18.650. Fundamentals of Statistics
Spring 2026. Recitation Sheet #2

Problem 1 (GPS localization error model (bivariate normal)).

For jointly Gaussian variables, $U = aX + bY$ is Gaussian, and

$$\mathbb{E}[U] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y], \qquad \mathbb{V}(U) = a^2\mathbb{V}(X) + b^2\mathbb{V}(Y) + 2ab\,\mathrm{Cov}(X, Y).$$

If U, V are jointly Gaussian and $\mathrm{Cov}(U, V) = 0$, then $U \perp V$ (equivalently, $\mathbb{E}[U \mid V] = \mathbb{E}[U]$). For $S = aX + bY$ with $\mathbb{V}(S) > 0$,

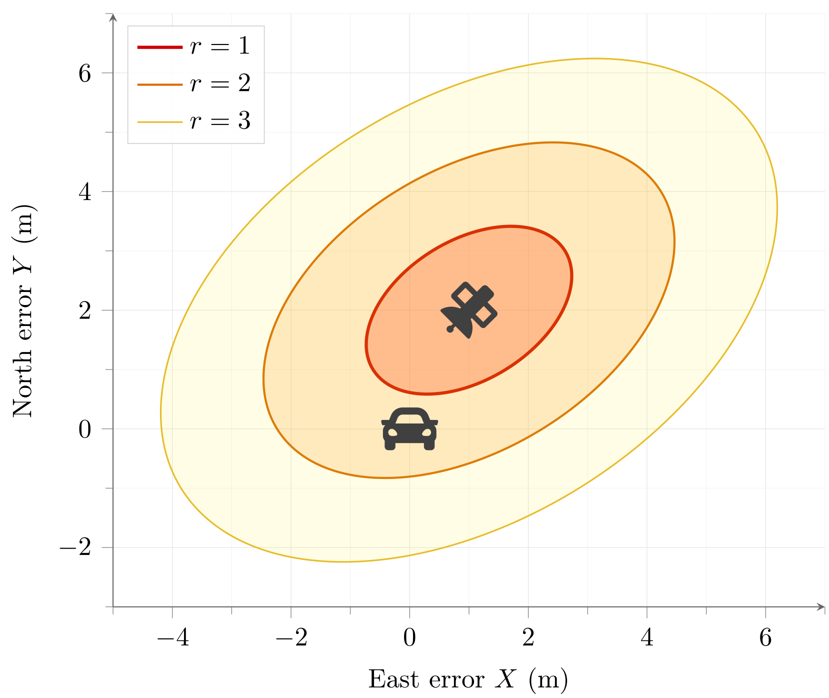
$$\mathbb{E}[X \mid S = s] = \mathbb{E}[X] + \frac{\mathrm{Cov}(X, S)}{\mathbb{V}(S)}(s - \mathbb{E}[S]),$$

and similarly for Y. Also $\mathbb{E}[X^2] = \mathbb{V}(X) + (\mathbb{E}[X])^2$ and $\mathbb{E}[Y^2] = \mathbb{V}(Y) + (\mathbb{E}[Y])^2$.

A car navigation system uses GPS to estimate its position. Let (X, Y) be the *horizontal estimation error* (east, north) in meters (estimated minus true). We model the error vector as jointly Gaussian:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma), \qquad \mu = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

(So the GPS has a systematic bias of $(1, 2)$ meters, and correlated noise.)



1. Compute $\mathbb{E}[X], \mathbb{E}[Y]$, $\mathbb{V}(X), \mathbb{V}(Y)$, $\mathrm{Cov}(X, Y)$, and $\mathrm{Corr}(X, Y)$.

2. (GPS accuracy metric) The horizontal position error magnitude is $R = \sqrt{X^2 + Y^2}$ (in m). Many GPS specs report a horizontal RMS error $\sqrt{\mathbb{E}[R^2]}$. Compute $\mathbb{E}[R^2] = \mathbb{E}[X^2 + Y^2]$ and hence $\sqrt{\mathbb{E}[R^2]}$.

3. (Directional GPS errors) In practice, GPS error matters most along certain directions (determined by map geometry and satellite geometry). Define two scalar summaries of the horizontal error:

   - *Cross-track error* relative to a straight road oriented 45° northeast:
   $$C = X - Y$$
   (up to a constant factor, this is the signed distance from the road).
   - A *satellite-geometry diagnostic score* (a known linear combination of the horizontal errors):
   $$S = X + 2Y$$
   (up to a constant factor, this is the component of the error in the direction $(1, 2)$).

   (a) Find the distributions of C and S (means and variances).

   (b) Compute $\mathrm{Cov}(C, S)$. Are C and S independent?

