Module – 5

Three Dimensional Geometric Transformations, Curves and Fractal Generation

Translation

Three dimensional transformation matrix for translation with homogeneous coordinates is as given below. It specifies three coordinates with their own translation factor.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

$$\therefore \qquad P' = P \cdot T$$

$$\therefore \; [x' \; y' \; z' \; 1] = [x \; y \; z \; 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

$$= [x + t_x \quad y + t_y \quad z + t_z \quad 1]$$

Like two dimensional transformations, an object is translated in three dimensions by transforming each vertex of the object.

6.3 Scaling

Three dimensional transformation matrix for scaling with homogeneous coordinates is as given below.

It specifies three coordinates with their own scaling factor.

$$S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \qquad P' = P \cdot S$$

$$\therefore \; [x' \; y' \; z' \; 1] = [x \; y \; z \; 1] \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x \cdot S_x & y \cdot S_y & z \cdot S_z & 1 \end{bmatrix}$$

A scaling of an object with respect to a selected fixed position can be represented with the following transformation sequence.

1. Translate the fixed point to the origin.
2. Scale the object
3. Translate the fixed point back to its original position.

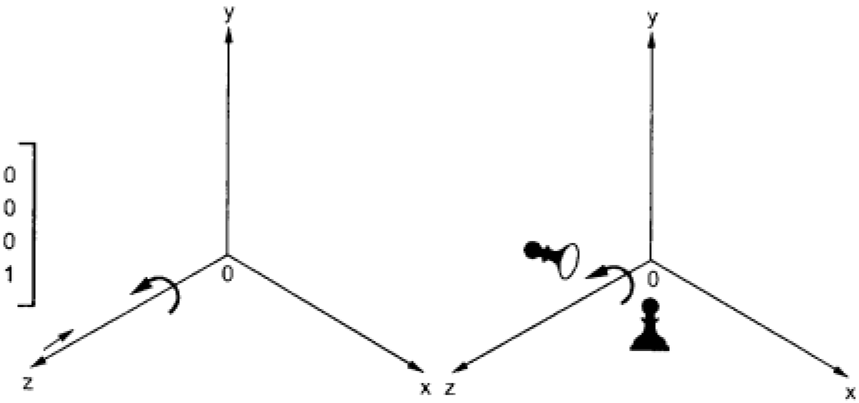
Rotation

Coordinate Axes Rotations

Three dimensional transformation matrix for each coordinate axes rotations with homogeneous coordinate are as given below

$$R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a)



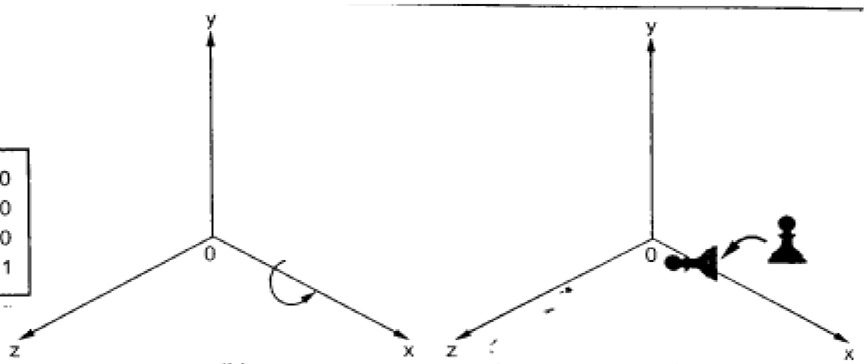
(b) (c)

Rotation about z axis

The positive value of angle θ indicates counterclockwise rotation. For clockwise rotation value of angle θ is negative.

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a)

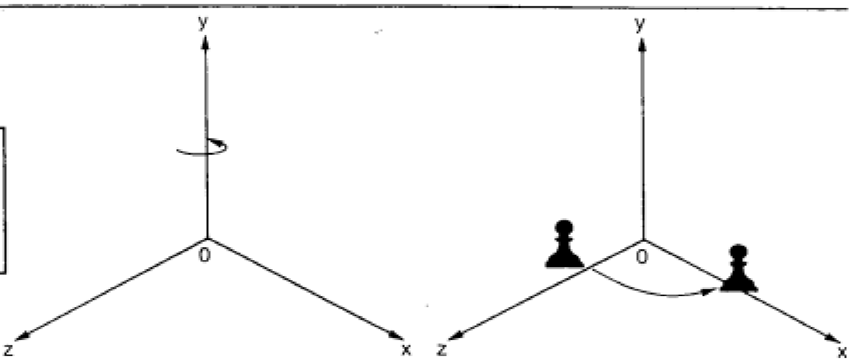


(b) (c)

Rotation about x axis

$$R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a)



(b) (c)

Rotation about y axis

Rotation about Arbitrary Axis

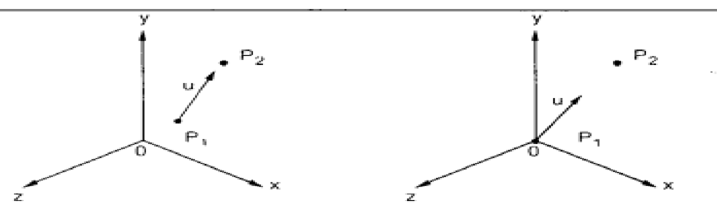
A rotation matrix for any axis that does not coincide with a coordinate axis can be set up as a composite transformation involving combinations of translations and the coordinate-axes rotations.

