18.650. Fundamentals of Statistics
Spring 2026. Problem Set 1

Due Wednesday, Feb 18

Problem 1 (Probability Review)**.** *Please use the accompanying bubble sheet for submitting your solutions for this problem.*

In what follows, Φ is the CDF of the standard Gaussian (Normal) distribution.

Let X be a random variable taking values between 0 and π, with pdf given by

$$f(x) = c \sin x, \qquad x \in [0, \pi].$$

1. What is the value of c? (a) π (b) $1/2$ (c) 2 (d) $1/\pi$

2. What is $\mathbb{E}[X]$? (a) $\pi/2$ (b) π (c) 1 (d) $1/2$

Let X be a Gaussian random variable with mean $\mu > 0$ and variance σ^2.

3. What is $\mathbb{E}[X]$? (a) 0 (b) σ^2 (c) $\mu^2 + \sigma^2$ (d) μ

4. What is $\mathbb{V}[X]$? (a) σ^2 (b) $\mu^2 + \sigma^2$ (c) μ^2 (d) σ

5. What is $\mathbb{E}[X^2]$? (a) μ^2 (b) $(\mu + \sigma)^2$ (c) $\mu^2 + \sigma^2$ (d) σ^2

6. What is $\mathbb{E}[X^3]$? (a) $\mu^3 + 3\mu\sigma$ (b) $\mu^3 + 3\mu\sigma^2$ (c) μ^3 (d) $3\mu\sigma^2$

7. What is $\mathbb{V}[X^2]$? (a) $4\mu^2\sigma^2$ (b) $(\mu^2 + \sigma^2)^2$ (c) $2\sigma^4$ (d) $4\mu^2\sigma^2 + 2\sigma^4$

8. What is $\mathbb{P}(X > 0)$ in terms of Φ?
   (a) $\Phi(\mu/\sigma)$ (b) $\Phi(\mu/\sigma^2)$ (c) $1 - \Phi(\mu/\sigma)$ (d) $\Phi(\mu)$

Let $X \sim \text{Lognormal}(\mu, \sigma^2)$, i.e., $\log X \sim \mathcal{N}(\mu, \sigma^2)$ with $\sigma > 0$.

9. What is $\mathbb{E}[X]$? (a) e^{μ} (b) $e^{\mu+\sigma^2/2}$ (c) $e^{\mu+\sigma^2}$ (d) $\mu + \sigma^2/2$

10. What is the median of X? (a) $e^{\mu+\sigma^2/2}$ (b) $e^{\mu-\sigma^2}$ (c) e^{μ} (d) μ

11. What is $\mathbb{P}(X > 1)$ in terms of Φ?
    (a) $1 - \Phi(\mu/\sigma)$ (b) $\Phi(\mu)$ (c) $\Phi(\mu/\sigma^2)$ (d) $\Phi(\mu/\sigma)$

Let $X, Y \sim \text{Lognormal}(0,1)$ be independent.

12. What is $\mathbb{P}(XY > 1)$? (a) $1/4$ (b) $1/2$ (c) $1/\sqrt{2\pi}$ (d) $1/e$

Let X be a random variable such that

$$X = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

for some $p \in [0,1]$.

13. What is $\mathbb{E}[X]$?

    (a) $-p$ (b) p (c) $1-2p$ (d) $2p-1$

14. What is $\mathbb{V}[X]$?

    (a) $p(1-p)$ (b) $4p-p^2$ (c) $4p(1-p)$ (d) $4p^2(1-p)$

15. For what p is $\mathbb{V}[X]$ maximized?

    (a) 1 (b) 0 (c) 0.5 (d) $1/\sqrt{2}$

16. What is $\mathbb{E}[X^k]$?

    (a) p^k (b) $p^k - (1-p)^k$ (c) $p(-1)^k + (1-p)$ (d) $p + (1-p)(-1)^k$

Let X, Y be two independent standard Gaussian random variables.

