

## TEST 1

**PLEASE DO NOT TURN THIS PAGE OR START  
ANSWERING QUESTIONS UNTIL YOU ARE  
INSTRUCTED TO DO SO.**

1. You are allowed a two-sided letter-sized cheat sheet.
2. No other notes or books are allowed
3. Calculators and connected devices like phones, laptops, or tablets are strictly forbidden.
4. The test starts at 10:05
5. The test ends at 10:55 regardless of your time of arrival.
6. All questions should be answered on the present exam sheet
7. Make sure to mark your name on the first page and **do not remove the staple**

Problem1	30	
Problem2	75	
TOTAL	100	

**Problem 1.** (30 pts) Check the (unique) correct answer. No justification is required. 5 points each.

1. An estimator of  $\theta$  whose MSE is equal to 0 is...
  - unbiased
  - deterministic
  - equal to  $\theta$  almost surely
  - All of the above
  
2. In a histogram with *unequal* bin widths, which statement is true?
  - Taller bars always correspond to larger proportions of observations.
  - The *area* of each bar equals the proportion of observations in its bin.
  - The *height* of each bar equals the proportion in its bin regardless of width.
  - The *area* of each bar is equal to  $1/k$  if there are  $k$  bins.
  
3. Which pair of summary statistics is most *robust* to outliers?
  - mean and standard deviation
  - median and interquartile range (IQR)
  - mean and IQR
  - median and standard deviation
  
4. Let  $X$  be the time (in minutes) it takes to find a free dock for a Blue Bike on MIT campus. Assume  $X \sim \mathcal{N}(10, 4)$ . What is the probability that it takes more than 9 minutes to dock a bike?
  - 0.6915
  - 0.3085
  - 0.5987
  - 0.5000

5. Let  $X_1, \dots, X_n \sim \text{Bernoulli}(1/4)$  be i.i.d., and let  $\hat{p} = \bar{X}_n$ . Define  $g(p) = 2\sqrt{p}$ . By the Delta Method,

$$\sqrt{n} (g(\hat{p}) - g(1/4)) \rightsquigarrow \mathcal{N}(0, \sigma^2).$$

What is  $\sigma^2$ ?

- .25
- 0.5
- .75
- 1

6. Let  $Z = (Z_1, Z_2)^\top$  be bivariate normal with mean  $\mu = (1, 1)^\top$  and covariance  $\Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ . What is the distribution of  $Y = Z_1 - 2Z_2$ ?

- $\mathcal{N}(-1, 6)$
- $\mathcal{N}(-1, 10)$
- $\mathcal{N}(0, 6)$
- $\mathcal{N}(-1, 5)$

**Problem 2.** (70 pts + 5 bonus pts)

Let  $X$  be a random vector with unknown mean  $\mu = (a, b)^\top \in \mathbb{R}^2$  and covariance matrix

$$\Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Define  $Y^+ = X^{(1)} + X^{(2)}$  and  $Y^- = X^{(1)} - X^{(2)}$ , where  $X = (X^{(1)}, X^{(2)})^\top$ .

Our goal in this exercise is to estimate the parameter  $\theta = a^2 - b^2$ . To that end, we observe  $X_1, \dots, X_n$  which are i.i.d. with the same distribution as  $X$  and we propose to use the estimator

$$\hat{\theta} = (\bar{Y}_n^+)(\bar{Y}_n^-),$$

where

$$\bar{Y}_n^\pm = \frac{1}{n} \sum_{i=1}^n (X_i^{(1)} \pm X_i^{(2)}).$$

1. (10 points) What is the mean of  $(Y^+, Y^-)^\top$ ?

2. (10 points) What is the covariance matrix of  $(Y^+, Y^-)^\top$ ?

3. (10 points) Is  $\hat{\theta}$  consistent? Why?
4. (10 points) Compute the bias of  $\hat{\theta}$ . Is it biased or unbiased?
5. (10 points) Write a multivariate central limit theorem for  $(\bar{Y}_n^+, \bar{Y}_n^-)^\top$ .
6. (10 points) Show that  $\hat{\theta}$  is asymptotically normal and compute its asymptotic variance  $\sigma^2$ .

7. (5 points) Propose a consistent estimator  $\hat{\sigma}^2$  of  $\sigma^2$ .
8. (10 points) Write a 99% confidence interval for  $\theta$  that is symmetric about  $\hat{\theta}$ .

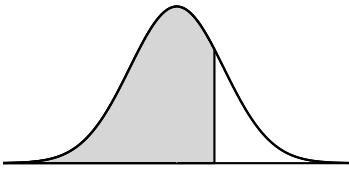


Table 1: The table lists  $P(Z \leq z)$  where  $Z \sim N(0, 1)$  for positive values of  $z$ .

Z	Second decimal place of $Z$									
	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

\*For  $Z \geq 3.50$ , the probability is greater than or equal to 0.9998.