In a special case where an object is to be rotated about an axis that is parallel to one of the coordinate axes we can obtain the resultant coordinates with the following transformation sequence.

1. Translate the object so that the rotation axis coincides with the parallel coordinate axis.
2. Perform the specified rotation about that axis.
3. Translate the object so that the rotation axis is moved back to its original position.

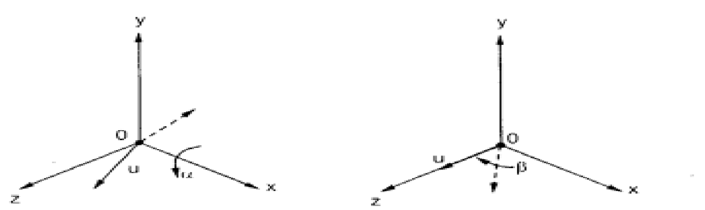
When an object is to be rotated about an axis that is not parallel to one of the coordinate axes, we have to perform some additional transformations. The sequence of these transformations is given below.

1. Translate the object so that rotation axis specified by unit vector u passes through the coordinate origin.
2. Rotate the object so that the axis of rotation coincides with one of the coordinate axes. Usually the z axis is preferred. To coincide the axis of rotation to z axis we have to first perform rotation of unit vector u about x axis to bring it into xz plane and then perform rotation about y axis to coincide it with z axis.
3. Perform the desired rotation θ about the z axis.
4. Apply the inverse rotation about y axis and then about x axis to bring the rotation axis back to its original orientation.
5. Apply the inverse translation to move the rotation axis back to its original position.



a) Axis of rotation defined by points P_1 and P_2

b) Translation of the rotation axis to the coordinate origin



c) Rotation of unit vector about x axis to bring it into xz plane

d) Rotation of unit vector u about y axis to align it with the z axis

As shown in the Fig (a) the rotation axis is defined with two coordinate points P_1 and P_2 and unit vector u is defined along the rotation of axis as

$$u = \frac{V}{|V|} = (a, b, c)$$

where V is the axis vector defined by two points P_1 and P_2 as

$$V = P_2 - P_1$$
$$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

The components a, b, and c of unit vector u are the direction cosines for the rotation axis and they can be defined as

$$a = \frac{x_2 - x_1}{|V|}, \quad b = \frac{y_2 - y_1}{|V|}, \quad c = \frac{z_2 - z_1}{|V|}$$

As mentioned earlier, the first step in the transformation sequence is to translate the object to pass the rotation axis through the coordinate origin. This can be accomplished by moving point P_1 to the origin. The translation is as given below

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x_1 & -y_1 & -z_1 & 1 \end{bmatrix}$$

Now we have to perform the rotation of unit vector u about x axis. The rotation of u around the x axis into the xz plane is accomplished by rotating u′ (0, b, c) through angle α into the z axis and the cosine of the rotation angle α can be determined from the dot product of u′ and the unit vector u_z (0, 0, 1) along the z axis.

$$\cos\alpha = \frac{u' \cdot u_z}{|u'||u_z|} = \frac{c}{d} \qquad \text{where } u'(0, b, c) = bJ + cK \text{ and}$$

$$u_z(0, 0, 1) = K$$

$$= \frac{c}{|u'||u_z|}$$

$$= \frac{c}{|u'|} \qquad \text{Since } |u_z| = 1$$

$$= \frac{c}{d}$$

where d is the magnitude of u′ :

$$d = \sqrt{b^2 + c^2}$$

Similarly, we can determine the sine of α from the cross product of u′ and u_z.

$$u' \times u_z = u_x \, |u'| \; |u_z| \sin\alpha \qquad \text{... (6.3)}$$

and the Cartesian form for the cross product gives us

$$u' \times u_z = u_x \cdot b \qquad \text{... (6.4)}$$

Equating the right sides of equations 6.3 and 6.4 we get

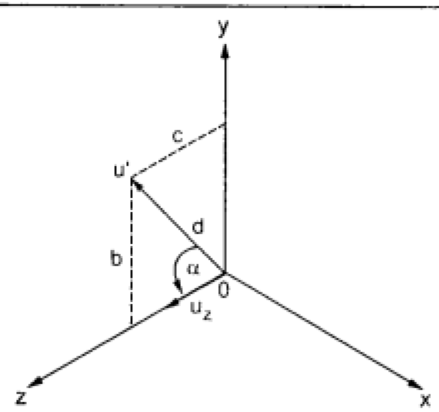
$$u_x \; |u'| \; |u_z| \; \sin\alpha = u_x \,.\, b$$

$$\therefore \quad |u'| \; |u_z| \; \sin\alpha = b$$

$$\therefore \quad \sin\alpha = \frac{b}{|u'|\,|u_z|}$$

$$= \frac{b}{d} \qquad \text{since } |u_z| = 1 \text{ and } |u'| = d$$

This can also be verified graphically as shown in Fig



Fig

By substituting values of $\cos\alpha$ and $\sin\alpha$ the rotation matrix R_x can be given as

