18.06 Cheat Sheet
(Linear Algebra)

Gatlen Culp (gculp@mit.edu) · May 2025

They don't actually have a cheatsheet, this is just for my review

1 Inverses

Gauss Jordan elimination for

$$(A \mid I) \to (I \mid A^{-1})$$

To solve system of linear equations you can use

$$(A \mid b) \to (I \mid c)$$

where c is the solution IF A is invertible.

Let $A \in \mathbb{F}^{n \times n}$ (must be square)

$$\text{Invertible(A)} \Leftrightarrow \text{rank}(A) = n$$
$$\Leftrightarrow C(A) = \mathbb{F}^n$$
$$\Leftrightarrow C(A^T) = \mathbb{F}^n$$

2 Linear Systems

- RREF can be broken down into sub matrix transformations

3 Subspaces

Requirements:
- Todo

4 Null Space $N(A)$

$$N(A) := \{x \mid \forall x \text{ s.t. } Ax = 0\}$$

Let $A \in \mathbb{F}^{m \times n}$
- $C(A) \in \text{Subspaces}(\mathbb{C}^m)$
- $N(A) \in \text{Subspaces}(\mathbb{C}^n)$

Rank nullity theorem / ???:

$$\dim C(A) + \dim N(A) = n$$

5 Properties & Common Symbols

- **Square**
  - ‣ **Invertible**? (which of these are?)
  - ‣ **Triangular**
    - – **Upper Triangular** R
    - – **Lower Triangular** L
      - • **Diagonal** D
        - ‣ **Diagonal Eigenvalue Matrix** Λ (Diag(eigenvalues))
          - – For the eigenvalues of A, $AV = V\Lambda$

- ‣ **(Positive/Negative) Definite** $A > 0$
  - – Square matrix with positive eigenvalues
  - – Alternatively: x^T A x > 0
  - – Invertible$(A) \Rightarrow A^T A$ is positive definite
  - – Can be factored with a full rank matrix R as $R^T R$
  - – **(Positive/Negative) Semi-Definite** $A \geq 0$ – Same, but can have some zero eigenvalues

- **Normal**: $AA^* = A^*A$
  - ‣ **Symmetric** S: $S^T = S$
    - – **Hermitian** H: $H^* = H$
      - • Diagonal entries are real
  - ‣ **Orthogonal** Q: $Q^T = Q^{-1}$
    - – **Unitary** U: $U^* = U^{-1}$
      - • ie: $U^*U = I$
      - • Eigenvalues sit on unit circle

Eigen
- **Eigenvector Matrix** E = Eigenvectors (as columns)????

Misc
- **Projection** P
  - ‣ **Indempotent**: $P^2 = P$

Frequency
- **Unitary Inverse DFT** F
  - ‣ $F^* = \overline{F} = F^{-1}$ (Unitary)
  - ‣ Typically x might be some time series data about a signal (ex: x_t is the reading at time t). So Fx would transform x to be written in the fourier (frequency) basis.
- **Forward DFT** $K = \sqrt{n}\overline{F}$
- **Circulant** C (right-shift in this class, C^T is left shift)
  - ‣ Can be defined as a sum of a vector and shift-permutation matrix $C = \text{circ}(c) = \sum_{k=0}^{n-1} c_k P^k$
  - ‣ Can be further broken down into $C = \sum_{k=0}^{n-1} c_k P^k = F^*\left(\sum_{k=0}^{n-1} c_k \Lambda^k\right)F = FDF^*$ (Type of spectral decomposition)
    - – Since $FPF^* = \text{diag}(1, \omega, \omega^2, ..., \omega^{n-1}) = \Lambda$
    - – Here D are diag(eigenvalues(C)) while Λ are diag(eigenvalues(P)). The eigenvalues of C are exactly $\lambda_k = (Kc)_k$

6 Projection Matrix

7 Decompositions

- **Eigen/Spectral Decomposition** $A = Q\Lambda Q^{-1}$
  - ‣ $A \in \mathbb{F}^{n \times n}$
  - ‣ Λ is capital λ hence "Eigenvalues" on the diagonal
  - ‣ Q are the eigenvectors (one per row) (not necessarily normalized)
  - ‣ Type of Schur form (don't need to know what this is)
  - ‣ **Diagonalizable** – A has n independent Eigenvectors (ie: You can find $E^{-1}AE = D$ where D is diagonal.)
  - ‣ **Spectral Theorem** $A = E\Lambda E^*$
    - – Spectral decomposition but Q is unitary
    - – Only possible if A is Hermitian

- **Singular Value Decomposition (SVD)**

- **QR Decomposition** $A = QR$
  - ‣ Q = Orthonormal
  - ‣ R = Upper Triangular
  - ‣ Achieved with **Gram Schmidt**

8 Least Squares

Minimizing $\| Ax - b \|^2$ is $x = (A^T A)^{-1} A^T b$

9 Pseudoinverse

$A^+ = V\Sigma^+ U^*$

10 Determinants

$\det(AB) = \det(A)\det(B)$

11 Misc

- If A is full-rank that just means there are no ZERO eigenvalues.

Godly video on the fundamental spaces: https://youtu.be/ZdlraR_7cMA

12 Quadratic Form

Linear Quadratic (LQ) form is $v^T A v \in \mathbb{F}$
- Conceptually, describes how much v relates with A. Or as distance where A dictates the geometric relationship, e.g. if $A = I$, then $v^T I v = v^T v = \sum v_i^2$
- Derivative:

$$\nabla_x(x^T A x) = (A + A^T)x$$

13 Vector Calculus

$$\frac{\partial}{\partial x}[uAx] = A^T u$$