Lagrangian and Hamiltonian Formulations for Simulating a Double Compound Pendulum with Uniform Rods

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This note analyzes a double compound pendulum consisting of two rods of equal length and uniform mass distribution. The kinetic and potential energies are given by

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}I(\dot{\varphi}_1^2 + \dot{\varphi}_2^2) \tag{1}$$

$$U = mg(y_1 + y_2) \tag{2}$$

The center of mass for each pendelum is located at its center. Thus

$$\begin{aligned}
\dot{x}_1^2 + \dot{y}_1^2 &= \frac{\dot{\varphi}_1^2 l^2}{4}(\sin^2(\varphi_1) + \cos^2(\varphi_1)) && (3)\\
&= \frac{\dot{\varphi}_1^2 l^2}{4} && (4)\\
\dot{x}_2^2 + \dot{y}_2^2 &= l^2((\dot{\varphi}_1\cos(\varphi_1) + \frac{1}{2}\dot{\varphi}_2\cos^2(\varphi_2)) + (\dot{\varphi}_1\sin(\varphi_1) + \frac{1}{2}\dot{\varphi}_2\sin^2(\varphi_2))) && (5)\\
&= l^2(\dot{\varphi}_1^2 + \frac{1}{4}\dot{\varphi}_2^2 + \dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1)\cos(\varphi_2) + \dot{\varphi}_1\dot{\varphi}_2\sin(\varphi_1)\sin(\varphi_2)) && (6)\\
&= l^2(\dot{\varphi}_1^2 + \frac{1}{4}\dot{\varphi}_2^2 + \dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2)) && (7)
\end{aligned}$$

Plugging in the coordinates with respect to the angle as well as the moment of inertia for a thin rod rotated about its axis we get

$$T = \frac{m}{2}\left(\frac{\dot{\varphi}_1^2 l^2}{4} + l^2(\dot{\varphi}_1^2 + \frac{1}{4}\dot{\varphi}_2^2 + \dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2))\right) + \frac{1}{2}\frac{1}{12}ml^2(\dot{\varphi}_1^2 + \dot{\varphi}_2^2) \tag{8}$$

$$U = -mgl(\frac{1}{2}\cos(\varphi_1) + \cos(\varphi_1) + \frac{1}{2}\cos(\varphi_2)) \tag{9}$$

The Lagrangian L is given by

$$\begin{aligned}
L &= T - U && (10)\\
&= \frac{ml^2}{2}\left(\frac{\dot{\varphi}_1^2}{4} + \dot{\varphi}_1^2 + \frac{1}{4}\dot{\varphi}_2^2 + \dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2)\right) + \frac{1}{24}ml^2(\dot{\varphi}_1^2 + \dot{\varphi}_2^2) + \frac{mgl}{2}(3\cos(\varphi_1) + \cos(\varphi_2)) && (11)\\
&= \frac{ml^2}{24}\left(15\dot{\varphi}_1^2 + 3\dot{\varphi}_2^2 + 12\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2)\right) + \frac{1}{24}ml^2(\dot{\varphi}_1^2 + \dot{\varphi}_2^2) + \frac{mgl}{2}(3\cos(\varphi_1) + \cos(\varphi_2)) && (12)\\
&= \frac{ml^2}{6}\left(4\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 3\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2)\right) + \frac{mgl}{2}(3\cos(\varphi_1) + \cos(\varphi_2)) && (13)
\end{aligned}$$

Using the Euler-Lagrange equations we could solve for the Equations of Motion in the form of Lagrangian mechanics. By this we would obtain two second-order differential equations. But solving for the Equations of Motion in Hamilton mechanics is better for simulating, because we only get first-order differential equations, which can more be solved more accurately numerically. Therefore we perform a Legendre-Transform on the Lagrangian, which turns it into the Hamiltonian.
With the generalized momenta $p = \frac{\partial L}{\partial \dot{\varphi}}$ we obtain

$$H = p\dot{\varphi} - L \tag{14}$$

where p and $\dot{\varphi}$ are 2-dimensional vectors. First we solve for the generalized momenta.

$$
\begin{aligned}
p_1 &= \frac{\partial L}{\partial \dot{\varphi}_1} && (15)\\
&= \frac{8ml^2}{6}\dot{\varphi}_1 + \frac{3ml^2}{6}\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) && (16)\\
p_2 &= \frac{\partial L}{\partial \dot{\varphi}_2} && (17)\\
&= \frac{2ml^2}{6}\dot{\varphi}_2 + \frac{3ml^2}{6}\dot{\varphi}_1 \cos(\varphi_1 - \varphi_2) && (18)
\end{aligned}
$$

With

$$
K = \begin{pmatrix} \frac{4ml^2}{3} & \frac{ml^2}{2}\cos(\varphi_1 - \varphi_2) \\ \frac{ml^2}{2}\cos(\varphi_1 - \varphi_2) & \frac{ml^2}{3} \end{pmatrix} \tag{19}
$$

this can be written as

$$
\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = K \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} \tag{20}
$$

Since

$$
\begin{aligned}
\det(K) &= m^2 l^4 (\frac{4}{9} - \frac{\cos^2(\varphi_1 - \varphi_2)}{4}) && (21)\\
&= m^2 l^4 (\frac{16 - 9\cos^2(\varphi_1 - \varphi_2)}{36}) && (22)\\
&\geq 0 && (23)
\end{aligned}
$$