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & b/d & 0 \\ 0 & -b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Next we have to perform the rotation of unit vector about y axis. This can be achieved by rotating u'' (a, 0, d) through angle β onto the z axis. Using similar equations we can determine $\cos\beta$ and $\sin\beta$ as follows.

We have angle of rotation $= -\beta$

$$\therefore \quad \cos(-\beta) = \cos\beta = \frac{u'' \cdot u_z}{|u''|\,|u_z|} \text{ where } u'' = aI + dK \text{ and}$$

$$u_z = K$$

$$= \frac{d}{|u''|\,|u_z|}$$

$$= \frac{d}{|u''|} \qquad \because |u_z| = 1$$

$$= \frac{d}{\sqrt{a^2 + d^2}}$$

Consider cross product of u'' and u_z.

$$u'' \times u_z = u_y \, |u''| \, |u_z| \, \sin(-\beta)$$

$$= -u_y \, |u''| \, |u_z| \, \sin\beta \qquad \because \sin(-\theta) = -\sin\theta$$

Cartesian form of cross product gives us

$$u'' \times u_z = u_y \, (+a)$$

Equating above equations,

$$-|u''| \, |u_z| \sin\beta = a$$

$$\therefore \quad \sin\beta = \frac{-a}{|u''||u_z|}$$

$$= \frac{-a}{|u''|} \qquad \because |u_z| = 1$$

$$= \frac{-a}{\sqrt{a^2 + d^2}}$$

but we have, $\quad d = \sqrt{b^2 + c^2}$

$$\therefore \quad \cos\beta = \frac{d}{\sqrt{a^2 + d^2}}$$

$$= \frac{\sqrt{b^2 + c^2}}{\sqrt{a^2 + b^2 + c^2}}$$

and $\quad \sin\beta = \frac{-a}{\sqrt{a^2 + d^2}}$

$$= \frac{-a}{\sqrt{a^2 + b^2 + c^2}}$$

By substituting values of $\cos\beta$ and $\sin\beta$ in the rotation matrix R_y can be given as

$$R_y = \begin{bmatrix} \frac{\sqrt{b^2 + c^2}}{\sqrt{a^2 + b^2 + c^2}} & 0 & \frac{+a}{\sqrt{a^2 + b^2 + c^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-a}{\sqrt{a^2 + b^2 + c^2}} & 0 & \frac{\sqrt{b^2 + c^2}}{\sqrt{a^2 + b^2 + c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $\quad \lambda = \sqrt{b^2 + c^2}$ and $|V| = \sqrt{a^2 + b^2 + c^2}$

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{c}{\lambda} & \frac{+b}{\lambda} & 0 \\ 0 & \frac{-b}{\lambda} & \frac{c}{\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_y = \begin{bmatrix} \frac{\lambda}{|V|} & 0 & \frac{+a}{|V|} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-a}{|V|} & 0 & \frac{\lambda}{|V|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Resultant rotation matrix $R_{xy} = R_x \cdot R_y$

$$\therefore \qquad R_{xy} = \begin{bmatrix} \frac{\lambda}{|V|} & 0 & \frac{a}{|V|} & 0 \\ \frac{-ab}{|V|\lambda} & \frac{c}{\lambda} & \frac{b}{|V|} & 0 \\ \frac{-ac}{|V|\lambda} & \frac{-b}{\lambda} & \frac{c}{|V|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have,

$$t_{ij}^{-1} = \frac{(-1)^{i+j} \det M_{ji}}{\det T}$$

Using above equation we get inverse of R_{xy} as

$$R_{xy}^{-1} = \begin{bmatrix} \frac{\lambda}{|V|} & \frac{-ab}{|V|\lambda} & \frac{-ac}{|V|\lambda} & 0 \\ 0 & \frac{c}{\lambda} & \frac{-b}{\lambda} & 0 \\ \frac{a}{|V|} & \frac{b}{|V|} & \frac{c}{|V|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse of translation matrix can be given as

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_1 & y_1 & z_1 & 1 \end{bmatrix}$$

With transformation matrices T and R_{xy} we can align the rotation axis with the positive z axis. Now the specified rotation with angle θ can be achieved by rotation transformation as given below

$$R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To complete the required rotation about the given axis, we have to transform the rotation axis back to its original position. This can be achieved by applying the inverse transformations T^{-1} and R_{xy}^{-1}. The overall transformation matrix for rotation about an arbitrary axis then can be expressed as the concatenation of five individual transformations.

