PS01
MIT 18.650 Stats

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Collaborators: Claude 4.6 Sonnet (Questions, not Solving) · Resources Used: None

1 Probability Review

Please use the accompanying bubble sheet for submitting your solutions for this problem. In what follows, Φ is the CDF of the standard Gaussian (Normal) distribution.

1.1 Sinusoidal PDF

Let X be a random variable taking values between 0 and π, with pdf given by

$$f(x) = c \sin x, \quad x \in [0, \pi]. \tag{1}$$

1. What is the value of c?
   (a) π (b) $1/2$ (c) 2 (d) $1/\pi$

For f to be a valid pdf:

$$c \int_0^\pi \sin x \, \mathrm{d}x = 1 \tag{2}$$

Solving:

$$\int_0^\pi \sin x \, \mathrm{d}x = [-\cos x]_0^\pi = 2 \tag{3}$$

$$\Rightarrow c = \frac{1}{2} \tag{4}$$

Answer: (b)

2. What is $\mathbb{E}[X]$?
   (a) $\pi/2$ (b) π (c) 1 (d) $1/2$

Integration by parts ($u = x$, $\mathrm{d}v = \sin x \, \mathrm{d}x$):

$$\mathbb{E}[X] = \frac{1}{2} \int_0^\pi x \sin x \, \mathrm{d}x \tag{5}$$

$$= \frac{1}{2}[-x \cos x + \sin x]_0^\pi \tag{6}$$

$$= \frac{1}{2} \cdot \pi = \frac{\pi}{2} \tag{7}$$

Answer: (a)

1.2 Gaussian Random Variable

Let X be a Gaussian random variable with mean $\mu > 0$ and variance σ^2.

3. What is $\mathbb{E}[X]$?
   (a) 0 (b) σ^2 (c) $\mu^2 + \sigma^2$ (d) μ

By definition, **Answer: (d)**

4. What is $\mathbb{V}[X]$?
   (a) σ^2 (b) $\mu^2 + \sigma^2$ (c) μ^2 (d) σ

By definition, **Answer: (a)**

5. What is $\mathbb{E}[X^2]$?
   (a) μ^2 (b) $(\mu + \sigma)^2$ (c) $\mu^2 + \sigma^2$ (d) σ^2

Using $\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$:

$$\mathbb{E}[X^2] = \mathbb{V}[X] + (\mathbb{E}[X])^2 = \sigma^2 + \mu^2 \tag{8}$$

Answer: (c)

6. What is $\mathbb{E}[X^3]$?
   (a) $\mu^3 + 3\mu\sigma$ (b) $\mu^3 + 3\mu\sigma^2$
   (c) μ^3 (d) $3\mu\sigma^2$

Write $X = \mu + \sigma Z$ with $Z \sim \mathcal{N}(0, 1)$:

$$\mathbb{E}[X^3] = \mathbb{E}[(\mu + \sigma Z)^3] \tag{9}$$

$$= \mu^3 + 3\mu^2\sigma\underbrace{\mathbb{E}[Z]}_{0} + 3\mu\sigma^2\underbrace{\mathbb{E}[Z^2]}_{1} + \sigma^3\underbrace{\mathbb{E}[Z^3]}_{0} \tag{10}$$

$$= \mu^3 + 3\mu\sigma^2 \tag{11}$$

Answer: (b)

7. What is $\mathbb{V}[X^2]$?
   (a) $4\mu^2\sigma^2$ (b) $(\mu^2 + \sigma^2)^2$
   (c) $2\sigma^4$ (d) $4\mu^2\sigma^2 + 2\sigma^4$

For a Gaussian, $\mathbb{E}[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$ (via MGF):

$$\mathbb{V}[X^2] = \mathbb{E}[X^4] - (\mathbb{E}[X^2])^2 \tag{12}$$

$$= (\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4) - (\mu^2 + \sigma^2)^2 \tag{13}$$

$$= 4\mu^2\sigma^2 + 2\sigma^4 \tag{14}$$

Answer: (d)

8. What is $\mathbb{P}(X > 0)$ in terms of Φ?
   (a) $\Phi(\mu/\sigma)$ (b) $\Phi(\mu/\sigma^2)$ (c) $1 - \Phi(\mu/\sigma)$ (d) $\Phi(\mu)$

Standardize:

$$\mathbb{P}(X > 0) = \mathbb{P}\left(\frac{X-\mu}{\sigma} > -\frac{\mu}{\sigma}\right) \tag{15}$$

$$= \mathbb{P}\left(Z > -\frac{\mu}{\sigma}\right) \tag{16}$$

$$= \Phi(\mu/\sigma) \tag{17}$$

by symmetry of the standard normal. **Answer: (a)**

1.3 Lognormal Distribution

Let $X \sim \text{Lognormal}(\mu, \sigma^2)$, i.e., $\log X \sim \mathcal{N}(\mu, \sigma^2)$ with $\sigma > 0$.

