18.650. Fundamentals of Statistics
Spring 2026. Recitation Sheet #1

Problem 1 (Gaussian standardization and Φ)**.** A standardized exam score X is modeled as

$$X \sim \mathcal{N}(100, 15^2).$$

Let Φ denote the CDF of $Z \sim \mathcal{N}(0,1)$. Express answers in terms of Φ and Φ^{-1} (no decimals needed).

1. (Standardization) Define $Z = (X - 100)/15$. What is the distribution of Z?

2. Compute the following probabilities:

   (a) $\mathbb{P}(X \leq 85)$

   (b) $\mathbb{P}(100 \leq X \leq 115)$

   (c) $\mathbb{P}(|X - 100| \geq 30)$

3. (Quantile) Find c such that $\mathbb{P}(X \geq c) = 0.10$ (top 10% cutoff).

4. (Affine transform / rescaling) The exam board also reports the "index score"

$$T \;=\; 50 + 10 \cdot Z \;=\; 50 + 10 \cdot \frac{X - 100}{15}.$$

   What is the distribution of T? Express $\mathbb{P}(T \geq 70)$ in terms of Φ.

5. (Conditional tail) Compute $\mathbb{P}(X \geq 130 \mid X \geq 115)$ in terms of Φ.

Problem 2. The mean weight of eggs produced by a farm is $\mathbf{60}g$ with a standard deviation of $\mathbf{4}g$.

1. Express $\mathbf{55}g$ in standard units (z -score).

2. How many standard deviations from the mean is a weight of 68 grams?

3. What weight corresponds to $\mathbf{1.5}$ standard deviations below the mean?

4. Find the probability that the weight, X, of eggs is less than $58g$, where $X \sim N\left(\mathbf{60}, \mathbf{4^2}\right)$.

Problem 3 (All-in gambling with rare catastrophic risk.)**.** A player starts with \$1 and repeatedly bets all their money for n rounds. Each round: with probability $1/2$ they win and their wealth doubles; with probability $1/2$ they lose and their wealth becomes 0 and they immediately go home. Let W_n be the player's final wealth (when they leave).

1. (Distribution) Find $\mathbb{P}(W_n = 2^n)$ and $\mathbb{P}(W_n = 0)$.

2. (Convergence) Does $W_n \to 0$ in probability? In distribution?

3. (Expectations vs convergence) Compute $\mathbb{E}[W_n]$. Does $W_n \to 0$ in L^1?

4. (Illegal casino: rare fine) Now suppose the casino is illegal in the current state. If the player loses, they go home and are not caught. If the player survives all n rounds (so they are still at the casino at the end), then with probability $1/n$ they are caught and must pay a fine of $\$\,4^n$. Let Y_n be the player's *net* wealth after the fine.

   (a) Find the distribution of Y_n.

   (b) Does $Y_n \to 0$ in probability? In distribution?

   (c) Compute $\mathbb{E}[Y_n]$ and describe its behavior as $n \to \infty$.

Problem 4 (Estimating noise level from repeated measurements.)**.** A device is used to measure the same fixed quantity over and over (e.g., a QC sample in a lab). We model the measurements as i.i.d. random variables

$$X_1, \ldots, X_n \text{ i.i.d.}, \qquad \mathbb{E}[X_1] = \mu, \qquad \mathbb{V}(X_1) = \sigma^2 \in (0, \infty).$$

Define

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

1. (Algebra) Show that

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X}_n)^2.$$

2. (Consistency; AoS 5.1) Show that $\hat{\sigma}_n^2 \xrightarrow{\mathbb{P}} \sigma^2$.

3. **Optional (Slutsky).** Assume also that the CLT applies to $\bar{X}_n$, i.e.

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \Rightarrow \mathcal{N}(0, 1).$$

Show that

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}_n} \Rightarrow \mathcal{N}(0, 1).$$

Appendix: The table lists $P(Z \leq z)$ where $Z \sim N(0,1)$ for positive values of z.

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998