Objective Function:

$$F=\frac{1}{2}(1-u)\sum_{i=1}^{N}w_i\left\|(x_i-z_i\sum_{t=1}^{T}p_{it}e_t-\sum_{k=1}^{M}p_{ik}e_k)\right\|_2^2+\frac{u}{2}\sum_{t=1}^{T}\|(e_t-\mu_0)\|_2^2+\frac{u}{2}\sum_{k=1}^{M}\|(e_k-\mu_0)\|_2^2+\sum_{k=1}^{M}\gamma_k\sum_{i=1}^{N}p_{ik}$$

where

$$w_i=\begin{cases}1,\text{ if } x_i \text{ is from negatively labeled bag}\\ \frac{\alpha N_n}{N_t},\text{ if } x_i \text{ is from positively labeled bag}\end{cases}$$

$$\gamma_k=\frac{\Gamma}{\sum_{i=1}^{N}p_{ik}^{old}}$$

M-stpe:

Take Expectation respect to z_i :

$$\sum_{i=1}^{N}\sum_{z_i\in\{0,1\}}\left[P(z_i\,|\,x_{i,}\theta^{(t-1)})\frac{1}{2}(1-u)w_i\left\|(x_i-z_i\sum_{t=1}^{T}p_{it}e_t-\sum_{k=1}^{M}p_{ik}e_k)\right\|_2^2\right]+\frac{u}{2}\sum_{t=1}^{T}\|(e_t-\mu_0)\|_2^2+\frac{u}{2}\sum_{k=1}^{M}\|(e_k-\mu_0)\|_2^2$$
$$+\sum_{k=1}^{M}\gamma_k\sum_{i=1}^{N}p_{ik}$$

P update equation:

For points from positive bags:

$$F^+=\sum_{i=1}^{N^+}\left[P(z_i=0)\frac{1}{2}(1-u)w_i\left\|(x_i-\sum_{k=1}^{M}p_{ik}e_k)\right\|_2^2+P(z_i=1)\frac{1}{2}(1-u)w_i\left\|(x_i-z_i\sum_{t=1}^{T}p_{it}e_t-\sum_{k=1}^{M}p_{ik}e_k)\right\|_2^2\right]$$
$$+\sum_{i}\lambda_i^+(\sum_{t=1}^{T}p_{it}+\sum_{k=1}^{M}p_{ik}-1)+\sum_{k=1}^{M}\gamma_k\sum_{i=1}^{N^+}p_{ik}$$

Define :

$$P_i=\begin{bmatrix}p_{it1}\\p_{it2}\\\cdots\\p_{itT}\\p_{i1}\\p_{i2}\\\cdots\\p_{iM}\end{bmatrix}=\begin{bmatrix}P_i^+\\P_i^-\end{bmatrix},\ E=[e_{t1}\ e_{t2}\ \cdots\ e_{tT}\ e_1\ e_2\cdots e_M],\ E^-=[e_1\ e_2\cdots e_M]$$

$$\frac{\partial F^+}{\partial p_{it}}=P(z_i=1)(1-u)w_i(-1)e_t^T(x_i-\sum_{t=1}^{T}p_{it}e_t-\sum_{k=1}^{M}p_{ik}e_k)+\lambda_i^+$$

$$\frac{\partial F^+}{\partial p_{ik}}=P(z_i=0)(1-u)w_i(-1)e_k^T(x_i-\sum_{k=1}^{M}p_{ik}e_k)$$

$$+P(z_i=1)(1-u)w_i(-1)e_k^T(x_i-\sum_{t=1}^{T}p_{it}e_t-\sum_{k=1}^{M}p_{ik}e_k)+\gamma_k+\lambda_i^+$$

Denote $a=(1-u)w_i(-1)$,

then rewrite the above two expressions in a consistent form:

$$\frac{\partial F^+}{\partial p_{it}}=aP(z_i=0)0_{d\times 1}^T(x_i-[0_{d\times T}\ E^-]P_i)+aP(z_i=1)e_t^T(x_i-EP_i)+0+\lambda_i^+$$

$$\frac{\partial F^+}{\partial p_{ik}}=aP(z_i=0)e_k^T(x_i-[0_{d\times T}\ E^-]P_i)+aP(z_i=1)e_k^T(x_i-EP_i)+\gamma_k+\lambda_i^+$$