$$R(\theta) = T \cdot R_{xy} \cdot R_z \cdot R_{xy}^{-1} \cdot T^{-1}$$

i.e.
$$R(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x_1 & -y_1 & -z_1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\lambda}{|V|} & 0 & \frac{a}{|V|} & 0 \\ \frac{-ab}{\lambda|V|} & \frac{c}{\lambda} & \frac{b}{|V|} & 0 \\ \frac{-ac}{\lambda|V|} & \frac{-b}{\lambda} & \frac{c}{|V|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

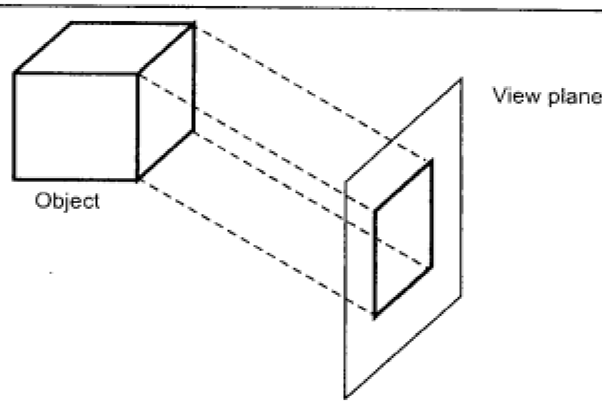
$$\begin{bmatrix} \frac{\lambda}{|V|} & \frac{-ab}{\lambda|V|} & \frac{-ac}{\lambda|V|} & 0 \\ 0 & \frac{c}{\lambda} & \frac{-b}{\lambda} & 0 \\ \frac{a}{|V|} & \frac{b}{|V|} & \frac{c}{|V|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_1 & y_1 & z_1 & 1 \end{bmatrix}$$

Projections

After converting the description of objects from world coordinates to viewing coordinates, we can project the three dimensional objects onto the two dimensional view plane. There are two basic ways of projecting objects onto the view plane : Parallel projection and Perspective projection.

Parallel Projection

In parallel projection, z - coordinate is discarded and parallel lines from each vertex on the object are extended until they intersect the view plane. The point of intersection is the projection of the vertex. We connect the projected vertices by line segments which correspond to connections on the original object.

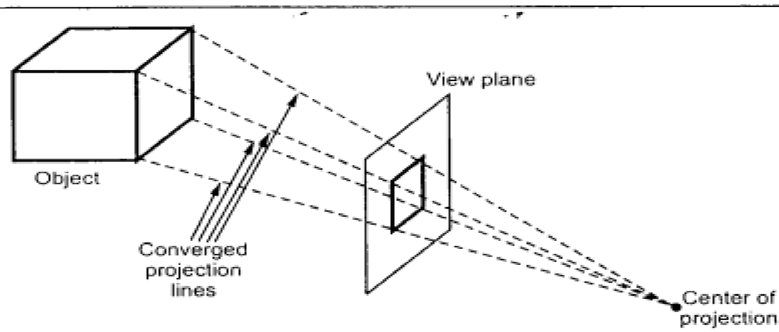


Parallel projection of an object to the view plane

As shown in the Fig a parallel projection preserves relative proportions of objects but does not produce the realistic views.

Perspective Projection

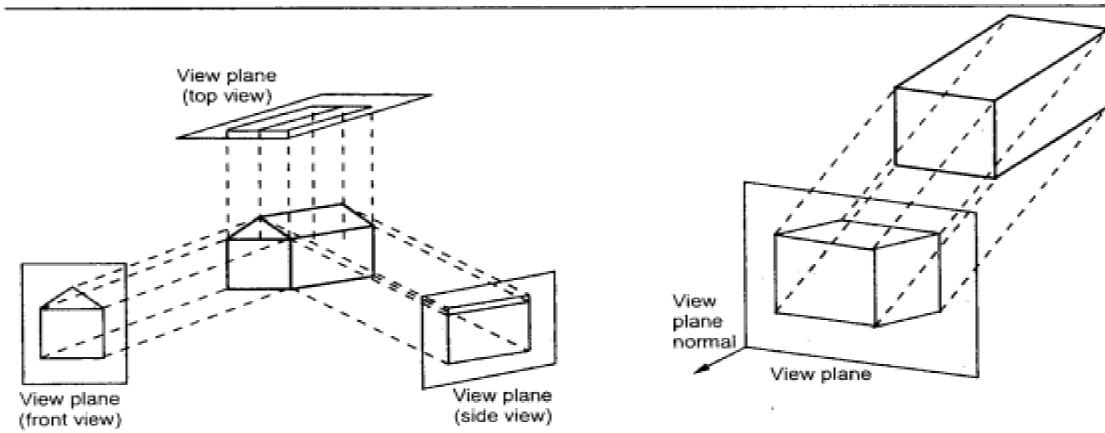
The perspective projection, on the other hand, produces realistic views but does not preserve relative proportions. In perspective projection, the lines of projection are not parallel. Instead, they all converge at a single point called the **center of projection** or **projection reference point**. The object positions are transformed to the view plane along these converged projection lines and the projected view of an object is determined by calculating the intersection of the converged projection lines with the view plane, as shown in the Fig.



