the particular case of an inertially spherical spacecraft.

However, several problems remain open. In particular, if inertial pointing is considered, the global stabilisation problems by means of full (or partial) state feedback is still theoretically unsolved. Note, in passing, that from a practical point of view these problems have an engineering solution, as demonstrated by the increasing number of applications of this approach to attitude control.

In the light of the above discussion, the aims of this paper can be summarised as follows.

- To obtain stability conditions for state feedback control laws achieving inertial pointing for magnetically actuated spacecraft. This result can be achieved by means of arguments similar to those in Wen and Kreutz-Delgado (1991), provided that time-varying feedback laws are used and that the control gains satisfy certain scaling properties. In particular, with respect to previous work dealing only with the case of a magnetically actuated, isoinertial spacecraft Lovera and Astolfi (2001), this paper deals with a generic magnetically actuated satellite. For this problem, an almost global [1] stabilisation result is given for the case of full state feedback.
- To extend the applicability of the partial (i.e., attitude only) state feedback results of Battilotti (1996), Lizarralde and Wen (1996) and Akella (2001) from the case of a spacecraft with three independent controls to the case of a magnetically controlled spacecraft. For the case of partial state feedback, however, almost global stabilisation can be guaranteed only in the case of isoinertial spacecraft.
- To show that these stability conditions, which can take also actuator saturation into account, provide very useful and novel guidelines for the design and tuning of magnetic attitude controllers.

Finally, note that the results presented herein do not rely on the (frequently adopted) periodicity assumption for the geomagnetic field along the considered orbit, which is correct only to first approximation (see, e.g., Wertz, 1978).

The paper is organised as follows. In Section 2 the model of the system is presented. In Section 3 some results on the state feedback stabilisation of magnetically actuated spacecraft are presented. In Section 4 the design of control laws using only attitude information, so avoiding the need for rate measurements in the control system, is discussed. Finally, Sections 5 and 6 present some simulation results and concluding remarks.

[1] Given a system $\dot{x} = f(x)$ we say that an equilibrium x_0 is almost globally asymptotically stable if it is locally asymptotically stable, all the trajectories of the system are bounded and the set of initial conditions giving rise to trajectories which do not converge to x_0 has zero Lebesgue measure.

2. Spacecraft model

The model of a rigid spacecraft with magnetic actuation can be described in various reference frames (Wertz, 1978). For the purpose of the present analysis, the following reference systems are adopted.

- Earth-centered inertial reference axes (ECI). The origin of these axes is in the Earth's centre. The X-axis is parallel to the line of nodes, that is the intersection between the Earth's equatorial plane and the plane of the ecliptic, and is positive in the Vernal equinox direction (Aries point). The Z-axis is defined as being parallel to the Earth's geographic north–south axis and pointing north. The Y-axis completes the right-handed orthogonal triad.
- Satellite body axes. The origin of these axes is in the satellite centre of mass; the axes are assumed to coincide with the body's principal inertia axes.

The attitude dynamics can be expressed by the well known Euler's equations (Wertz, 1978).

$$I\dot{\omega} = S(\omega)I\omega + T_{\text{coils}} + T_{\text{dist}}, \tag{1}$$

where $\omega \in \mathbb{R}^3$ is the vector of spacecraft angular rates, expressed in body frame, $I \in \mathbb{R}^{3\times 3}$ is the inertia matrix, $S(\omega)$ is given by

$$S(\omega) = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}, \tag{2}$$

$T_{\text{coils}} \in \mathbb{R}^3$ is the vector of external torques induced by the magnetic coils and $T_{\text{dist}} \in \mathbb{R}^3$ is the vector of external disturbance torques. As the focus of this paper is on the stabilisation problem, disturbance torques will be taken into account only in the simulation study (Section 5), in which their role in affecting the closed-loop system's behaviour will be assessed. Indeed, when the control system of a real spacecraft has to deal with attitude and angular rate errors large enough to require a nonlinear treatment of the control problem, it turns out that the required control torques are usually large enough to render disturbance torques negligible for all practical purposes. In actual implementations, only local control laws, for nominal equilibrium operation of the spacecraft, are optimised for disturbance attenuation; note, however, that magnetic actuators introduce performance limitations in this respect and make explicit disturbance compensation a difficult task (see, e.g., Pittelkau (1993), Wisniewski and Markley (1999), Psiaki (2001) and Lovera et al. (2002)).

In turn, the attitude kinematics can be described by means of a number of possible parameterisations (see, e.g., Wertz, 1978). The most common parameterisation is given by the four Euler parameters (or quaternions), which lead to the following representation for the attitude kinematics

$$\dot{\boldsymbol{q}} = W(\omega)\boldsymbol{q}, \tag{3}$$

where $\boldsymbol{q} = [q_1\ q_2\ q_3\ q_4]^{\mathrm{T}} = [\boldsymbol{q}^{\mathrm{T}}\ q_4]^{\mathrm{T}}$ is the vector of unit norm ($\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q} = 1$) Euler parameters and

$$W(\omega) = \frac{1}{2}\begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix}. \tag{4}$$

It is useful to point out that Eq. (3) can be equivalently written as

$$\dot{\boldsymbol{q}} = \tilde{W}(\boldsymbol{q})\omega, \tag{5}$$

where

$$\tilde{W}(\boldsymbol{q}) = \frac{1}{2}\begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}. \tag{6}$$

Note that the attitude of inertially pointing spacecraft is usually referred to the ECI reference frame.