$\Rightarrow$ combine into vector form:

$$\frac{\partial F^+}{\partial P_i}=aP(z_i=0)[0_{d\times T}\ E^-]^T(x_i-[0_{d\times T}\ E^-]P_i)+aP(z_i=1)E^T(x_i-EP_i)+\begin{bmatrix}0_{T\times 1}\\ V\end{bmatrix}+\lambda_i^+1_{(T+M)\times 1}$$

$$=\left[aP(z_i=0)[0_{d\times T}\ E^-]^T+aP(z_i=1)E^T\right]x_i-\left\{aP(z_i=0)[0_{d\times T}\ E^-]^T[0_{d\times T}\ E^-]+aP(z_i=1)E^TE\right\}P_i$$

$$+\begin{bmatrix}0_{T\times 1}\\ V\end{bmatrix}+\lambda_i^+1_{(T+M)\times 1}=0,\ \text{where } V=\begin{bmatrix}\gamma_1\\ \gamma_2\\ \dots\\ \gamma_k\end{bmatrix}$$

$$P_i=\left\{P(z_i=0)[0_{d\times T}\ E^-]^T[0_{d\times T}\ E^-]+P(z_i=1)E^TE\right\}^{-1}\cdot$$

$$\left\{\left[P(z_i=0)[0_{d\times T}\ E^-]^T+P(z_i=1)E^T\right]x_i+\frac{1}{a}\begin{bmatrix}0_{T\times 1}\\ V\end{bmatrix}+1_{(T+M)\times 1}\frac{\lambda_i^+}{a}\right\}---------(1)$$

$$1_{1\times(T+M)}P_i=1$$

$$\Rightarrow\lambda_i^+=\frac{a\left(1-1_{1\times(T+M)}\left\{P(z_i=0)[0_{d\times T}\ E^-]^T[0_{d\times T}\ E^-]+P(z_i=1)E^TE\right\}^{-1}\cdot\left\{\left[P(z_i=0)[0_{d\times T}\ E^-]^T+P(z_i=1)E^T\right]x_i+\frac{1}{a}\begin{bmatrix}0_{T\times 1}\\ V\end{bmatrix}\right\}\right)}{1_{1\times(T+M)}\left\{P(z_i=0)[0_{d\times T}\ E^-]^T[0_{d\times T}\ E^-]+P(z_i=1)E^TE\right\}^{-1}1_{(T+M)\times 1}}$$

$$-----------(2)$$

$(2)\rightarrow(1)$

$$P_i=\left\{P(z_i=0)[0_{d\times T}\ E^-]^T[0_{d\times T}\ E^-]+P(z_i=1)E^TE\right\}^{-1}\cdot$$

$$\left\{\begin{aligned}&\left[P(z_i=0)[0_{d\times T}\ E^-]^T+P(z_i=1)E^T\right]x_i+\frac{1}{a}\begin{bmatrix}0_{T\times 1}\\ V\end{bmatrix}+\\ &1_{(T+M)\times 1}\frac{1-1_{1\times(T+M)}\left\{P(z_i=0)[0_{d\times T}\ E^-]^T[0_{d\times T}\ E^-]+P(z_i=1)E^TE\right\}^{-1}\cdot\left\{\left[P(z_i=0)[0_{d\times T}\ E^-]^T+P(z_i=1)E^T\right]x_i+\frac{1}{a}\begin{bmatrix}0_{T\times 1}\\ V\end{bmatrix}\right\}}{1_{1\times(T+M)}\left\{P(z_i=0)[0_{d\times T}\ E^-]^T[0_{d\times T}\ E^-]+P(z_i=1)E^TE\right\}^{-1}1_{(T+M)\times 1}}\end{aligned}\right\}$$