Perspective projection of an object to the view plane

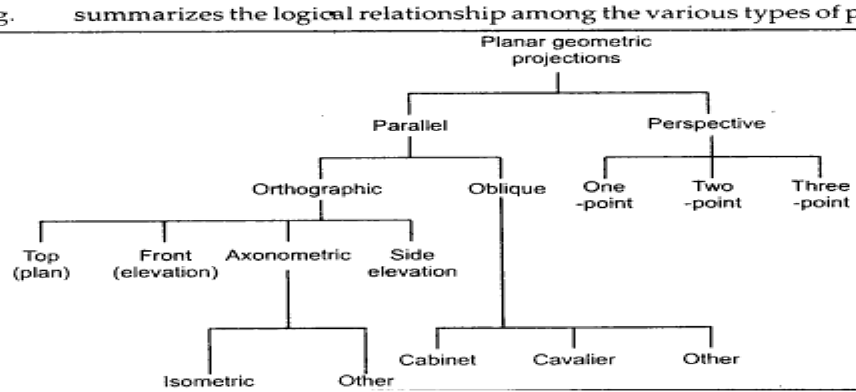
Types of Parallel Projections

Parallel projections are basically categorized into two types, depending on the relation between the direction of projection and the normal to the view plane. When the direction of the projection is normal (perpendicular) to the view plane, we have an orthographic parallel projection. Otherwise, we have an oblique parallel projection. Fig illustrates the two types of parallel projection.



(a) Orthographic parallel projection **(b) Oblique parallel projection**

The Fig. summarizes the logical relationship among the various types of projections



Transformation Matrix for Perspective Projection

Let us consider the center of projection is at $P_c(x_c, y_c, z_c)$ and the point on object is $P_1(x_1, y_1, z_1)$, then the parametric equation for line containing these points can be given as

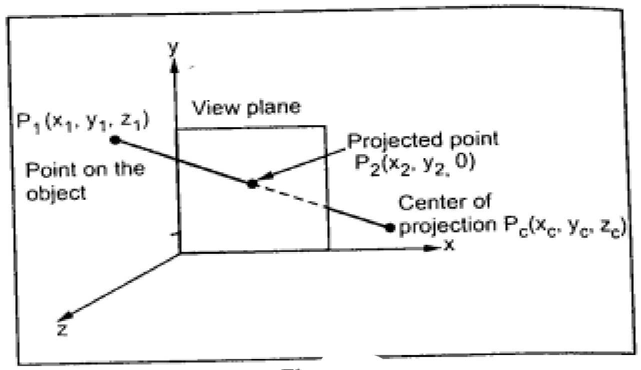


Fig.

$$x_2 = x_c + (x_1 - x_c)\, u$$

$$y_2 = y_c + (y_1 - y_c)\, u$$

$$z_2 = z_c + (z_1 - z_c)\, u$$

For projected point z_2 is 0, therefore, the third equation can be written as

$$0 = z_c + (z_1 - z_c)\, u$$

$$\therefore \quad u = -\frac{z_c}{z_1 - z_c}$$

Substituting the value of u in first two equations we get,

$$x_2 = x_c - z_c \frac{x_1 - x_c}{z_1 - z_c}$$

$$= \frac{x_c z_1 - x_c z_c - x_1 z_c + x_c z_c}{z_1 - z_c}$$

$$= \frac{x_c z_1 - x_1 z_c}{z_1 - z_c} \quad \text{and}$$

$$y_2 = y_c - z_c \frac{y_1 - y_c}{z_1 - z_c}$$

$$= \frac{y_c z_1 - y_c z_c - y_1 z_c + y_c z_c}{z_1 - z_c}$$

$$= \frac{y_c z_1 - y_1 z_c}{z_1 - z_c}$$

The above equations can be represented in the homogeneous matrix form as given below :

$$[x_2 \quad y_2 \quad z_2 \quad 1] = [x_1 \quad y_1 \quad z_1 \quad 1] \begin{bmatrix} -z_c & 0 & 0 & 0 \\ 0 & -z_c & 0 & 0 \\ x_c & y_c & 0 & 1 \\ 0 & 0 & 0 & -z_c \end{bmatrix}$$