The magnetic attitude control torques are generated by a set of three magnetic coils, aligned with the spacecraft principal inertia axes, which generate torques according to the law

$$T_{\text{coils}} = m_{\text{coils}} \times \tilde{b}(t), \tag{7}$$

where $\times$ denotes the vector cross product, $m_{\text{coils}} \in \mathbb{R}^3$ is the vector of magnetic dipoles for the three coils (which represent the actual control variables for the coils), $\tilde{b}(t) \in \mathbb{R}^3$ is the vector formed with the components of the Earth's magnetic field in the body frame of reference. Note that the vector $\tilde{b}(t)$ can be expressed in terms of the attitude matrix $A(\boldsymbol{q})$ (see Wertz, 1978 for details) and of the magnetic field vector expressed in the ECI coordinates, namely $\tilde{b}_0(t)$, as

$$\tilde{b}(t) = A(\boldsymbol{q})\tilde{b}_0(t) \tag{8}$$

and that the orthogonality of $A(\boldsymbol{q})$ implies $\|\tilde{b}(t)\| = \|\tilde{b}_0(t)\|$. The dynamics of the magnetic coils reduce to a very short electrical transient and can be neglected. The cross product in Eq. (7) can be expressed more simply as a matrix-vector product as

$$T_{\text{coils}} = S(\tilde{b}(t))m_{\text{coils}}. \tag{9}$$

Note that since $S(\tilde{b}(t))$ is structurally singular, as mentioned in the Introduction, magnetic actuators do not provide full controllability of the system at each time instant. In particular, it is easy to see that rank $(S(\tilde{b}(t)))=2$ (since $\|\tilde{b}_0(t)\| \neq 0$ along all orbits of practical interest for magnetic control) and that the kernel of $S(\tilde{b}(t))$ is given by the vector $\tilde{b}(t)$ itself, i.e., at each time instant it is *not* possible to apply a control torque along the direction of $\tilde{b}(t)$.

If a preliminary feedback of the form

$$m_{\text{coils}} = \frac{1}{\|\tilde{b}_0(t)\|^2} S^{\mathrm{T}}(\tilde{b}(t))u \tag{10}$$

is applied to the system, where $u \in \mathbb{R}^3$ is a new control vector, the overall dynamics can be written as

$$\begin{aligned} \dot{\boldsymbol{q}} &= \tilde{W}(\boldsymbol{q})\omega, \\ I\dot{\omega} &= S(\omega)I\omega + \Gamma(t)u, \end{aligned} \tag{11}$$

where $\Gamma(t)=S(b(t))S^{\mathrm{T}}(b(t)) \geqslant 0$ and $b(t)=1/\|\tilde{b}_0(t)\|\tilde{b}(t)=1/\|\tilde{b}(t)\|\tilde{b}(t)$. Similarly, let $\Gamma_0(t) = S(b_0(t))S^{\mathrm{T}}(b_0(t)) \geqslant 0$ and $b_0(t) = 1/\|\tilde{b}_0(t)\|\tilde{b}_0(t)$. Note, also, that $\Gamma(t)$ can be written as $\Gamma(t) = \mathscr{I}_3 - b(t)b(t)^{\mathrm{T}}$, where $\mathscr{I}_3$ is the 3×3 identity matrix and $\Gamma(t) \geqslant 0$. We now prove a preliminary result which will be exploited in the next section.

Lemma 1. *Consider system (11) and assume that the considered orbit for the spacecraft satisfies the condition*

$$\bar{\Gamma}_0 = \lim_{T\to\infty} \frac{1}{T}\int_0^T S(b_0(t))S^{\mathrm{T}}(b_0(t))\,\mathrm{d}t > 0.$$

Then, there exists $\omega_M > 0$ such that if $\|\omega\| < \omega_M$ for all $t > \bar{t}$, for some $0 < \bar{t} < \infty$, then

$$\bar{\Gamma} = \lim_{T\to\infty} \frac{1}{T}\int_0^T S(b(t))S^{\mathrm{T}}(b(t))\,\mathrm{d}t > 0 \tag{12}$$

along the trajectories of system (11).

Proof. Consider first the particular case $\omega=0$, which implies that $\boldsymbol{q} = \bar{\boldsymbol{q}} = \text{const}$. If $\bar{\Gamma}$ is singular there exists a nonzero vector $\bar{v}$ such that

$$\bar{v}^{\mathrm{T}}\bar{\Gamma}\bar{v} = 0 \tag{13}$$

and $v_0 = A(\bar{\boldsymbol{q}})^{\mathrm{T}}\bar{v}$. However, (13) and (8) imply that

$$v_0^{\mathrm{T}}\bar{\Gamma}_0 v_0 = 0, \tag{14}$$

which contradicts the assumption. Finally, continuity arguments suffice to guarantee that (12) holds provided that ω is sufficiently small for all $t > \bar{t}$. $\square$

Lemma 1 lends itself to a very simple physical interpretation. Condition $\det(\bar{\Gamma}) = 0$ defines the set of all trajectories along which average controllability is lost. Clearly, this represents a nongeneric condition, as it implies that the combination of the natural, on-orbit variability of $b_0(t)$ with the attitude motion of the satellite gives rise to a *constant* magnetic field vector ($b(t) = \bar{b}$) in the body reference frame. Such a condition, however, can only arise whenever the angular rate of the spacecraft is sufficiently large, hence average controllability in the sense of (12) is guaranteed for sufficiently small ω.

3. State feedback stabilisation

In this section, a general stabilisation result for a spacecraft with magnetic actuators is given in the case of full state feedback (attitude and rate). Without loss of generality, in the following we assume that the equilibrium to be stabilised is given by $(\bar{\boldsymbol{q}}, 0)$, where $\bar{\boldsymbol{q}} = [0\ 0\ 0\ 1]^{\mathrm{T}}$ and we denote by $CN(A)$ the condition number of the matrix A.

Proposition 1. *Consider the magnetically actuated spacecraft described by* (11) *and the control law*

$$u = -(\varepsilon^2 k_p \mathbf{q} + \varepsilon k_v I\omega). \tag{15}$$

Suppose that $0 < \bar{\Gamma}_0 < \mathscr{I}_3$. *Then there exist* $\varepsilon^\star > 0$, $k_p > 0$ *and* $k_v > 0$ *with*

$$k_v^2 > k_p \frac{\sigma^2_{\min}(\bar{\Gamma})}{\sigma_{\min}(I)} \sqrt{CN(\bar{\Gamma})} \tag{16}$$

such that for any $0 < \varepsilon < \varepsilon^\star$ *the control law* (15) *ensures that* $(\bar{\boldsymbol{q}}, 0)$ *is a locally exponentially stable equilibrium for the closed-loop system* (11)–(15). *Moreover, all trajectories of* (11)–(15) *are such that* $\mathbf{q} \to 0$ *and* $\omega \to 0$.