For points from negative bags:

$$F^- = \sum_{i=1}^{N^-}\left[\frac{(1-u)}{2}\left\|(x_i - \sum_{k=1}^{M} p_{ik}e_k)\right\|_2^2\right] + \sum_{k=1}^{M}\gamma_k \sum_{i=1}^{N} p_{ik} + \sum_i \lambda_i^- (\sum_{k=1}^{M} p_{ik} - 1)$$

$$\frac{\partial F^-}{\partial p_{ik}} = (1-u)(-1)e_k^T(x_i - \sum_{k=1}^{M} p_{ik}e_k) + \gamma_k + \lambda_i^-$$

vector form:

$$\frac{\partial F^-}{\partial P_i^-} = (1-u)(-1)E^{-T}(x_i - E^- P_i^-) + V + \lambda_i^- 1_{M\times 1} = 0$$

$$P_i^- = (E^{-T}E^-)^{-1}\left(E^{-T}x_i - \frac{1}{1-u}V\right) - \frac{\lambda_i^-}{1-u}(E^{-T}E^-)^{-1}\cdot 1_{M\times 1}$$

$$1_{1\times M}P_i^- = 1, \text{ assume } \; a = -(1-u)$$

$$\Rightarrow \lambda_i^- = -\frac{(1-u)\left(1 - 1_{1\times M}(E^{-T}E^-)^{-1}(E^{-T}x_i + \frac{1}{a}V\right)}{1_{1\times M}(E^{-T}E^-)^{-1}1_{M\times 1}}$$

$$\Rightarrow P_i^- = (E^{-T}E^-)^{-1}\left[E^{-T}x_i + \frac{1}{a}V + 1_{M\times 1}\frac{1 - 1_{1\times M}(E^{-T}E^-)^{-1}(E^{-T}x_i + \frac{1}{a}V)}{1_{1\times M}(E^{-T}E^-)^{-1}1_{M\times 1}}\right]$$

Matrix form:

$$P^- = (E^{-T}E^-)^{-1}\left[E^{-T}X^- + \frac{1}{a}\mathrm{repmat}(V, 1, N^-) + 1_{M\times 1}\cdot\frac{1 - 1_{1\times M}\cdot(E^{-T}E^-)^{-1}\left(E^{-T}X^- + \frac{1}{a}\mathrm{repmat}(V, 1, N^-)\right)}{1_{1\times M}\cdot(E^{-T}E^-)^{-1}\cdot 1_{M\times 1}}\right]$$

$$\Rightarrow \text{In each iteration:}$$

$$P = \begin{bmatrix} P^+ & \begin{matrix} 0_{T\times N^-} \\ P^- \end{matrix} \end{bmatrix}_{(T+M)\times N}$$

E update:

Define as previous :

$$F=\sum_{i=1}^{N}\left[P(z_i=0)\frac{1}{2}(1-u)w_i\left\|(x_i-\sum_{k=1}^{M}p_{ik}e_k)\right\|_2^2+P(z_i=1)\frac{1}{2}(1-u)w_i\left\|(x_i-\sum_{t=1}^{T}p_{it}e_t-\sum_{k=1}^{M}p_{ik}e_k)\right\|_2^2\right]$$

$$\frac{u}{2}\sum_{t=1}^{T}\|(e_t-\mu_0)\|_2^2+\frac{u}{2}\sum_{k=1}^{M}\|(e_k-\mu_0)\|_2^2$$

$$=\sum_{i=1}^{N^+}\left[P(z_i=0)\frac{1}{2}(1-u)w_i\left\|(x_i-\sum_{k=1}^{M}p_{ik}e_k)\right\|_2^2+P(z_i=1)\frac{1}{2}(1-u)w_i\left\|(x_i-\sum_{t=1}^{T}p_{it}e_t-\sum_{k=1}^{M}p_{ik}e_k)\right\|_2^2\right]$$

$$+\frac{1}{2}(1-u)\sum_{i=1}^{N^-}\left[\left\|(x_i-\sum_{k=1}^{M}p_{ik}e_k)\right\|_2^2\right]+\frac{u}{2}\sum_{t=1}^{T}\|(e_t-\mu_0)\|_2^2+\frac{u}{2}\sum_{k=1}^{M}\|(e_k-\mu_0)\|_2^2$$

$$\frac{\partial F}{\partial e_t}=\sum_{i=1}^{N^+}\left[P(z_i=1)(-1)(1-u)w_ip_{it}(x_i-\sum_{t=1}^{T}p_{it}e_t-\sum_{k=1}^{M}p_{ik}e_k)\right]+u(e_t-\mu_0)$$

$$\frac{\partial F}{\partial e_k}=\sum_{i=1}^{N^+}\left[P(z_i=0)(-1)(1-u)w_ip_{ik}(x_i-\sum_{k=1}^{M}p_{ik}e_k)+P(z_i=1)(-1)(1-u)w_ip_{ik}(x_i-\sum_{t=1}^{T}p_{it}e_t-\sum_{k=1}^{M}p_{ik}e_k)\right]$$

$$+(1-u)\sum_{i=1}^{N^-}\left[-p_{ik}(x_i-\sum_{k=1}^{M}p_{ik}e_k)\right]+u(e_k-\mu_0)$$

