18.650. Spring 2025

Date: 2/26/25

Name: SOLUTIONS

ID#: ______________________________

Test 1

PLEASE DO NOT TURN THIS PAGE OR START ANSWERING QUESTIONS UNTIL YOU ARE INSTRUCTED TO DO SO.

1. You are allowed a hand-written two-sided cheat sheet on a standard-sized piece of paper (8.5" x 11").
2. No other notes are allowed.
3. Calculators and connected devices like phones, laptops, or tablets are strictly forbidden.
4. The test starts at 1:05.
5. The test ends at 1:55 regardless of your time of arrival.
6. All questions should be answered on the present exam sheet.
7. Make sure to mark your name and ID in the spaces provided above and **do not remove the staple**.

Problem 1 (20 pts). Circle the correct answer. No justification required. 5 points each.

1. The set of all cdfs given by step functions with exactly two jumps is a

    A. parametric statistical model

    B. nonparametric statistical model

2. Let $X \sim N(-1, 4)$. What is $\mathbb{P}(|X| \leq 1)$? Refer to the table on page 6.

    A. 0.6915

    B. 0.3413

    C. 0.1915

    D. 0.8413

Let $X_1, \ldots, X_n$ be i.i.d. with mean $\mathbb{E}[X_1] = \theta$. Consider the following estimator of θ:

$$\hat{\theta} = \begin{cases} \bar{X}_n, & \text{with prob. } 1 - 1/n, \\ 2^n, & \text{with prob. } 1/n \end{cases}$$

3. $\hat{\theta}$ is a consistent estimator of θ.

    A. TRUE

    B. FALSE

4. $\hat{\theta}$ is asymptotically unbiased.

    A. TRUE

    B. FALSE

Problem 2 (80 pts).

We are given n samples $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_X^2)$ and n samples $Y_1, \ldots, Y_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_Y^2)$. The X_i's are independent of the Y_i's. In this problem, we are interested in estimating θ given by the ratio of the two variances:

$$\theta = \sigma_X^2 / \sigma_Y^2.$$

Consider the following estimator of θ:

$$\hat{\theta} = \frac{\hat{\sigma}_X^2}{\hat{\sigma}_Y^2}, \qquad \text{where} \quad \hat{\sigma}_X^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2, \qquad \hat{\sigma}_Y^2 = \frac{1}{n} \sum_{i=1}^{n} Y_i^2.$$

1. (10 points) Show that $\hat{\theta}$ is consistent.

By LLN, $\hat{\sigma}_X^2 \xrightarrow{\mathbb{P}} \mathbb{E}[X_i^2] = \sigma_X^2$ and similarly $\hat{\sigma}_Y^2 \xrightarrow{\mathbb{P}} \sigma_Y^2$.

By the Continuous Mapping Theorem applied with the continuous function $g(t) = 1/t$, we have $1/\hat{\sigma}_Y^2 \xrightarrow{\mathbb{P}} 1/\sigma_Y^2$.

Finally, using that convergence in probability is preserved under products, we conclude

$$\hat{\theta} = \frac{\hat{\sigma}_X^2}{\hat{\sigma}_Y^2} = \hat{\sigma}_X^2 \cdot \frac{1}{\hat{\sigma}_Y^2} \xrightarrow{\mathbb{P}} \frac{\sigma_X^2}{\sigma_Y^2} = \theta,$$ and this is the defn of consistency.

2. (10 points) Is $\hat{\sigma}_X^2$ a biased or unbiased estimator of σ_X^2, and why?

$$\text{bias}(\hat{\sigma}_X^2) = \mathbb{E}[\hat{\sigma}_X^2] - \sigma_X^2$$
$$= \frac{1}{n}\sum_{i=1}^{n} \underbrace{\mathbb{E}[X_i^2]}_{\sigma_X^2} - \sigma_X^2 = 0,$$

using that $\mathbb{E}[X_i^2] = \mathbb{V}[X_i^2] = \sigma_X^2$, since $\mathbb{E}[X_i] = 0$.

Therefore by defn, $\hat{\sigma}_X^2$ is unbiased.

3. (10 points) Is $\hat{\theta}$ a biased or unbiased estimate of θ? A yes/no answer suffices here.

$\hat{\theta}$ is biased (because $\mathbb{E}[\hat{\sigma}_X^2/\hat{\sigma}_Y^2] \neq \frac{\mathbb{E}[\hat{\sigma}_X^2]}{\mathbb{E}[\hat{\sigma}_Y^2]}$)

4. (10 points) Compute $\mathbb{V}[X_1^2]$, $\mathbb{V}[Y_1^2]$, and $\mathrm{Cov}(X_1^2, Y_1^2)$. You may use that $\mathbb{E}[Z^4] = 3$ for a standard Gaussian $Z \sim \mathcal{N}(0,1)$. *Write solution on next page*

Since $X_1 \sim \mathcal{N}(0, \sigma_X^2)$ we can write $X_1 = \sigma_X Z$ where $Z \sim \mathcal{N}(0,1)$.

Therefore, $\mathbb{V}[X_1^2] = \mathbb{V}[\sigma_X^2 Z^2] = \sigma_X^4 \mathbb{V}[Z^2] = \sigma_X^4 (\underbrace{\mathbb{E}[Z^4]}_{3} - \underbrace{\mathbb{E}[Z^2]^2}_{1}) = 2\sigma_X^4$

By same argument, $\mathbb{V}[Y_1^2] = 2\sigma_Y^4$.