9. What is $\mathbb{E}[X]$?
   (a) e^{μ} (b) $e^{\mu+\sigma^2/2}$ (c) $e^{\mu+\sigma^2}$ (d) $\mu + \sigma^2/2$

Standard lognormal result from the MGF of the normal:

$$\mathbb{E}[X] = e^{\mu+\sigma^2/2} \tag{18}$$

Answer: (b)

10. What is the median of X?
    (a) $e^{\mu+\sigma^2/2}$ (b) $e^{\mu-\sigma^2}$ (c) e^{μ} (d) μ

The median of $\log X$ is μ (dist is symmetric). Since $x \mapsto e^x$ is monotone:

$$\text{median}(X) = e^{\text{median}(\log X)} = e^{\mu} \tag{19}$$

Answer: (c)

11. What is $\mathbb{P}(X > 1)$ in terms of Φ?
    (a) $1 - \Phi(\mu/\sigma)$ (b) $\Phi(\mu)$ (c) $\Phi(\mu/\sigma^2)$ (d) $\Phi(\mu/\sigma)$

Standardize:

$$\mathbb{P}(X > 1) = \mathbb{P}(\log X > 0) \tag{20}$$

$$= \mathbb{P}\left(Z > \frac{0 - \mu}{\sigma}\right) \tag{21}$$

$$= \mathbb{P}\left(Z > -\frac{\mu}{\sigma}\right) = \Phi(\mu/\sigma) \tag{22}$$

Answer: (d)

Let $X, Y \sim \text{Lognormal}(0, 1)$ be independent.

12. What is $\mathbb{P}(XY > 1)$?
    (a) $1/4$ (b) $1/2$ (c) $1/\sqrt{2\pi}$ (d) $1/e$

Since $\log X, \log Y \sim \mathcal{N}(0, 1)$ independently:

$$\log(XY) = \log X + \log Y \sim \mathcal{N}(0, 2) \tag{23}$$

By symmetry of the normal about 0:

$$\mathbb{P}(XY > 1) = \mathbb{P}(\log(XY) > 0) = \frac{1}{2} \tag{24}$$

Answer: (b)

1.4 Bernoulli ± 1

Let X be a random variable such that

$$X = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases} \tag{25}$$

for some $p \in [0,1]$.

13. What is $\mathbb{E}[X]$?
    (a) $-p$ (b) p (c) $1-2p$ (d) $2p-1$

$$\mathbb{E}[X] = p \cdot 1 + (1-p) \cdot (-1) \tag{26}$$
$$= 2p - 1 \tag{27}$$

Answer: (d)

14. What is $\mathbb{V}[X]$?
    (a) $p(1-p)$ (b) $4p - p^2$ (c) $4p(1-p)$ (d) $4p^2(1-p)$

$$\mathbb{E}[X^2] = p \cdot 1 + (1-p) \cdot 1 \tag{28}$$
$$= 1 \tag{29}$$

Thus:

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \tag{30}$$
$$= 1 - (2p-1)^2 \tag{31}$$
$$= 1 - (4p^2 - 4p + 1) \tag{32}$$
$$= 4p(1-p) \tag{33}$$

Answer: (c)

15. For what p is $\mathbb{V}[X]$ maximized?
    (a) 1 (b) 0 (c) 0.5 (d) $1/\sqrt{2}$

$\mathbb{V}[X] = 4p(1-p) = -4(p^2 - p)$ is maximized at $p = 1/2$.
Answer: (c)

16. What is $\mathbb{E}[X^k]$?
    (a) p^k (b) $p^k - (1-p)^k$ (c) $p(-1)^k + (1-p)$ (d) $p + (1-p)(-1)^k$

$$\mathbb{E}[X^k] = p \cdot 1^k + (1-p) \cdot (-1)^k \tag{34}$$
$$= p + (1-p)(-1)^k \tag{35}$$

Answer: (d)

1.5 Independent Standard Gaussians

Let X, Y be two independent standard Gaussian random variables.

