Lec 3: Convergence of Random Variables

Convergence of sequences

$x_1, x_2, x_3, \ldots$

$\lim_{n\to\infty} x_n = x^*$ $\quad x_n \xrightarrow[n\to\infty]{} x^*$

$X_1, X_2, X_3, \ldots$

$X_n \to X$

almost sure

① $X_n(\omega) \to X(\omega) \ \forall \omega \in \Omega$ (almost all $\omega \in \Omega$)

② the pdf of X_n

↓ conv.

the pdf of X.

Ω sample space

$\mathbb{P}$

$X : \Omega \to \mathbb{R}$

$\omega \in \Omega$

Ω, $X_i : \Omega \to \mathbb{R}$

(imagine each X_i is a feature of smthing (like X_1 = temp, X_2 = date, ...))

$\omega \in \Omega$

↑ specifies the values "all at once"

now RV values map to $\mathbb{R}$

ex: $X_1, X_2, \ldots$ are iid, $\mathbb{E}[X_1] = \mu$, $\mathbb{V}[X_1] = \sigma^2$

$$\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \overline{X_n} \to \mu \qquad \text{(SLLN)}$$

↳ strong law...

Convergence in Probability:

def: $X_n \xrightarrow{\mathbb{P}} X$ if $\forall \varepsilon > 0$

$$\mathbb{P}(|X_n - X| > \varepsilon) \xrightarrow[n\to\infty]{} 0$$

↳ takes whole sample space into account.

if n = large, look at X_n & X & see how far apart they are.

ex: $X_1, X_2, \ldots,$ $X_n \overset{iid}{\sim} \text{Ber}(1/2)$ → coin toss

$X \sim \text{Ber}(1/2)$

X and $\{X_n\}$ are independent

$X_n \xrightarrow{\mathbb{P}} X$?

for $\varepsilon > 0$, (have to choose ε wisely bc $X \in [0,1]$)

$$\mathbb{P}(|\underbrace{X_n - X}| > \varepsilon) = \mathbb{P}(\{X_n = 0\} \cap \{X = 1\}) \cup (\{X_n = 1\} \cap \{X = 0\})$$

must be diff or else $\mathbb{P} = 0$.

↳ proper notation

$$= \mathbb{P}(X_n = 0, X = 1) + P(X_n = 1, X = 0) \leftarrow \text{bc disjoint}$$

$$= \mathbb{P}(X_n = 0)\,\mathbb{P}(X = 1) + P(X_n = 1)\,P(X = 0)$$

$$= 1/2 \cdot 1/2 + 1/2 \cdot 1/2 = 1/2 \neq 0$$

(WLLN) exercise prove this for X_n iid w/ $\mathbb{E}[X_n] = \mu$

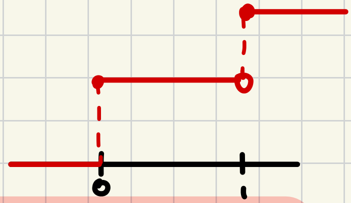
weak (Markov) $\mathbb{V}[X_n] = \sigma^2 < \infty$

Convergence in dist:

$X_n \rightsquigarrow X$

$F_X(x) = \mathbb{P}(X \le x)$

CDF of X



If CDF of X_n converges to the CDF of x at all continuity points of CDF of x.

can show dist. of var.

ex: (CLT) (central limit thm)

$\text{Bin}(n, 1/2)$ (take gaussian of CDF & compare)

how to write: $\Rightarrow$ or $\xrightarrow{(d)}$ or $\xrightarrow{d}$ or $\xrightarrow{\mathcal{L}}$ or $\mathcal{L}(X_n) \to \mathcal{L}(X)$

Thm: if $X_n \xrightarrow{\mathbb{P}} X$ then $X_n \rightsquigarrow X$.

conv. in dist.

Lem: if $X_n \rightsquigarrow c$ (constant) then $X_n \xrightarrow{\mathbb{P}} c$

exercise

let $\varepsilon > 0$:

$\mathbb{P}(|X_n - c| > \varepsilon) > \ldots$

$\star \quad \dfrac{e^{\tan \frac{1}{n^2}}}{\cos \frac{1}{n}} \to 1$

Operation:

- if $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} Y$:

$$X_n + Y_n \xrightarrow{P} X + Y$$

$$X_n Y_n \xrightarrow{P} XY$$

- $X_n \rightsquigarrow X$, $Y_n \xrightarrow{P} c$ then $X_n + Y_n \rightsquigarrow X + c$, $X_n Y_n \rightsquigarrow cX$

- $Y_n = -X_n$

$$X, X_1, X_2, \ldots \overset{iid}{\sim} N(0,1)$$

$$X_n \rightsquigarrow X$$

$$Y_n \rightsquigarrow \underbrace{X}_{}$$

can be anything with same dist. as X, so X_1, $-X$, etc.

Continuous Mapping Thm:

$$X_n \xrightarrow{P} X \rightsquigarrow$$

$$g(X_n) \xrightarrow{P} g(X) \rightsquigarrow (\bar{X}_n)^2 ?$$

★ go thru lec notes (didn't finish in lec)

↑

reading

Delta Method:

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \rightsquigarrow \frac{N(0,1)}{Z} \rightarrow \left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right)^2 \rightsquigarrow Z^2$$

(not good bc want)

$$\frac{(\bar{X}_n)^2 - 0}{0} \rightsquigarrow 0$$