17. What is $\mathbb{E}[X^2 Y]$?

    (a) 0 (b) 1 (c) 2 (d) 3

18. What is $\mathbb{V}(X+Y)$?

    (a) 0 (b) 1 (c) 2 (d) 3

19. What is $\mathbb{V}(XY)$?

    (a) 0 (b) 1 (c) 2 (d) 3

20. What is $\text{Cov}(X, X+Y)$?

    (a) 0 (b) 1 (c) 2 (d) 3

21. What is $\text{Cov}(X, XY)$?

    (a) 0 (b) 1 (c) 2 (d) 3

Let X be an exponential random variable with parameter $1/2$ that models the lifetime (in years) of a lightbulb.[1]

22. What is (approximately) the probability that the lightbulb will last at least 2 years?

    (a) 0.002 (b) 0.018 (c) 0.180 (d) 0.810

[1] We use the convention from AoS for the parameter of an exponential distribution.

23. Given that the lightbulb has already lasted for at least 3 years, what is (approximately) the probability that it will last for at least two more years?

    (a) 0.002 (b) 0.018 (c) 0.180 (d) 0.810

Let $X_1, \ldots, X_n$ be i.i.d with mean μ and variance σ^2.

24. What is $\mathbb{E}[\sum_{i=1}^n X_i]$? (a) μ (b) $n\sigma$ (c) $n\mu$ (d) σ

25. What is $\mathbb{V}[\sum_{i=1}^n X_i]$? (a) $n^2\sigma^2$ (b) $n\sigma^2$ (c) $n\sigma^2 + n^2\mu^2$ (d) $n\mu$

26. What is $\mathbb{E}\big[\big(\sum_{i=1}^n X_i\big)^2\big]$? (a) $n^2\mu^2$ (b) $n\sigma^2$ (c) $n\mu$ (d) $n\sigma^2 + n^2\mu^2$

27. What is $\mathbb{E}[\frac{1}{n}\sum_{i=1}^n X_i]$? (a) σ (b) $n\sigma^2$ (c) $n\mu$ (d) μ

28. What is $\mathbb{V}[\frac{1}{n}\sum_{i=1}^n X_i]$? (a) μ (b) σ^2/n (c) σ^2 (d) $n\mu$

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The following problems are of "show-your-work" type. You get complete points for a solution if you show your complete work. That includes all computations that lead to your answer with appropriate reasoning. Please upload your work to Gradescope.

Problem 2. Let $X_n \sim \mathrm{Unif}\big(-\frac{1}{n}, \frac{1}{n}\big)$ and let X be a random variable such that $\mathbb{P}(X = 0) = 1$.

1. Compute and draw the CDF $F_n(x)$ and $F(x)$ of X_n and X respectively.

2. Does $X_n \xrightarrow{\mathbb{P}} X$? (prove or disprove)

3. Does $X_n \rightsquigarrow X$? (prove or disprove)

Problem 3. Let $X \sim \mathcal{N}(2, 1.44)$. Compute the following probabilities:

1. $\mathbb{P}(2X - 1 < 0)$

2. $\mathbb{P}\left(\frac{7}{5} \leq X \leq \frac{16}{5}\right)$

3. $\mathbb{P}\left(X > \frac{16}{5} \mid X > \frac{7}{5}\right)$

4. $\mathbb{P}\left(X \leq \frac{4}{5} \text{ or } X \geq \frac{16}{5}\right)$

Problem 4. Let

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\right)$$

Compute the following quantities (show your work):

1. $\mathbb{V}[X]$

2. $\mathbb{E}[Y^2 + X]$

3. $\mathbb{E}[(X-Y)^2]$

4. $\mathbb{V}[X+2Y]$

5. Find $\alpha > 0$ such that $\alpha X = Y$ with probability 1 or prove that no such α exists.

Problem 5. We are testing n lightbulbs. Each bulb independently *passes* some quality check with probability $p \in (0,1)$ and *fails* with probability $1-p$. Let $X_i \in \{0,1\}$ indicate the outcome, where $X_i = 1$ means the bulb passes.

Conditioned on X_i, the lifetime Y_i of bulb i is exponentially distributed:

$$Y_i \mid (X_i = 1) \sim \mathrm{Exp}(\lambda_1), \qquad Y_i \mid (X_i = 0) \sim \mathrm{Exp}(\lambda_0),$$

where $\lambda_0, \lambda_1 > 0$. Assume the pairs (X_i, Y_i) are i.i.d. across i.

1. Let (X, Y) be a copy of (X_1, Y_1) and let $T := XY$. Compute the following quantities in terms of p, λ_0, λ_1:

   $\mathbb{E}[Y]$, $\mathbb{V}(Y)$, $\mathrm{Cov}(X,Y)$, $\mathbb{E}[T]$, $\mathbb{V}(T)$, and $\mathrm{Cov}(X,T)$.

