Lec 5: Multidimensional

$X = (x_1, \dots, x_k)$

$\mathbb{E}[X] = (\mathbb{E}[x_1], \dots, \mathbb{E}[x_k])$

$\mathbb{V}[X] = (\mathbb{V}[x_1], \dots, \mathbb{V}[x_k])$ ← Not good, which makes sense bc $Var = E[x^2] - E[x]^2$

↳ $Cov(x_i, x_j) = \mathbb{E}[(x_i - \mathbb{E}[x_i])(x_j - \mathbb{E}[x_j])]$

$= \mathbb{E}[x_i x_j] - \mathbb{E}[x_i]\mathbb{E}[x_j]$ → from 18.600

$\mathbb{V}[x_j] = Cov(x_j, x_j)$

$\Sigma_{i,j} = cov(x_i, x_j)$ Is it positive semidefinite? ← 18.06 :)

$X \in \mathbb{R}^k$

$$XX^T = \begin{array}{c c} & \scriptstyle j \\ \scriptstyle i \rightarrow & \begin{bmatrix} & & \\ & x_i x_j & \\ & & \end{bmatrix} \end{array}$$

$\mathbb{E}[XX^T] \rightsquigarrow \mathbb{E}[x_i x_j] = Cov(x_i, x_j) + \underbrace{\mathbb{E}[x_i]}_{\mu_i}\underbrace{\mathbb{E}[x_j]}_{\mu_j}$

$\mathbb{E}[X] = \mu \quad (\in \mathbb{R}^k)$ $\quad = \Sigma + \mu\mu^T$

$\Sigma = \mathbb{E}[XX^T] - \mu\mu^T$

$= \mathbb{E}[(X-\mu)(X-\mu)^T]$

↳ $XX^T \underbrace{- \mu^T X - \mu X^T}_{-2\mu\mu^T} + \mu\mu^T$

$-2\mu\mu^T \longrightarrow -\mu\mu^T$

$\Sigma = \mathbb{V}(X)$

↓

covariance mtx = variance

$a \in \mathbb{R}^k$ $\quad X$ is a round vector in $\mathbb{R}^k$, $\mathbb{E}[X] = \mu$, $\mathbb{V}[X] = \Sigma$,

① $\mathbb{E}[a^T X] = \mathbb{E}[\langle a, X \rangle]$

(diff notation)

$$\mathbb{E}[a^T X] = \mathbb{E}\left[\sum_{i=1}^{k} a_i X_i\right] \xrightarrow{\hspace{3em}} \text{linearity of } \mathbb{E}$$

← dot prod.

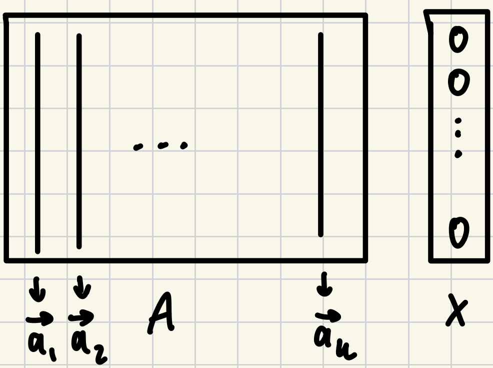
$$= \sum_{i=1}^{k} a_i \mathbb{E}[X_i] = \sum_{i=1}^{k} a_i \mu_i = a^T \mu = \langle a, \mu \rangle$$

$\left.\begin{matrix} \int x f(x)\,dx \\ \int x \, d\mathbb{P}(x) \end{matrix}\right\}$ proves linear?

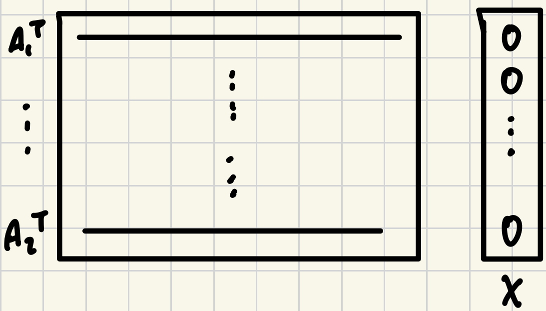
$$\mathbb{E}[a^T X] = a^T \mathbb{E}[X]$$

② if $A \in \mathbb{R}^{m \times k}$ fixed matrix

$$\mathbb{E}[AX] = A\,\mathbb{E}[X]$$



$$= x_1 \vec{a}_1 + x_2 \vec{a}_2 + \ldots + x_k \vec{a}_k$$



$$= \begin{bmatrix} A_1^T X \\ A_2^T X \\ \vdots \\ A_k^T X \end{bmatrix}$$

Example:

Suppose Q is a rand. square mtx.

$$\mathbb{E}[\mathrm{tr}(Q)] = \mathrm{tr}[\mathbb{E}(Q)]$$

trace = linear operation, so can move out of $\mathbb{E}[x]$.

3) $\mathbb{V}[a^Tx] = \mathbb{E}[(a^Tx)^2] - (\mathbb{E}[a^Tx])^2$

$\mathbb{E} = ?$

$$(a^Tx)^2 = (a^Tx)(a^Tx)$$
$$= a^Tx \; x^Ta \quad ?$$
$$= \underbrace{a^T(xx^T)a}_{\text{quadratic form}}$$

$$\mathbb{E}[\ldots]$$
$$= a^T\mathbb{E}[(xx^T)a]$$
$$= a^T\mathbb{E}[xx^T]a$$
$$= a^T\Sigma a + a^T\mu\mu^Ta$$
$$= a^T\Sigma a + \cancel{(a^T\mu)^2}$$

(x^TAx)

A is a square matrix $(k \times k)$

$f(x) = x^T\underline{\underline{A}}x, \quad f: \mathbb{R}^k \to \mathbb{R}$

$$= \sum_{i,j=1}^{k} A_{ij}x_ix_j$$

$g(x) = x^TAx + b^Tx + c$

$\boxed{\begin{array}{l}(A \text{ is p.s.d if } f \geq 0) \\ \iff \text{all eigenvalues of } A \geq 0\end{array}}$

Two Examples:

1) let B be any matrix
B^TB

2) BB^T