18.650: Fundamentals of Statistics February 9, 2026

Lecture 4 — The Gaussian distribution

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This lecture is on the Gaussian distribution. But first, here is a summary of distributions we'll see in this class. You should be familiar with the ones above the dotted line.

discrete	continuous
Bernoulli	Exponential
Binomial	Uniform
Poisson	Gaussian/normal
	Gamma
	Beta
	Student t
	Chi-squared (*)

* The chi-squared distribution will be covered more extensively later.

1 The Gaussian Distribution

The Gaussian distribution is denoted $X \sim \mathcal{N}(\mu, \sigma^2)$. It is parameterized by its mean $\mu = \mathbb{E}[X]$ and variance $\sigma^2 = \mathbb{E}[(X-\mu)^2]$. The Gaussian distribution will most frequently arise as a limit of a sequence of normalized averages (thanks to the CLT). The probability density function (pdf) corresponding to $X \sim \mathcal{N}(\mu, \sigma^2)$ is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

As you can see from Figure 1, it's a bell-shaped curve.

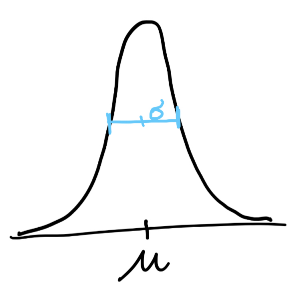

Figure 1: the pdf of the normal distribution

Figure 2 compares the pdfs of $\mathcal{N}(0,1)$ and $\mathcal{N}(10,4)$. The latter is shifted farther to the right, and has a wider spread.

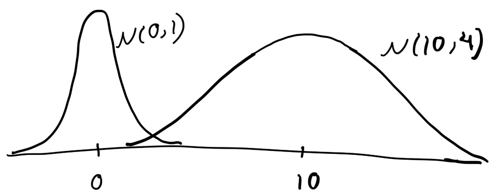

Figure 2: comparison between two normal pdfs

1.1 The 68–95–99.7 rule.

The 68–95–99.7 rule tells you what percentage of the Gaussian density is contained within one, two, and three standard deviations of the mean, i.e., in the intervals $(\mu-\sigma,\mu+\sigma)$, $(\mu-2\sigma,\mu+2\sigma)$, and $(\mu-3\sigma,\mu+3\sigma)$, respectively. Formally, these numbers are given by the integral under the curve $f(x)$ over each of the three intervals. For example, $\int_{\mu-\sigma}^{\mu+\sigma} f(x)\,dx \approx 0.68$.

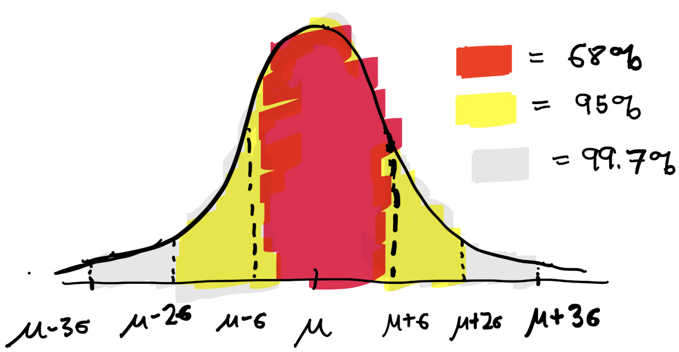

Figure 3: The area contained within one, two, and three standard deviations of the mean.

1.2 Transformation and standardization

If $X \sim \mathcal{N}(\mu,\sigma^2)$ then $aX+b \sim \mathcal{N}(a\mu+b, a^2\sigma^2)$. In particular, any Gaussian can be transformed into the standard normal Gaussian as follows:

$$X \mapsto Z = (X-\mu)/\sigma \sim \mathcal{N}(0,1).$$

Equivalently,

$$X = \mu + \sigma Z.$$

We call $Z = (X-\mu)/\sigma$ the “z-score” or standardization of X. Standardizing X is useful because it means we only need to know how to compute probabilities of Z.

For example,

$$\mathbb{P}(a \le X \le b) = \mathbb{P}\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right), \quad Z = (X-\mu)/\sigma \sim \mathcal{N}(0,1). \quad (1)$$

Definition 1.1: pdf and cdf of standard Gaussian distribution

The standard Gaussian/normal distribution $Z \sim \mathcal{N}(0,1)$ has pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}.$$

The cdf is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} dt,$$

and it has *no closed form*! Need computer/calculator/table to compute it.

Note that Z is symmetric! If $Z \sim \mathcal{N}(0,1)$ then $-Z \sim \mathcal{N}(0,1)$. So

$$\mathbb{P}(Z > t) = \mathbb{P}(-Z > t) = \mathbb{P}(Z < -t) = \phi(-t).$$

At the same time, $\mathbb{P}(Z > t) = 1 - \mathbb{P}(Z < t) = 1 - \Phi(t)$, so we have shown that $\Phi(-t) = 1 - \Phi(t)$.

Example.

1. $\mathbb{P}(Z \le 1) = \Phi(1)$
2. $\mathbb{P}(Z \ge -1) = \Phi(1)$
3. $\mathbb{P}(|Z| > 1) = \mathbb{P}(Z \le -1) + \mathbb{P}(Z \ge 1) = \Phi(-1) + 1 - \Phi(1) = 1 - \Phi(1) + 1 - \Phi(1) = 2 - 2\Phi(1)$.

Visualizing these probabilities (as in Figure 4) is always a good idea.

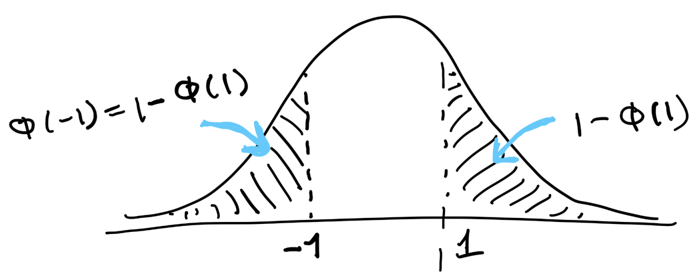

Figure 4: the probability $\mathbb{P}(|Z| > 1)$ is a sum of the areas of two symmetric regions.

1.3 Reading the table

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The table lists $\mathbb{P}(Z \leq z)$ where $Z \sim \mathcal{N}(0,1)$ for positive values of z.

To get the value of $\Phi(2.34)$, say, go down to row 2.3, and across to column 0.04. We get $\Phi(2.34) = 0.9904$. For negative numbers we use the Gaussian symmetry; e.g. $\Phi(-2.34) = 1 - 0.9904 = 0.0096$.