18.650. Fundamentals of Statistics
Spring 2026. Recitation Sheet #2

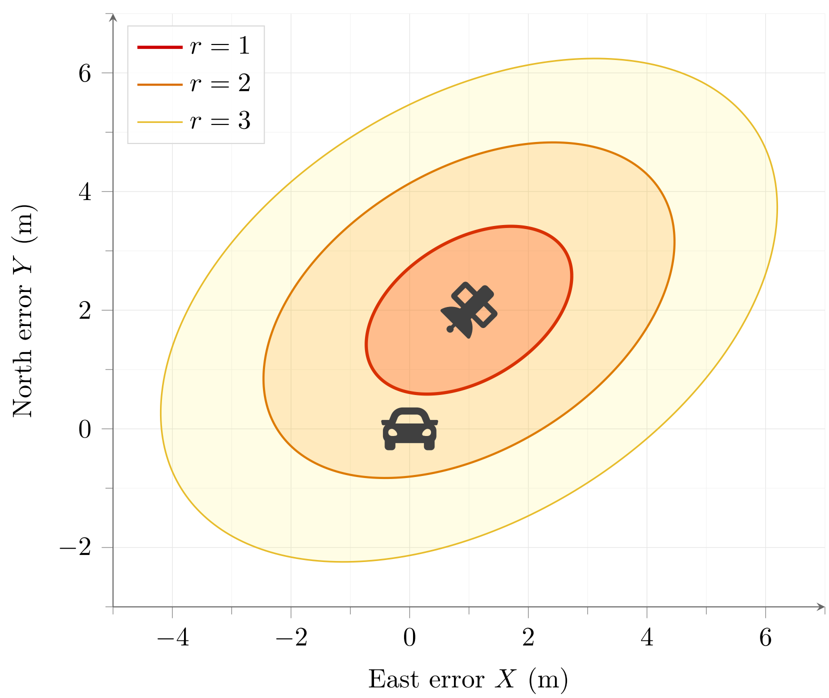
Problem 1 (GPS localization error model (bivariate normal)).

> For jointly Gaussian variables, $U = aX + bY$ is Gaussian, and
>
> $$\mathbb{E}[U] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y], \qquad \mathbb{V}(U) = a^2\mathbb{V}(X) + b^2\mathbb{V}(Y) + 2ab\ \mathrm{Cov}(X, Y).$$
>
> If U, V are jointly Gaussian and $\mathrm{Cov}(U, V) = 0$, then $U \perp V$ (equivalently, $\mathbb{E}[U \mid V] = \mathbb{E}[U]$). For $S = aX + bY$ with $\mathbb{V}(S) > 0$,
>
> $$\mathbb{E}[X \mid S = s] = \mathbb{E}[X] + \frac{\mathrm{Cov}(X, S)}{\mathbb{V}(S)}(s - \mathbb{E}[S]),$$
>
> and similarly for Y. Also $\mathbb{E}[X^2] = \mathbb{V}(X) + (\mathbb{E}[X])^2$ and $\mathbb{E}[Y^2] = \mathbb{V}(Y) + (\mathbb{E}[Y])^2$.

A car navigation system uses GPS to estimate its position. Let (X, Y) be the *horizontal estimation error* (east, north) in meters (estimated minus true). We model the error vector as jointly Gaussian:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma), \qquad \mu = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

(So the GPS has a systematic bias of $(1, 2)$ meters, and correlated noise.)



1. Compute $\mathbb{E}[X], \mathbb{E}[Y]$, $\mathbb{V}(X), \mathbb{V}(Y)$, $\mathrm{Cov}(X,Y)$, and $\mathrm{Corr}(X,Y)$.

