18.650. Fundamentals of Statistics
Spring 2026. Recitation Sheet #1

Problem 1 (Gaussian standardization and Φ)**.** A standardized exam score X is modeled as

$$X \sim \mathcal{N}(100, 15^2).$$

Let Φ denote the CDF of $Z \sim \mathcal{N}(0,1)$. Express answers in terms of Φ and Φ^{-1} (no decimals needed).

1. (Standardization) Define $Z = (X - 100)/15$. What is the distribution of Z?

   Solution. $Z \sim \mathcal{N}(0,1)$.

2. Compute the following probabilities:

   (a) $\mathbb{P}(X \le 85)$

   Solution.

   $$\mathbb{P}(X \le 85) = \Phi\left(\frac{85 - 100}{15}\right) = \Phi(-1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587.$$

   (b) $\mathbb{P}(100 \le X \le 115)$

   Solution.

   $$\mathbb{P}(100 \le X \le 115) = \Phi(1) - \Phi(0) = 0.8413 - 0.5000 = 0.3413.$$

   (c) $\mathbb{P}(|X - 100| \ge 30)$

   Solution.

   $$\mathbb{P}(|X - 100| \ge 30) = \mathbb{P}(|Z| \ge 2) = 2(1 - \Phi(2)) = 2(1 - 0.9772) = 0.0456.$$

3. (Quantile) Find c such that $\mathbb{P}(X \ge c) = 0.10$ (top 10% cutoff).

   Solution. We have $\mathbb{P}(X \le c) = 0.90$. Therefore $c = 100 + 15\,\Phi^{-1}(0.90)$. The value 0.9000 is not listed exactly in the table, but

   $$\Phi(1.28) = 0.8997, \qquad \Phi(1.29) = 0.9015,$$

   so $\Phi^{-1}(0.90) \approx 1.282$ (linear interpolation) and hence $c \approx 100 + 15(1.282) = 119.2$.

4. (Affine transform / rescaling) The exam board also reports the "index score"

$$T \;=\; 50 + 10 \cdot Z \;=\; 50 + 10 \cdot \frac{X - 100}{15}.$$

What is the distribution of T? Express $\mathbb{P}(T \geq 70)$ in terms of Φ.

Solution. Since $T = 50 + 10Z$ and $Z \sim \mathcal{N}(0,1)$, we have $T \sim \mathcal{N}(50, 10^2)$. Then

$$\mathbb{P}(T \geq 70) = 1 - \Phi\left(\frac{70-50}{10}\right) = 1 - \Phi(2) = 1 - 0.9772 = 0.0228.$$

5. (Conditional tail) Compute $\mathbb{P}(X \geq 130 \mid X \geq 115)$ in terms of Φ.

Solution.

$$\mathbb{P}(X \geq 130 \mid X \geq 115) = \frac{\mathbb{P}(X \geq 130)}{\mathbb{P}(X \geq 115)} = \frac{1 - \Phi\left(\frac{130-100}{15}\right)}{1 - \Phi\left(\frac{115-100}{15}\right)} = \frac{1-\Phi(2)}{1-\Phi(1)}.$$

Using the table: $1 - \Phi(2) = 1 - 0.9772 = 0.0228$ and $1 - \Phi(1) = 1 - 0.8413 = 0.1587$, so the answer is $\frac{0.0228}{0.1587} \approx 0.1437$.

Problem 2. The mean weight of eggs produced by a farm is $\mathbf{60}g$ with a standard deviation of $\mathbf{4}g$.

1. Express $\mathbf{55}g$ in standard units (z -score).

   Solution. $z = \frac{55-60}{4} = \frac{-5}{4} = -1.25$. Hence, $55g$ is -1.25 in standard units (1.25 standard deviations below the mean).

2. How many standard deviations from the mean is a weight of 68 grams?

   Solution. $z = \frac{x-\mu}{\sigma} = \frac{68-60}{4} = \frac{8}{4} = 2$. Hence, 68g is 2 standard deviations above the mean.

3. What weight corresponds to $\mathbf{1.5}$ standard deviations below the mean?

   Solution. First compute the amount below the mean: $1.5 \times 4 = 6$. Then subtract from the mean: $60 - 6 = 54$. Thus, a weight 1.5 standard deviations below the mean is $54g$.

4. Find the probability that the weight, X, of eggs is less than $58g$, where $X \sim N\left(\mathbf{60}, \mathbf{4^2}\right)$.

   Solution. $z = \frac{58-60}{4} = -0.5$. $\Phi(-0.5) = 1 - \Phi(0.5) = 0.3085$.

Problem 3 (All-in gambling with rare catastrophic risk.)**.** A player starts with \$1 and repeatedly bets all their money for n rounds. Each round: with probability $1/2$ they win and their wealth doubles; with probability $1/2$ they lose and their wealth becomes 0 and they immediately go home. Let W_n be the player's final wealth (when they leave).

1. (Distribution) Find $\mathbb{P}(W_n = 2^n)$ and $\mathbb{P}(W_n = 0)$.

   Solution. The player ends with 2^n dollars if and only if they win all n rounds, so

   $$\mathbb{P}(W_n = 2^n) = \left(\frac{1}{2}\right)^n.$$

   Otherwise they lose at some point and end with 0, so $\mathbb{P}(W_n = 0) = 1 - (1/2)^n$.

2. (Convergence) Does $W_n \to 0$ in probability? In distribution?

