$$\{a_n\} = \{a_1, a_2, a_3, \dots\}$$

$$S = a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$$

$$S_n = a_1 + a_2 + \dots a_n \rightarrow \lim_{n\to\infty} S_n$$

$\lim_{n\to\infty} S_n$ → finite: convergent

→ $\pm\infty$, DNE: Divergent

1. Geometric Series $\sum_{n=0}^{\infty} ar^n$

Convergent $\frac{a}{1-r}$ $|r|<1$

Divergent $|r|\geq 1$

2. Telescopic Series

3. P-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Convergent $p>1$

Divergent $p\leq 1$

Tests of Convergence

1) Divergence test

If $\sum_{n=1}^{\infty} a_n$ Convergent $\Rightarrow \lim_{n\to\infty} a_n = 0$

$\not\Leftarrow$

If $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ Divergent

$a_n = \frac{1}{n} \Rightarrow \lim_{n\to\infty} \frac{1}{n} = 0$

$\sum_{n=1}^{\infty} \frac{1}{n}$ Divergent

$$\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4} \longrightarrow \text{Divergent}$$

a_n

$$\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0$$

$$\sum_{n=2}^{\infty}\left(\frac{n+1}{n-1}\right)^n \rightarrow \text{Divergent}$$

a_n

$$\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n = \lim_{n\to\infty}\frac{\cancel{n^n}\left(1+\frac{1}{n}\right)^n}{\cancel{n^n}\left(1-\frac{1}{n}\right)^n} = \frac{e^1}{e^{-1}} = e^2 \neq 0$$

2 Ratio test

$\sum a_n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \to \begin{cases} 0<L<1 \to \text{Series Convergent} \\ L>1 \to \text{Series Divergent} \\ L=1 \to \text{test fails.} \end{cases}$$

$n!$

$(a_n)^n$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$$

a_n

Converge

$$a_n = (-1)^n \frac{n^3}{3^n}$$

$$a_{n+1} = (-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{\cancel{3^{n+1}}\,3} \cdot \frac{\cancel{3^n}}{n^3}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{(n+1)^3}{n^3} = \frac{1}{3} < 1$$

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!\, n!}$$

Divergent

$$a_n = \frac{(2n)!}{n!\, n!} \qquad a_{n+1} = \frac{(2n+2)!}{(n+1)!\,(n+1)!}$$

$$(n+1)! = (n+1)\, n!$$

$$(2n+2)! = (2n+2)(2n+1)(2n)!$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\cancel{(2n+2)!}}{\cancel{(n+1)!}\;\cancel{(n+1)!}} \cdot \frac{\cancel{n!}\;\cancel{n!}}{\cancel{(2n)!}}$$

$$(2n+2)(2n+1) \qquad n+1 \qquad n+1$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4 > 1$$

$\boxed{3}$ n^{th}-root test

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

- $0 < L < 1$ → Series Convergent
- $L > 1$ → Series Divergent
- $L = 1$ → test fails

$$\sum_{n=1}^{\infty} \underbrace{\left(\frac{2n+3}{3n+2}\right)^n}_{a_n} \longrightarrow \text{Convergent}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1$$

4 Basic Comparison test

$\sum a_n$, $\sum b_n$ $\qquad 0 \leq a_n \leq b_n$

1. $\sum b_n$ Convergent $\rightarrow$ $\sum a_n$ Convergent
2. $\sum a_n$ Divergent $\rightarrow$ $\sum b_n$ Divergent

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$$

$$0 < \frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

a_n b_n

$$\therefore \sum \frac{5}{2n^2} = \frac{5}{2} \sum \frac{1}{n^2}$$ Convergent

$$\therefore \sum \frac{5}{2n^2+4n+3}$$ Convergent

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \quad \Bigg| \quad \frac{2n^2+3n}{\sqrt{5+n^5}} \geq \frac{2n^2}{\sqrt{5+n^5}} \geq \frac{2n^2}{\sqrt{5n^5+n^5}}$$

b

$\therefore \frac{2}{\sqrt{6}} \sum \frac{1}{\sqrt{n}}$ Divergent $\quad \sqrt{n^5}$

$$\frac{2n^2}{n^{5/2}\sqrt{6}}$$

$$\frac{2}{\sqrt{6}} \quad \frac{1}{\sqrt{n}}$$

a_n

$\therefore \sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ is Divergent

$$\sum \frac{n^2+3n}{\sqrt{n^5}}$$ bn

Diverge

$$\frac{2n^2+3n}{\sqrt{5+n^5}}$$

$$\frac{2n^2}{n^{5/2}} + \frac{3n}{n^{5/2}} \quad \frac{1}{n^2} = \frac{2}{[illegible]}$$

$$\frac{1}{\sqrt{n}} + \frac{\quad}{\sqrt{n^3}}$$

5 Limit Comparison test

$\sum a_n \qquad \sum b_n \qquad , \; a_n, b_n > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} \begin{cases} \to c \neq 0 \\ \to 0 \\ \to \infty \end{cases}$$

a_n, b_n have the same behaviour of Convergent and divergent

If $\sum b_n$ Convergent $\Rightarrow \sum a_n$ is Convergent

If $\sum b_n$ Diverget $\Rightarrow \sum a_n$ is Diverget

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} \quad a_n$$

$$\frac{2n+1}{(n+1)^2} \xrightarrow[n\to\infty]{} \frac{2n}{n^2} = \frac{2}{n} \quad b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n+1}{(n+1)^2} \cdot \frac{n}{2} = \frac{2}{2} = 1$$

$\therefore \sum \frac{2}{n}$ Divergent $\Rightarrow \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$ is Divergent