Proof. To begin with we prove that for all $k_p > 0$ and $k_v > 0$ there exists $\varepsilon > 0$ such that for the closed-loop system (11)–(15) $\bar{\Gamma} > 0$. For, consider the ω-sub-system only and the function

$$V_1 = \frac{\lambda}{2}\,\omega^{\mathrm{T}} I^2 \omega - \frac{1}{2}\,\omega^{\mathrm{T}} I A(\boldsymbol{q}) M(t) A(\boldsymbol{q})^{\mathrm{T}} I\omega, \tag{17}$$

where $\lambda > 0$,

$$M(t) = \int_0^t (b_0(\tau) b_0(\tau)^{\mathrm{T}} - N)\,\mathrm{d}\tau \tag{18}$$

and $N \geqslant 0$ is a constant matrix. The assumption $\bar{\Gamma}_0 < \mathscr{I}_3$ implies that it is possible to select N such that $-\sigma\mathscr{I}_3 \leqslant M(t) \leqslant \sigma\mathscr{I}_3$ for some positive σ. Note that V_1 is positive definite for sufficiently large λ. The time derivative of V_1 is given by

$$\begin{aligned}\dot{V}_1 = &-\omega^{\mathrm{T}} I A(\boldsymbol{q}) Q A(\boldsymbol{q})^{\mathrm{T}} I\omega \\ &-\varepsilon^2 k_p \omega^{\mathrm{T}} I(\lambda\mathscr{I}_3 - A(\boldsymbol{q}) M(t) A(\boldsymbol{q})^{\mathrm{T}})\Gamma(t)\mathbf{q},\end{aligned} \tag{19}$$

where

$$\begin{aligned}Q = \bigg(&\varepsilon k_v \lambda \Gamma_0(t) - \frac{\varepsilon k_v}{2} M(t)\Gamma_0(t) \\ &- \frac{\varepsilon k_v}{2}\Gamma_0(t) M(t) + b_0 b_0^{\mathrm{T}} - N\bigg).\end{aligned} \tag{20}$$

Introduce now the time varying vectors $b_1(t)$ and $b_2(t)$ such that $b_i^{\mathrm{T}} b_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta and $i, j = 0, 1, 2$, and let

$$\tilde{Q} = \begin{bmatrix} b_0^{\mathrm{T}} \\ b_1^{\mathrm{T}} \\ b_2^{\mathrm{T}} \end{bmatrix} Q [b_0\ \ b_1\ \ b_2]. \tag{21}$$

Then, it is easy to show that for any $\varepsilon > 0$ there exists a $\lambda > 0$ and sufficiently large such that $\tilde{Q}$ (and, therefore, Q) is positive definite. As a result

$$\begin{aligned}\dot{V}_1 \leqslant &-\psi(\varepsilon,\lambda)\omega^{\mathrm{T}}\omega + \frac{\varepsilon^2 k_p \lambda}{2}\,\omega^{\mathrm{T}} I^2 \omega \\ &+ \frac{\varepsilon^2 k_p \lambda}{2}\,\mathbf{q}^{\mathrm{T}} \Gamma^2(t)\mathbf{q} + \frac{\varepsilon^2 k_p}{2}\,\omega^{\mathrm{T}} I^2 \omega \\ &+ \frac{\varepsilon^2 k_p}{2}\,\mathbf{q}^{\mathrm{T}} \Gamma(t) A(\boldsymbol{q}) M^2(t) A(\boldsymbol{q})^{\mathrm{T}} \Gamma(t)\mathbf{q},\end{aligned} \tag{22}$$

where $\psi(\varepsilon, \lambda) > \varepsilon$ for all $\varepsilon > 0$ and $\lambda > 0$ and sufficiently large, from which one has

$$\begin{aligned}\dot{V}_1 \leqslant &-(\varepsilon - \varepsilon^2 k_p \lambda \sigma^2_{\max}(I) - \varepsilon^2 k_p \sigma^2_{\max}(I))\omega^{\mathrm{T}}\omega \\ &+ \frac{\varepsilon^2 k_p \lambda}{2} + \frac{\varepsilon^2 k_p \sigma^2}{2}.\end{aligned} \tag{23}$$

This, in turn, implies that for any $\omega_M > 0$ there exists $\varepsilon > 0$ such that $\|\omega\| < \omega_M$ for sufficiently large t, and therefore, by Lemma 1, $\bar{\Gamma} > 0$.

Introduce now the coordinates transformation

$$z_1 = \boldsymbol{q} \qquad z_2 = \frac{\omega}{\varepsilon} \tag{24}$$

(so that $\mathbf{z}_1 = \mathbf{q}$ and $z_{14} = q_4$) in which system (11) is described by the equations

$$\begin{aligned}\dot{\boldsymbol{z}}_1 &= \varepsilon\tilde{W}(\boldsymbol{z}_1) z_2, \\ I\dot{z}_2 &= \varepsilon S(z_2) I z_2 + \varepsilon\Gamma(t)(-k_p \mathbf{z}_1 - k_v I z_2).\end{aligned} \tag{25}$$

System (25) satisfies all the hypotheses [2] for the applicability of the generalised averaging theory (see Khalil, 2001, Theorem 10.5), which yields the averaged system

$$\begin{aligned}\dot{\boldsymbol{z}}_1 &= \varepsilon\tilde{W}(\boldsymbol{z}_1) z_2, \\ I\dot{z}_2 &= \varepsilon S(z_2) I z_2 + \varepsilon\bar{\Gamma}(-k_p \mathbf{z}_1 - k_v I z_2).\end{aligned} \tag{26}$$

As a result, there exists $\varepsilon^\star > 0$ such that for any $0 < \varepsilon < \varepsilon^\star$ the trajectories of system (26) are close to the trajectories

[2] In particular, it is easy to verify that the Jacobian of the difference between the right-hand sides of (25) and (26) has zero average.

of system (25). Consider now the positive definite function

$$V_2(z_2) = \frac{1}{2} z_2^{\mathrm{T}} I^2 z_2, \tag{27}$$

and its time derivative

$$\dot{V}_2 = \varepsilon z_2^{\mathrm{T}} \bar{\Gamma}(-k_p \mathbf{z}_1 - k_v I z_2) \tag{28}$$

and note that, for any $\alpha > 0$,

$$\dot{V}_2 \leqslant -\varepsilon \left(k_v - \frac{k_p}{2\alpha} \right) z_2^{\mathrm{T}} I \bar{\Gamma} I z_2 + \varepsilon k_p \frac{\alpha}{2} \mathbf{z}_1^{\mathrm{T}} \bar{\Gamma} \mathbf{z}_1. \tag{29}$$