K is invertible. The inverse matrix of a 2×2-matrix can be computed through

$$
\begin{aligned}
K^{-1} &= \frac{1}{m^2 l^4 (\frac{4}{9} - \frac{\cos^2(\varphi_1 - \varphi_2)}{4})} \begin{pmatrix} \frac{ml^2}{3} & -\frac{ml^2}{2}\cos(\varphi_1 - \varphi_2) \\ -\frac{ml^2}{2}\cos(\varphi_1 - \varphi_2) & \frac{4ml^2}{3} \end{pmatrix} && (24)\\
&= \frac{36}{ml^2(16 - 9\cos^2(\varphi_1 - \varphi_2))} \begin{pmatrix} \frac{1}{3} & -\frac{1}{2}\cos(\varphi_1 - \varphi_2) \\ -\frac{1}{2}\cos(\varphi_1 - \varphi_2) & \frac{4}{3} \end{pmatrix} && (25)
\end{aligned}
$$

Because $H = T + U$ and L=$T - U$, $T = \frac{1}{2}p\dot{\varphi}$. Therefore T can be written as

$$
T = \frac{1}{2} p K^{-1} p \tag{26}
$$

The Hamiltonian can now be written as

$$
\begin{aligned}
H &= \frac{1}{2} p K^{-1} p + U && (27)\\
&= \frac{1}{2} p K^{-1} p - \frac{mgl}{2}(3\cos(\varphi_1) + \cos(\varphi_2)) && (28)\\
&= \frac{18}{ml^2(16 - 9\cos^2(\varphi_1 - \varphi_2))}(\frac{1}{3}p_1^2 - \cos(\varphi_1 - \varphi_2)p_1 p_2 + \frac{4}{3}p_2^2) - \frac{mgl}{2}(3\cos(\varphi_1) + \cos(\varphi_2)) && (29)\\
&= \frac{6(p_1^2 - 3\cos(\varphi_1 - \varphi_2)p_1 p_2 + 4p_2^2)}{ml^2(16 - 9\cos^2(\varphi_1 - \varphi_2))} - \frac{mgl}{2}(3\cos(\varphi_1) + \cos(\varphi_2)) && (30)
\end{aligned}
$$

Now we can use the Hamilton equations to get the Equations of Motion as first-order differential equations. With $\Delta = \varphi_1 - \varphi_2$

$$
\begin{aligned}
\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} &= \frac{\partial H}{\partial p_1} && (31)\\
&= \frac{6(2p_1 - 3\cos(\Delta)p_2)}{ml^2(16 - 9\cos^2(\Delta))} && (32)\\
\frac{\mathrm{d}\varphi_2}{\mathrm{d}t} &= \frac{\partial H}{\partial p_2} && (33)\\
&= \frac{6(8p_2 - 3\cos(\Delta)p_1)}{ml^2(16 - 9\cos^2(\Delta))} && (34)\\
\frac{\mathrm{d}p_1}{\mathrm{d}t} &= -\frac{\partial H}{\partial \varphi_1} && (35)\\
&= -\frac{18(p_1p_2\sin(\Delta)(16 - 9\cos^2(\Delta)) - 6\sin(\Delta)\cos(\Delta)(p_1^2 + 4p_2^2 - 3p_1p_2\cos(\Delta)))}{ml^2(16 - 9\cos^2(\Delta))^2} - \frac{3mgl\sin(\varphi_1)}{2} && (36)\\
\frac{\mathrm{d}p_2}{\mathrm{d}t} &= -\frac{\partial H}{\partial \varphi_2} && (37)\\
&= \frac{18(p_1p_2\sin(\Delta)(16 - 9\cos^2(\Delta)) - 6\sin(\Delta)\cos(\Delta)(p_1^2 + 4p_2^2 - 3p_1p_2\cos(\Delta)))}{ml^2(16 - 9\cos^2(\Delta))^2} + \frac{mgl\sin(\varphi_2)}{2} && (38)
\end{aligned}
$$

Finally we can simulate a trajectorie of the system given initial conditions using Runge-Kutta 4 (RK4).

```python
import numpy as np
from math import *

import matplotlib.pyplot as plt

def rk4(f, h, v, param):
    k_1=f(v, param)
    k_2=f(v+h/2*k_1, param)
    k_3=f(v+h/2*k_2, param)
    k_4=f(v+h*k_3, param)
    return v+h/6*(k_1+2*k_2+2*k_3+k_4)

def double_pendulum(v, param=[1,1,9.81]):
    phi1, phi2, p1, p2 = v
    m, l, g = np.array(param)
    cos_delta = cos(phi1-phi2)
    sin_delta = sin(phi1-phi2)
    c = (16-9*cos_delta*cos_delta)
    a = 6/(m*l*l*c)
    b = 3*a/c*(p1*p2*sin_delta*c-6*sin_delta*cos_delta*(p1**2+4*p2**2-3*p1*p2*cos_delta))
    return np.array([
        a*(2*p1-3*cos_delta*p2),
        a*(8*p2-3*cos_delta*p1),
        -b-3*m*g*l*sin(phi1)/2,
        b+m*g*l*sin(phi2)/2
    ])

def generate_trajectorie_rk4(f, h, n, v, param):
    traj = [v]
    for x in range(n):
        v = rk4(f, h, v, param)
        traj.append(v)
    return np.array(traj)

traj = generate_trajectorie_rk4(double_pendulum, 0.01, 10000, [pi/4,pi/4,1,1], [1,1,9.81])
print(traj)
plt.plot(traj[:,1],traj[:,2])
plt.show()
```

ImplementationRK4.py

It's trivial to see, that the double pendulum stays bounded on the φ_1-φ_2-plane. This can be useful when analysing the phase-space of the system.