   Solution. From μ and Σ,

   $$\mathbb{E}[X] = 1,\ \mathbb{E}[Y] = 2, \qquad \mathbb{V}(X) = 3,\ \mathbb{V}(Y) = 2, \qquad \mathrm{Cov}(X,Y) = 1.$$

   Therefore
   $$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathbb{V}(X)\mathbb{V}(Y)}} = \frac{1}{\sqrt{3\cdot 2}} = \frac{1}{\sqrt{6}}.$$

2. (GPS accuracy metric) The horizontal position error magnitude is $R = \sqrt{X^2+Y^2}$ (in m). Many GPS specs report a horizontal RMS error $\sqrt{\mathbb{E}[R^2]}$. Compute $\mathbb{E}[R^2] = \mathbb{E}[X^2+Y^2]$ and hence $\sqrt{\mathbb{E}[R^2]}$.

   Solution. Using $\mathbb{E}[X^2] = \mathbb{V}(X) + (\mathbb{E}[X])^2$ and $\mathbb{E}[Y^2] = \mathbb{V}(Y) + (\mathbb{E}[Y])^2$,

   $$\mathbb{E}[R^2] = \mathbb{E}[X^2+Y^2] = \mathbb{V}(X) + \mathbb{V}(Y) + (\mathbb{E}[X])^2 + (\mathbb{E}[Y])^2 = 3 + 2 + 1^2 + 2^2 = 10.$$

   Therefore the RMS horizontal error is $\sqrt{\mathbb{E}[R^2]} = \sqrt{10}$ meters.

3. (Directional GPS errors) In practice, GPS error matters most along certain directions (determined by map geometry and satellite geometry). Define two scalar summaries of the horizontal error:

   - *Cross-track error* relative to a straight road oriented 45° northeast:
     $$C = X - Y$$
     (up to a constant factor, this is the signed distance from the road).
   - A *satellite-geometry diagnostic score* (a known linear combination of the horizontal errors):
     $$S = X + 2Y$$
     (up to a constant factor, this is the component of the error in the direction $(1,2)$).

   (a) Find the distributions of C and S (means and variances).

   (b) Compute $\mathrm{Cov}(C,S)$. Are C and S independent?

   (c) As a quick check, compute $\mathbb{E}[C \mid S = 8]$.

   Solution. **(a)** Since linear combinations of jointly Gaussian variables are Gaussian, both C and S are Normal. Their means:

   $$\mathbb{E}[C] = \mathbb{E}[X] - \mathbb{E}[Y] = 1 - 2 = -1, \qquad \mathbb{E}[S] = \mathbb{E}[X] + 2\mathbb{E}[Y] = 1 + 4 = 5.$$

   Their variances:

   $$\mathbb{V}(C) = \mathbb{V}(X - Y) = \mathbb{V}(X) + \mathbb{V}(Y) - 2\,\mathrm{Cov}(X,Y) = 3 + 2 - 2\cdot 1 = 3,$$

$$\mathbb{V}(S) = \mathbb{V}(X + 2Y) = \mathbb{V}(X) + 4\mathbb{V}(Y) + 4\,\mathrm{Cov}(X, Y) = 3 + 8 + 4 = 15.$$

Therefore

$$C \sim \mathcal{N}(-1, 3), \qquad S \sim \mathcal{N}(5, 15).$$

(b) The covariance is

$$\mathrm{Cov}(C, S) = \mathrm{Cov}(X - Y,\ X + 2Y) = \mathbb{V}(X) + \mathrm{Cov}(X, Y) - 2\mathbb{V}(Y) = 3 + 1 - 4 = 0.$$

Since (C, S) is jointly Gaussian and uncorrelated, C and S are independent.

(c) By independence, $\mathbb{E}[C \mid S = 8] = \mathbb{E}[C] = -1$.

4. (Conditional expectation) On a particular run, the receiver reports the diagnostic score $S = 8$ (i.e. $X + 2Y = 8$). Compute $\mathbb{E}[X \mid S = 8]$ and $\mathbb{E}[Y \mid S = 8]$.