The positive definiteness of $\bar{\Gamma}$ and the boundedness of z_1 imply that, for a proper selection of k_p, k_v and α

$$\dot{V}_2 \leqslant -rV_2 + d \tag{30}$$

for some constants $r > 0$ and $d > 0$. In particular, for[3] $\alpha = \alpha^\star = k_p/k_v$, Eq. (30) implies that along the trajectories of the closed-loop system one has

$$\|z_2(t)\|^2 \leqslant \frac{k_p^2}{k_v^2} \frac{\mathrm{CN}(\bar{\Gamma})}{\sigma_{\min}^2(I)} = K^2, \tag{31}$$

for all $t \geqslant t^\star$ and for some $0 \leqslant t^\star \leqslant \infty$. As a result, for any $K > 0$ the set

$$Z_K = \{(z_1, z_2) \colon \|z_2\| < K\} \tag{32}$$

is attractive and positively invariant. Observe that K can be made arbitrarily small by a suitable choice of k_p and k_v.

We now prove that all trajectories of system (26) starting in set (32) are such that $\mathbf{z}_1 \to 0$ and $z_2 \to 0$. For, consider the Lyapunov function

$$V_3(z_1, z_2) = \frac{1}{2} k_p(\mathbf{z}_1^{\mathrm{T}} \mathbf{z}_1 + (z_{14} - 1)^2) + \frac{1}{2} z_2^{\mathrm{T}} \bar{\Gamma}^{-1} z_2, \tag{33}$$

and its time derivative

$$\dot{V}_3 = \varepsilon z_2^{\mathrm{T}} \bar{\Gamma}^{-1} S(z_2) I z_2 - \varepsilon k_v z_2^{\mathrm{T}} I z_2. \tag{34}$$

Note that

$$\dot{V}_3 \leqslant \varepsilon K \sigma_{\min}^2(\bar{\Gamma}) z_2^{\mathrm{T}} I z_2 - \varepsilon k_v z_2^{\mathrm{T}} I z_2, \tag{35}$$

which is negative if condition (16) holds. As a result, $z_2 \to 0$ and, applying La Salle invariance principle, $\mathbf{z}_1 \to 0$.

Finally, consider the linear approximation of system (26) around the equilibrium $(\bar{\boldsymbol{q}}, 0)$, which is given by

$$\begin{aligned} \dot{\mathbf{z}}_1 &= \frac{1}{2} \varepsilon z_2, \\ I\dot{z}_2 &= -\varepsilon \bar{\Gamma}(k_p \mathbf{z}_1 + k_v I z_2). \end{aligned} \tag{36}$$

It is easy to verify that

$$V_L(z_1, z_2) = 2k_p \mathbf{z}_1^{\mathrm{T}} \mathbf{z}_1 + z_2^{\mathrm{T}} \bar{\Gamma}^{-1} z_2 \tag{37}$$

is a Lyapunov function for the linear system (36), so the convergence of the trajectories of the closed-loop system is locally exponential. □

[3] It is easy to see that this selection of α is *optimal*.

Remark 2. Proposition 1 clarifies the main difference between magnetic attitude control and the fully actuated case. It has been shown in Wen and Kreutz-Delgado (1991) that whenever three independent torques are available, the state feedback problem can be solved via a PD control law and that almost global stability of the closed-loop system can be guaranteed for *any* choice of $k_p > 0$ and $k_v > 0$. This is not the case for magnetic attitude control, as the proportional and derivative actions must meet the scaling condition defined by ε in order to guarantee closed-loop stability. In this respect, this result provides a very useful guideline for the design of magnetic controllers in practical cases, as it combines the simplicity of a state feedback control law (as in Wen and Kreutz-Delgado (1991)) with an explicit stability condition. On the other hand, the choice of a suitable value for ε cannot be carried out on the basis of Proposition 1 only, but is likely to require some iterations of the tuning process.

Remark 3. The coordinate transformation (24) used in the proof of Proposition 1, and particularly the parameter ε, can be simply interpreted as a scaling of the time variable which enables the reformulation of the dynamics of the closed-loop system (25) as a small perturbation with respect to the averaged system (26).

When considering the problem of nonlinear attitude control, actuators saturations usually play an important role. In this respect it is worth pointing out that the above state feedback magnetic control law can be readily modified to deal with saturation of the magnetic coils, as expressed in the following statement.

Corollary 4. *Consider system* (11) *and the state feedback control law*[4]

$$u = -\varepsilon^2 k_p \mathbf{q} - \varepsilon \beta \,\mathrm{sat}\left(k_v \frac{I\omega}{\beta} \right), \tag{38}$$

with $\beta > 0$ and suppose that $0 < \bar{\Gamma}_0 < \mathscr{I}_3$. Then for any $\rho > 0$ there exist $\varepsilon^\star > 0$, $k_p > 0$ and $k_v > 0$, satisfying condition (16), *such that for any $0 < \varepsilon < \varepsilon^\star$ the control law* (38) *ensures that $(\bar{\boldsymbol{q}}, 0)$ is a locally exponentially stable equilibrium for the closed-loop system* (11)–(38), *all trajectories of* (11)–(38) *are such that $\mathbf{q} \to 0$, $\omega \to 0$ and*

$$|u_i| \leqslant \rho. \tag{39}$$

Proof. The proof of the first two statements is similar to the proof of Proposition 1. To prove bound (39), note that

$$|u_i| \leqslant \varepsilon^2 k_p + \varepsilon \beta, \tag{40}$$

and this can be made arbitrarily small by a proper selection of the design parameters. □

[4] By sat(·) we indicate a continuous saturating function limited between −1 and 1.

Remark 5. Bound (39) on the signals u implies the bound

$$|m_{\text{coils}_i}| \leqslant \|b_o\|\rho \tag{41}$$

on the actual control inputs m_{coils}.

Remark 6. The parameter β in the control law (38) is used only to assign the amplitude of the saturation function.