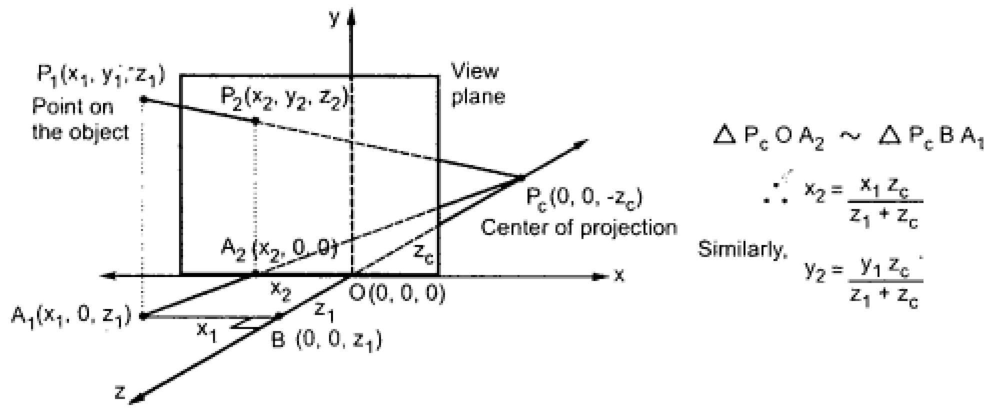
Here, we have taken the center of projection as $P_c(x_c, y_c, z_c)$. If we take the center of projection on the negative z-axis such that

$$x = 0$$

$$y = 0$$

$$z = -z_c$$

i.e. $\quad P_c(0, 0, -z_c)$ then we have



$$x_2 = \frac{z_c x_1}{z_c + z_1}$$

$$y_2 = \frac{z_c y_1}{z_c + z_1}$$

$$z_2 = 0$$

Thus we get the homogeneous perspective transformation matrix as

$$[x_2 \quad y_2 \quad z_2 \quad 1] = [x_1 \quad y_1 \quad z_1 \quad 1] \begin{bmatrix} z_c & 0 & 0 & 0 \\ 0 & z_c & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & z_c \end{bmatrix}$$

Bezier Curves

Bezier curve is an another approach for the construction of the curve. A Bezier curve is determined by a defining polygon. Bezier curves have a number of properties that make them highly useful and convenient for curve and surface design. They are also easy to implement. Therefore Bezier curves are widely available in various CAD systems and in general graphic packages. In this section we will discuss the cubic Bezier curve. The reason for choosing cubic Bezier curve is that they provide reasonable design flexibility and also avoid the large number of calculations.

Properties of Bezier curve

1. The basis functions are real.
2. Bezier curve always passes through the first and last control points i.e. curve has same end points as the guiding polygon.
3. The degree of the polynomial defining the curve segment is one less than the number of defining polygon point. Therefore, for 4 control points, the degree of the polynomial is three, i.e. cubic polynomial.
4. The curve generally follows the shape of the defining polygon.
5. The direction of the tangent vector at the end points is the same as that of the vector determined by first and last segments.
6. The curve lies entirely within the convex hull formed by four control points.
7. The convex hull property for a Bezier curve ensures that the polynomial smoothly follows the control points.
8. The curve exhibits the variation diminishing property. This means that the curve does not oscillate about any straight line more often than the defining polygon.
9. The curve is invariant under an affine transformation.

In cubic Bezier curve four control points are used to specify complete curve. Unlike the B-spline curve, we do not add intermediate points and smoothly extend Bezier curve, but we pick four more points and construct a second curve which can be attached to the first.

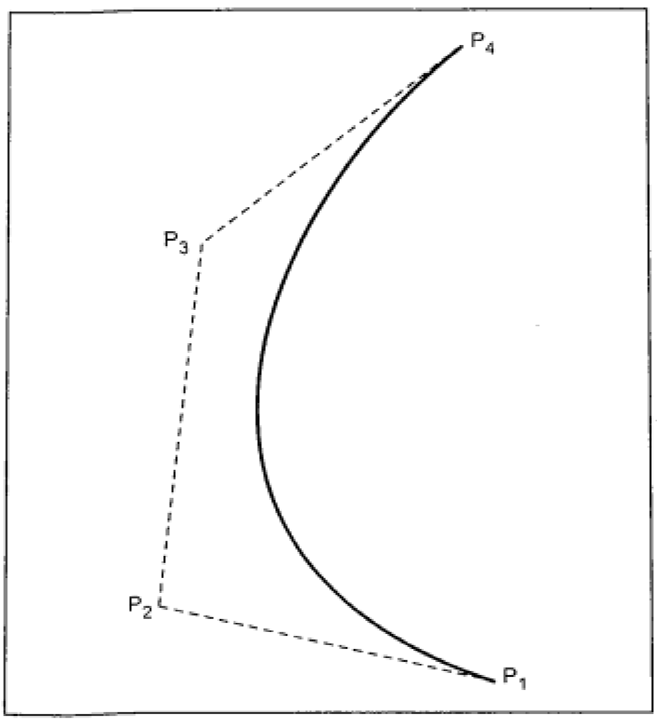


Fig. A cubic Bezier spline

The second curve can be attached to the first curve smoothly by selecting appropriate control points.