   Solution. First compute $\mathrm{Cov}(X, S)$ and $\mathrm{Cov}(Y, S)$:

$$\mathrm{Cov}(X, S) = \mathrm{Cov}(X, X + 2Y) = \mathbb{V}(X) + 2\,\mathrm{Cov}(X, Y) = 3 + 2 = 5,$$

$$\mathrm{Cov}(Y, S) = \mathrm{Cov}(Y, X + 2Y) = \mathrm{Cov}(X, Y) + 2\mathbb{V}(Y) = 1 + 4 = 5.$$

   Using the conditional mean formula with $\mathbb{E}[S] = 5$ and $\mathbb{V}(S) = 15$,

$$\mathbb{E}[X \mid S = 8] = 1 + \frac{5}{15}(8 - 5) = 1 + 1 = 2, \qquad \mathbb{E}[Y \mid S = 8] = 2 + \frac{5}{15}(8 - 5) = 2 + 1 = 3.$$

5. (Averaging independent fixes; optional) Suppose we take 3 independent GPS readings with the same error distribution and average them. Let $(\bar{X}, \bar{Y})$ be the averaged error and $\bar{C} = \bar{X} - \bar{Y}$. Find the distribution of $\bar{C}$ and compute $\mathbb{P}(\bar{C} > 0)$ using the Φ-table.

   Solution. We can write $\bar{C} = \frac{1}{3}\sum_{i=1}^{3}(X_i - Y_i)$. Each $C_i = X_i - Y_i \sim \mathcal{N}(-1, 3)$, i.i.d. Therefore

$$\bar{C} \sim \mathcal{N}\left(-1, \frac{3}{3}\right) = \mathcal{N}(-1, 1).$$

   Hence

$$\mathbb{P}(\bar{C} > 0) = \mathbb{P}\left(\frac{\bar{C} - (-1)}{1} > 1\right) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587.$$

   (Averaging reduces variance but does not remove the bias -1.)

Problem 2 (Sensor fusion: what correlation really changes)**.**

For random variables X, Y and constants a, b, c, d,

$$\mathrm{Cov}(X + c, Y + d) = \mathrm{Cov}(X, Y), \qquad \mathrm{Cov}(aX, bY) = ab\ \mathrm{Cov}(X, Y).$$

$$\mathrm{Cov}(X, Y) = \mathrm{Corr}(X, Y)\sqrt{\mathbb{V}(X)\mathbb{V}(Y)}.$$

$$\mathbb{V}(aX + bY) = a^2\mathbb{V}(X) + b^2\mathbb{V}(Y) + 2ab\ \mathrm{Cov}(X, Y).$$

A device has two sensors that measure the same unknown scalar quantity θ. The sensor readings are

$$T_1 = \theta + E_1, \qquad T_2 = \theta + E_2,$$

where $\mathbb{E}[E_1] = \mathbb{E}[E_2] = 0$, $\mathbb{V}(E_1) = 9$, $\mathbb{V}(E_2) = 4$, and $\mathrm{Corr}(E_1, E_2) = \frac{1}{2}$. For a weight $w \in [0, 1]$, define the fused estimator

$$T(w) = wT_1 + (1 - w)T_2.$$

1. (Covariance matrix) Compute $\mathrm{Cov}(E_1, E_2)$ and write the covariance matrix $\Sigma_T = \mathrm{Cov}\left(\left(\begin{smallmatrix} T_1 \\ T_2 \end{smallmatrix}\right)\right)$.

   Solution. Since $\mathrm{Corr}(E_1, E_2) = \frac{1}{2}$,

   $$\mathrm{Cov}(E_1, E_2) = \frac{1}{2}\sqrt{\mathbb{V}(E_1)\mathbb{V}(E_2)} = \frac{1}{2}\sqrt{9 \cdot 4} = 3.$$

   Adding constants does not change covariance, so $\mathbb{V}(T_1) = \mathbb{V}(E_1) = 9$, $\mathbb{V}(T_2) = \mathbb{V}(E_2) = 4$, and $\mathrm{Cov}(T_1, T_2) = \mathrm{Cov}(E_1, E_2) = 3$. Therefore

   $$\Sigma_T = \begin{pmatrix} 9 & 3 \\ 3 & 4 \end{pmatrix}.$$

2. (Unbiasedness) Show that $T(w)$ is unbiased for θ.