17. What is $\mathbb{E}[X^2Y]$?
(a) 0 (b) 1 (c) 2 (d) 3

By independence:

$$\begin{aligned} \mathbb{E}[X^2Y] &= \mathbb{E}[X^2] \cdot \mathbb{E}[Y] && (36) \\ &= 1 \cdot 0 && (37) \\ &= 0 && (38) \end{aligned}$$

Answer: (a)

18. What is $\mathbb{V}(X+Y)$?
(a) 0 (b) 1 (c) 2 (d) 3

By independence:

$$\mathbb{V}(X+Y) = \mathbb{V}(X) + \mathbb{V}(Y) = 1 + 1 = 2 \tag{39}$$

Answer: (c)

19. What is $\mathbb{V}(XY)$?
(a) 0 (b) 1 (c) 2 (d) 3

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = 0 \tag{40}$$

$$\mathbb{E}[(XY)^2] = \mathbb{E}[X^2]\mathbb{E}[Y^2] = 1 \tag{41}$$

$$\Rightarrow \mathbb{V}(XY) = 1 - 0 = 1 \tag{42}$$

Answer: (b)

20. What is $\mathrm{Cov}(X, X+Y)$?
(a) 0 (b) 1 (c) 2 (d) 3

$$\mathrm{Cov}(X, X+Y) = \mathrm{Cov}(X, X) + \mathrm{Cov}(X, Y) = \mathbb{V}(X) + 0 = 1 \tag{43}$$

Answer: (b)

21. What is $\mathrm{Cov}(X, XY)$?
(a) 0 (b) 1 (c) 2 (d) 3

$$\mathrm{Cov}(X, XY) = \mathbb{E}[X^2Y] - \mathbb{E}[X]\mathbb{E}[XY] = 0 - 0 = 0 \tag{44}$$

Answer: (a)

1.6 Exponential Lifetime

Let X be an exponential random variable with parameter $1/2$ that models the lifetime (in years) of a lightbulb.[1]

> 22. What is (approximately) the probability that the lightbulb will last at least 2 years?
>   (a) 0.002 (b) 0.018 (c) 0.180 (d) 0.810

Under the AoS convention, $\text{Exp}(\beta)$ has mean β, so $\mathbb{E}[X] = 1/2$ and rate $= 2$, we can use the survival function $\mathbb{P}(X \geq x) = e^{-x/\beta}$:

$$\mathbb{P}(X \geq 2) = e^{-2 \cdot 2} = e^{-4} \approx 0.018 \tag{45}$$

Answer: (b)

> 23. Given that the lightbulb has already lasted for at least 3 years, what is (approximately) the probability that it will last for at least two more years?
>   (a) 0.002 (b) 0.018 (c) 0.180 (d) 0.810

By the memoryless property of the exponential:

$$\mathbb{P}(X \geq 5 \mid X \geq 3) = \mathbb{P}(X \geq 2) = e^{-4} \approx 0.018 \tag{46}$$

Answer: (b)

[1]We use the convention from AoS for the parameter of an exponential distribution.

1.7 i.i.d. Sample Statistics

Let $X_1, \ldots, X_n$ be i.i.d with mean μ and variance σ^2.

24. What is $\mathbb{E}\left[\sum_{i=1}^n X_i\right]$?
(a) μ (b) $n\sigma$ (c) $n\mu$ (d) σ

By linearity of expectation:

$$\mathbb{E}\left[\sum_{i=1}^n X_i\right] = n\mu \tag{47}$$

Answer: (c)

25. What is $\mathbb{V}\left[\sum_{i=1}^n X_i\right]$?
(a) $n^2\sigma^2$ (b) $n\sigma^2$ (c) $n\sigma^2 + n^2\mu^2$ (d) $n\mu$

Since the X_i are independent:

$$\mathbb{V}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{V}[X_i] = n\sigma^2 \tag{48}$$

Answer: (b)

26. What is $\mathbb{E}\left[\left(\sum_{i=1}^n X_i\right)^2\right]$?
(a) $n^2\mu^2$ (b) $n\sigma^2$ (c) $n\mu$ (d) $n\sigma^2 + n^2\mu^2$

$$\mathbb{E}\left[\left(\sum X_i\right)^2\right] = \mathbb{V}\left[\sum X_i\right] + \left(\mathbb{E}\left[\sum X_i\right]\right)^2 = n\sigma^2 + n^2\mu^2 \tag{49}$$