Re write the above two expressions into consistent forms:

$$\frac{\partial F}{\partial e_t}=\sum_{i=1}^{N^+}\left[P(z_i=0)(-1)(1-u)w_i 0\cdot(x_i-E\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix})+P(z_i=1)(-1)(1-u)w_i p_{it}(x_i-EP_i)\right]$$

$$+(1-u)\sum_{i=1}^{N^-}\left[-0\cdot(x_i-E\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}\right]+u(e_t-\mu_0)$$

$$\frac{\partial F}{\partial e_k}=\sum_{i=1}^{N^+}\left[P(z_i=0)(-1)(1-u)w_i p_{ik}(x_i-E\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix})+P(z_i=1)(-1)(1-u)w_i p_{ik}(x_i-EP_i)\right]$$

$$+(1-u)\sum_{i=1}^{N^-}\left[-p_{ik}\cdot(x_i-E\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}\right]+u(e_k-\mu_0)$$

Write into matrix form:

$$\frac{\partial F}{\partial E}=\sum_{i=1}^{N^+}\left[P(z_i=0)(-1)(1-u)w_i(x_i-E\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix})\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T+P(z_i=1)(-1)(1-u)w_i(x_i-EP_i)P_i^T\right]$$

$$+(1-u)\sum_{i=1}^{N^-}\left[-(x_i-E\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix})\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T\right]+u(E-\mathrm{repmat}(\mu_0,1,T+M))=0$$

$$\Rightarrow(1-u)w_i\sum_{i=1}^{N^+}\left[-P(z_i=0)x_i\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T+P(z_i=0)E\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T-P(z_i=1)x_iP_i^T+P(z_i=1)EP_iP_i^T\right]$$

$$+(1-u)\sum_{i=1}^{N^-}\left[-x_i\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T\right]+(1-u)\sum_{i=1}^{N^-}\left[E\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T\right]+uE-u\cdot\mathrm{repmat}(\mu_0,1,T+M)=0$$

$$\Rightarrow(1-u)w_i\sum_{i=1}^{N^+}\left[P(z_i=0)E\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T+P(z_i=1)EP_iP_i^T\right]+(1-u)\sum_{i=1}^{N^-}\left[E\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T\right]+uE$$

$$=(1-u)w_i\sum_{i=1}^{N^+}\left[P(z_i=0)x_i\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T+P(z_i=1)x_iP_i^T\right]+(1-u)\sum_{i=1}^{N^-}\left[x_i\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T\right]+u\cdot\mathrm{repmat}(\mu_0,1,T+M)$$

$$\Rightarrow E\cdot\left\{(1-u)w_i\sum_{i=1}^{N^+}\left[P(z_i=0)\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T+P(z_i=1)P_iP_i^T\right]+(1-u)\sum_{i=1}^{N^-}\left[\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T\right]+u\right\}$$

$$=(1-u)w_i\sum_{i=1}^{N^+}\left[P(z_i=0)x_i\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T+P(z_i=1)x_iP_i^T\right]+(1-u)\sum_{i=1}^{N^-}\left[x_i\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T\right]+u\cdot\mathrm{repmat}(\mu_0,1,T+M)$$

$$\Rightarrow E=\left\{(1-u)w_i\sum_{i=1}^{N^+}\left[P(z_i=0)x_i\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T+P(z_i=1)x_iP_i^T\right]+(1-u)\sum_{i=1}^{N^-}\left[x_i\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T\right]+u\cdot\mathrm{repmat}(\mu_0,1,T+M)\right\}$$

$$\cdot\left\{(1-u)w_i\sum_{i=1}^{N^+}\left[P(z_i=0)\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T+P(z_i=1)P_iP_i^T\right]+(1-u)\sum_{i=1}^{N^-}\left[\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}\begin{bmatrix}0_{T\times1}\\P_i^-\end{bmatrix}^T\right]+u\right\}^{-1}$$