Remark 7. An interesting particular case is the one of a spacecraft that has an inertia matrix which is proportional to the identity matrix $\mathscr{I}_3$, i.e.

$$I = \kappa \mathscr{I}_3 \tag{42}$$

for some $\kappa > 0$, so that $S(\omega)I\omega = 0$ for all ω. In this case one can achieve convergence of the trajectories of the closed-loop system (11)–(15) for any positive k_p and k_v and $0 < \varepsilon < \varepsilon^\star$, as the derivative of the Lyapunov function V_3 reduces to

$$\dot{V}_3 = -\varepsilon k_v z_2^{\mathrm{T}} I z_2. \tag{43}$$

The same considerations apply to the closed-loop system (11)–(38), i.e., in the presence of saturations.

4. Stabilisation without rate feedback

The ability of ensuring attitude regulation without rate feedback is of great importance from a practical point of view. In this section, an approach similar to the one used in Akella (2001) for the case of a fully actuated spacecraft is followed for the case of magnetic attitude control, and an *almost global* result is given in the case of an isoinertial spacecraft. In addition, a local stability result is derived for a generic satellite.

Proposition 8. *Consider system* (11) *with I such that* (42) *holds, and the control law*

$$\begin{aligned}
\dot{\delta} &= \alpha(\boldsymbol{q} - \varepsilon\lambda\delta),\\
u &= -\varepsilon^2(k_p \mathbf{q} + k_v \alpha\lambda \tilde{W}^{\mathrm{T}}(\boldsymbol{q})(\boldsymbol{q} - \varepsilon\lambda\delta)).
\end{aligned} \tag{44}$$

Suppose that $0 < \bar{\Gamma}_0 < \mathscr{I}_3$. *Then there exists* $\varepsilon^\star > 0$, $k_p > 0$, $k_v > 0$, $\alpha > 0$ *and* $\lambda > 0$ *such that for any* $0 < \varepsilon < \varepsilon^\star$ *the control law renders the equilibrium* $(\bar{\boldsymbol{q}}, 0, \frac{1}{\varepsilon\lambda}\bar{\boldsymbol{q}})$ *of the closed-loop system* (11)–(44) *locally exponentially stable. Moreover, the equilibrium is almost globally asymptotically stable.*

Proof. As in the case of the proof of Proposition 1, introduce the coordinates transformation

$$z_1 = \boldsymbol{q} \quad z_2 = \frac{\omega}{\varepsilon} \quad z_3 = \varepsilon\delta, \tag{45}$$

in which system (11)–(44) is described by the equations

$$\begin{aligned}
\dot{\boldsymbol{z}}_1 &= \varepsilon \tilde{W}(\boldsymbol{z}_1) z_2,\\
\dot{z}_2 &= -\frac{\varepsilon}{\kappa}\Gamma(t)(k_p \mathbf{z}_1 + k_v \alpha\lambda \tilde{W}^{\mathrm{T}}(\boldsymbol{z}_1)(\boldsymbol{z}_1 - \lambda z_3)),\\
\dot{z}_3 &= \varepsilon\alpha(\boldsymbol{z}_1 - \lambda z_3).
\end{aligned} \tag{46}$$

System (46) satisfies all the hypotheses for the applicability of generalised averaging theory (Khalil (2001), Theorem 10.5), which leads to the averaged system

$$\begin{aligned}
\dot{\boldsymbol{z}}_1 &= \varepsilon \tilde{W}(\boldsymbol{z}_1) z_2,\\
\dot{z}_2 &= -\frac{\varepsilon}{\kappa}\bar{\Gamma}(k_p \mathbf{z}_1 + k_v \alpha\lambda \tilde{W}^{\mathrm{T}}(\boldsymbol{z}_1)(\boldsymbol{z}_1 - \lambda z_3)),\\
\dot{z}_3 &= \varepsilon\alpha(\boldsymbol{z}_1 - \lambda z_3).
\end{aligned} \tag{47}$$

Consider now the Lyapunov function

$$\begin{aligned}
V_4(z_1, z_2, z_3) &= \frac{1}{2} k_p(\mathbf{z}_1^{\mathrm{T}}\mathbf{z}_1 + (z_{14} - 1)^2) + \frac{1}{2} z_2^{\mathrm{T}} \bar{\Gamma}^{-1} z_2\\
&\quad + \frac{1}{2} k_v (\boldsymbol{z}_1 - \lambda z_3)^{\mathrm{T}}(\boldsymbol{z}_1 - \lambda z_3),
\end{aligned} \tag{48}$$

yielding

$$\dot{V}_4 = -k_v\lambda(\boldsymbol{z}_1 - \lambda z_3)^{\mathrm{T}}(\boldsymbol{z}_1 - \lambda z_3). \tag{49}$$

As a result, $\boldsymbol{z}_1 - \lambda z_3 \to 0$, hence $z_2 \to 0$ and $\mathbf{z}_1 \to 0$, provided that

$$\bar{\Gamma} = \lim_{T\to\infty} \frac{1}{T}\int_0^T S(b(t))S^{\mathrm{T}}(b(t))\,\mathrm{d}t > 0.$$

Note now that, by Lemma 1, any trajectory starting sufficiently close to the equilibrium $(\bar{q}, 0, \frac{1}{\varepsilon\lambda}\bar{q})$ is such that $\bar{\Gamma} > 0$, and this together with the above Lyapunov arguments proves local exponential stability of the equilibrium.