Since X_1 is independent of Y_1, we also have that X_1^2 is independent of Y_1^2.

Therefore, $Cov(X_1^2, Y_1^2) = 0$.

5. (10 points) Write a multivariate central limit theorem for $\begin{pmatrix} \hat{\sigma}_X^2 \\ \hat{\sigma}_Y^2 \end{pmatrix}$.

Note that $\begin{pmatrix} \hat{\sigma}_X^2 \\ \hat{\sigma}_Y^2 \end{pmatrix} = \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} X_i^2 \\ Y_i^2 \end{pmatrix}$. Therefore CLT implies

$$\sqrt{n}\left(\begin{pmatrix} \hat{\sigma}_X^2 \\ \hat{\sigma}_Y^2 \end{pmatrix} - \begin{pmatrix} \mathbb{E}[X_1^2] \\ \mathbb{E}[Y_1^2] \end{pmatrix} \right) \rightsquigarrow \mathcal{N}_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbb{V}[X_1^2] & Cov(X_1^2, Y_1^2) \\ Cov(X_1^2, Y_1^2) & \mathbb{V}[Y_1^2] \end{pmatrix} \right)$$

Using above computations we get

$$\sqrt{n}\left(\begin{pmatrix} \hat{\sigma}_X^2 \\ \hat{\sigma}_Y^2 \end{pmatrix} - \begin{pmatrix} \sigma_X^2 \\ \sigma_Y^2 \end{pmatrix} \right) \rightsquigarrow \mathcal{N}_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\sigma_X^4 & 0 \\ 0 & 2\sigma_Y^4 \end{pmatrix} \right)$$

6. (10 points) Show $\hat{\theta}$ is asymptotically normal with asymptotic variance $4\sigma_X^4/\sigma_Y^4$. [Hint: when taking derivatives with respect to σ_X^2, treat it as a single unit. Same for σ_Y^2.]
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We apply Multivariate Delta Method with $g(a,b) = a/b$.

Note $\nabla g(a,b) = (1/b, -a/b^2)$. Therefore $\nabla g(\sigma_X^2, \sigma_Y^2) = \left(\frac{1}{\sigma_Y^2}, -\frac{\sigma_X^2}{\sigma_Y^4}\right)$

Delta Method & CLT from part 5 give

$$\sqrt{n}(\hat{\theta} - \theta) = \sqrt{n}\left(\frac{\hat{\sigma}_X^2}{\hat{\sigma}_Y^2} - \frac{\sigma_X^2}{\sigma_Y^2}\right) \rightsquigarrow N(0, \tau^2) \text{ where}$$

$$\tau^2 = \begin{pmatrix} \frac{1}{\sigma_Y^2} & \frac{-\sigma_X^2}{\sigma_Y^4} \end{pmatrix} \begin{pmatrix} 2\sigma_X^4 & 0 \\ 0 & 2\sigma_Y^4 \end{pmatrix} \begin{pmatrix} 1/\sigma_Y^2 \\ -\frac{\sigma_X^2}{\sigma_Y^4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2\sigma_X^4}{\sigma_Y^2} & -2\sigma_X^2 \end{pmatrix} \begin{pmatrix} 1/\sigma_Y^2 \\ -\frac{\sigma_X^2}{\sigma_Y^4} \end{pmatrix} = \frac{2\sigma_X^4}{\sigma_Y^4} + \frac{2\sigma_X^4}{\sigma_Y^4} = \frac{4\sigma_X^4}{\sigma_Y^4}.$$

Thus we have shown

$\sqrt{n}(\hat{\theta} - \theta) \rightsquigarrow N(0, \frac{4\sigma_X^4}{4\sigma_Y^4})$, proving θ is asymptotically normal with asymptotic variance $4\sigma_X^4/\sigma_Y^4$!

7. (10 points) What is the standard error of $\hat{\theta}$, approximately?

Using part 6,

$$se(\hat{\theta})^2 = \mathbb{V}(\hat{\theta}) \approx \frac{4\sigma_X^4/\sigma_Y^4}{n} = \frac{4\theta^2}{n}$$

Thus $se(\hat{\theta}) \approx \frac{2\sigma_X^2/\sigma_Y^2}{\sqrt{n}} = \frac{2\theta}{\sqrt{n}}$

8. (10 points) Write a 95% confidence interval for σ_X^2/σ_Y^2.

Using that $z_{2.5\%} = 1.96$, an approximate 95% CI is given by

$$\hat{\theta} \pm 1.96 \frac{se(\hat{\theta})}{\sqrt{n}} \approx \hat{\theta} \pm 1.96 \cdot \frac{2\theta}{\sqrt{n}}$$

But since θ is unknown we replace it with $\hat{\theta}$, to get

$$\hat{\theta}\left(1 \pm \frac{2 \cdot 1.96}{\sqrt{n}}\right) = \frac{\hat{\sigma}_X^2}{\hat{\sigma}_Y^2}\left(1 \pm \frac{2 \cdot 1.96}{\sqrt{n}}\right)$$

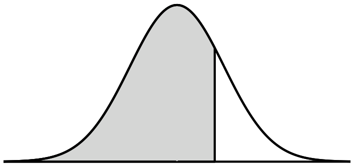

Table 1: The table lists $P(Z \leq z)$ where $Z \sim N(0,1)$ for positive values of z.

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

*For $Z \geq 3.50$, the probability is greater than or equal to 0.9998.