Answer: (d)

27. What is $\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n X_i\right]$?
(a) σ (b) $n\sigma^2$ (c) $n\mu$ (d) μ

$$\mathbb{E}\left[\bar{X}_n\right] = \frac{1}{n} \cdot n\mu = \mu \tag{50}$$

Answer: (d)

28. What is $\mathbb{V}\left[\frac{1}{n}\sum_{i=1}^n X_i\right]$?
(a) μ (b) σ^2/n (c) σ^2 (d) $n\mu$

$$\mathbb{V}\left[\bar{X}_n\right] = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n} \tag{51}$$

Answer: (b)

2 Convergence of Uniforms

Let $X_n \sim \text{Unif}(-1/n, 1/n)$ and let X be a random variable such that $\mathbb{P}(X = 0) = 1$.

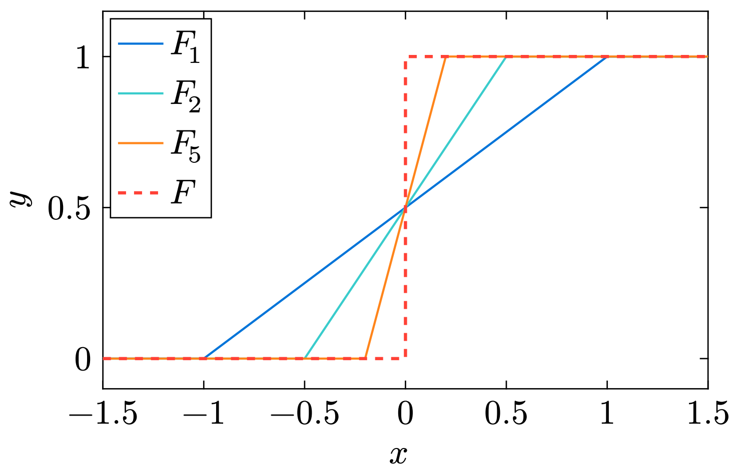
1. Compute and draw the CDF $F_n(x)$ and $F(x)$ of X_n and X respectively.

$$F_n(x) = \begin{cases} 0 & \text{if } x < -1/n \\ \frac{nx+1}{2} & \text{if} -1/n \leq x \leq 1/n \\ 1 & \text{if } x > 1/n \end{cases} \tag{52}$$

This is a linear ramp from 0 to 1 on $[-1/n, 1/n]$.

$$F(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \end{cases} \tag{53}$$

This is a unit step function at $x = 0$.



2. Does $X_n \xrightarrow{P} X$? (prove or disprove)

Yes. Fix $\varepsilon > 0$. For $n > 1/\varepsilon$:

$$X_n \in (-1/n, 1/n) \subset (-\varepsilon, \varepsilon) \tag{54}$$

So $|X_n| < \varepsilon$. Therefore $\mathbb{P}(|X_n| > \varepsilon) = 0$ for all $n > 1/\varepsilon$.

3. Does $X_n \rightsquigarrow X$? (prove or disprove)

Yes. By definition, $X_n \rightsquigarrow X$ iff $F_n(x) \to F(x)$ at every continuity point of F.
The only discontinuity of F is at $x = 0$, so we check $x \neq 0$:
Case $x < 0$: Pick $N > 1/|x|$. Then for all $n \geq N$:

$$x < -1/n \quad \Rightarrow \quad F_n(x) = 0 = F(x) \checkmark \tag{55}$$

Case $x > 0$: Pick $N > 1/x$. Then for all $n \geq N$:

$$x > 1/n \quad \Rightarrow \quad F_n(x) = 1 = F(x) \checkmark \tag{56}$$

In both cases $F_n(x) = F(x)$ eventually, so $F_n(x) \to F(x)$.
Note: Since convergence in probability implies convergence in distribution, this also follows directly from part 2.

3 Gaussian Probabilities

Let $X \sim \mathcal{N}(2, 1.44)$. Note $\sigma = \sqrt{1.44} = 1.2$.