To complete the proof, we need to show that the set of initial conditions yielding *bad* trajectories, i.e., trajectories that do not converge to the equilibrium $(\bar{\boldsymbol{q}}, 0, \frac{1}{\varepsilon\lambda}\bar{\boldsymbol{q}})$, has zero (Lebesgue) measure. These *bad* trajectories are those converging to the equilibrium $(-\bar{\boldsymbol{q}}, 0, -\frac{1}{\varepsilon\lambda}\bar{\boldsymbol{q}})$ and those for which $\bar{\Gamma}$ is singular. Note that the equilibrium $(-\bar{\boldsymbol{q}}, 0, -\frac{1}{\varepsilon\lambda}\bar{\boldsymbol{q}})$ is unstable, hence the set of initial conditions yielding trajectories converging to $(-\bar{\boldsymbol{q}}, 0, -\frac{1}{\varepsilon\lambda}\bar{\boldsymbol{q}})$ is composed only by the associated stable manifold, which has zero measure. Finally, the trajectories such that $\bar{\Gamma}$ is singular are nongeneric, hence the equilibrium is almost globally asymptotically stable. □

Remark 9. The signal u generated by the output feedback control law (44) is bounded, provided that $\delta(0)$ is properly selected. Indeed, if

$$\varepsilon\lambda\delta_i(0) \in [-1, 1], \tag{50}$$

then

$$|\varepsilon\lambda\delta_i(t)| \in [-1, 1], \tag{51}$$

and therefore

$$|u_i(t)| \leqslant \varepsilon^2(k_p + 2k_v). \tag{52}$$

In particular, condition (50) holds if $\delta(0) = \boldsymbol{q}(0)$ and $\varepsilon\lambda \leqslant 1$, or if $\delta(0) = 0$.

Proposition 8 holds only in the case of an isoinertial spacecraft, i.e., if I is such that (42) holds. In the general case, it is possible to prove the following weaker result.

Proposition 10. *Consider system* (11) *and the control law*

$$\dot{\delta} = \alpha(\boldsymbol{q} - \varepsilon\lambda\delta),$$
$$u = -\varepsilon^2(k_p I^{-1}\mathbf{q} + k_v\alpha\lambda\tilde{W}^{\mathrm{T}}(\boldsymbol{q})(\boldsymbol{q} - \varepsilon\lambda\delta)), \tag{53}$$

and suppose that $0 < \bar{\Gamma}_0 < \mathscr{I}_3$. *Then there exists* $\varepsilon^\star > 0$ *such that for any* $0 < \varepsilon < \varepsilon^\star$ *the control law renders the equilibrium* $(\bar{\boldsymbol{q}}, 0, \frac{1}{\varepsilon\lambda}\bar{\boldsymbol{q}})$ *of the closed-loop system* (11)–(53) *locally exponentially stable.*

Proof. The claim can be proved by introducing the coordinates transformation (45) and considering the Lyapunov function

$$V_5(\mathbf{z}_1, z_2, z_3) = \frac{1}{2}k_p\mathbf{z}_1^{\mathrm{T}}\mathbf{z}_1 + \frac{1}{2}z_2^{\mathrm{T}}I\bar{\Gamma}^{-1}Iz_2 + \frac{1}{2}k_v\alpha\lambda(\mathbf{z}_1 - \varepsilon\lambda z_{3r})^{\mathrm{T}}I(\mathbf{z}_1 - \varepsilon\lambda z_{3r})$$

for the linear approximation of system (11)–(53) around the equilibrium $(\bar{\boldsymbol{q}}, 0, \frac{1}{\varepsilon\lambda}\bar{\boldsymbol{q}})$. □

5. Simulation results

In order to assess the performance of the magnetic attitude control laws discussed in this paper, a number of simulated case studies has been considered. The simulations presented herein have been carried out using the tools presented in Annoni, De Marchi, Diani, Lovera, and Morea (1999) and Lovera (2003), on the basis of the models for the space environment described in the classical references Sidi (1997) and Wertz (1978).

5.1. State feedback control

The considered spacecraft has an inertia matrix given by $I = \mathrm{diag}[27, 17, 25]$ kg m^2, and operates in a near polar (87° inclination) orbit with an altitude of 450 km and a corresponding orbit period of about 5600 s. It is worth, first of all, to check that the assumption $0 < \bar{\Gamma}_0 < \mathscr{I}_3$, which plays a major role in the formulation of the magnetic attitude control problem, is satisfied in practice. In order to illustrate this, in Fig. 1 a time history of the eigenvalues of $\frac{1}{T}\int_0^T \Gamma_0(t)\,\mathrm{d}t$ computed for the considered orbit is presented. As can be seen from the Fig. 1, $\frac{1}{T}\int_0^T \Gamma_0(t)\,\mathrm{d}t$ converges to a $\bar{\Gamma}_0$ which satisfies the assumption.

For the considered spacecraft, a simulation related to the acquisition of the target attitude $\bar{\boldsymbol{q}}$ from an initial condition characterised by a high initial angular rate has been carried out. In order to take into account the effect of disturbance torques on the behaviour of the controlled spacecraft, a residual magnetic dipole $m_0 = [0.5\ 0.5\ 0.5]^{\mathrm{T}}$ (chosen according to the guidelines given in Chobotov (1991)) has been considered, together with the effect of gravity gradient torques. The results of a simulation of the attitude acquisition of the desired attitude $\bar{q}$ from an initial condition characterised by

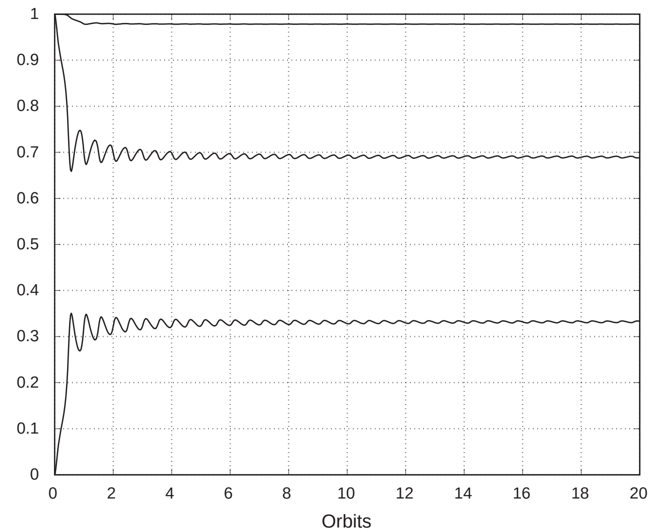

Fig. 1. Eigenvalues of $\frac{1}{T}\int_0^T \Gamma_0(t)\,\mathrm{d}t$ for the considered orbit.

very high angular rate are displayed in Fig. 2, from which the good performance of the unsaturated, state feedback control law can be seen. The controller parameters are given by $k_p = k_v = 50$ and $\varepsilon = 0.001$. Note, in particular, that, as expected, the disturbance torques affect only the steady state behaviour of the system. In particular, the steady state offset in the desired attitude can be eliminated locally via a suitable nominal (linear) control law with integral action (see, for example, the approaches proposed in Pittelkau (1993), Psiaki (2001) and Lovera et al. (2002)).