   Solution. We have $\mathbb{E}[T_1] = \theta + \mathbb{E}[E_1] = \theta$ and $\mathbb{E}[T_2] = \theta + \mathbb{E}[E_2] = \theta$. Hence

   $$\mathbb{E}[T(w)] = w\,\mathbb{E}[T_1] + (1 - w)\,\mathbb{E}[T_2] = w\theta + (1 - w)\theta = \theta.$$

3. (Variance as a function of w) Derive $\mathbb{V}(T(w))$ in terms of w.

   Solution. Using $\mathbb{V}(T_1) = 9$, $\mathbb{V}(T_2) = 4$, and $\mathrm{Cov}(T_1, T_2) = 3$,

   $$\mathbb{V}(T(w)) = w^2\mathbb{V}(T_1) + (1-w)^2\mathbb{V}(T_2) + 2w(1-w)\,\mathrm{Cov}(T_1, T_2) = 9w^2 + 4(1-w)^2 + 6w(1-w).$$

   Expanding gives

   $$\mathbb{V}(T(w)) = 4 - 2w + 7w^2.$$

4. (Optimal fusion weight) Find the minimizer $w^\star = \arg\min_{w\in[0,1]} \mathbb{V}(T(w))$ and compute the minimized variance $\mathbb{V}(T(w^\star))$.

Solution. Since $\mathbb{V}(T(w))$ is a quadratic in w with positive leading coefficient, the unconstrained minimizer satisfies $\frac{d}{dw}\mathbb{V}(T(w)) = 0$:

$$-2 + 14w = 0 \quad \Rightarrow \quad w^\star = \frac{1}{7}.$$

This lies in $[0, 1]$, so it is also the constrained minimizer. Plugging in:

$$\mathbb{V}(T(w^\star)) = 4 - 2 \cdot \frac{1}{7} + 7\left(\frac{1}{7}\right)^2 = \frac{27}{7}.$$

5. (Compare to the naive average) Compute $\mathbb{V}(T(1/2))$ and compare it to $\mathbb{V}(T(w^\star))$.

Solution. Since $T(1/2) = (T_1 + T_2)/2$,

$$\mathbb{V}(T(1/2)) = \frac{1}{4}\mathbb{V}(T_1) + \frac{1}{4}\mathbb{V}(T_2) + \frac{1}{2}\,\mathrm{Cov}(T_1, T_2) = \frac{9}{4} + \frac{4}{4} + \frac{1}{2}(3) = \frac{19}{4}.$$

Therefore

$$\mathbb{V}(T(1/2)) = \frac{19}{4} > \frac{27}{7} = \mathbb{V}(T(w^\star)).$$

So the optimized weight improves on the naive average.

6. **Optional (Gaussian probability via Φ).** Assume (E_1, E_2) is jointly Gaussian. Using the optimized estimator $T(w^\star)$, compute $\mathbb{P}(|T(w^\star) - \theta| \le 2)$ in terms of Φ.

Solution. From part 4, $w^\star = 1/7$ and $\mathbb{V}(T(w^\star)) = 27/7$. Since (E_1, E_2) is jointly Gaussian, the error $T(w^\star) - \theta$ is Normal with mean 0 and variance $27/7$. Let $\sigma = \sqrt{27/7}$. Then

$$\mathbb{P}(|T(w^\star) - \theta| \le 2) = \mathbb{P}\left(\left|\frac{T(w^\star) - \theta}{\sigma}\right| \le \frac{2}{\sigma}\right) = 2\Phi\left(\frac{2}{\sigma}\right) - 1 = 2\Phi\left(2\sqrt{\frac{7}{27}}\right) - 1.$$

Using the Φ table, $2\sqrt{7/27} \approx 1.02$, so the probability is approximately $2\Phi(1.02) - 1 \approx 2(0.846) - 1 \approx 0.69$.