Define

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i, \qquad \text{and} \qquad \bar{T}_n = \frac{1}{n}\sum_{i=1}^{n} X_i Y_i.$$

Write a central limit theorem for each of the following quantities in the form

$$\sqrt{n}(Z_n - \mu) \rightsquigarrow \mathcal{N}(0, \sigma^2) \quad \text{or} \quad \sqrt{n}(Z_n - \mu) \rightsquigarrow \mathcal{N}(0, \Sigma),$$

depending on whether Z_n is a random variable or a random vector.

2. $Z_n = \begin{pmatrix} \bar{X}_n \\ \bar{T}_n \end{pmatrix}$.

3. $Z_n = \log(\bar{Y}_n)$.

Define the average lifetime among passed bulbs

$$\widehat{\lambda}_{1,n} = \begin{cases} \dfrac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i} = \dfrac{\bar{T}_n}{\bar{X}_n}, & \text{if } \sum_{i=1}^n X_i > 0, \\ 0, & \text{if } \sum_{i=1}^n X_i = 0, \end{cases}$$

and the corresponding rate estimator

$$\widehat{\rho}_{1,n} = \begin{cases} \dfrac{1}{\widehat{\lambda}_{1,n}} = \dfrac{\bar{X}_n}{\bar{T}_n}, & \text{if } \sum_{i=1}^n X_i > 0, \\ 0, & \text{if } \sum_{i=1}^n X_i = 0. \end{cases}$$

(Note that $\mathbb{P}(\sum_{i=1}^n X_i = 0) = (1-p)^n \to 0$, so this convention does not affect any CLT/delta-method limits.)

Write a CLT for each of the following choices of Z_n:

4. $Z_n = \widehat{\lambda}_{1,n}$.

5. $Z_n = \widehat{\rho}_{1,n}$.

Extra probability practice (not graded)

Problem 6. Let X be a random variable with pmf given by

$$\mathbb{P}(X = k) = \frac{c\lambda^k}{k!}, k = 0, 1, 2, \ldots$$

for some $\lambda > 0$.

1. What is the value of c? (a) 1 (b) λ (c) $e^{-\lambda}$ (d) e^{λ}
2. What is $\mathbb{E}[X]$? (a) 1 (b) λ (c) $e^{-\lambda}$ (d) e^{λ}
3. What is $\mathbb{V}[X]$? (a) 1 (b) λ (c) $e^{-\lambda}$ (d) e^{λ}

Let X be a uniform random variable in the interval $[2, 8]$.

4. What is $\mathbb{E}[X]$?

   (a) 2 (b) 3 (c) 5 (d) 8
5. What is $\mathbb{V}[X]$?

   (a) 2 (b) 3 (c) 5 (d) 8
6. What[2] is $\mathbb{P}[\log(X) \leq 1]$ approximately?

   (a) .12 (b) 0.8 (c) $-.1$ (d) 0

Let X be an exponential random variable with parameter 3 and Y be a Poisson random variable with parameter 2. Assume that X and Y are independent.

7. What is $\mathbb{E}[X^2 + Y^2]$?

   (a) 12 (b) 23 (c) 24 (d) 36
8. What is $\mathbb{E}[X^2 Y]$?

   (a) 12 (b) 23 (c) 24 (d) 36
9. What is $\mathbb{V}(2X + 3Y)$?

   (a) 24 (b) 34 (c) 44 (d) 54

Let $X \geq 0$ be a positive random variable such that $\mathbb{E}[X] = \lambda$.

10. Which is correct?

    (a) $\mathbb{E}[1/X] = 1/\lambda$ (b) $\mathbb{E}[1/X] \geq 1/\lambda$ (c) $\mathbb{E}[1/X] \leq 1/\lambda$

Let X and Y be two random variables such that X is a Bernoulli random variable with parameter $p \in (0, 1)$, and $Y^2 + 2XY = 3X^2$ almost surely.

[2] all logs are natural (base e) unless specified otherwise

11. What is $\mathbb{E}[Y]$?

(a) 0 (b) $-3p$ (c) X (d) $-3X$ (e) Some number in $[-3, 1]$

Let X and Y be two independent, identically distributed random variables.

12. Compute the conditional expectation $\mathbb{E}[X|X+Y=x]$.

(a) $x/2$ (b) x (c) $-x$ (d) 0