1. $\mathbb{P}(2X - 1 < 0)$

$\mathbb{P}(2X - 1 < 0) = \mathbb{P}(X < 1/2)$. Standardize:

$$\mathbb{P}(X < 1/2) = \Phi\left(\frac{0.5 - 2}{1.2}\right) \tag{57}$$

$$= \Phi(-1.25) \tag{58}$$

$$= 1 - \Phi(1.25) \approx 0.1056 \tag{59}$$

2. $\mathbb{P}(7/5 \le X \le 16/5)$

Standardize:

$$z_1 = \frac{1.4 - 2}{1.2} = -0.5, \quad z_2 = \frac{3.2 - 2}{1.2} = 1 \tag{60}$$

Thus:

$$\mathbb{P} = \Phi(1) - \Phi(-0.5) \tag{61}$$

$$= 0.8413 - 0.3085 = 0.5328 \tag{62}$$

3. $\mathbb{P}(X > 16/5 \mid X > 7/5)$

Using the z-scores from part 2:

$$\mathbb{P}(X > 3.2 \mid X > 1.4) = \frac{\mathbb{P}(X > 3.2)}{\mathbb{P}(X > 1.4)} \tag{63}$$

$$= \frac{1 - \Phi(1)}{\Phi(0.5)} \tag{64}$$

$$= \frac{0.1587}{0.6915} \approx 0.2295 \tag{65}$$

4. $\mathbb{P}(X \leq 4/5 \text{ or } X \geq 16/5)$

Standardize:

$$z_1 = \frac{0.8 - 2}{1.2} = -1, \quad z_2 = \frac{3.2 - 2}{1.2} = 1 \tag{66}$$

These are symmetric about 0:

$$\mathbb{P} = \Phi(-1) + [1 - \Phi(1)] \tag{67}$$

$$= 2(1 - \Phi(1)) \approx 2(0.1587) = 0.3174 \tag{68}$$

4 Bivariate Normal

Let

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \right) \tag{69}$$

So $\mathbb{E}[X] = 1$, $\mathbb{E}[Y] = 0$, $\mathbb{V}[X] = 1$, $\mathbb{V}[Y] = 2$, $\mathrm{Cov}(X, Y) = 1$.

1. $\mathbb{V}[X]$

From the $(1, 1)$ entry of the covariance matrix:

$$\mathbb{V}[X] = 1 \tag{70}$$

2. $\mathbb{E}\left[Y^2 + X\right]$

$$\begin{aligned}
\mathbb{E}[Y^2 + X] &= \mathbb{E}[Y^2] + \mathbb{E}[X] && (71)\\
&= \left(\mathbb{V}[Y] + (\mathbb{E}[Y])^2\right) + \mathbb{E}[X] && (72)\\
&= (2 + 0) + 1 = 3 && (73)
\end{aligned}$$

3. $\mathbb{E}\big[(X-Y)^2\big]$

$$\mathbb{E}\big[(X-Y)^2\big] = \mathbb{V}[X-Y] + (\mathbb{E}[X-Y])^2 \tag{74}$$

$$= \mathbb{V}[X] + \mathbb{V}[Y] - 2\text{Cov}(X,Y) + (\mathbb{E}[X] - \mathbb{E}[Y])^2 \tag{75}$$

$$= 1 + 2 - 2 + 1 \tag{76}$$

$$= 2 \tag{77}$$

4. $\mathbb{V}[X + 2Y]$

$$\mathbb{V}[X + 2Y] = \mathbb{V}[X] + 4\mathbb{V}[Y] + 4\ \mathrm{Cov}(X, Y) \tag{78}$$
$$= 1 + 8 + 4 \tag{79}$$
$$= 13 \tag{80}$$

5. Find $\alpha > 0$ such that $\alpha X = Y$ with probability 1 or prove that no such α exists.

No such $\alpha > 0$ exists.
If $Y = \alpha X$:

$$\begin{aligned} \mathbb{E}[Y] = \alpha \mathbb{E}[X] &\Rightarrow 0 = \alpha \cdot 1 \quad (81) \\ &\Rightarrow \alpha = 0 \quad (82) \end{aligned}$$

This contradicts $\alpha > 0$.

5 Lightbulb Quality Testing

5.1 Setup

We are testing n lightbulbs. Each bulb independently passes some quality check with probability $p \in (0, 1)$ and fails with probability $1 - p$. Let $X_i \in \{0, 1\}$ indicate the outcome, where $X_i = 1$ means the bulb passes.