In order to illustrate the performance of the state feedback controller in the presence of saturation on the control action, the simulation related to attitude acquisition has been repeated, using the control law given in Eq. (38) with $\beta = 0.15$. From the results, shown in Fig. 3, it appears that the saturated control law can still guarantee the convergence of the closed-loop system to the desired equilibrium. Clearly, the transient behaviour is much slower than in the case of Fig. 2, but the amplitude of the control inputs is significantly smaller.

5.2. Output feedback control

The output feedback attitude control law has been applied to a spacecraft with an inertia matrix given by $I = \mathrm{diag}[10, 10, 10]$ kg m^2, operating along the same orbit as in the case discussed in Section 5.1. The results of the simulations which have been carried out for this case are illustrated in Fig. 4, taking into account the effect of a magnetic disturbance torque as induced by a residual dipole of strength $m_0 = [0.1\ 0.1\ 0.1]^{\mathrm{T}}$. The controller parameters are given by $k_p = 50$, $k_v = 100$, $\lambda = 5$ and $\varepsilon = 0.001$. In particular, concerning the attitude acquisition, notice that the control action is actually bounded, as described in Remark 5. This leads to an acquisition transient which is clearly separated in a

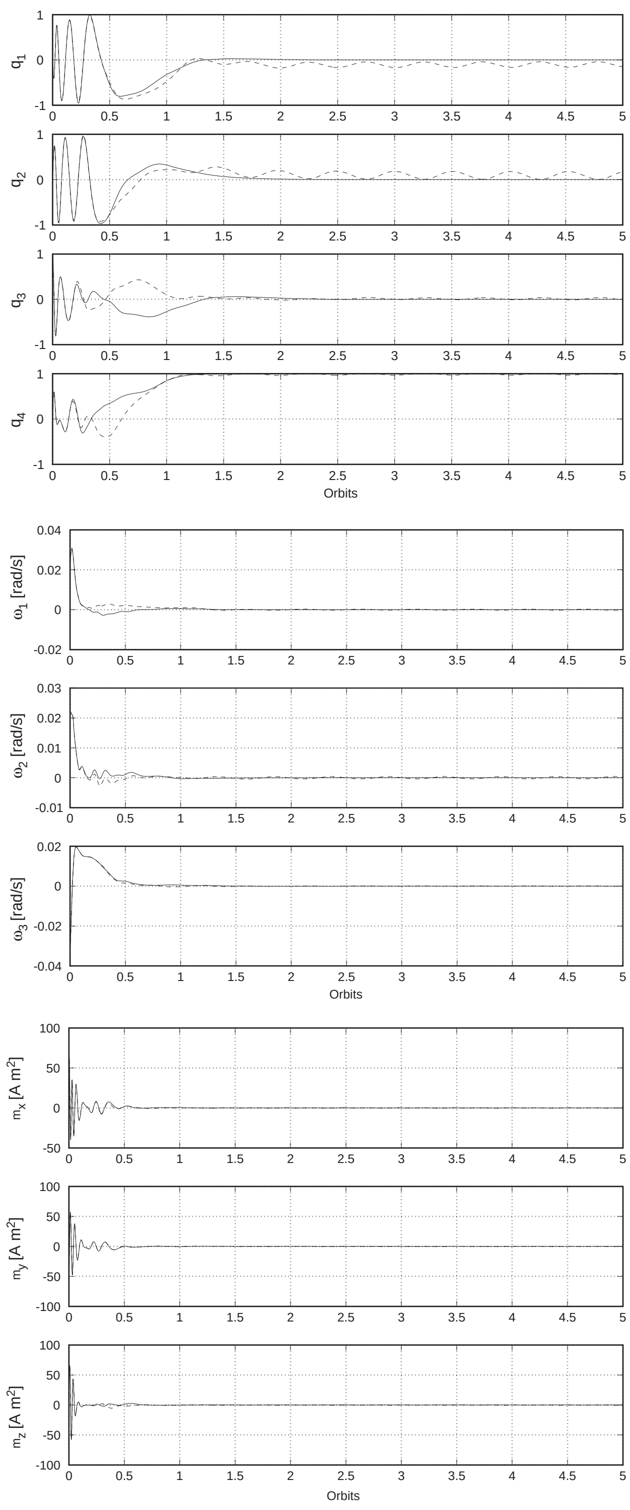

Fig. 2. Quaternion, angular rates and control dipole moments for the attitude acquisition: state feedback controller (simulations without (solid lines) and with (dashed lines) disturbance torques).