   Solution. Fix $\varepsilon > 0$. For all large n, we have $2^n > \varepsilon$ and hence

   $$\mathbb{P}(|W_n - 0| > \varepsilon) = \mathbb{P}(W_n = 2^n) = \left(\frac{1}{2}\right)^n \to 0.$$

   Therefore $W_n \xrightarrow{\mathbb{P}} 0$. This implies $W_n \Rightarrow 0$ as well.

3. (Expectations vs convergence) Compute $\mathbb{E}[W_n]$. Does $W_n \to 0$ in L^1?

   Solution.
   $$\mathbb{E}[W_n] = 2^n \cdot \mathbb{P}(W_n = 2^n) = 2^n \left(\frac{1}{2}\right)^n = 1.$$

   Therefore $W_n \not\to 0$ in L^1, since $\mathbb{E}|W_n - 0| = \mathbb{E}[W_n] = 1 \not\to 0$.

4. (Illegal casino: rare fine) Now suppose the casino is illegal in the current state. If the player loses, they go home and are not caught. If the player survives all n rounds (so they are still at the casino at the end), then with probability $1/n$ they are caught and must pay a fine of \$ 4^n. Let Y_n be the player's *net* wealth after the fine.

   (a) Find the distribution of Y_n.

   (b) Does $Y_n \to 0$ in probability? In distribution?

   (c) Compute $\mathbb{E}[Y_n]$ and describe its behavior as $n \to \infty$.

   Solution. **(a)** The player wins all n rounds with probability $(1/2)^n$. Conditional on that, with probability $1/n$ they are caught and pay the fine. Thus

   $$Y_n = \begin{cases} 0, & \text{with prob. } 1 - (1/2)^n, \\ 2^n, & \text{with prob. } (1/2)^n \left(1 - \frac{1}{n}\right), \\ 2^n - 4^n, & \text{with prob. } (1/2)^n \left(\frac{1}{n}\right). \end{cases}$$

(b) Fix $\varepsilon > 0$. The only way $|Y_n| > \varepsilon$ is if the player wins all n rounds, so

$$\mathbb{P}(|Y_n| > \varepsilon) \leq \mathbb{P}(\text{win all } n \text{ rounds}) = \left(\frac{1}{2}\right)^n \to 0.$$

Hence $Y_n \xrightarrow{\mathbb{P}} 0$, and therefore $Y_n \Rightarrow 0$.

(c) We can compute

$$\mathbb{E}[Y_n] = \mathbb{E}[W_n] - 4^n \cdot \mathbb{P}(\text{caught and win all } n \text{ rounds}) = 1 - 4^n \left(\frac{1}{2}\right)^n \left(\frac{1}{n}\right) = 1 - \frac{2^n}{n} \to -\infty.$$

Problem 4 (Estimating noise level from repeated measurements.)**.** A device is used to measure the same fixed quantity over and over (e.g., a QC sample in a lab). We model the measurements as i.i.d. random variables

$$X_1, \ldots, X_n \text{ i.i.d.}, \qquad \mathbb{E}[X_1] = \mu, \qquad \mathbb{V}(X_1) = \sigma^2 \in (0, \infty).$$

Define

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

1. (Algebra) Show that

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X}_n)^2.$$

Solution. Expand and simplify:

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}\Big(X_i^2 - 2X_i\bar{X}_n + (\bar{X}_n)^2\Big) = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \frac{2\bar{X}_n}{n}\sum_{i=1}^{n} X_i + (\bar{X}_n)^2.$$

Since $\sum_{i=1}^{n} X_i = n\bar{X}_n$, the middle term is $2(\bar{X}_n)^2$, hence

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X}_n)^2.$$

2. (Consistency; AoS 5.1) Show that $\hat{\sigma}_n^2 \xrightarrow{\mathbb{P}} \sigma^2$.

Solution. By the LLN,

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{\mathbb{P}} \mathbb{E}[X_1^2], \qquad \bar{X}_n \xrightarrow{\mathbb{P}} \mathbb{E}[X_1] = \mu.$$

By the continuous mapping theorem, $(\bar{X}_n)^2 \xrightarrow{\mathbb{P}} \mu^2$. Using the identity from part (a) and taking limits gives

$$\hat{\sigma}_n^2 \xrightarrow{\mathbb{P}} \mathbb{E}[X_1^2] - \mu^2 = \mathbb{V}(X_1) = \sigma^2.$$

3. **Optional (Slutsky).** Assume also that the CLT applies to $\bar{X}_n$, i.e.

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \Rightarrow \mathcal{N}(0, 1).$$

Show that

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}_n} \Rightarrow \mathcal{N}(0, 1).$$

Solution. From part (b), $\hat{\sigma}_n^2 \xrightarrow{\mathbb{P}} \sigma^2$, and since $\sigma^2 > 0$,

$$\hat{\sigma}_n = \sqrt{\hat{\sigma}_n^2} \xrightarrow{\mathbb{P}} \sigma \quad \Rightarrow \quad \frac{\sigma}{\hat{\sigma}_n} \xrightarrow{\mathbb{P}} 1.$$

Then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}_n} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \cdot \frac{\sigma}{\hat{\sigma}_n} \Rightarrow \mathcal{N}(0,1) \cdot 1 = \mathcal{N}(0,1)$$

by Slutsky's theorem.

Appendix: The table lists $P(Z \leq z)$ where $Z \sim N(0,1)$ for positive values of z.

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998