   (c) As a quick check, compute $\mathbb{E}[C \mid S = 8]$.

4. (Conditional expectation) On a particular run, the receiver reports the diagnostic score $S = 8$ (i.e. $X + 2Y = 8$). Compute $\mathbb{E}[X \mid S = 8]$ and $\mathbb{E}[Y \mid S = 8]$.

5. (Averaging independent fixes; optional) Suppose we take 3 independent GPS readings with the same error distribution and average them. Let $(\bar{X}, \bar{Y})$ be the averaged error and $\bar{C} = \bar{X} - \bar{Y}$. Find the distribution of $\bar{C}$ and compute $\mathbb{P}(\bar{C} > 0)$ using the Φ-table.

Problem 2 (Sensor fusion: what correlation really changes)**.**

For random variables X, Y and constants a, b, c, d,

$$\mathrm{Cov}(X+c, Y+d) = \mathrm{Cov}(X,Y), \qquad \mathrm{Cov}(aX, bY) = ab\ \mathrm{Cov}(X,Y).$$

$$\mathrm{Cov}(X,Y) = \mathrm{Corr}(X,Y)\sqrt{\mathbb{V}(X)\mathbb{V}(Y)}.$$

$$\mathbb{V}(aX+bY) = a^2\mathbb{V}(X) + b^2\mathbb{V}(Y) + 2ab\ \mathrm{Cov}(X,Y).$$

A device has two sensors that measure the same unknown scalar quantity θ. The sensor readings are

$$T_1 = \theta + E_1, \qquad T_2 = \theta + E_2,$$

where $\mathbb{E}[E_1] = \mathbb{E}[E_2] = 0$, $\mathbb{V}(E_1) = 9$, $\mathbb{V}(E_2) = 4$, and $\mathrm{Corr}(E_1, E_2) = \frac{1}{2}$. For a weight $w \in [0,1]$, define the fused estimator

$$T(w) = wT_1 + (1-w)T_2.$$

1. (Covariance matrix) Compute $\mathrm{Cov}(E_1, E_2)$ and write the covariance matrix $\Sigma_T = \mathrm{Cov}\left(\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\right)$.

2. (Unbiasedness) Show that $T(w)$ is unbiased for θ.

3. (Variance as a function of w) Derive $\mathbb{V}(T(w))$ in terms of w.

4. (Optimal fusion weight) Find the minimizer $w^\star = \arg\min_{w\in[0,1]} \mathbb{V}(T(w))$ and compute the minimized variance $\mathbb{V}(T(w^\star))$.

5. (Compare to the naive average) Compute $\mathbb{V}(T(1/2))$ and compare it to $\mathbb{V}(T(w^\star))$.

6. **Optional (Gaussian probability via Φ).** Assume (E_1, E_2) is jointly Gaussian. Using the optimized estimator $T(w^\star)$, compute $\mathbb{P}(|T(w^\star) - \theta| \leq 2)$ in terms of Φ.

Problem 3 (Uncorrelated is not independent (latency score model)).

For random variables A, B,

$$\mathrm{Cov}(A,B) = \mathbb{E}[AB] - \mathbb{E}[A]\mathbb{E}[B], \qquad \mathrm{Corr}(A,B) = \frac{\mathrm{Cov}(A,B)}{\sqrt{\mathbb{V}(A)\mathbb{V}(B)}}.$$

If A and B are independent, then for any events E, F with $\mathbb{P}(E) > 0$,

$$\mathbb{P}(F \mid E) = \mathbb{P}(F).$$

For $Z \sim \mathcal{N}(0,1)$, let $\Phi(t) = \mathbb{P}(Z \le t)$.

A monitoring system tracks minute-level latency via a standardized score Z. It reports two summary features:

$$X = Z, \qquad Y = Z^2 - 1.$$

Here X captures direction of deviation, while Y captures deviation magnitude (centered at 0).

1. Compute $\mathbb{E}[X]$ and $\mathbb{V}(X)$.

2. Compute $\mathbb{E}[Y]$ and $\mathbb{V}(Y)$.

3. Compute $\mathrm{Cov}(X,Y)$ and $\mathrm{Corr}(X,Y)$.

4. Let $E = \{|X| < 1/2\}$ and $F = \{Y > 0\}$. Show $\mathbb{P}(E) > 0$, and compute $\mathbb{P}(F)$ in terms of Φ.

5. Compute $\mathbb{P}(F \mid E)$ and conclude whether X and Y are independent.

Problem 4 (Sample variance consistency (AoS 5.1) and Slutsky)**.** *Estimating noise level from repeated measurements.* A device is used to measure the same fixed quantity over and over (e.g., a QC sample in a lab). We model the measurements as i.i.d. random variables

$$X_1, \ldots, X_n \text{ i.i.d.}, \qquad \mathbb{E}[X_1] = \mu, \qquad \mathbb{V}(X_1) = \sigma^2 \in (0, \infty).$$

Define

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \qquad \hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2.$$

1. (Algebra) Show that

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2 - (\bar{X}_n)^2.$$

2. (Consistency; AoS 5.1) Show that $\hat{\sigma}_n^2 \xrightarrow{\mathbb{P}} \sigma^2$.

