Lagrangian and Hamiltonian Formulations for Simulating a Double Compound Pendulum with Uniform Rods

Gatze :3

The kinetic and potential energies are given by

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{x_1^2} + \dot{y_1^2} + \dot{x_2^2} + \dot{y_2^2}) + \frac{1}{2}I(\dot{\varphi_1^2} + \dot{\varphi_2^2}) \\
U &= mg(y_1 + y_2)
\end{aligned}
$$

The center of mass for each pendelum is located at its center. Thus

$$
\begin{aligned}
\dot{x_1^2} + \dot{y_1^2} &= \frac{\dot{\varphi_1^2} l^2}{4}(\sin^2(\varphi_1) + \cos^2(\varphi_1)) \\
&= \frac{\dot{\varphi_1^2} l^2}{4} \\
\dot{x_2^2} + \dot{y_2^2} &= l^2((\dot{\varphi}_1 \cos(\varphi_1) + \frac{1}{2}\dot{\varphi}_2 \cos^2(\varphi_2)) + (\dot{\varphi}_1 \sin(\varphi_1) + \frac{1}{2}\dot{\varphi}_2 \sin^2(\varphi_2))) \\
&= l^2(\dot{\varphi_1^2} + \frac{1}{4}\dot{\varphi_2^2} + \dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1)\cos(\varphi_2) + \dot{\varphi}_1\dot{\varphi}_2 \sin(\varphi_1)\sin(\varphi_2)) \\
&= l^2(\dot{\varphi_1^2} + \frac{1}{4}\dot{\varphi_2^2} + \dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2))
\end{aligned}
$$

Plugging in the coordinates with respect to the angle as well as the moment of inertia for a thin rod rotated about its axis we get

$$
\begin{aligned}
T &= \frac{m}{2}\left(\frac{\dot{\varphi_1^2} l^2}{4} + l^2(\dot{\varphi_1^2} + \frac{1}{4}\dot{\varphi_2^2} + \dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2))\right) + \frac{1}{2}\frac{1}{12}ml^2(\dot{\varphi_1^2} + \dot{\varphi_2^2}) \\
U &= -mgl(\frac{1}{2}\cos(\varphi_1) + \cos(\varphi_1) + \frac{1}{2}\cos(\varphi_2))
\end{aligned}
$$

The Lagrangian L is given by

$$
\begin{aligned}
L &= T - U \\
&= \frac{ml^2}{2}\left(\frac{\dot{\varphi_1^2}}{4} + \dot{\varphi_1^2} + \frac{1}{4}\dot{\varphi_2^2} + \dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)\right) + \frac{1}{24}ml^2(\dot{\varphi_1^2} + \dot{\varphi_2^2}) + \frac{mgl}{2}(3\cos(\varphi_1) + \cos(\varphi_2)) \\
&= \frac{ml^2}{24}\left(15\dot{\varphi_1^2} + 3\dot{\varphi_2^2} + 12\dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)\right) + \frac{1}{24}ml^2(\dot{\varphi_1^2} + \dot{\varphi_2^2}) + \frac{mgl}{2}(3\cos(\varphi_1) + \cos(\varphi_2)) \\
&= \frac{ml^2}{6}\left(4\dot{\varphi_1^2} + \dot{\varphi_2^2} + 3\dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)\right) + \frac{mgl}{2}(3\cos(\varphi_1) + \cos(\varphi_2))
\end{aligned}
$$

Using the Euler-Lagrange equations we could solve for the Equations of Motion in the form of Lagrangian mechanics. But these we yield a second-order differential equation. Solving for the Equations of Motion in Hamilton mechanics is better for simulating, because you only get first-order differential equations, which can more be solved more accurately numerically. Therefore we perform a Legendre-Transform on the Lagrangian, which turns it into the Hamiltonian.
With the generalized momenta $p = \frac{\partial L}{\partial \dot{\varphi}}$ we obtain

$$H = p\dot{\varphi} - L$$

First we solve for the generalized momenta.

$$
\begin{aligned}
p_1 &= \frac{\partial L}{\partial \dot{\varphi}_1} \\
&= \frac{8ml^2}{6}\dot{\varphi}_1 + \frac{3}{6}\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \\
p_2 &= \frac{\partial L}{\partial \dot{\varphi}_2} \\
&= \frac{2ml^2}{6}\dot{\varphi}_2 + \frac{3}{6}\dot{\varphi}_1 \cos(\varphi_1 - \varphi_2)
\end{aligned}
$$

Because $H = T + U$ and L$=T - U$, $T = \frac{1}{2}p\dot{\varphi}$. Here these are 2-dimensional vectors.