Fig. shows the Bezier curve and its four control points. As shown in the Fig. 9.6, Bezier curve begins at the first control point and ends at the fourth control point. This means that if we want to connect two Bezier curves, we have to make the first control point of the second Bezier curve match the last control point of the first curve. We can also observe that at the start of the curve, the curve is tangent to the line connecting first and second control points. Similarly at the end of curve, the curve is tangent to the line connecting the third and fourth control point. This means that, to join two Bezier curves smoothly we have to place the third and the fourth control points of the first curve on the same line specified by the first and the second control points of the second curve.

The Bezier matrix for periodic cubic polynomial is

$$M_B = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad P(u) = U \cdot M_B \cdot G_B$$

where $$G_B = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

and the product $P(u) = U.M_B.G_B$ is

$$P(u) = (1-u)^3 P_1 + 3u(1-u)^2 P_2 + 3u^2(1-u) P_3 + u^3 P_4$$

Ex. *Construct the Bezier curve of order 3 and with 4 polygon vertices A(1, 1), B(2, 3), C(4, 3) and D(6, 4).*

Sol. : The equation for the Bezier curve is given as

$$P(u) = (1-u)^3 P_1 + 3u(1-u)^2 P_2 + 3u^2(1-u) P_3 + u^3 P_4$$

for $0 \le u \le 1$

where $P(u)$ is the point on the curve P_1, P_2, P_3, P_4

Let us take $u = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$

$$\therefore \quad P(0) = P_1 = (1, 1)$$

$$\therefore \quad P\left(\frac{1}{4}\right) = \left(1-\frac{1}{4}\right)^3 P_1 + 3\frac{1}{4}\left(1-\frac{1}{4}\right)^2 P_2 + 3\left(\frac{1}{4}\right)^2\left(1-\frac{1}{4}\right)P_3 + \left(\frac{1}{4}\right)^3 P_4$$

$$= \frac{27}{64}(1,1) + \frac{27}{64}(2,3) + \frac{9}{64}(4,3) + \frac{1}{64}(6,4)$$

$$= \left[\frac{27}{64}\times 1 + \frac{27}{64}\times 2 + \frac{9}{64}\times 4 + \frac{1}{64}\times 6, \quad \frac{27}{64}\times 1 + \frac{27}{64}\times 3 + \frac{9}{64}\times 3 + \frac{1}{64}\times 4\right]$$

$$= \left[\frac{123}{64}, \frac{139}{64}\right]$$

$$= (1.9218, 2.1718)$$

$$\therefore \quad P\left(\frac{1}{2}\right) = \left(1-\frac{1}{2}\right)^3 P_1 + 3\frac{1}{2}\left(1-\frac{1}{2}\right)^2 P_2 + 3\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)P_3 + \left(\frac{1}{2}\right)^3 P_4$$

$$= \frac{1}{8}(1, 1) + \frac{3}{8}(2, 3) + \frac{3}{8}(4, 3) + \frac{1}{8}(6, 4)$$

$$= \left[\frac{1}{8}\times 1 + \frac{3}{8}\times 2 + \frac{3}{8}\times 4 + \frac{1}{8}\times 6, \ \frac{1}{8}\times 1 + \frac{3}{8}\times 3 + \frac{3}{8}\times 3 + \frac{1}{8}\times 4\right]$$

$$= \left[\frac{25}{8}, \frac{23}{8}\right]$$

$$= (3.125, \quad 2.875)$$

$$\therefore \quad P\left(\frac{3}{4}\right) = \left(1-\frac{3}{4}\right)^3 P_1 + 3\frac{3}{4}\left(1-\frac{3}{4}\right)^2 P_2 + 3\left(\frac{3}{4}\right)^2\left(1-\frac{3}{4}\right)P_3 + \left(\frac{3}{4}\right)^3 P_4$$

$$= \frac{1}{64}P_1 + \frac{9}{64}P_2 + \frac{27}{64}P_3 + \frac{27}{64}P_4$$

$$= \frac{1}{64}(1,1) + \frac{9}{64}(2,3) + \frac{27}{64}(4,3) + \frac{27}{64}(6,4)$$

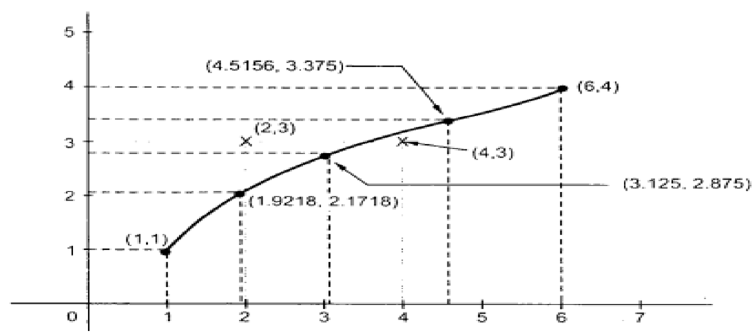
$$= \left[\frac{1}{64}\times 1 + \frac{9}{64}\times 2 + \frac{27}{64}\times 4 + \frac{27}{64}\times 6,\ \frac{1}{64}\times 1 + \frac{9}{64}\times 3 + \frac{27}{64}\times 3 + \frac{27}{64}\times 4\right]$$

$$= \left[\frac{289}{64}, \frac{217}{64}\right]$$

$$= (4.5156,\ 3.375)$$

$$P(1) = P_3 = (6, 4)$$

The Fig. shows the calculated points of the Bezier curve and curve passing through it