3. **Optional (Slutsky).** Assume also that the CLT applies to $\bar{X}_n$, i.e.

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \Rightarrow \mathcal{N}(0, 1).$$

   Show that

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}_n} \Rightarrow \mathcal{N}(0, 1).$$

Problem 5 (From I/Q samples to log-amplitude and polar coordinates (delta method))**.** A receiver records i.i.d. in-phase/quadrature samples

$$W_i = (I_i, Q_i) \in \mathbb{R}^2, \qquad i = 1, \dots, n,$$

with

$$\mathbb{E}[W_i] = m, \qquad \mathrm{Cov}(W_i) = \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Assume

$$m = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \qquad A = \|m\|_2 = 5, \qquad \phi = \mathrm{atan2}(4, 3) \approx 0.9273 \text{ rad } (\approx 53.13^\circ).$$

Let

$$\bar{W}_n = (\bar{I}_n, \bar{Q}_n) = \frac{1}{n} \sum_{i=1}^{n} W_i.$$

Define

$$\widehat{A}_n = \sqrt{\bar{I}_n^2 + \bar{Q}_n^2}, \qquad \widehat{\phi}_n = \mathrm{atan2}(\bar{Q}_n, \bar{I}_n), \qquad \widehat{L}_n = \log(\widehat{A}_n).$$

(Assume $\bar{W}_n \neq 0$ for large n; asymptotics are evaluated at $m \neq 0$.)

Multivariate delta method. If $\sqrt{n}(\bar{X}_n - \mu) \Rightarrow \mathcal{N}(0, \Sigma)$ for $\bar{X}_n \in \mathbb{R}^k$ and $G : \mathbb{R}^k \to \mathbb{R}^m$ is differentiable at μ with Jacobian $J_G(\mu)$, then

$$\sqrt{n}\,(G(\bar{X}_n) - G(\mu)) \Rightarrow \mathcal{N}\Big(0,\ J_G(\mu)\Sigma J_G(\mu)^\top\Big).$$

1. (Jacobian of the polar map; for use below) Define

$$G(w) = \begin{pmatrix} a(w) \\ p(w) \end{pmatrix} = \begin{pmatrix} \sqrt{w_1^2 + w_2^2} \\ \mathrm{atan2}(w_2, w_1) \end{pmatrix}, \qquad w = (w_1, w_2),\ w \neq 0.$$

Show that the Jacobian has the form

$$J_G(w) = \begin{pmatrix} \dfrac{w_1}{\sqrt{w_1^2 + w_2^2}} & \dfrac{w_2}{\sqrt{w_1^2 + w_2^2}} \\ -\dfrac{w_2}{w_1^2 + w_2^2} & \dfrac{w_1}{w_1^2 + w_2^2} \end{pmatrix}.$$

(You may use this formula directly in the next parts.)

2. (2D delta method for the polar map) Using G (and its Jacobian) from part (a), and starting from the CLT for $\bar{W}_n$, find the joint limiting distribution of

$$\sqrt{n}\begin{pmatrix} \widehat{A}_n - A \\ \widehat{\phi}_n - \phi \end{pmatrix},$$

and give its asymptotic covariance matrix explicitly.

3. (CLTs for amplitude and phase) Using part (b), write CLTs for

$$\sqrt{n}(\widehat{A}_n - A), \qquad \sqrt{n}(\widehat{\phi}_n - \phi).$$

> **1D delta method.** If $\sqrt{n}(T_n - \theta) \Rightarrow \mathcal{N}(0, \sigma^2)$ and $g : \mathbb{R} \to \mathbb{R}$ is differentiable at θ, then
> $$\sqrt{n}\,(g(T_n) - g(\theta)) \Rightarrow \mathcal{N}\Big(0, \big(g'(\theta)\big)^2 \sigma^2\Big).$$

4. (CLT for log-amplitude) Use part (c) and $g(x) = \log x$ to find the limiting distribution of

$$\sqrt{n}(\widehat{L}_n - \log A).$$

Appendix: The table lists $P(Z \leq z)$ where $Z \sim N(0, 1)$ for positive values of z.

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998