Problem 3 (Uncorrelated is not independent (latency score model))**.**

For random variables A, B,

$$\mathrm{Cov}(A,B) = \mathbb{E}[AB] - \mathbb{E}[A]\mathbb{E}[B], \qquad \mathrm{Corr}(A,B) = \frac{\mathrm{Cov}(A,B)}{\sqrt{\mathbb{V}(A)\mathbb{V}(B)}}.$$

If A and B are independent, then for any events E, F with $\mathbb{P}(E) > 0$,

$$\mathbb{P}(F \mid E) = \mathbb{P}(F).$$

For $Z \sim \mathcal{N}(0,1)$, let $\Phi(t) = \mathbb{P}(Z \le t)$.

A monitoring system tracks minute-level latency via a standardized score Z. It reports two summary features:

$$X = Z, \qquad Y = Z^2 - 1.$$

Here X captures direction of deviation, while Y captures deviation magnitude (centered at 0).

1. Compute $\mathbb{E}[X]$ and $\mathbb{V}(X)$.

   Solution. Since $X = Z$ and $Z \sim \mathcal{N}(0,1)$,

   $$\mathbb{E}[X] = 0, \qquad \mathbb{V}(X) = 1.$$

2. Compute $\mathbb{E}[Y]$ and $\mathbb{V}(Y)$.

   Solution.

   $$\mathbb{E}[Y] = \mathbb{E}[Z^2 - 1] = \mathbb{E}[Z^2] - 1 = 1 - 1 = 0.$$

   Also,

   $$\mathbb{V}(Y) = \mathbb{V}(Z^2 - 1) = \mathbb{V}(Z^2) = \mathbb{E}[Z^4] - \big(\mathbb{E}[Z^2]\big)^2 = 3 - 1 = 2.$$

3. Compute $\mathrm{Cov}(X,Y)$ and $\mathrm{Corr}(X,Y)$.

   Solution.

   $$\mathrm{Cov}(X,Y) = \mathrm{Cov}(Z, Z^2 - 1) = \mathbb{E}[Z^3] - \mathbb{E}[Z]\mathbb{E}[Z^2 - 1] = 0 - 0 = 0.$$

   Therefore

   $$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathbb{V}(X)\mathbb{V}(Y)}} = 0.$$

4. Let $E = \{|X| < 1/2\}$ and $F = \{Y > 0\}$. Show $\mathbb{P}(E) > 0$, and compute $\mathbb{P}(F)$ in terms of Φ.

   Solution. Since $X = Z$,

   $$\mathbb{P}(E) = \mathbb{P}(|Z| < 1/2) = 2\Phi(1/2) - 1 > 0.$$

   Also $F = \{Z^2 - 1 > 0\} = \{|Z| > 1\}$, so

   $$\mathbb{P}(F) = \mathbb{P}(|Z| > 1) = 2\big(1 - \Phi(1)\big).$$

5. Compute $\mathbb{P}(F \mid E)$ and conclude whether X and Y are independent.

   Solution. On event E, we have $|Z| < 1/2$, so $Z^2 < 1/4$ and hence

   $$Y = Z^2 - 1 < 0.$$

   Therefore

   $$\mathbb{P}(F \mid E) = 0.$$

   Since $\mathbb{P}(F) = 2(1-\Phi(1)) > 0$, we get $\mathbb{P}(F \mid E) \neq \mathbb{P}(F)$. Hence X and Y are *not* independent, even though $\mathrm{Corr}(X, Y) = 0$.

Appendix: The table lists $P(Z \leq z)$ where $Z \sim N(0, 1)$ for positive values of z.

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998