Mid-point approach to construct Bezier curve

Another approach to construct the Bezier curve is called midpoint approach. In this approach the Bezier curve can be constructed simply by taking midpoints. In midpoint approach midpoints of the lines connecting four control points (A, B, C, D) are determined (AB, BC, CD). These midpoints are connected by line segments and their midpoints ABC and BCD are determined. Finally these two midpoints are connected by line segments and its midpoint ABCD is determined. This is illustrated in Fig

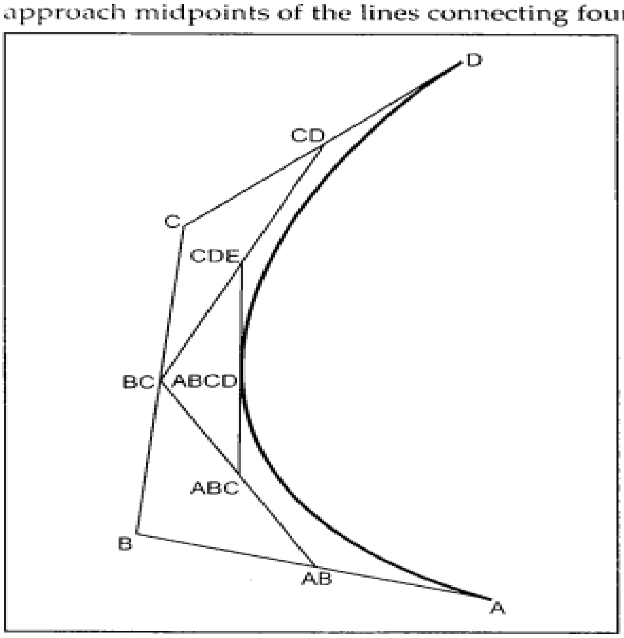


Fig. Subdivision of a Bezier spline

The point ABCD on the Bezier curve divides the original curve into two sections. This makes the points A, AB, ABC and ABCD are the control points for the first section and the points ABCD, BCD, CD and D are the control points for the second section. By considering two sections separately we can get two more sections for each separate section i.e. the original Bezier curve gets divided into four different curves. This process can be repeated to split the curve into smaller sections until we have sections so short that they can be replaced by straight lines or even until the sections are not bigger than individual pixels.

Algorithm

1. Get four control points say A (x_A, y_A), B (x_B, y_B), C (x_C, y_C), D (x_D, y_D)
2. Divide the curve represented by points A, B, C and D in two sections

   $x_{AB} = (x_A + x_B) / 2$

   $y_{AB} = (y_A + y_B) / 2$

   $x_{BC} = (x_B + x_C) / 2$

   $y_{BC} = (y_B + y_C) / 2$

   $x_{CD} = (x_C + x_D) / 2$

   $y_{CD} = (y_C + y_D) / 2$

   $x_{ABC} = (x_{AB} + x_{BC}) / 2$

   $y_{ABC} = (y_{AB} + y_{BC}) / 2$

   $x_{BCD} = (x_{BC} + x_{CD}) / 2$

   $y_{BCD} = (y_{BC} + y_{CD}) / 2$

   $x_{ABCD} = (x_{ABC} + x_{BCD}) / 2$

   $y_{ABCD} = (y_{ABC} + y_{BCD}) / 2$
3. Repeat the step 2 for section A, AB, ABC and ABCD and section ABCD, BCD, CD and D
4. Repeat step 3 until we have sections so short that they can be replaced by straight lines.
5. Replace small sections by straight lines.
6. Stop

B-Spline Curves

Properties of B-spline curve

- The sum of the B-spline basis functions for any parameter value u is 1.

  i.e. $\sum_{i=1}^{n+1} N_{i,k}(u) \equiv 1$
- Each basis function is positive or zero for all parameter values, i.e., $N_{i,k} \geq 0$.
- Except for k = 1 each basis function has precisely one maximum value.
- The maximum order of the curve is equal to the number of vertices of defining polygon.
- The degree of B-spline polynomial is independent on the number of vertices of defining polygon (with certain limitations).
- B-spline allows local control over the curve surface because each vertex affects the shape of a curve only over a range of parameter values where its associated basis function is nonzero.
- The curve exhibits the variation diminishing property. Thus the curve does not oscillate about any straight line move often than its defining polygon.
- The curve generally follows the shape of defining polygon.
- Any affine transformation can be applied to the curve by applying it to the vertices of defining polygon.
- The curve line within the convex hull of its defining polygon.