Conditioned on X_i, the lifetime Y_i of bulb i is exponentially distributed:

$$Y_i \mid (X_i = 1) \sim \mathrm{Exp}(\lambda_1), \quad Y_i \mid (X_i = 0) \sim \mathrm{Exp}(\lambda_0), \tag{83}$$

where $\lambda_0, \lambda_1 > 0$. Assume the pairs (X_i, Y_i) are i.i.d. across i.

1. Let (X, Y) be a copy of (X_1, Y_1) and let $T := XY$. Compute the following quantities in terms of p, λ_0, λ_1:

$$\mathbb{E}[Y], \quad \mathbb{V}(Y), \quad \mathrm{Cov}(X, Y), \quad \mathbb{E}[T], \quad \mathbb{V}(T), \quad \text{and} \quad \mathrm{Cov}(X, T). \tag{84}$$

Using $\mathrm{Exp}(\lambda)$ with rate λ (mean $1/\lambda$, variance $1/\lambda^2$):

$\mathbb{E}[Y]$:

$$\mathbb{E}[Y] = p \cdot \frac{1}{\lambda_1} + (1 - p) \cdot \frac{1}{\lambda_0} \tag{85}$$

$\mathbb{V}(Y)$: By law of total variance, $\mathbb{V}(Y) = \mathbb{E}[\mathbb{V}(Y|X)] + \mathbb{V}(\mathbb{E}[Y|X])$.

$$\mathbb{E}[\mathbb{V}(Y|X)] = p \cdot \mathbb{V}(Y|X = 1) + (1 - p) \cdot \mathbb{V}(Y|X = 0) \tag{86}$$

$$= \frac{p}{\lambda_1^2} + \frac{1 - p}{\lambda_0^2} \tag{87}$$

$$\mathbb{V}(\mathbb{E}[Y|X]) = \mathbb{E}\left[(\mathbb{E}[Y|X])^2\right] - (\mathbb{E}[\mathbb{E}[Y|X]])^2 \tag{88}$$

$$= p \cdot \frac{1}{\lambda_1^2} + (1 - p) \cdot \frac{1}{\lambda_0^2} - \left(\frac{p}{\lambda_1} + \frac{1 - p}{\lambda_0}\right)^2 \tag{89}$$

$$= p(1 - p)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_0}\right)^2 \tag{90}$$

$$\mathbb{V}(Y) = \frac{p}{\lambda_1^2} + \frac{1 - p}{\lambda_0^2} + p(1 - p)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_0}\right)^2 \tag{91}$$

$\mathrm{Cov}(X, Y)$:

$$\mathbb{E}[XY] = \mathbb{P}(X = 1) \cdot \mathbb{E}[Y \mid X = 1] \tag{92}$$

$$= \frac{p}{\lambda_1} \tag{93}$$

$$\mathrm{Cov}(X, Y) = \frac{p}{\lambda_1} - p\left(\frac{p}{\lambda_1} + \frac{1 - p}{\lambda_0}\right) \tag{94}$$

$$= p(1 - p)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_0}\right) \tag{95}$$

$\mathbb{E}[T]$:

- $T = Y$ when $X = 1$
- $T = 0$ when $X = 0$:

$$\mathbb{E}[T] = p \cdot \mathbb{E}[Y \mid X = 1] \tag{96}$$
$$= \frac{p}{\lambda_1} \tag{97}$$

$\mathbb{V}(T)$:

$$\mathbb{E}[T^2] = p \cdot \mathbb{E}[Y^2 \mid X = 1] \tag{98}$$
$$= p \cdot \frac{2}{\lambda_1^2} \tag{99}$$

$$\mathbb{V}(T) = 2\frac{p}{\lambda_1^2} - \frac{p^2}{\lambda_1^2} \tag{100}$$
$$= \frac{p(2-p)}{\lambda_1^2} \tag{101}$$

$\mathrm{Cov}(X, T)$:

$$\mathbb{E}[XT] = \mathbb{E}[X^2 Y] = \frac{p}{\lambda_1} \tag{102}$$

$$\mathrm{Cov}(X, T) = \frac{p}{\lambda_1} - p \cdot \frac{p}{\lambda_1} = \frac{p(1-p)}{\lambda_1} \tag{103}$$

5.2 Central Limit Theorems

Define

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i, \quad \text{and} \quad \bar{T}_n = \frac{1}{n}\sum_{i=1}^{n} X_i Y_i. \tag{104}$$

Write a central limit theorem for each of the following quantities in the form

$$\sqrt{n}(Z_n - \mu) \rightsquigarrow \mathcal{N}(0, \sigma^2) \quad \text{or} \quad \sqrt{n}(Z_n - \mu) \rightsquigarrow \mathcal{N}(0, \Sigma), \tag{105}$$

depending on whether Z_n is a random variable or a random vector.