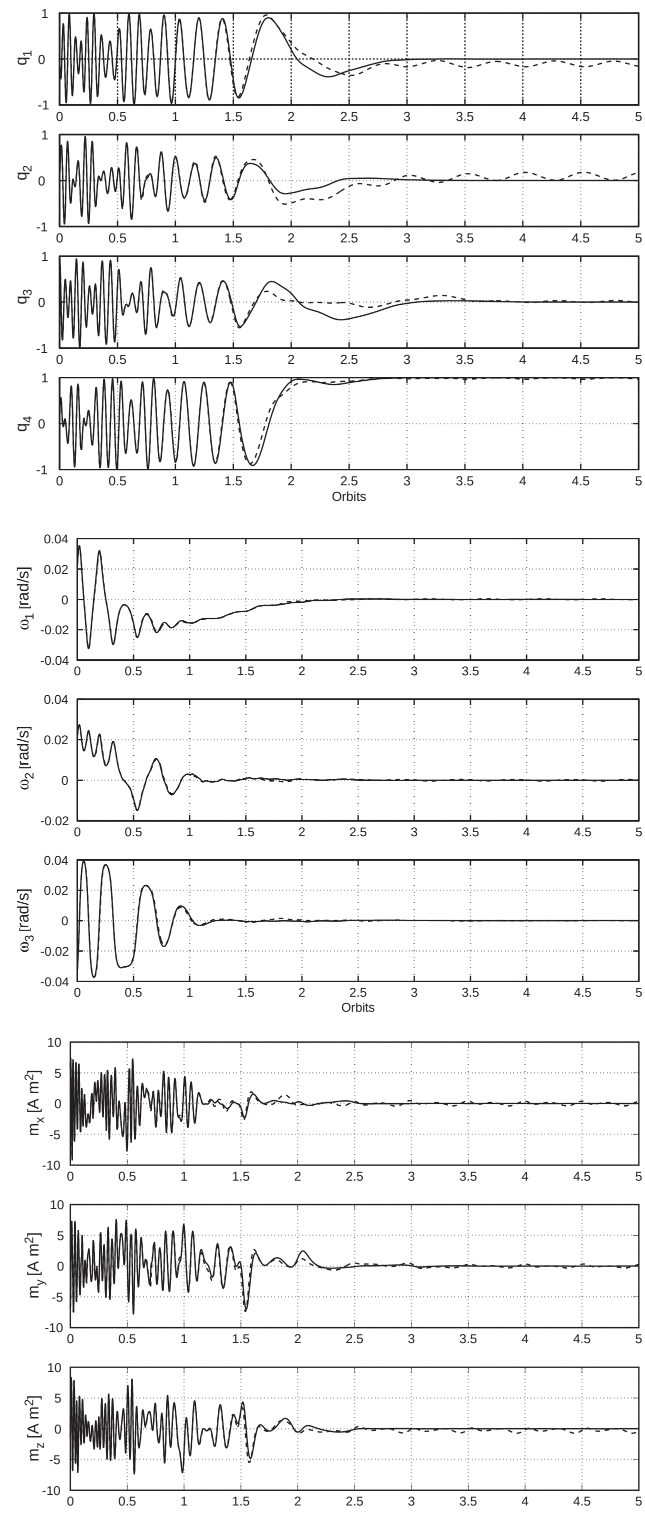

Fig. 3. Quaternion, angular rates and control dipole moments for the attitude acquisition: state feedback controller with saturation (simulations without (solid lines) and with (dashed lines) disturbance torques).

detumbling phase (saturated derivative action) and a reorientation phase (linear operation of the controller).

Similarly, it is possible to demonstrate the ability of the proposed output feedback control law to deal (at least locally) with the case of a nonisoinertial spacecraft, however the simulation results have been omitted for brevity.

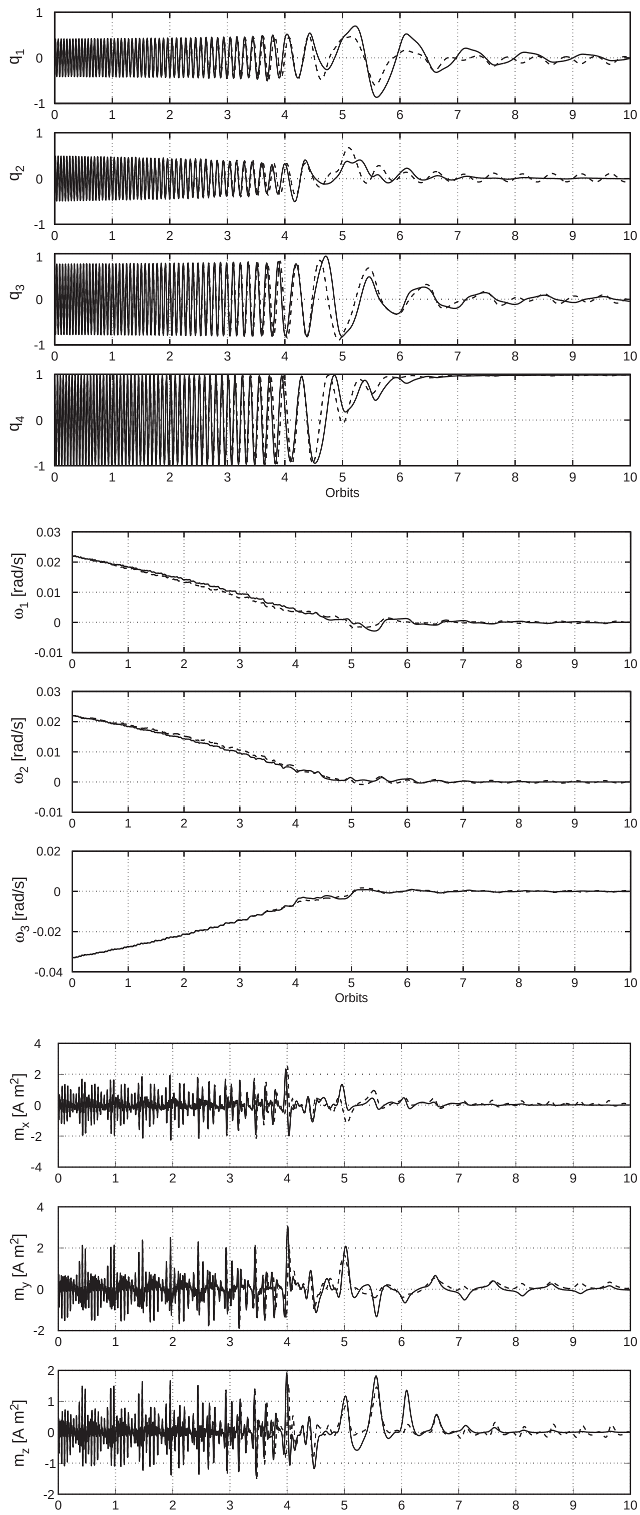

Fig. 4. Quaternion, angular rates and control dipole moments for the attitude acquisition: output feedback controller, $I = \mathrm{diag}[10, 10, 10]$ kg m^2 (simulations without (solid lines) and with (dashed lines) disturbance torques).

6. Concluding remarks

The problem of inertial attitude regulation for a small spacecraft using only magnetic coils as actuators has been analysed and a global solution to the problem has been proposed, based on static attitude and rate feedback. A local solution based on dynamic attitude feedback has also been presented. Attitude regulation can be achieved even in the absence of additional active or passive attitude control actuators such as momentum wheels or gravity gradient booms.

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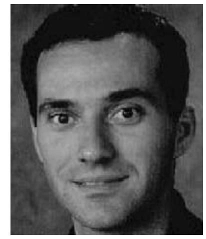

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