2.

$$Z_n = \begin{pmatrix} \bar{X}_n \\ \bar{T}_n \end{pmatrix} \tag{106}$$

By the multivariate CLT:

$$\sqrt{n}(Z_n - \mu) \rightsquigarrow \mathcal{N}(0, \Sigma) \tag{107}$$

where $\mu = \binom{p}{p/\lambda_1}$ and Σ is the covariance matrix of (X, T):

$$\Sigma = \begin{pmatrix} p(1-p) & p(1-p)/\lambda_1 \\ p(1-p)/\lambda_1 & p(2-p)/\lambda_1^2 \end{pmatrix} \tag{108}$$

3. $Z_n = \log(\bar{Y}_n)$.

By the CLT:

$$\sqrt{n}(\bar{Y}_n - \mu_Y) \rightsquigarrow \mathcal{N}(0, \sigma_Y^2) \tag{109}$$

with $\mu_Y = \mathbb{E}[Y]$ and $\sigma_Y^2 = \mathbb{V}(Y)$.
Apply the delta method with $g(y) = \log(y)$, $g'(\mu_Y) = 1/\mu_Y$:

$$\sqrt{n}(\log(\bar{Y}_n) - \log(\mu_Y)) \rightsquigarrow \mathcal{N}\left(0, \frac{\sigma_Y^2}{\mu_Y^2}\right) \tag{110}$$

where $\mu_Y = p/\lambda_1 + (1-p)/\lambda_0$ and $\sigma_Y^2 = \mathbb{V}(Y)$ from part 1.

5.3 Delta Method Estimators

Define the average lifetime among passed bulbs

$$\hat{\lambda}_{1,n} = \begin{cases} \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i} = \frac{\bar{T}_n}{\bar{X}_n} & \text{if} \sum_{i=1}^{n} X_i > 0 \\ 0 & \text{if} \sum_{i=1}^{n} X_i = 0 \end{cases} \tag{111}$$

and the corresponding rate estimator

$$\hat{\rho}_{1,n} = \begin{cases} \frac{1}{\hat{\lambda}_{1,n}} = \frac{\bar{X}_n}{\bar{T}_n} & \text{if} \sum_{i=1}^{n} X_i > 0 \\ 0 & \text{if} \sum_{i=1}^{n} X_i = 0. \end{cases} \tag{112}$$

(Note that $\mathbb{P}\left(\sum_{i=1}^{n} X_i = 0\right) = (1-p)^n \to 0$, so this convention does not affect any CLT/delta-method limits.)

Write a CLT for each of the following choices of Z_n:

4. $Z_n = \hat{\lambda}_{1,n}$.

$\hat{\lambda}_{1,n} = g\left(\bar{X}_n, \bar{T}_n\right)$ where $g(a,b) = b/a$. The gradient:

$$\nabla g(a,b) = \left(-b/a^2, 1/a\right) \tag{113}$$

Evaluated at $(p, p/\lambda_1)$:

$$\nabla g = \left(-\frac{1}{p\lambda_1}, \frac{1}{p}\right) \tag{114}$$

Computing the asymptotic variance:

$$\Sigma \nabla g = \begin{pmatrix} 0 \\ \frac{1}{\lambda_1^2} \end{pmatrix} \Rightarrow \nabla g^\top \Sigma \nabla g = \frac{1}{p\lambda_1^2} \tag{115}$$

Therefore:

$$\sqrt{n}\left(\hat{\lambda}_{1,n} - \frac{1}{\lambda_1}\right) \rightsquigarrow \mathcal{N}\left(0, \frac{1}{p\lambda_1^2}\right) \tag{116}$$

5. $Z_n = \hat{\rho}_{1,n}$.

Apply the delta method to $h(x) = 1/x$ at $x = 1/\lambda_1$, where $h'(1/\lambda_1) = -\lambda_1^2$:

$$\sqrt{n}(\hat{\rho}_{1,n} - \lambda_1) \rightsquigarrow \mathcal{N}\left(0, \lambda_1^4 \cdot \frac{1}{p\lambda_1^2}\right) = \mathcal{N}\left(0, \frac{\lambda_1^2}{p}\right) \tag{117}$$